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THE ELEMENTARY THEORY OF LARGE FIELDS OF TOTALLY $\mathfrak{S}$-ADIC NUMBERS

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Abstract We analyze the elementary theory of certain fields $K^{\mathfrak{S}}(\boldsymbol{\sigma})$ of totally $\mathfrak{S}$-adic algebraic numbers that were introduced and studied by Geyer and Jarden and by Haran, Jarden, and Pop. In particular, we provide an axiomatization of these theories and prove their decidability, thereby giving a common generalization of classical decidability results of Jarden and Kiehne, Fried, Haran, and Völklein, and Ershov.

Keywords: totally S-adic numbers; absolute Galois group; model theory of profinite groups; decidability of fields

1. Introduction

Let $\mathfrak{S}$ be a finite set of absolute values on a number field K. By $K^{\mathfrak{S}}$ we denote the field of totally $\mathfrak{S}$-adic numbers – the maximal Galois extension of K in which the elements of $\mathfrak{S}$ are totally split. For an integer $e \geqslant 0$ and an e-tuple $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_e) \in \mathrm{Gal}(K)^e$ of elements of the absolute Galois group of K, we let $K^{\mathfrak{S}}(\boldsymbol{\sigma})$ be the fixed field of the group $\langle \sigma_1, \ldots, \sigma_e \rangle \leqslant \mathrm{Gal}(K)$ inside $K^{\mathfrak{S}}$.

These fields $K^{\mathfrak{S}}(\boldsymbol{\sigma})$ were studied by Jarden and Razon [25], Geyer and Jarden [18], and recently in a series of papers by Haran, Jarden, and Pop [20, 21]. In particular, these authors prove that, for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$, in the sense of Haar measure on the compact group $\mathrm{Gal}(K)^e$, the field $K^{\mathfrak{S}}(\boldsymbol{\sigma})$ satisfies a local–global principle for rational points on varieties, and its absolute Galois group has a nice description as a free product of local factors.

Combining these results, we are able to give an axiomatization of the theory $T_{\mathrm{almost},\mathfrak{S},e}$ of first-order sentences (in the language of rings with constants from K) that hold in almost all $K^{\mathfrak{S}}(\boldsymbol{\sigma})$ (Theorem 10.11), and we prove the decidability of this theory.

This work builds on Chapters 3 and 4 of the author's Ph.D. thesis [13] supervised by M. Jarden at Tel Aviv University. As such, it was supported by the European Commission under contract MRTN-CT-2006-035495.

Theorem 1.1. *Let $\mathfrak{S}$ be a finite set of absolute values on a number field K, and let $e \geqslant 0$. Then the first-order theory $T_{\mathrm{almost},\mathfrak{S},e}$ of almost all $K^{\mathfrak{S}}(\boldsymbol{\sigma})$, $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$, is decidable.*

This theorem is a common generalization of classical decidability results of Jarden and Kiehne [23] (the case $\mathfrak{S} = \emptyset$), Fried, Haran, and Völklein [15] (the case $K = \mathbb{Q}$, $e = 0$, and $\mathfrak{S}$ consisting only of the Archimedean absolute value), and Ershov [11] (the case $K = \mathbb{Q}$, $e = 0$, and $\mathfrak{S}$ consisting only of p-adic absolute values).

In fact, we prove a more general and stronger statement; see Theorem 11.12. However, although we can work for example with more general fields K, it seems difficult to allow also infinite sets of absolute values $\mathfrak{S}$; see the discussion in Remark 11.14.

The main part of the proof consists of an analysis of the absolute Galois group of $K^{\mathfrak{S}}(\boldsymbol{\sigma})$ together with local data, and the model theory of such structures.

2. Preliminaries on profinite groups and spaces

We assume that the reader is familiar with the basic theory of profinite groups, as presented in [16, Ch. 1] and [35, Ch. 2].

We always consider profinite groups as topological groups, so in particular homomorphisms between profinite groups are continuous group homomorphisms. By $H \leqslant G$ (respectively, $H \trianglelefteq G$) we indicate that H is a closed (respectively, normal closed) subgroup of G. If $X \subseteq G$, we denote by $\langle X \rangle$ the closed subgroup generated by X in G. We use the symbol 1 to denote both the unit element of G and the trivial subgroup $\{1\} \leqslant G$.

In the category of profinite groups, direct products, inverse limits, and fiber products exist [16, 22.2.1]. For the notion of the *rank* $\mathrm{rk}(G)$ of a profinite group G and the notion of a *free* profinite group, see [16, Ch. 17]. We denote by $\hat{F}_e$ the free profinite group of rank e.

Lemma 2.1. *Let $\pi: G \to H$ be an epimorphism of profinite groups. Let $e \geqslant 0$, let $N \trianglelefteq G$ be a closed normal subgroup with $\mathrm{rk}(G/N) \leqslant e$, and let $h_1, \ldots, h_e \in H$ such that $H = \langle h_1, \ldots, h_e, \pi(N) \rangle$. Then there exist $g_1, \ldots, g_e \in G$ such that $G = \langle g_1, \ldots, g_e, N \rangle$ and $\pi(g_i) = h_i$, $i = 1, \ldots, e$.*

Proof. Let $\bar{G} = G/N$, $\bar{H} = H/\pi(N)$, and let $\bar{\pi}: \bar{G} \to \bar{H}$ be the induced epimorphism. Then $\bar{H} = \langle \bar{h}_1, \ldots, \bar{h}_e \rangle$, so Gaschütz' lemma [16, 17.7.2] implies that there are $g_1, \ldots, g_e \in G$ such that $\bar{G} = \langle \bar{g}_1, \ldots, \bar{g}_e \rangle$ and $\bar{\pi}(\bar{g}_i) = \bar{h}_i$, $i = 1, \ldots, e$. So, $G = \langle g_1, \ldots, g_e, N \rangle$, and there are $n_1, \ldots, n_e \in N$ such that $\pi(g_i) = h_i \pi(n_i)$, $i = 1, \ldots, e$. Thus, setting $g_i' = g_i n_i^{-1}$, we have $G = \langle g_1', \ldots, g_e', N \rangle$ and $\pi(g_i') = h_i$, $i = 1, \ldots, e$. □

A *profinite space* is a totally disconnected compact Hausdorff space. Profinite spaces can be characterized as inverse limits of finite discrete spaces, or as zero-dimensional compact Hausdorff spaces [35, 1.1.12]. Any product and any finite coproduct (i.e., direct sum) of profinite spaces is a profinite space, and a subspace of a profinite space is profinite if and only if it is closed. Since profinite spaces are compact Hausdorff, any continuous map between profinite spaces is closed, and any continuous bijection of profinite spaces is a homeomorphism.

3. Group piles

The notion of group piles was introduced in [20] to enrich profinite groups with extra local data (see Remark 3.6 for some history concerning such structures). We recall this notion and extend it. Our main innovation is the introduction of a certain quotient $\bar{\mathbf{G}}$ that measures the failure of a deficient group pile $\mathbf{G}$ to be self-generated.

Fix a finite set $\mathfrak{S}$ not containing the symbol 0, and let $e \geqslant 0$.

Definition 3.1. Let $G = \varprojlim_N G/N$ be a profinite group, where N runs over all open normal subgroups of G. Then the set $\mathrm{Subgr}(G)$ of all closed subgroups of G is equipped with a profinite topology, induced by $\mathrm{Subgr}(G) = \varprojlim_N \mathrm{Subgr}(G/N)$. The group G acts continuously on $\mathrm{Subgr}(G)$ by conjugation. A homomorphism $\alpha\colon G \to H$ of profinite groups induces a map $\mathrm{Subgr}(\alpha)\colon \mathrm{Subgr}(G) \to \mathrm{Subgr}(H)$ given by $\Gamma \mapsto \alpha(\Gamma)$.

Lemma 3.2. *The map* Subgr *is a covariant functor from the category of profinite groups (with homomorphisms) to the category of profinite spaces (with continuous maps).*

Proof. It is easy to check that, if $\alpha\colon G \to H$ is a homomorphism of profinite groups, then the induced map $\mathrm{Subgr}(G) \to \mathrm{Subgr}(H)$ is continuous. □

Lemma 3.3. *If H is a closed subgroup of a profinite group G, then* $\mathrm{Subgr}(H)$ *is a closed subspace of* $\mathrm{Subgr}(G)$.

Proof. By Lemma 3.2, the inclusion $\iota\colon \mathrm{Subgr}(H) \to \mathrm{Subgr}(G)$ is continuous. Since both spaces are compact Hausdorff, ι is closed, and is thus a topological embedding. □

Definition 3.4. A *group pile* is a structure $\mathbf{G} = (G, \mathcal{G}_0, \mathcal{G}_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}}$ consisting of

(1) a profinite group G,

(2) a nonempty G-invariant closed subset $\mathcal{G}_0 \subseteq \mathrm{Subgr}(G)$ such that the elements of $\mathcal{G}_0$ are pairwise conjugate in G, and

(3) a G-invariant closed subset $\mathcal{G}_\mathfrak{p} \subseteq \mathrm{Subgr}(G)$ for each $\mathfrak{p} \in \mathfrak{S}$.

The *order* of $\mathbf{G}$ is the order of G. The *rank* $\mathrm{rk}(\mathbf{G})$ of $\mathbf{G}$ is the rank $\mathrm{rk}(G)$ of G. A *finite* group pile is a group pile of finite order. Let $\mathcal{G} = \bigcup_{\mathfrak{p}\in\mathfrak{S}} \mathcal{G}_\mathfrak{p}$. We call $\mathbf{G}$ *self-generated* if there exists $G_0 \in \mathcal{G}_0$ such that $G = \langle G_0, \mathcal{G}\rangle$, i.e., G is generated by G_0 and the groups in $\mathcal{G}_\mathfrak{p}$, $\mathfrak{p} \in \mathfrak{S}$. It is called *bare* if $\mathcal{G} = \{1\}$, and *deficient* if $\mathcal{G}_0 = \{1\}$. The *deficient reduct* of $\mathbf{G}$ is $\mathbf{G}^\circ = (G, \{1\}, \mathcal{G}_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}}$. Instead of $(G, \{1\}, \mathcal{G}_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}}$, we also write $(G, \mathcal{G}_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}}$. We call $\mathbf{G}$ *separated* if the sets $\mathcal{G}_\mathfrak{p}$, $\mathfrak{p} \in \{0\} \cup \mathfrak{S}$, are disjoint, and *reduced* if there are no nontrivial inclusions among the elements of $\mathcal{G}$.

Remark 3.5. Note that the notion of a group pile depends on the fixed set of primes $\mathfrak{S}$. Also note that, if $\mathbf{G}$ is self-generated, then $G = \langle G_0, \mathcal{G}\rangle$ for *any* $G_0 \in \mathcal{G}_0$. Condition (2) says that $\mathcal{G}_0$ consists of a single G-orbit in $\mathrm{Subgr}(G)$; i.e., there exists $G_0 \in \mathcal{G}_0$ such that $\mathcal{G}_0 = (G_0)^G := \{(G_0)^g : g \in G\}$. Hence, our notion of group piles coincides with the group piles of [20], except for a small difference in notation concerning $\mathcal{G}_0$.

Remark 3.6. The idea of using structures similar to group piles has a long history, going back for example to the *Artin–Schreier structures* in [19], the *involutory structures* in [15], or the Δ^*-*groups* in [9], [10], and [12].

Definition 3.7. A *homomorphism* of group piles

$$f: (G, \mathcal{G}_0, \mathcal{G}_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}} \to (H, \mathcal{H}_0, \mathcal{H}_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}}$$

is a homomorphism of profinite groups $f: G \to H$ such that $f(\mathcal{G}_\mathfrak{p}) \subseteq \mathcal{H}_\mathfrak{p}$ for each $\mathfrak{p} \in \{0\} \cup \mathfrak{S}$. It is an *epimorphism* if $f: G \to H$ is surjective and $f(\mathcal{G}_\mathfrak{p}) = \mathcal{H}_\mathfrak{p}$ for each $\mathfrak{p} \in \{0\} \cup \mathfrak{S}$. It is an *isomorphism* if in addition $f: G \to H$ is an isomorphism. The homomorphism f is called *rigid* if $f|_\Gamma$ is injective for each $\Gamma \in \mathcal{G}$. If N is a closed normal subgroup of G, define the *quotient* $\mathbf{G}/N = (G/N, \mathcal{G}_{0,N}, \mathcal{G}_{\mathfrak{p},N})_{\mathfrak{p}\in\mathfrak{S}}$ by $\mathcal{G}_{\mathfrak{p},N} = \{\Gamma N/N : \Gamma \in \mathcal{G}_\mathfrak{p}\} \subseteq \mathrm{Subgr}(G/N)$. This is again a group pile, the quotient map $G \to G/N$ extends to an epimorphism of group piles $\mathbf{G} \to \mathbf{G}/N$, and every epimorphism of group piles is of this form.

Remark 3.8. We identify the category of bare deficient group piles (with homomorphisms) with the category of profinite groups (with homomorphisms) via the forgetful functor $(G, \mathcal{G}_0, \mathcal{G}_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}} \mapsto G$.

Lemma 3.9. *In the category of group piles with epimorphisms, inverse limits exist.*

Proof. For a directed set I and an inverse family $\mathbf{G}_i = (G_i, \mathcal{G}_{i,0}, \mathcal{G}_{i,\mathfrak{p}})_{\mathfrak{p}\in\mathfrak{S}}$, $i \in I$, of group piles, $\mathbf{G} = (G, \mathcal{G}_0, \mathcal{G}_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}}$ with $G := \varprojlim_{i\in I} G_i$ and $\mathcal{G}_\mathfrak{p} := \varprojlim_{i\in I} \mathcal{G}_{i,\mathfrak{p}} \subseteq \mathrm{Subgr}(G)$, $\mathfrak{p} \in \mathfrak{S} \cup \{0\}$, is an inverse limit. □

Definition 3.10. For a group pile $\mathbf{G} = (G, \mathcal{G}_0, \mathcal{G}_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}}$, let $G' := \langle \mathcal{G} \rangle$ be the closed subgroup generated by the subgroups in $\mathcal{G}_\mathfrak{p}$, $\mathfrak{p} \in \mathfrak{S}$. Let $\mathbf{G}' := (G', \mathcal{G}_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}}$ and $\bar{\mathbf{G}} := \mathbf{G}/G'$. We say that $\mathbf{G}$ is *e-generated* if $\mathrm{rk}(\bar{\mathbf{G}}) \leqslant e$, and *e-bounded* if $\mathbf{G}$ is self-generated and $\mathrm{rk}(G_0) \leqslant e$ for all $G_0 \in \mathcal{G}_0$.

Lemma 3.11. $\mathbf{G}'$ *is a self-generated and deficient group pile, and* $\bar{\mathbf{G}}$ *is a bare group pile.*

Proof. By Lemma 3.3, $\mathbf{G}'$ is a group pile. The other claims are obvious. □

Lemma 3.12. *If* $\varphi: \mathbf{G} \to \mathbf{H}$ *is an epimorphism of group piles, then* $\varphi(G') = H'$, *so* φ *induces epimorphisms* $\varphi': \mathbf{G}' \to \mathbf{H}'$ *and* $\bar{\varphi}: \bar{\mathbf{G}} \to \bar{\mathbf{H}}$.

Proof. By the definition of an epimorphism of group piles, $\varphi(\mathcal{G}) = \mathcal{H}$. Hence, since φ is continuous and closed, $\varphi(\langle \mathcal{G} \rangle) = \langle \mathcal{H} \rangle$, as claimed. □

Lemma 3.13. *The map* $\mathbf{G} \mapsto \mathbf{G}'$ *(respectively,* $\mathbf{G} \mapsto \bar{\mathbf{G}}$*) is a covariant functor from the category of group piles with epimorphisms to the category of self-generated deficient group piles with epimorphisms (respectively, the category of bare group piles with epimorphisms).*

Proof. This follows from Lemma 3.12. □

Lemma 3.14. *Let* $\mathbf{G} = (G, \mathcal{G}_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}}$ *be a deficient group pile and* $\mathbf{A} = A$ *a bare deficient group pile. Then the map* $\varphi \mapsto \bar{\varphi}$ *gives a bijection between the epimorphisms from* $\mathbf{G}$ *to* $\mathbf{A}$ *and the epimorphisms from* $\bar{G}$ *to* A*.*

Proof. If $\varphi\colon \mathbf{G} \to \mathbf{A}$ is an epimorphism, then $\bar{\varphi}\colon \bar{\mathbf{G}} \to \bar{\mathbf{A}} = \mathbf{A}$ is an epimorphism. Conversely, given an epimorphism $\bar{\varphi}\colon \bar{G} \to A$, the composition $\bar{\varphi} \circ \pi\colon \mathbf{G} \to \mathbf{A}$, where $\pi\colon \mathbf{G} \to \bar{\mathbf{G}}$ is the quotient map, is an epimorphism. These two operations are inverse to each other. □

Remark 3.15. Note that a deficient group pile is self-generated if and only if it is 0-generated. Every e-bounded group pile is e-generated.

Lemma 3.16. *Let* $\varphi\colon \mathbf{G} \to \mathbf{H}$ *be an epimorphism of group piles. If* $\mathbf{G}$ *is* e*-generated, then* $\mathbf{H}$ *is* e*-generated. If* $\mathbf{G}$ *is* e*-bounded, then* $\mathbf{H}$ *is* e*-bounded.*

Proof. The induced map $\bar{\varphi}\colon \bar{\mathbf{G}} \to \bar{\mathbf{H}}$ is an epimorphism by Lemma 3.13, so $\mathrm{rk}(\bar{H}) \leqslant \mathrm{rk}(\bar{G})$. If $\mathbf{G}$ is self-generated, then $\mathbf{H}$ is also self-generated. Since $\varphi(\mathcal{G}_0) = \mathcal{H}_0$, for $H_0 \in \mathcal{H}_0$ there exists $G_0 \in \mathcal{G}_0$ with $\varphi(G_0) = H_0$, and thus $\mathrm{rk}(H_0) \leqslant \mathrm{rk}(G_0)$. □

Proposition 3.17. *A deficient group pile* $\mathbf{G}$ *is* e*-generated if and only if every finite quotient of* $\mathbf{G}$ *is* e*-generated.*

Proof. If $\mathbf{G}$ is e-generated, then every finite quotient of $\mathbf{G}$ is e-generated by Lemma 3.16. Conversely, suppose that $\mathbf{G}$ is not e-generated. Then there is an epimorphism $\bar{G} \to A$ onto a finite group A with $\mathrm{rk}(A) > e$ (see, for example, [35, 2.5.3]), so A is a finite quotient of $\mathbf{G}$ (Lemma 3.14) which is not e-generated. □

Lemma 3.18. *Let* $\mathbf{A}$ *be an* e*-bounded group pile, and let* $\tilde{\mathbf{B}} = (B, \mathcal{B}_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}}$ *be an* e*-generated deficient group pile. For every epimorphism* $\pi\colon \tilde{\mathbf{B}} \to \mathbf{A}^\circ$ *there exists an* e*-bounded self-generated group pile* $\mathbf{B}$ *with* $\mathbf{B}^\circ = \tilde{\mathbf{B}}$ *such that* $\pi\colon \mathbf{B} \to \mathbf{A}$ *is an epimorphism.*

Proof. Let $A_0 \in \mathcal{A}_0$, and choose $a_1, \dots, a_e \in A$ with $A_0 = \langle a_1, \dots, a_e \rangle$. By Lemma 3.12, $A = \langle a_1, \dots, a_e, A' \rangle$ and $A' = \pi(\tilde{B}')$, so Lemma 2.1 gives $b_1, \dots, b_e \in B$ with $B = \langle b_1, \dots, b_e, \tilde{B}' \rangle$ and $\pi(b_i) = a_i$. Let $B_0 = \langle b_1, \dots, b_e \rangle$ and $\mathbf{B} = (B, (B_0)^B, \mathcal{B}_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}}$. Then $\mathbf{B}$ is e-bounded, and $\pi\colon \mathbf{B} \to \mathbf{A}$ is an epimorphism. □

4. Embedding problems

We recall the notion of embedding problems for group piles from [20, §4], and rephrase some results in terms of e-bounded group piles.

Definition 4.1. Let $\mathbf{G}$ be a group pile. An *embedding problem* for $\mathbf{G}$ is a pair

$$EP = (\varphi\colon \mathbf{G} \to \mathbf{A},\ \alpha\colon \mathbf{B} \to \mathbf{A})$$

of epimorphisms of group piles. It is called *finite*, *self-generated*, *e-generated*, *e-bounded*, *deficient*, or *bare*, if $\mathbf{B}$ has this property. It is called *rigid* if α is rigid. A *solution* of the embedding problem (φ, α) is an epimorphism $\gamma\colon \mathbf{G} \to \mathbf{B}$ such that $\alpha \circ \gamma = \varphi$. The embedding problem *EP* is *locally solvable* if, writing $\mathbf{G} = (G, \mathcal{G}_0, \mathcal{G}_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}}$ and $\mathbf{B} = (B, \mathcal{B}_0, \mathcal{B}_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}}$, the following holds for every $\mathfrak{p} \in \{0\} \cup \mathfrak{S}$.

For every $\Gamma \in \mathcal{G}_\mathfrak{p}$ there is a $\Delta \in \mathcal{B}_\mathfrak{p}$, and for every $\Delta \in \mathcal{B}_\mathfrak{p}$ there is a $\Gamma \in \mathcal{G}_\mathfrak{p}$, such that there exists an epimorphism $\gamma_\Gamma\colon \Gamma \to \Delta$ with $\alpha \circ \gamma_\Gamma = \varphi|_\Gamma$. $\quad (LS)$

Lemma 4.2. *If there exist $G_0 \in \mathcal{G}_0$ and $B_0 \in \mathcal{B}_0$ and an epimorphism $\gamma_0\colon G_0 \to B_0$ with $\alpha \circ \gamma_0 = \varphi|_{G_0}$, then (LS) holds for $\mathfrak{p} = 0$.*

Proof. If $g \in G$ and $\Gamma = (G_0)^g \in \mathcal{G}_0$, choose $b \in B$ with $\alpha(b) = \varphi(g)$, and let $\Delta = (B_0)^b$. If $b \in B$ and $\Delta = (B_0)^b \in \mathcal{B}_0$, choose $g \in G$ with $\varphi(g) = \alpha(b)$, and let $\Gamma = (G_0)^g$. Define $\gamma_\Gamma\colon \Gamma \to \Delta$ by $\gamma_\Gamma(x) = \gamma_0(x^{g^{-1}})^b$. Then $\alpha(\gamma_\Gamma(x)) = \varphi(x^{g^{-1}})^{\varphi(g)} = \varphi(x)$ for all $x \in \Gamma$. □

Lemma 4.3. *Every rigid embedding problem satisfies (LS) for every $\mathfrak{p} \in \mathfrak{S}$.*

Proof. Suppose that *EP* is rigid. If $\Gamma \in \mathcal{G}_\mathfrak{p}$, choose $\Delta \in \mathcal{B}_\mathfrak{p}$ with $\alpha(\Delta) = \varphi(\Gamma)$. If $\Delta \in \mathcal{B}_\mathfrak{p}$, choose $\Gamma \in \mathcal{G}_\mathfrak{p}$ with $\varphi(\Gamma) = \alpha(\Delta)$. Since α is rigid, $\gamma_\Gamma = (\alpha|_\Delta)^{-1} \circ \varphi|_\Gamma$ maps Γ onto Δ and satisfies $\alpha \circ \gamma_\Gamma = \varphi|_\Gamma$. □

Proposition 4.4. *Every rigid deficient embedding problem is locally solvable.*

Proof. Suppose that *EP* is rigid and deficient. By Lemma 4.3, (LS) holds for $\mathfrak{p} \in \mathfrak{S}$. Since $\mathbf{B}$ is deficient, so is $\mathbf{A}$; hence if $G_0 \in \mathcal{G}_0$, then $\varphi(G_0) = 1$. Thus, (LS) is satisfied for $\mathfrak{p} = 0$. □

Definition 4.5. Let $\varphi\colon \mathbf{G} \to \mathbf{A}$ and $\alpha\colon \mathbf{B} \to \mathbf{A}$ be homomorphisms of deficient group piles. Define the (asymmetric) *rigid product* of B and G over A as $\mathbf{B} \times^{\mathrm{rig}}_{\mathbf{A}} \mathbf{G} = (H, \mathcal{H}_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}}$, where $H = B \times_A G$ is the fiber product and

$$\mathcal{H}_\mathfrak{p} = \{\Gamma \in \mathrm{Subgr}(H)\colon \beta(\Gamma) \in \mathcal{G}_\mathfrak{p},\ \pi(\Gamma) \in \mathcal{B}_\mathfrak{p},\ \beta|_\Gamma \text{ is injective}\},$$

with $\pi : H \to B$ and $\beta : H \to G$ the natural projection maps.

Lemma 4.6. *Let $EP = (\varphi\colon \mathbf{G} \to \mathbf{A},\ \alpha\colon \mathbf{B} \to \mathbf{A})$ be a locally solvable embedding problem of finite deficient group piles. Then $\mathbf{B} \times^{\mathrm{rig}}_{\mathbf{A}} \mathbf{G}$ is a deficient group pile, $\pi : \mathbf{B} \times^{\mathrm{rig}}_{\mathbf{A}} \mathbf{G} \to \mathbf{B}$ is an epimorphism, and $\beta : \mathbf{B} \times^{\mathrm{rig}}_{\mathbf{A}} \mathbf{G} \to \mathbf{G}$ is a rigid epimorphism.*

Proof. Since $\mathcal{G}_\mathfrak{p}$ is G-invariant and $\mathcal{B}_\mathfrak{p}$ is B-invariant, $\mathcal{H}_\mathfrak{p}$ is H-invariant. Since H is finite, $\mathcal{H}_\mathfrak{p}$ is closed. The projections β and π are surjective and, by the definition of $\mathcal{H}_\mathfrak{p}$, homomorphisms of group piles. The definition of $\mathcal{H}_\mathfrak{p}$ also gives that β is rigid. Given $G_1 \in \mathcal{G}_\mathfrak{p}$, there is $B_1 \in \mathcal{B}_\mathfrak{p}$ and there is an epimorphism $\gamma\colon G_1 \to B_1$ with $\alpha \circ \gamma = \varphi|_{G_1}$. It defines a homomorphism $\hat{\gamma}\colon G_1 \to H$ with $\beta \circ \hat{\gamma} = \mathrm{id}_{G_1}$ and $\pi \circ \hat{\gamma} = \gamma$. Let $H_1 = \hat{\gamma}(G_1)$.

Then $\beta(H_1) = G_1 \in \mathcal{G}_\mathfrak{p}$ and $\pi(H_1) = \gamma(G_1) = B_1 \in \mathcal{B}_\mathfrak{p}$. Furthermore, since $\beta \circ \hat{\gamma} = \mathrm{id}_{G_1}$, $\beta|_{H_1}$ is injective, so $H_1 \in \mathcal{H}_\mathfrak{p}$. Similarly, given $B_1 \in \mathcal{B}_\mathfrak{p}$, there is $H_1 \in \mathcal{H}_\mathfrak{p}$ with $\pi(H_1) = B_1$. Therefore, β and π are epimorphisms of group piles. □

Remark 4.7. The rigid product can be seen as a canonical version of [20, Lemma-Construction 4.2]. One could define it as a subgroup pile of the fiber product in the category of deficient group piles, which always exists.

Lemma 4.8. *Let* $(\varphi: \mathbf{G} \to \mathbf{A},\ \alpha: \mathbf{B} \to \mathbf{A})$ *be a locally solvable embedding problem. Then, for every normal subgroup* $N \trianglelefteq B$, *the induced embedding problem* $(\mathbf{G} \to \mathbf{A}/\alpha(N), \mathbf{B}/N \to \mathbf{A}/\alpha(N))$ *is also locally solvable.*

Proof. Let $\tilde{\mathbf{B}} = (\tilde{B}, \tilde{\mathcal{B}}_0, \tilde{\mathcal{B}}_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}} = \mathbf{B}/N$ and $\tilde{\mathbf{A}} = (\tilde{A}, \tilde{\mathcal{A}}_0, \tilde{\mathcal{A}}_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}} = \mathbf{A}/\alpha(N)$, and let $\pi: \mathbf{B} \to \tilde{\mathbf{B}}$, $\tilde{\pi}: \mathbf{A} \to \tilde{\mathbf{A}}$ be the quotient maps and $\tilde{\alpha}: \tilde{\mathbf{B}} \to \tilde{\mathbf{A}}$ the induced epimorphism. Then $\tilde{\pi} \circ \alpha = \tilde{\alpha} \circ \pi$. We have to prove that the embedding problem $(\tilde{\pi} \circ \varphi, \tilde{\alpha})$ is locally solvable.

Let $\mathfrak{p} \in \{0\} \cup \mathfrak{S}$, and let $\Gamma \in \mathcal{G}_\mathfrak{p}$ be given. Then there is a $\Delta \in \mathcal{B}_\mathfrak{p}$ and there is an epimorphism $\gamma_\Gamma: \Gamma \to \Delta$ with $\alpha \circ \gamma_\Gamma = \varphi|_\Gamma$. Let $\Lambda = \pi(\Delta) \in \tilde{\mathcal{B}}_\mathfrak{p}$. Then $\pi \circ \gamma_\Gamma: \Gamma \to \Lambda$ is an epimorphism with $\tilde{\alpha} \circ (\pi \circ \gamma_\Gamma) = \tilde{\pi} \circ \alpha \circ \gamma_\Gamma = (\tilde{\pi} \circ \varphi)|_\Gamma$.

Conversely, let $\Lambda \in \tilde{\mathcal{B}}_\mathfrak{p}$ be given. Choose $\Delta \in \mathcal{B}_\mathfrak{p}$ with $\pi(\Delta) = \Lambda$. Then there is a $\Gamma \in \mathcal{G}_\mathfrak{p}$ and there is an epimorphism $\gamma_\Gamma: \Gamma \to \Delta$ with $\alpha \circ \gamma_\Gamma = \varphi|_\Gamma$. Hence, $\pi \circ \gamma_\Gamma: \Gamma \to \Lambda$ is an epimorphism with $\tilde{\alpha} \circ (\pi \circ \gamma_\Gamma) = \tilde{\pi} \circ \alpha \circ \gamma_\Gamma = (\tilde{\pi} \circ \varphi)|_\Gamma$. □

Lemma 4.9. *Let* $(\varphi: \mathbf{G} \to \mathbf{A},\ \alpha: \mathbf{B} \to \mathbf{A})$ *be a locally solvable finite embedding problem. Then there exists an open normal subgroup* $N \trianglelefteq G$ *with* $N \leqslant \mathrm{Ker}(\varphi)$ *such that the induced embedding problem* $(\mathbf{G}/N \to \mathbf{A}, \mathbf{B} \to \mathbf{A})$ *is locally solvable.*

Proof. This is a special case of [20, Lemma 4.1]. □

The following proposition is closely related to [20, Lemma 4.3], which we need to reprove because we have to take e-boundedness into consideration.

Proposition 4.10. *Let* $\mathbf{G}$ *be an* e*-bounded group pile and let* $(\varphi: \mathbf{G} \to \mathbf{A},\ \alpha: \mathbf{B} \to \mathbf{A})$ *be a locally solvable* e*-bounded finite embedding problem for* $\mathbf{G}$*. Then it can be dominated by a rigid* e*-bounded finite embedding problem; i.e., there exist epimorphisms* $\hat{\alpha}: \hat{\mathbf{B}} \to \hat{\mathbf{A}}$, $\hat{\varphi}: \mathbf{G} \to \hat{\mathbf{A}}$, $\hat{\beta}: \hat{\mathbf{A}} \to \mathbf{A}$, *and* $\beta: \hat{\mathbf{B}} \to \mathbf{B}$ *such that* $\varphi = \hat{\beta} \circ \hat{\varphi}$ *and* $\hat{\beta} \circ \hat{\alpha} = \alpha \circ \beta$, *and* $(\hat{\varphi}, \hat{\alpha})$ *is a rigid* e*-bounded finite embedding problem.*

Proof. By Lemma 4.9, there is a finite group pile $\hat{\mathbf{A}}$ and there are epimorphisms $\hat{\varphi}: \mathbf{G} \to \hat{\mathbf{A}}$, $\hat{\beta}: \hat{\mathbf{A}} \to \mathbf{A}$ with $\varphi = \hat{\beta} \circ \hat{\varphi}$ such that $(\hat{\beta}, \alpha)$ is a locally solvable embedding problem. Since $\mathbf{G}$ is e-bounded, $\hat{\mathbf{A}}$ is also e-bounded (Lemma 3.16).

Let $\tilde{\mathbf{B}} = \mathbf{B}^\circ \times^{\mathrm{rig}}_{\mathbf{A}^\circ} \hat{\mathbf{A}}^\circ$ be the rigid product, and let $\tilde{\alpha}: \tilde{\mathbf{B}} \to \hat{\mathbf{A}}^\circ$ and $\tilde{\beta}: \tilde{\mathbf{B}} \to \mathbf{B}^\circ$ be the projections. By Lemma 4.6, $\tilde{\alpha}$ is a rigid epimorphism and $\tilde{\beta}$ is an epimorphism. Choose $\hat{A}_0 \in \hat{\mathcal{A}}_0$ and $B_0 \in \mathcal{B}_0$ and an epimorphism $\gamma_0: \hat{A}_0 \to B_0$ with $\alpha \circ \gamma_0 = \hat{\beta}|_{\hat{A}_0}$. Then γ_0 defines a homomorphism $\hat{\gamma}_0: \hat{A}_0 \to \tilde{B}$ with $\tilde{\alpha} \circ \hat{\gamma}_0 = \mathrm{id}_{\hat{A}_0}$ and $\tilde{\beta} \circ \hat{\gamma}_0 = \gamma_0$. Let $\tilde{B}_0 = \hat{\gamma}_0(\hat{A}_0)$, and note that $\tilde{\alpha}(\tilde{B}_0) = \hat{A}_0$ and $\tilde{\beta}(\tilde{B}_0) = B_0$.

We have that $\mathrm{rk}(\tilde{B}_0) \leqslant \mathrm{rk}(\hat{A}_0) \leqslant e$, and, since $\hat{\mathbf{A}}$ and $\mathbf{B}$ are self-generated, $\hat{A} = \langle \hat{A}_0, \hat{A}' \rangle$ and $B = \langle B_0, B' \rangle$. Let $\hat{B} = \langle \tilde{B}_0, \tilde{B}' \rangle \leqslant \tilde{B}$ and $\hat{\mathcal{B}}_0 = (\tilde{\mathcal{B}}_0)^{\hat{B}}$. Then $\hat{\mathbf{B}} = (\hat{B}, \hat{\mathcal{B}}_0, \tilde{\mathcal{B}}_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}}$ is a self-generated group pile, and $\tilde{\alpha}(\hat{B}) = \langle \hat{A}_0, \hat{A}' \rangle = \hat{A}$ and $\tilde{\beta}(\hat{B}) = \langle B_0, B' \rangle = B$ by Lemma 3.12. Since $\hat{\mathcal{A}}_0 = (\hat{A}_0)^{\hat{A}}$ and $\mathcal{B}_0 = (B_0)^B$, $\tilde{\alpha}(\hat{\mathcal{B}}_0) = \hat{\mathcal{A}}_0$ and $\tilde{\beta}(\hat{\mathcal{B}}_0) = \mathcal{B}_0$, so $\tilde{\alpha}|_{\hat{B}}$ and $\tilde{\beta}|_{\hat{B}}$ are epimorphisms of group piles. Therefore, with $\hat{\alpha} = \tilde{\alpha}|_{\hat{B}}$ and $\beta = \tilde{\beta}|_{\hat{B}}$, $(\hat{\varphi}, \hat{\alpha})$ is a rigid e-bounded finite embedding problem which dominates (φ, α). □

5. Model theory of group piles

This section extends the comodel theory of profinite groups in [6] (see also [5]) to group piles. A similar construction can be found in [8]. We will give full definitions, but focus our proofs on the necessary extensions to the classical theory. For more on the comodel theory of profinite groups, see [2–4] or [17].

Definition 5.1. The *colanguage* $\mathcal{L}_{\mathrm{co},\mathfrak{S}} = \{\leqslant, \sqsubseteq, P, (G_n)_{n\in\mathbb{N}}, (\mathcal{G}_{\mathfrak{p},n})_{\mathfrak{p}\in\mathfrak{S},n\in\mathbb{N}}\}$ consists of unary relation symbols $G_n (n \in \mathbb{N})$, binary relation symbols $\leqslant$ and $\sqsubseteq$, a ternary relation symbol P, and n-ary relation symbols $\mathcal{G}_{\mathfrak{p},n}$ ($\mathfrak{p} \in \mathfrak{S}$, $n \in \mathbb{N}$).

Definition 5.2. To a deficient group pile $\mathbf{G} = (G, \mathcal{G}_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}}$ assign an $\mathcal{L}_{\mathrm{co},\mathfrak{S}}$-structure $S(\mathbf{G}) = (S^\mathbf{G}, \leqslant^\mathbf{G}, \sqsubseteq^\mathbf{G}, P^\mathbf{G}, (G_n^\mathbf{G})_{n\in\mathbb{N}}, (\mathcal{G}_{\mathfrak{p},n}^\mathbf{G})_{\mathfrak{p}\in\mathfrak{S},n\in\mathbb{N}})$ as follows.

(1) $S^\mathbf{G} = \dot{\bigcup}_N G/N$, where N runs over all open normal subgroups of G.
(2) $x_1N_1 \leqslant^\mathbf{G} x_2N_2$ if and only if $N_1 \subseteq N_2$.
(3) $x_1N_1 \sqsubseteq^\mathbf{G} x_2N_2$ if and only if $x_1N_1 \subseteq x_2N_2$.
(4) $(x_1N_1, x_2N_2, x_3N_3) \in P^\mathbf{G}$ if and only if $N_1 = N_2 = N_3$ and $x_1x_2N_1 = x_3N_1$.
(5) $xN \in G_n^\mathbf{G}$ if and only if $(G : N) \leqslant n$.
(6) $(x_1N_1, \ldots, x_nN_n) \in \mathcal{G}_{\mathfrak{p},n}^\mathbf{G}$ if and only if $N_1 = \cdots = N_n \in G_n^\mathbf{G}$ and there is $\Gamma \in \mathcal{G}_\mathfrak{p}$ such that $\Gamma N_1/N_1 = \{x_1N_1, \ldots, x_nN_n\}$.

Definition 5.3. An $\mathcal{L}_{\mathrm{co},\mathfrak{S}}$-structure $\mathbf{S} = (S, \leqslant, \sqsubseteq, P, (G_n)_{n\in\mathbb{N}}, (\mathcal{G}_{\mathfrak{p},n})_{\mathfrak{p}\in\mathfrak{S},n\in\mathbb{N}})$ is an *inverse system (of group piles)* if the following statements hold.

(1) $\leqslant$ is a pre-order with a unique largest element.
(2) If $\sim$ denotes the equivalence relation defined by $\leqslant$, and $[x]$ denotes the equivalence class of $x \in S$ with respect to $\sim$, then the induced partial order on $\{[x] : x \in S\} = S/{\sim}$ is directed downwards.
(3) $G_n = \{x \in S : |[x]| \leqslant n\}$.
(4) $(x, y, z) \in P$ implies that $[x] = [y] = [z]$, and, for each $x \in S$, $P \cap [x]^3$ is the graph of a binary operation making $[x]$ into a group.
(5) If $(x_1, \ldots, x_n) \in \mathcal{G}_{\mathfrak{p},n}$, then $x_1, \ldots, x_n \in G_n$ and $[x_1] = \cdots = [x_n]$. If moreover $y_1, \ldots, y_n \in G_n$ and $\{y_1, \ldots, y_n\} = \{x_1, \ldots, x_n\}$, then $(y_1, \ldots, y_n) \in \mathcal{G}_{\mathfrak{p},n}$. If $x \in G_n$, then, with
$$\mathcal{G}_{\mathfrak{p},x} = \{\{x_1, \ldots, x_n\} \subseteq [x] : (x_1, \ldots, x_n) \in \mathcal{G}_{\mathfrak{p},n}\},$$
$[\![x]\!] = ([x], \mathcal{G}_{\mathfrak{p},x})_{\mathfrak{p}\in\mathfrak{S}}$ is a (finite) deficient group pile.

(6) $x \sqsubseteq y$ implies that $x \leqslant y$, and, for each $x, y \in S$ with $x \leqslant y$, $\sqsubseteq$ defines an epimorphism of group piles $\pi_{x,y}$ from $[\![x]\!]$ to $[\![y]\!]$, depending only on $[x]$ and $[y]$.

(7) For all $x \leqslant y \leqslant z$, $\pi_{x,x} = \mathrm{id}_{[\![x]\!]}$ and $\pi_{x,z} = \pi_{y,z} \circ \pi_{x,y}$.

(8) If N is a normal subgroup of $[x]$, then there is a unique $[y]$ such that $x \leqslant y$ and $N = \ker(\pi_{x,y})$.

(9) $S = \bigcup_{n\in\mathbb{N}} G_n$.

Remark 5.4. Note that, for $\mathfrak{S} = \emptyset$, (5) is vacant, and the remaining axioms are exactly the ones used in [6]. Also note that (1)–(8) are $\mathcal{L}_{\mathrm{co},\mathfrak{S}}$-elementary statements, but (9) is not.

Lemma 5.5. *If* $\mathbf{G}$ *is a deficient group pile, then* $S(\mathbf{G})$ *is an inverse system.*

Proof. This can be checked directly from the definitions. □

Definition 5.6. If $\varphi : \mathbf{G} \to \mathbf{H}$ is an epimorphism of deficient group piles, $\mathbf{G} = (G, \mathcal{G}_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}}$, $\mathbf{H} = (H, \mathcal{H}_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}}$, define a map φ^* of $S(\mathbf{H})$ into $S(\mathbf{G})$ by

$$\varphi^*(hN) = g\varphi^{-1}(N) \in G/\varphi^{-1}(N),$$

where $N \trianglelefteq H$ is open, $h \in H$, and $g \in G$ satisfies $\varphi(g) = h$.

Lemma 5.7. *The map* φ^* *is an embedding of* $\mathcal{L}_{\mathrm{co},\mathfrak{S}}$*-structures.*

Proof. First of all note that, since φ is a surjective homomorphism, φ^* is injective and it preserves the relations $\leqslant$, $\sqsubseteq$, P, and $(G_n)_{n\in\mathbb{N}}$.

Let $h_1N_1, \ldots, h_nN_n \in S^{\mathbf{H}}$ and $\mathfrak{p} \in \mathfrak{S}$. Then $(h_1N_1, \ldots, h_nN_n) \in \mathcal{G}^{\mathbf{H}}_{\mathfrak{p},n}$ if and only if $N_1 = \cdots = N_n \in G^{\mathbf{H}}_n$ and there is $\Delta \in \mathcal{H}_\mathfrak{p}$ such that $\Delta N_1/N_1 = \{h_1N_1, \ldots, h_nN_n\}$. Since $\varphi(\mathcal{G}_\mathfrak{p}) = \mathcal{H}_\mathfrak{p}$, this is the case if and only if $\varphi^*(N_1) = \cdots = \varphi^*(N_n) \in G^{\mathbf{G}}_n$ and there is $\Gamma \in \mathcal{G}_\mathfrak{p}$ such that $\Gamma\varphi^{-1}(N_1)/\varphi^{-1}(N_1) = \{\varphi^*(h_1N_1), \ldots, \varphi^*(h_nN_n)\}$. This is equivalent to $(\varphi^*(h_1N_1), \ldots, \varphi^*(h_nN_n)) \in \mathcal{G}^{\mathbf{G}}_{\mathfrak{p},n}$, so φ^* also preserves the relations $(\mathcal{G}_{\mathfrak{p},n})_{\mathfrak{p}\in\mathfrak{S},n\in\mathbb{N}}$. □

Definition 5.8. We assign to each inverse system $\mathbf{S} = (S, \leqslant, \sqsubseteq, P, (G_n)_{n\in\mathbb{N}}, (\mathcal{G}_{\mathfrak{p},n})_{\mathfrak{p}\in\mathfrak{S},n\in\mathbb{N}})$ a deficient group pile $G(\mathbf{S})$ as follows. By axioms (1), (2), and (4), the $[x]$ are a family of groups, which by (3) and (9) are all finite. By (5), the $[\![x]\!]$ are finite groups piles. By axioms (6) and (7), the maps $\pi_{x,y}$ turn these group piles into an inverse system in the category of group piles with epimorphisms. Let $G(\mathbf{S}) = (G^{\mathbf{S}}, \mathcal{G}^{\mathbf{S}}_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}} := \varprojlim_{x\in S/\sim} [\![x]\!]$ be the inverse limit; cf. Lemma 3.9.

Lemma 5.9. *The maps* S *and* G *are quasi-inverse to each other; i.e., for every deficient group pile* $\mathbf{G}$ *there is a natural isomorphism* $G(S(\mathbf{G})) \cong G$*, and for every inverse system* $\mathbf{S}$ *there is a natural isomorphism* $S(G(\mathbf{S})) \cong \mathbf{S}$.

Proof. Let $\mathbf{G} = (G, \mathcal{G}_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}}$ be a deficient group pile. Since $\mathcal{G}_\mathfrak{p}$ is closed, $\mathcal{G}_\mathfrak{p} = \varprojlim_N \mathcal{G}_{\mathfrak{p},N}$, where N runs over all open normal subgroups of G. Therefore $G(S(\mathbf{G})) \cong \mathbf{G}$.

Conversely, let $\mathbf{S}$ be an inverse system. Given $x \in S$, let N_x be the kernel of the natural projection $G^{\mathbf{S}} \to [\![x]\!]$. Define $\psi : \mathbf{S} \to S(G(\mathbf{S}))$ to send x to its image under the natural isomorphism $[\![x]\!] \cong G^{\mathbf{S}}/N_x \subseteq S^{G(\mathbf{S})}$. Then ψ is injective and it preserves the relations $\leqslant$, $\sqsubseteq$, P, G_n, and $\mathcal{G}_{\mathfrak{p},n}$. By axiom (5.3), ψ is surjective. Thus ψ gives an isomorphism $\mathbf{S} \cong S(G(\mathbf{S}))$. □

Remark 5.10. Given the preceding lemma, we will sometimes identify an inverse system $\mathbf{S}$ with $S(G(\mathbf{S}))$. In particular, we will treat elements of $\mathbf{S}$ as cosets xN of open normal subgroups of the group pile $G(\mathbf{S})$.

Definition 5.11. If $\psi : \mathbf{T} \to \mathbf{S}$ is an embedding of inverse systems, define an epimorphism of profinite groups $\psi^* : G^{\mathbf{S}} \to G^{\mathbf{T}}$ as follows.

Since ψ preserves the relation $\leqslant$, it also preserves the relation $\sim$, i.e., $\psi([x]) \leqslant [\psi(x)]$. Since ψ preserves the relations G_n ($n \in \mathbb{N}$), ψ gives, for every $x \in T$, a bijection between $[x]$ and $[\psi(x)]$. Since ψ preserves the relation P, this bijection is an isomorphism of groups. Since ψ preserves the relation $\sqsubseteq$, the inverse system of finite groups $([\psi(x)])_{x\in T/\sim}$ with homomorphisms $\pi_{\psi(x),\psi(y)}$ is isomorphic to the inverse system of finite groups $([x])_{x\in T/\sim}$ with homomorphisms $\pi_{x,y}$. This gives a natural epimorphism

$$\psi^* : G^{\mathbf{S}} = \varprojlim_{y\in S/\sim} [x] \to \varprojlim_{x\in T/\sim} [\psi(x)] \cong \varprojlim_{x\in T/\sim} [x] = G^{\mathbf{T}}.$$

Lemma 5.12. *The map ψ^* induces an epimorphism $\psi^* : G(\mathbf{S}) \to G(\mathbf{T})$ of group piles.*

Proof. Let $G(\mathbf{S}) = (G, \mathcal{G}_{\mathfrak{p}})_{\mathfrak{p}\in\mathfrak{S}}$, $G(\mathbf{T}) = (H, \mathcal{H}_{\mathfrak{p}})_{\mathfrak{p}\in\mathfrak{S}}$, $\mathfrak{p} \in \mathfrak{S}$ and $\Gamma \in \mathcal{G}_{\mathfrak{p}}$. For an open normal subgroup M of H one may check that $\psi(\psi^*(\Gamma)M/M) = \Gamma N/N \in \mathcal{G}_{\mathfrak{p},N}$, where $\psi(H/M) = G/N$. Since ψ preserves the relations $\mathcal{G}_{\mathfrak{p},n}$, this implies that $\psi^*(\Gamma)M/M \in \mathcal{H}_{\mathfrak{p},M}$. So $\psi^*(\Gamma) \in \varprojlim_N \mathcal{H}_{\mathfrak{p},N} = \mathcal{H}_{\mathfrak{p}}$.

Conversely, let $\Gamma \in \mathcal{H}_{\mathfrak{p}}$ be given. The family $\psi(\Gamma M/M) \in \mathcal{G}_{\mathfrak{p},N}$, $M \trianglelefteq H$ open and $\psi(H/M) = G/N$, is compatible with respect to the $\pi_{x,y}$; hence there exists $\Delta \in \mathcal{G}_{\mathfrak{p}}$ with $\psi(\Gamma M/M) = \Delta N/N$ for all $M \trianglelefteq H$ open, where $\psi(H/M) = G/N$, and thus $\psi^*(\Delta) = \Gamma$. Thus $\psi^*(\mathcal{G}_{\mathfrak{p}}) = \mathcal{H}_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathfrak{S}$, so ψ^* is an epimorphism of group piles. □

Proposition 5.13. *The map S defines an equivalence of categories between the category of deficient group piles (with epimorphisms) and the category of inverse systems (with embeddings), with quasi-inverse G.*

Proof. Note that S and G are in fact functorial. Now apply the previous lemmas. □

Definition 5.14. A *coformula* is an $\mathcal{L}_{\mathrm{co},\mathfrak{S}}$-formula that is bounded; i.e., all quantifiers are of the form $(\exists v \in G_n)$. A *cosentence* is a coformula without free variables.

A group pile $\mathbf{G}$ *cosatisfies* a set Σ of coformulas with free variables V (or is a *comodel* of Σ) if there are elements $x_v \in S(\mathbf{G})$, $v \in V$, such that $S(\mathbf{G}) \models \varphi(x_v, v \in V)$ for all $\varphi \in \Sigma$. A *cotheory* T is a set of cosentences.

The *cocardinality* of a group pile $\mathbf{G}$ is the cardinality of $S(\mathbf{G})$. A set Σ of coformulas with parameters in some inverse system $\mathbf{S}$ is *ranked* if, for every variable v that occurs in some formula $\varphi \in \Sigma$, also $G_n(v) \in \Sigma$ for some $n \in \mathbb{N}$. A group pile $\mathbf{G}$ is κ-*cosaturated*, for a cardinal number κ, if every ranked set Σ of coformulas with parameters in $S(\mathbf{G})$ with $|\Sigma| < \kappa$ is cosatisfied in $\mathbf{G}$, provided that every finite subset of Σ is cosatisfied in $\mathbf{G}$.

Remark 5.15. An inverse system $\mathbf{S}$ in the language $\mathcal{L}_{\mathrm{co},\mathfrak{S}}$ can also be viewed as an ω-sorted structure $\mathbf{S}^\omega$ in a language $\mathcal{L}^\omega_{\mathrm{co},\mathfrak{S}}$, where the nth sort consists of the $s \in G_n \smallsetminus G_{n-1}$, and for each k-ary relation relation R in $\mathcal{L}_{\mathrm{co},\mathfrak{S}}$ we have an ω^k-family of k-ary relations $R^{n_1,\dots,n_k}$ in $\mathcal{L}^\omega_{\mathrm{co},\mathfrak{S}}$; cf. [3, §1]. Clearly, every $\mathcal{L}_{\mathrm{co},\mathfrak{S}}$-formula can be translated to a corresponding $\mathcal{L}^\omega_{\mathrm{co},\mathfrak{S}}$-formula, and vice versa.

6. e-free C-piles

In this section, we generalize the *Cantor group piles* of [20]. Recall that for a group pile $\mathbf{G}$ we defined the deficient reduct $\mathbf{G}^\circ$, the subgroup $G' = \langle \mathcal{G} \rangle$ of G, and the quotient $\bar{G} = G/G'$; see §3. Moreover, recall that $\hat{F}_e$ denotes the free profinite group of rank e; cf. §2.

Definition 6.1. An e-*free C-pile* is an e-generated deficient group pile $\mathbf{G}$ for which every rigid e-generated deficient finite embedding problem is solvable (cf. Definition 3.10).

Lemma 6.2. *If* $\mathbf{G}$ *is an* e*-free C-pile, then* $\bar{G} \cong \hat{F}_e$.

Proof. If B is a finite group with $\mathrm{rk}(B) \leqslant e$, then $(\mathbf{G} \to 1, B \to 1)$ is a rigid e-generated deficient finite embedding problem for $\mathbf{G}$, so it has a solution by assumption. Therefore, by Lemma 3.14, every finite group B with $\mathrm{rk}(B) \leqslant e$ is a quotient of $\bar{G}$. Since $\mathrm{rk}(\bar{G}) \leqslant e$, this implies that $\bar{G} \cong \hat{F}_e$; cf. [16, 16.10.7]. □

Lemma 6.3. *Let* $\mathbf{G}$ *be an* e*-bounded group pile, and let* $(\varphi\colon \mathbf{G}^\circ \to \tilde{\mathbf{A}},\ \alpha\colon \tilde{\mathbf{B}} \to \tilde{\mathbf{A}})$ *be a locally solvable* e*-generated deficient finite embedding problem. If* $G_0 \cong \hat{F}_e$ *for* $G_0 \in \mathcal{G}_0$*, then there exist* $\mathbf{A}$ *and* $\mathbf{B}$ *with* $\mathbf{A}^\circ = \tilde{\mathbf{A}}$ *and* $\mathbf{B}^\circ = \tilde{\mathbf{B}}$ *such that* $(\varphi\colon \mathbf{G} \to \mathbf{A},\ \alpha\colon \mathbf{B} \to \mathbf{A})$ *is a locally solvable* e*-bounded finite embedding problem.*

Proof. Let $G_0 \in \mathcal{G}_0$ and $A_0 = \varphi(G_0)$. Then $G = \langle G_0, G' \rangle$ implies that $\tilde{A} = \langle A_0, \tilde{A}' \rangle$ (Lemma 3.12), so $\mathbf{A} = (\tilde{A}, (A_0)^{\tilde{A}}, \tilde{\mathcal{A}}_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}}$ is e-bounded. By Lemma 3.18, there exists an e-bounded group pile $\mathbf{B} = (\tilde{B}, (B_0)^{\tilde{B}}, \tilde{\mathcal{B}}_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}}$ such that $\alpha\colon \mathbf{B} \to \mathbf{A}$ is an epimorphism. Without loss of generality, assume that $\alpha(B_0) = A_0$. We claim that $EP = (\varphi\colon \mathbf{G} \to \mathbf{A},\ \alpha\colon \mathbf{B} \to \mathbf{A})$ is locally solvable. Clearly it satisfies (*LS*) for $\mathfrak{p} \in \mathfrak{S}$. Since $G_0 \cong \hat{F}_e$ and $\mathbf{B}$ is e-bounded, there exists an epimorphism $\gamma_0\colon G_0 \to B_0$ with $\alpha \circ \gamma_0 = \varphi|_{G_0}$, cf. [16, 17.7.3]. Thus, by Lemma 4.2, EP satisfies (*LS*) for $\mathfrak{p} = 0$. □

Proposition 6.4. *Every locally solvable* e*-generated deficient finite embedding problem for an* e*-free C-pile* $\mathbf{G}$ *is solvable.*

Proof. Let $EP = (\varphi: \mathbf{G} \to \tilde{\mathbf{A}},\ \alpha: \tilde{\mathbf{B}} \to \tilde{\mathbf{A}})$ be a locally solvable e-generated deficient finite embedding problem for $\mathbf{G}$. By Lemma 6.2, $\bar{G} \cong \hat{F}_e$. Let $G_0 \leqslant G$ be a subgroup of rank at most e that under the quotient map $G \to \bar{G}$ maps onto $\bar{G} \cong \hat{F}_e$. Since every finite group generated by e elements is a quotient of $\hat{F}_e$, it is also a quotient of G_0, and thus $G_0 \cong \hat{F}_e$; cf. [16, 16.10.7]. Moreover, $\mathbf{G}^* = (G, (G_0)^G, \mathcal{G}_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}}$ is e-bounded. By Lemma 6.3, there exist $\mathbf{A}$ and $\mathbf{B}$ with $\mathbf{A}^\circ = \tilde{\mathbf{A}}$ and $\mathbf{B}^\circ = \tilde{\mathbf{B}}$ such that $EP_1 = (\varphi: \mathbf{G}^* \to \mathbf{A},\ \alpha: \mathbf{B} \to \mathbf{A})$ is locally solvable and e-bounded. By Proposition 4.10, EP_1 can be dominated by a rigid e-bounded finite embedding problem EP_2. The deficient reduct of EP_2 is a rigid e-generated deficient finite embedding problem, and hence has a solution. It induces a solution of EP. □

Example 6.5. For each $\mathfrak{p} \in \mathfrak{S}$, let $\Gamma_\mathfrak{p}$ be a profinite group and $T_\mathfrak{p}$ a profinite space, and let $\Gamma_0 = \hat{F}_e$ be the free profinite group of rank e. Then [20, §5] constructs from this data a certain group pile $\mathbf{G}_T$, which we call the *e-free semi-constant group pile* of $(\Gamma_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}}$ over $(T_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}}$. We do not repeat this definition but rely on the properties of $\mathbf{G}_T$ proven in [20].

Lemma 6.6. *The e-free semi-constant group pile $\mathbf{G}$ of $(\Gamma_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}}$ over $(T_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}}$ is an e-bounded self-generated group pile.*

Proof. By [20, Proposition 5.3(c)], $\mathbf{G}$ is self-generated. By the construction, every $G_0 \in \mathcal{G}_0$ is isomorphic to $\Gamma_0 = \hat{F}_e$; hence $\mathbf{G}$ is e-bounded. □

Proposition 6.7. *Let $\mathbf{G}$ be an e-free semi-constant group pile of nontrivial profinite groups $(\Gamma_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}}$ over perfect profinite spaces $(T_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}}$. Then the deficient reduct $\mathbf{G}^\circ$ of $\mathbf{G}$ is an e-free C-pile.*

Proof. By Lemma 6.6, $\mathbf{G}$ is e-bounded, so $\mathbf{G}^\circ$ is e-generated. Let $EP = (\varphi: \mathbf{G}^\circ \to \tilde{\mathbf{A}}, \alpha: \tilde{\mathbf{B}} \to \tilde{\mathbf{A}})$ be a rigid e-generated deficient finite embedding problem for $\mathbf{G}^\circ$. By Lemma 4.4, EP is locally solvable. By Lemma 6.3, there exist $\mathbf{A}$ and $\mathbf{B}$ with $\mathbf{A}^\circ = \tilde{\mathbf{A}}$ and $\mathbf{B}^\circ = \tilde{\mathbf{B}}$ such that $(\varphi: \mathbf{G} \to \mathbf{A},\ \alpha: \mathbf{B} \to \mathbf{A})$ is a locally solvable e-bounded (and hence self-generated) embedding problem. By [20, Proposition 5.3(h)], this embedding problem has a solution, which in turn induces a solution of EP. □

Definition 6.8. Let the cotheory $T^{\mathrm{co}}_{\mathrm{C},\mathfrak{S},e}$ consist of the following.

(1) For $n \in \mathbb{N}$, a cosentence about a group pile $\mathbf{G} = (G, \mathcal{G}_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}}$ stating that, for each $N \trianglelefteq G$ with $(G : N) \leqslant n$, the finite quotient $\mathbf{G}/N$ is e-generated.

(2) For $n, k \in \mathbb{N}$, a cosentence about a group pile $\mathbf{G} = (G, \mathcal{G}_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}}$ stating that, for every $N \trianglelefteq G$ with $(G : N) \leqslant n$ and every rigid epimorphism $\alpha: \mathbf{B} \to \mathbf{G}/N$ with $\mathbf{B}$ an e-generated deficient group pile of order k, there is an $M \trianglelefteq G$ with $(G : M) \leqslant k$ and $M \leqslant N$ and an isomorphism $\beta : \mathbf{G}/M \to \mathbf{B}$ such that $\alpha \circ \beta$ is the natural map $\mathbf{G}/M \to \mathbf{G}/N$.

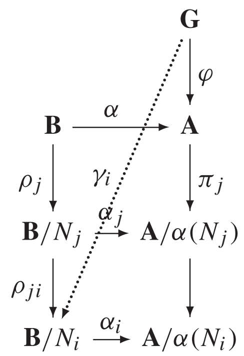
(Such sentences can easily be written down, since for every n and k there exist, up to isomorphism, only finitely many group piles $\mathbf{A}$ and $\mathbf{B}$ of order at most n (respectively, k) and only finitely many rigid epimorphisms $\alpha : \mathbf{B} \to \mathbf{A}$.)

Proposition 6.9. *A deficient group pile* **G** *is an e-free C-pile if and only if it cosatisfies* $T^{\mathrm{co}}_{\mathrm{C},\mathfrak{S},e}$.

Proof. By Proposition 3.17, **G** cosatisfies (1) if and only if **G** is e-generated. And (2) just says that all rigid e-generated deficient finite embedding problems for **G** are solvable. □

Proposition 6.10. *Let* **G** *be an* $\aleph_1$*-cosaturated e-free C-pile. Then every rigid e-generated deficient embedding problem* $(\varphi\colon \mathbf{G} \to \mathbf{A}, \alpha\colon \mathbf{B} \to \mathbf{A})$ *with* $\mathrm{rk}(\mathbf{B}) \leqslant \aleph_0$ *is solvable.*

Proof. Since $\mathrm{rk}(B) \leqslant \aleph_0$, there is a descending sequence of open normal subgroups $N_i \trianglelefteq B$, $i \in \mathbb{N}$, with $\bigcap_{i\in\mathbb{N}} N_i = 1$; cf. [16, 17.1.7(a)]. For each $i \in \mathbb{N}$, let $\alpha_i\colon \mathbf{B}/N_i \to \mathbf{A}/\alpha(N_i)$ be the epimorphism induced by α, and, for $i \leqslant j \in \mathbb{N}$, let $\pi_i\colon \mathbf{A} \to \mathbf{A}/\alpha(N_i)$, $\rho_i\colon \mathbf{B} \to \mathbf{B}/N_i$, and $\rho_{ji}\colon \mathbf{B}/N_j \to \mathbf{B}/N_i$ be the quotient maps. Then $\alpha_i \circ \rho_i = \pi_i \circ \alpha$.



By Lemma 4.4, the rigid deficient embedding problem (φ, α) is locally solvable; hence the induced embedding problem $(\pi_i \circ \varphi, \alpha_i)$ is locally solvable by Lemma 4.8. Since **B** is e-generated, $\mathbf{B}/N_i$ is e-generated by Lemma 3.16. Hence, $(\pi_i \circ \varphi, \alpha_i)$ is a locally solvable e-generated deficient finite embedding problem for **G**. Since **G** is an e-free C-pile, this embedding problem has a solution $\gamma_i\colon \mathbf{G} \to \mathbf{B}/N_i$ by Proposition 6.4.

For each i, fix an enumeration $\mathbf{B}/N_i = \{b_{i,1}, \dots, b_{i,n_i}\}$, and let $a_{i,\nu} = \alpha_i(b_{i,\nu}) \in \mathbf{A}/\alpha(N_i) \subseteq S(\mathbf{A})$. View $S(\mathbf{A})$ as a subset of $S(\mathbf{G})$ via φ^*, and let Σ be the following set of bounded $\mathcal{L}_{\mathrm{co},\mathfrak{S}}$-formulas in the variables $x_{i,\nu}$, $i \in \mathbb{N}$, $1 \leqslant \nu \leqslant n_i$, with constants from $S(\mathbf{A})$.

(1) For each i and each $1 \leqslant \nu \leqslant n_i$, the $\mathcal{L}_{\mathrm{co},\mathfrak{S}}$-formula $G_{n_i}(x_{i,\nu})$.

(2) For each i, an $\mathcal{L}_{\mathrm{co},\mathfrak{S}}$-formula stating that $[x_{i,1}] = \{x_{i,1}, \dots, x_{i,n_i}\}$ and that the map $x_{i,\nu} \mapsto b_{i,\nu}$ is an isomorphism of group piles $\beta_i : [\![x_{i,1}]\!] \to \mathbf{B}/N_i$.

(3) For each i, an $\mathcal{L}_{\mathrm{co},\mathfrak{S}}$-formula with constants from $S(\mathbf{A})$ stating that $x_{i,1} \leqslant a_{i,1}$ and $\pi_{x_{i,1},a_{i,1}}(x_{i,\nu}) = a_{i,\nu}$ for all $1 \leqslant \nu \leqslant n_i$.

(4) For each $i \leqslant j$, an $\mathcal{L}_{\mathrm{co},\mathfrak{S}}$-formula stating that $x_{j,1} \leqslant x_{i,1}$ and $\beta_i \circ \pi_{x_{j,1}x_{i,1}} = \rho_{ji} \circ \beta_j$.

Every finite subset Σ_0 of Σ is cosatisfied in **G**. Let j be the maximal index of a variable $x_{j,\nu}$ appearing in Σ, and, for $i \leqslant j$, $1 \leqslant \nu \leqslant n_i$, let $g_{i,\nu} = \gamma_j^*(\rho_{ji}^*(b_{i,\nu}))$. Then $(g_{i,\nu})_{1\leqslant i\leqslant j, 1\leqslant\nu\leqslant n_i}$ satisfies Σ_0. By (1), Σ is ranked.

Thus, since $\mathbf{G}$ is $\aleph_1$-cosaturated, there are $(\tilde{g}_{i,\nu})_{1\leqslant i,1\leqslant\nu\leqslant n_i}$ in $\mathbf{G}$ that satisfy Σ. For each i, the map $b_{i,\nu}\mapsto\tilde{g}_{i,\nu}$ gives an isomorphism $\psi_i\colon\mathbf{B}/N_i\to[\![\tilde{g}_{i,1}]\!]$ by (2), and hence an epimorphism $\tilde{\gamma}_i=\psi_i^*\colon\mathbf{G}\to\mathbf{B}/N_i$, which satisfies $\alpha_i\circ\tilde{\gamma}_i=\pi_i\circ\varphi$ by (3). By (4), these epimorphisms are compatible, giving rise to an epimorphism $\tilde{\gamma}=\varprojlim_i\tilde{\gamma}_i\colon\mathbf{G}\to\varprojlim_i\mathbf{B}/N_i=\mathbf{B}$, which then satisfies $\alpha\circ\tilde{\gamma}=\varphi$. □

7. Model theory of P$\mathfrak{S}$CC fields

In the next section, we will let the finite set $\mathfrak{S}$ be a set of *primes*, and associate to each field F a group pile $\mathbf{Gal}_{\mathfrak{S}}(F)$, which extends the absolute Galois group $\mathrm{Gal}(F)$ with $\mathfrak{S}$-local data. The notion of *prime* and the corresponding local–global principle P$\mathfrak{S}$CC we use is the one developed in [14]. We now briefly recall the main definitions and results, but refer to [14, §§2–3] for further details,[1] some history, and references to special cases of these results that were proven before. Basics on real closed and p-adically closed fields are summarized in Appendices A and B.

Definition 7.1. For a field F of characteristic zero, we denote by $\tilde{F}$ a fixed algebraic closure of F, and by $\mathrm{Gal}(F)=\mathrm{Gal}(\tilde{F}/F)$ the absolute Galois group of F. For a subfield E of F, we let $\tilde{E}$ be algebraic closure of E contained in $\tilde{F}$.

Definition 7.2. A *prime*[2] of a field K is either an ordering of K or an equivalence class of p-valuations on K, for some prime number p. It is *local* if the ordering is Archimedean (respectively, if the value group is isomorphic to $\mathbb{Z}$). If $\mathfrak{P}$ is a prime of K, we denote by $\mathrm{CC}(K,\mathfrak{P})$ (for *classical closures*) the set of all real (respectively, p-adic) closures of $(K,\mathfrak{P})$ inside $\tilde{K}$. If $\mathfrak{p}$ is a prime of K and F/K is a field extension, we denote by $\mathcal{S}_{\mathfrak{p}}(F)$ the set of all primes $\mathfrak{P}$ of F that lie above $\mathfrak{p}$ (we write this as $\mathfrak{P}|_K=\mathfrak{p}$) and are of the same type, and by $\mathrm{CC}_{\mathfrak{p}}(F)$ the union of all $\mathrm{CC}(F,\mathfrak{P})$, $\mathfrak{P}\in\mathcal{S}_{\mathfrak{p}}(F)$; cf. [14, Definitions 3.4, 3.7, 4.2].

Setting 7.3. *For the rest of this work, let $\mathfrak{S}$ be a finite set of local primes of a field K of characteristic zero, and let F/K be a field extension. For $\mathfrak{p}\in\mathfrak{S}$, fix a closure $K_{\mathfrak{p}}\in\mathrm{CC}_{\mathfrak{p}}(K)$.*

Lemma 7.4. *Let $K\subseteq E\subseteq F$, $\mathfrak{p}\in\mathfrak{S}$, $\mathfrak{Q}\in\mathcal{S}_{\mathfrak{p}}(F)$, and $\mathfrak{P}=\mathfrak{Q}|_E\in\mathcal{S}_{\mathfrak{p}}(E)$. If $F'\in\mathrm{CC}(F,\mathfrak{Q})$, then $E':=F'\cap\tilde{E}\in\mathrm{CC}(E,\mathfrak{P})$, and* $\mathrm{res}\colon\mathrm{Gal}(F')\to\mathrm{Gal}(E')$ *is an isomorphism. In particular, $F'\cap\tilde{E}\in\mathrm{CC}_{\mathfrak{p}}(E)$ for any $F'\in\mathrm{CC}_{\mathfrak{p}}(F)$.*

Proof. The field E' is algebraically closed in the real closed (respectively, p-adically closed) field F', so it is real closed (respectively, p-adically closed) itself; see Lemmas A.2 and B.1. Let $\mathfrak{Q}'$ be the unique prime of F' over $\mathfrak{Q}$. Then $\mathfrak{P}'=\mathfrak{Q}'|_{E'}$ is the unique prime

[1] The reader who wants to check these details should be aware of the fact that, in the notation of [14], here we consider only the case of relative type $\tau=(1,1)$, so for example the P$\mathfrak{S}$CC property is there called PS^τCC with $S=\mathfrak{S}$ and $\tau=(1,1)$.
[2] This is called a *classical prime* in [14].

of E' of the same type as $\mathfrak{p}$, so $E' \in \mathrm{CC}(E, \mathfrak{P}'|_E)$. Since $\mathfrak{P}'|_E = \mathfrak{Q}'|_E = \mathfrak{P}$, it follows that $E' \in \mathrm{CC}(E, \mathfrak{P})$.

Since $E' \equiv F'$ by model completeness (Propositions A.5 and B.3), and $\mathrm{Gal}(F')$ is finitely generated (Propositions A.3 and B.5), $\mathrm{Gal}(E') \cong \mathrm{Gal}(F')$ by [16, 20.4.6]. Thus the epimorphism $\mathrm{res}\colon \mathrm{Gal}(F') \to \mathrm{Gal}(E')$ is an isomorphism by [16, 16.10.8]. □

Definition 7.5. We say that F is **P$\mathfrak{S}$CC** (for *pseudo-$\mathfrak{S}$ classically closed*) if it satisfies the following local–global principle for any smooth absolutely irreducible F-variety V: $V(F) \neq \emptyset$ iff $V(F') \neq \emptyset$ for all $F' \in \mathrm{CC}_\mathfrak{p}(F)$, $\mathfrak{p} \in \mathfrak{S}$.

Definition 7.6. A prime $\mathfrak{P}$ is *quasi-local* if it is an ordering or a p-valuation with value group a $\mathbb{Z}$-group (cf. [14, Definition 3.5, Remark 3.6]), and F is $\mathfrak{S}$-*quasi-local* if all $\mathfrak{P} \in \mathcal{S}_\mathfrak{p}(F)$, $\mathfrak{p} \in \mathfrak{S}$, are quasi-local. The field F is $\mathfrak{S}$-*SAP* if it satisfies the strong approximation property of [14, Definition 10.1]. (We will not make use of the precise definition.)

Proposition 7.7. *If F/K is algebraic, or F is* P$\mathfrak{S}$CC, *then F is $\mathfrak{S}$-quasi-local and $\mathfrak{S}$-SAP.*

Proof. See [14, Lemma 4.8, Proposition 4.9, Lemma 10.5, Proposition 10.7]. □

Definition 7.8. An extension M/F is *totally $\mathfrak{S}$-adic* if the restriction map $\mathcal{S}_\mathfrak{p}(M) \to \mathcal{S}_\mathfrak{p}(F)$, $\mathfrak{P} \mapsto \mathfrak{P}|_F$, is surjective for all $\mathfrak{p} \in \mathfrak{S}$, cf. [14, Definition 11.1]. For $\mathfrak{p} \in \mathfrak{S}$ we let $R_\mathfrak{p}(F) = \bigcap_{\mathfrak{P} \in \mathcal{S}_\mathfrak{p}(F)} \mathcal{O}_\mathfrak{P}$, where $\mathcal{O}_\mathfrak{P}$ is the positive cone (respectively, the valuation ring) of $\mathfrak{P}$; cf. [14, Definition 4.2].

Proposition 7.9. *If F is $\mathfrak{S}$-SAP, then the following statements are equivalent for every extension M/F.*

(1) *M/F is totally $\mathfrak{S}$-adic.*
(2) *$R_\mathfrak{p}(M) \cap F = R_\mathfrak{p}(F)$ for every $\mathfrak{p} \in \mathfrak{S}$.*
(3) *$R_\mathfrak{p}(M) \cap F \subseteq R_\mathfrak{p}(F)$ for every $\mathfrak{p} \in \mathfrak{S}$.*

Proof. See [14, Lemma 11.4]. □

We now recall some results on the model theory of P$\mathfrak{S}$CC fields.

Definition 7.10. Let $\mathcal{L}_{\mathrm{ring}} = \{+, -, \cdot, 0, 1\}$ be the language of rings, $\mathcal{L}_{\mathrm{ring},\mathfrak{S}} = \mathcal{L}_{\mathrm{ring}} \cup \{R_\mathfrak{p} : \mathfrak{p} \in \mathfrak{S}\}$, where each $R_\mathfrak{p}$ is a unary predicate symbol, and $\mathcal{L}_{\mathrm{ring},\mathfrak{p}} = \mathcal{L}_{\mathrm{ring},\{\mathfrak{p}\}}$. For a language $\mathcal{L}$ we denote by $\mathcal{L}(K) = \mathcal{L} \cup \{c_a : a \in K\}$ the augmentation by constants from K.

Proposition 7.11. *There is a recursive $\mathcal{L}_{\mathrm{ring}}(K)$-theory $T_{\mathrm{P}\mathfrak{S}\mathrm{CC}}$ such that F satisfies $T_{\mathrm{P}\mathfrak{S}\mathrm{CC}}$ if and only if F is* P$\mathfrak{S}$CC.

Proof. See [14, Proposition 9.3]. □

Proposition 7.12. *For each* $\mathfrak{p} \in \mathfrak{S}$ *there exists an* $\mathcal{L}_{\mathrm{ring}}(K)$*-formula* $\varphi_{R,\mathfrak{p}}$ *that defines* $R_{\mathfrak{p}}(F)$ *in* F *for each* P$\mathfrak{S}$CC *field* F*.*

Proof. See [14, Theorem 1.2]. □

Proposition 7.13. *For every* $\mathfrak{p} \in \mathfrak{S}$ *there exists a recursive map* $\varphi(\mathbf{x}) \mapsto \hat{\varphi}_{\mathfrak{p},\exists}(\mathbf{x})$ *from* $\mathcal{L}_{\mathrm{ring}}$*-formulas to* $\mathcal{L}_{\mathrm{ring},\mathfrak{p}}(K)$*-formulas such that, for* $F \supseteq K$ *which is* $\mathfrak{S}$*-quasi-local and* $a_1, \ldots, a_m \in F$*, one has* $(F, R_{\mathfrak{p}}(F)) \models \hat{\varphi}_{\mathfrak{p},\exists}(\mathbf{a})$ *iff* $F' \models \varphi(\mathbf{a})$ *for some* $F' \in \mathrm{CC}_{\mathfrak{p}}(F)$*.*

Proof. Apply [14, Lemma 8.3] to $\neg\varphi(\mathbf{x})$. □

Proposition 7.14. *If* F *is* P$\mathfrak{S}$CC *and* $F \prec M$ *is an elementary extension, then* M/F *is regular and totally* $\mathfrak{S}$*-adic.*

Proof. See [14, Corollary 11.5]. □

The following embedding theorem will play a central role in §10.

Proposition 7.15 (Pop). *Let* $L \supseteq K$*, and let* E/L *and* F/L *be regular extensions, where* E *is countable and* F *is* $\aleph_1$*-saturated and* P$\mathfrak{S}$CC*. Then, for every homomorphism* $\gamma\colon \mathrm{Gal}(F) \to \mathrm{Gal}(E)$ *with* $\mathrm{res}_{\tilde{E}/\tilde{L}} \circ \gamma = \mathrm{res}_{\tilde{F}/\tilde{L}}|_{\mathrm{Gal}(F)}$*, there exists an* L*-embedding* $\tilde{E} \to \tilde{F}$ *such that* $\gamma(\tau) = \tau|_{\tilde{E}}$ *for all* $\tau \in \mathrm{Gal}(F)$*.*

Proof. This follows from [31, 6.1], as P$\mathfrak{S}$CC fields are pseudo-classically closed; see also [14, Proposition 4.6]. A proof in the present setting with all details can be found in [13, 2.11.5]. The basic idea for such an embedding theorem (namely the case $\mathfrak{S} = \emptyset$) was introduced already in [23]. □

8. $\mathfrak{S}$-adic absolute Galois group piles

We continue to work in Setting 7.3. We now define the group pile $\mathbf{Gal}_{\mathfrak{S}}(F)$ and prove some of its basic properties.

Definition 8.1. The $\mathfrak{S}$*-adic absolute Galois group pile* of F is the group pile

$$\mathbf{Gal}_{\mathfrak{S}}(F) = (\mathrm{Gal}(F), \mathcal{G}_{\mathfrak{p}})_{\mathfrak{p}\in\mathfrak{S}},$$

where $\mathcal{G}_{\mathfrak{p}} = \{\mathrm{Gal}(F')\colon F' \in \mathrm{CC}_{\mathfrak{p}}(F)\}$. For a Galois extension E/F, let $\mathbf{Gal}_{\mathfrak{S}}(E/F) = \mathbf{Gal}_{\mathfrak{S}}(F)/\mathrm{Gal}(E)$ be the $\mathfrak{S}$*-adic Galois group pile* of E/F.

In order to prove that $\mathbf{Gal}_{\mathfrak{S}}(F)$ is a indeed group pile, we will make use of the following group theoretical lemma.

Lemma 8.2. *Let* G *be a profinite group and* Γ *a finitely generated profinite group. Then* $\mathcal{G} = \{H \leqslant G\colon H$ *is a quotient of* $\Gamma\} \subseteq \mathrm{Subgr}(G)$ *is closed.*

Proof. We prove that $\mathrm{Subgr}(G) \smallsetminus \mathcal{G}$ is open. Let $H \leqslant G$ such that H is not a quotient of Γ. Since Γ is finitely generated, by [16, 16.10.7(a)] there exists an open normal subgroup $H_0 \trianglelefteq H$ such that H/H_0 is not a quotient of Γ. Let $N \trianglelefteq G$ be an open normal subgroup with $N \cap H \leqslant H_0$. Since H/H_0 is not a quotient of Γ, $H/(N \cap H)$ is also not a quotient of Γ. If $H' \leqslant G$ and $H'N = HN$, then $H'/(N \cap H') \cong H'N/N \cong H/(N \cap H)$, and hence H' is not a quotient of Γ. The set of such H' forms an open neighborhood of H. □

The following statement is similar to [20, Lemma 10.3(c–d)], which, however, is concerned with fields instead of group piles, and is restricted to certain subfields of $K^{\mathfrak{S}}$.

Proposition 8.3. *The $\mathfrak{S}$-adic absolute Galois group pile $\mathbf{Gal}_{\mathfrak{S}}(F)$ is a separated reduced deficient group pile.*

Proof. Let $\mathbf{G} = (G, \mathcal{G}_{\mathfrak{p}})_{\mathfrak{p}\in\mathfrak{S}} = \mathbf{Gal}_{\mathfrak{S}}(F)$.

We first prove that $\mathbf{G}$ is a group pile. Let $\mathbf{Gal}_{\mathfrak{S}}(K) = (H, \mathcal{H}_{\mathfrak{p}})_{\mathfrak{p}\in\mathfrak{S}}$, and fix $\mathfrak{p} \in \mathfrak{S}$. We have to show that $\mathcal{G}_{\mathfrak{p}}$ is closed in $\mathrm{Subgr}(G)$. Let $\Gamma = \mathrm{Gal}(K_{\mathfrak{p}})$. Since $\mathfrak{p}$ is local, we have $\mathcal{H}_{\mathfrak{p}} = \Gamma^H$, cf. [14, Remark 3.6], and hence $\mathcal{H}_{\mathfrak{p}}$ is closed in $\mathrm{Subgr}(H)$. By Propositions A.3 and B.5, Γ is finitely generated.

Let $G_0 \leqslant G$. We claim that $G_0 \in \mathcal{G}_{\mathfrak{p}}$ if and only if G_0 is a quotient of Γ and $\mathrm{res}_{\tilde{F}/\tilde{K}}(G_0) \in \mathcal{H}_{\mathfrak{p}}$. Indeed, if $G_0 \in \mathcal{G}_{\mathfrak{p}}$, then $\mathrm{res}_{\tilde{F}/\tilde{K}}(G_0) \in \mathcal{H}_{\mathfrak{p}}$ and $G_0 \cong \mathrm{res}_{\tilde{F}/\tilde{K}}(G_0) \cong \Gamma$ by Lemma 7.4. Conversely, if $\mathrm{res}_{\tilde{F}/\tilde{K}}(G_0) \in \mathcal{H}_{\mathfrak{p}} = \Gamma^H$, then Γ is a quotient of G_0. Hence, if G_0 is also a quotient of Γ, then $G_0 \cong \Gamma$ by [16, 16.10.7]. Therefore, by Propositions A.3 and B.6, the fixed field F' of G_0 is real closed (respectively, p-adically closed) of the same type as $\mathfrak{p}$. In addition, $\mathrm{res}_{\tilde{F}/\tilde{K}}(G_0) \in \mathcal{H}_{\mathfrak{p}}$ implies that $F' \in \mathrm{CC}_{\mathfrak{p}}(F)$; i.e., $G_0 \in \mathcal{G}_{\mathfrak{p}}$.

By Lemma 8.2, the set of $G_0 \leqslant G$ such that G_0 is a quotient of Γ is closed. Since $\mathcal{H}_{\mathfrak{p}} = \Gamma^H$ is closed, and $\mathrm{res}_{\tilde{F}/\tilde{K}}\colon \mathrm{Subgr}(G) \to \mathrm{Subgr}(H)$ is continuous by Lemma 3.2, the set of $G_0 \leqslant G$ with $\mathrm{res}_{\tilde{F}/\tilde{K}}(G_0) \in \mathcal{H}_{\mathfrak{p}}$ is closed. Therefore, $\mathcal{G}_{\mathfrak{p}}$ is closed.

We now prove that $\mathbf{G}$ is separated and reduced. Let $\mathfrak{p}, \mathfrak{q} \in \mathfrak{S}$, $\Gamma \in \mathcal{G}_{\mathfrak{p}}$, $\Gamma_1 \in \mathcal{G}_{\mathfrak{q}}$, and assume that $\Gamma \subseteq \Gamma_1$.

If $\mathfrak{p}$ or $\mathfrak{q}$ is an ordering, then both are orderings and $\Gamma = \Gamma_1$, since the absolute Galois group of a real closed field is finite (Proposition A.3), and the absolute Galois group of a p-adically closed field is nontrivial and torsion free (Proposition B.5). So, since the ordering of a real closed field is unique, $\mathfrak{p} = \mathfrak{q}$.

If $\mathfrak{p}$ is a p-valuation and $\mathfrak{q}$ is a q-valuation, let F' and F_1' be the fixed fields of Γ (respectively, Γ_1), and let $K' = \tilde{K} \cap F'$ and $K_1' = \tilde{K} \cap F_1'$. Then $K_1' \subseteq K'$, and $K' \in \mathrm{CC}_{\mathfrak{p}}(K)$ and $K_1' \in \mathrm{CC}_{\mathfrak{q}}(K)$ by Lemma 7.4. Thus, since $\mathfrak{p}$ and $\mathfrak{q}$ are local, K' is Henselian with respect to two rank-one valuations, which must be equivalent by Schmidt's theorem; cf. [7, 4.4.1]. In particular, $p = q$. Thus the restriction of the unique p-valuation of F' to F_1' is the unique p-valuation of F_1', so $\mathfrak{p} = \mathfrak{q}$. Therefore, by the maximality of p-adically closed fields (of the same type), $F' = F_1'$, and hence $\Gamma = \Gamma_1$. □

Remark 8.4. If $(N_i)_{i\in I}$ is a directed family of closed normal subgroups of a group pile $\mathbf{G}$ with $\bigcap_{i\in I} N_i = 1$, then $\mathbf{G} \cong \varprojlim_{i\in I} \mathbf{G}/N_i$. In particular, $\mathbf{Gal}_{\mathfrak{S}}(F) = \varprojlim_E \mathbf{Gal}_{\mathfrak{S}}(E/F)$, where E ranges over all finite Galois extensions of F.

Lemma 8.5. *Suppose that F is $\mathfrak{S}$-quasi-local. Let M/F be an extension, and let*

$$\mathrm{res}_{\tilde{M}/\tilde{F}}\colon \mathbf{Gal}_{\mathfrak{S}}(M) \to \mathbf{Gal}_{\mathfrak{S}}(F)$$

be the restriction map. Then $\mathrm{res}_{\tilde{M}/\tilde{F}}$ is a homomorphism of group piles, and the following are equivalent.

(1) $\mathrm{res}_{\tilde{M}/\tilde{F}}$ *is an epimorphism.*
(2) $\mathrm{res}_{\tilde{M}/\tilde{F}}$ *is a rigid epimorphism.*
(3) M/F *is regular and totally $\mathfrak{S}$-adic.*

Proof. Let $\mathfrak{p} \in \mathfrak{S}$. If $M' \in \mathrm{CC}_{\mathfrak{p}}(M)$, then $F' = M' \cap \tilde{F} \in \mathrm{CC}_{\mathfrak{p}}(F)$ by Lemma 7.4. Thus, $\mathrm{res}_{\tilde{M}/\tilde{F}}\colon \mathrm{Gal}(M) \to \mathrm{Gal}(F)$ indeed induces a homomorphism of group piles $\mathrm{res}_{\tilde{M}/\tilde{F}}$: $\mathbf{Gal}_{\mathfrak{S}}(M) \to \mathbf{Gal}_{\mathfrak{S}}(F)$.

Proof of (1) $\Rightarrow$ (3). Suppose that $\mathrm{res}_{\tilde{M}/\tilde{F}}\colon \mathbf{Gal}_{\mathfrak{S}}(M) \to \mathbf{Gal}_{\mathfrak{S}}(F)$ is an epimorphism of group piles. Then $\mathrm{res}_{\tilde{M}/\tilde{F}}\colon \mathrm{Gal}(M) \to \mathrm{Gal}(F)$ is surjective, so M/F is regular. Let $\mathfrak{p} \in \mathfrak{S}$, $\mathfrak{P} \in \mathcal{S}_{\mathfrak{p}}(F)$, and $F' \in \mathrm{CC}(F, \mathfrak{P})$. Then there exists $M' \in \mathrm{CC}_{\mathfrak{p}}(M)$ with $M' \cap \tilde{F} = F'$. Let $\mathfrak{Q}'$ be the unique prime of M' lying over $\mathfrak{p}$, and let $\mathfrak{Q} = \mathfrak{Q}'|_M$. Then $\mathfrak{Q} \in \mathcal{S}_{\mathfrak{p}}(M)$ and $\mathfrak{Q}|_F = \mathfrak{P}$. Therefore, M/F is totally $\mathfrak{S}$-adic.

Proof of (3) $\Rightarrow$ (2). Since M/F is regular, $\mathrm{res}_{\tilde{M}/\tilde{F}}\colon \mathrm{Gal}(M) \to \mathrm{Gal}(F)$ is surjective. Consider $\mathfrak{p} \in \mathfrak{S}$, $\mathfrak{P} \in \mathcal{S}_{\mathfrak{p}}(F)$, and $F' \in \mathrm{CC}(F, \mathfrak{P})$. Since M/F is totally $\mathfrak{S}$-adic, there exists $\mathfrak{Q} \in \mathcal{S}_{\mathfrak{p}}(M)$ lying over $\mathfrak{P}$. If $M'' \in \mathrm{CC}(M, \mathfrak{Q})$, then $F'' = M'' \cap \tilde{F} \in \mathrm{CC}(F, \mathfrak{P})$ by Lemma 7.4. Since $\mathfrak{P}$ is quasi-local, F' and F'' are F-isomorphic; cf. [14, Remark 3.6]. Since $\mathrm{res}_{\tilde{M}/\tilde{F}}$ is surjective, there exists a conjugate $M' \in \mathrm{CC}(M, \mathfrak{Q})$ of M'' with $M' \cap \tilde{F} = F'$. Therefore, $\mathrm{res}_{\tilde{M}/\tilde{F}}\colon \mathbf{Gal}_{\mathfrak{S}}(M) \to \mathbf{Gal}_{\mathfrak{S}}(F)$ is an epimorphism of group piles. By Lemma 7.4, $\mathrm{res}\colon \mathrm{Gal}(M') \to \mathrm{Gal}(F')$ is an isomorphism, so $\mathrm{res}_{\tilde{M}/\tilde{F}}$ is rigid.

Proof of (2) $\Rightarrow$ (1). This is trivial. □

We now explain how to interpret statements about $\mathbf{Gal}_{\mathfrak{S}}(F)$ in F. Due to lack of a suitable reference we present the classical case $\mathfrak{S} = \emptyset$ in Appendix C. Here we only explain how to extend this to general $\mathfrak{S}$. The following proposition generalizes Proposition C.9.

Proposition 8.6. *To every ranked set of coformulas Σ in the variables $v_1, v_2, \ldots$ we can assign recursively a set Σ_{ring} of $\mathcal{L}_{\mathrm{ring},\mathfrak{S}}$-formulas such that, for every $F \supseteq K$ which is $\mathfrak{S}$-quasi-local, Σ is cosatisfied in $\mathbf{Gal}_{\mathfrak{S}}(F)$ if and only if Σ_{ring} is satisfied in $(F, R_{\mathfrak{p}}(F))_{\mathfrak{p}\in\mathfrak{S}}$.*

Proof. Building on the case $\mathfrak{S} = \emptyset$, we only have to explain how to translate statements of the form $(v_1, \ldots, v_n) \in \mathcal{G}_{\mathfrak{p},n}$. Let $\mathbf{Gal}_{\mathfrak{S}}(F) = (G, \mathcal{G}_{\mathfrak{p}})_{\mathfrak{p}\in\mathfrak{S}}$, let $N = \mathrm{Gal}(E) \trianglelefteq G$ be an open subgroup, and let $\Delta N/N \in \mathrm{Subgr}(G/N)$. By definition, $\Delta N/N \in \mathcal{G}_{\mathfrak{p},N}$ if and only if there is some $F' \in \mathrm{CC}_{\mathfrak{p}}(F)$ such that, with $\Gamma = \mathrm{Gal}(F') \in \mathcal{G}_{\mathfrak{p}}$, $\Gamma N/N = \Delta N/N$. So $(x_1N_1, \ldots, x_nN_n) \in \mathcal{G}_{\mathfrak{p},n}$ if and only if $N_1 = \cdots = N_n$ and $H = \{x_1N_1, \ldots, x_nN_n\}$ is a subgroup of $G/N_1 = \mathrm{Gal}(E/F)$ that corresponds to a field $F \subseteq E' \subseteq E$ that is the intersection of some $F' \in \mathrm{CC}_{\mathfrak{p}}(F)$ with E. This is equivalent to the fact that every polynomial $f \in F[X]$ of degree bounded by $[E : F]$ that has a root in E has a root in F' if and only if it has a root in E'. By Proposition 7.13, the existence of such an F' can be expressed by an $\mathcal{L}_{\mathrm{ring},\mathfrak{S}}$-formula in $(F, R_{\mathfrak{p}}(F))_{\mathfrak{p}\in\mathfrak{S}}$. □

Corollary 8.7. *There is a recursive map $\varphi \mapsto \varphi_{\mathrm{ring}}$ from cosentences to $\mathcal{L}_{\mathrm{ring}}(K)$-sentences such that, for every* P$\mathfrak{S}$CC *field F and every cosentence φ, we have that $F \models \varphi_{\mathrm{ring}}$ iff $S(\mathbf{Gal}_{\mathfrak{S}}(F)) \models \varphi$.*

Proof. A P$\mathfrak{S}$CC field F is $\mathfrak{S}$-quasi-local (Proposition 7.7) and the $R_{\mathfrak{p}}(F)$ are K-definable (Proposition 7.12) by some formula $\varphi_{R,\mathfrak{p}}$, so the claim follows from the special case $\Sigma = \{\varphi\}$ of Proposition 8.6 by replacing all occurrences of the predicates $R_{\mathfrak{p}}$ by $\varphi_{R,\mathfrak{p}}$. □

Corollary 8.8. *If F is* P$\mathfrak{S}$CC *and $\aleph_1$-saturated, then $\mathbf{Gal}_{\mathfrak{S}}(F)$ is $\aleph_1$-cosaturated.*

Proof. In Corollary C.11 of the Appendix C, this statement is proven in the special case $\mathfrak{S} = \emptyset$. The same proof goes through in the general case if we replace the use of Proposition C.9 by Proposition 8.6, and, like in the proof of the previous corollary, use Propositions 7.7 and 7.12 to replace all occurrences of the predicates $R_{\mathfrak{p}}$ by $\varphi_{R,\mathfrak{p}}$. □

Definition 8.9. If we apply Corollary 8.7 to the sentences of the cotheory $T^{\mathrm{co}}_{\mathrm{C},\mathfrak{S},e}$ (Definition 6.8), we get an $\mathcal{L}_{\mathrm{ring}}(K)$-theory, which we denote by $T^{\mathrm{ring}}_{\mathrm{C},\mathfrak{S},e}$.

9. Subfields of $K^{\mathfrak{S}}$

We now turn to the fields $K^{\mathfrak{S}}(\boldsymbol{\sigma})$ mentioned in § 1. We can define $K^{\mathfrak{S}} = \bigcap_{\mathfrak{p}\in\mathfrak{S}} \bigcap \mathrm{CC}_{\mathfrak{p}}(K)$; cf. [18, 0.1], [20].

Lemma 9.1. *Let $K \subseteq L \subseteq K^{\mathfrak{S}}$ be a field, and let $\mathfrak{p} \in \mathfrak{S}$. Then the following hold.*

(1) *L/K is totally $\mathfrak{S}$-adic.*
(2) $\mathrm{CC}_{\mathfrak{p}}(L) = \mathrm{CC}_{\mathfrak{p}}(K)$.
(3) $R_{\mathfrak{p}}(L) = L \cap R_{\mathfrak{p}}(K^{\mathfrak{S}}) = L \cap \bigcap_{K'\in\mathrm{CC}_{\mathfrak{p}}(K)} R_{\mathfrak{p}}(K')$.

Proof. (1). For $\mathfrak{p} \in \mathfrak{S}$, take $K' \in \mathrm{CC}_{\mathfrak{p}}(K)$ and $\mathfrak{P} \in \mathcal{S}_{\mathfrak{p}}(K')$. Since $L \subseteq K^{\mathfrak{S}} \subseteq K'$, we can restrict $\mathfrak{P}$ to $\mathfrak{P}|_L \in \mathcal{S}_{\mathfrak{p}}(L)$, which lies over $\mathfrak{p}$.

(2). Since L/K is algebraic, it is clear that $\mathrm{CC}_{\mathfrak{p}}(L) \subseteq \mathrm{CC}_{\mathfrak{p}}(K)$. Conversely, if $K' \in \mathrm{CC}_{\mathfrak{p}}(K)$, then $L \subseteq K^{\mathfrak{S}} \subseteq K'$, and therefore $K' \in \mathrm{CC}_{\mathfrak{p}}(L)$.

(3). By (2), $\mathrm{CC}_{\mathfrak{p}}(L) = \mathrm{CC}_{\mathfrak{p}}(K) = \mathrm{CC}_{\mathfrak{p}}(K^{\mathfrak{S}})$. Thus,

$$\begin{aligned} R_{\mathfrak{p}}(L) &= \bigcap_{\mathfrak{P}\in\mathcal{S}_{\mathfrak{p}}(L)} \mathcal{O}_{\mathfrak{P}} = \bigcap_{L'\in\mathrm{CC}_{\mathfrak{p}}(L)} R_{\mathfrak{p}}(L') \cap L = L \cap \bigcap_{K'\in\mathrm{CC}_{\mathfrak{p}}(K)} R_{\mathfrak{p}}(K') \\ &= L \cap \bigcap_{K'\in\mathrm{CC}_{\mathfrak{p}}(K^{\mathfrak{S}})} R_{\mathfrak{p}}(K') \cap K^{\mathfrak{S}} = L \cap \bigcap_{\mathfrak{P}\in\mathcal{S}_{\mathfrak{p}}(K^{\mathfrak{S}})} \mathcal{O}_{\mathfrak{P}} = L \cap R_{\mathfrak{p}}(K^{\mathfrak{S}}). \end{aligned}$$ □

Lemma 9.2. *Let $K \subseteq E \subseteq F \subseteq K^{\mathfrak{S}}$. If E is* P$\mathfrak{S}$CC, *then F is* P$\mathfrak{S}$CC.

Proof. [3] Let V be a smooth absolutely irreducible variety defined over F with $V(F') \neq \emptyset$ for all $F' \in \mathrm{CC}_{\mathfrak{p}}(F)$, $\mathfrak{p} \in \mathfrak{S}$. Since, by Propositions A.4 and B.2, $V(F')$ is in fact Zariski-dense in V for all $F' \in \mathrm{CC}_{\mathfrak{p}}(F)$, $\mathfrak{p} \in \mathfrak{S}$, we can assume without loss of generality that V is affine. Since F/E is algebraic, V is defined over a finite subextension F_0 of F/E. Let $W = \mathrm{res}_{F_0/E}(V)$ be the Weil restriction of V, and let F_1 be the Galois closure of F_0/E. Then W is a variety defined over E, and there are $\sigma_1, \ldots, \sigma_n \in \mathrm{Gal}(E)$ with $\sigma_1 = \mathrm{id}_{\tilde{E}}$ such that W is isomorphic over F_1 to $\prod_{i=1}^n V^{\sigma_i}$, and the projection onto the first factor $W \to V^{\sigma_1} = V$ is defined over F_0; cf. [16, 10.6.2]. Since V is smooth, W is also smooth.

Since $E \subseteq F \subseteq K^{\mathfrak{S}}$ and $E \subseteq F_1 \subseteq K^{\mathfrak{S}}$, Lemma 9.1(2) implies that $\mathrm{CC}_{\mathfrak{p}}(E) = \mathrm{CC}_{\mathfrak{p}}(F) = \mathrm{CC}_{\mathfrak{p}}(F_1)$ for all $\mathfrak{p} \in \mathfrak{S}$. In particular, if $E' \in \mathrm{CC}_{\mathfrak{p}}(E)$, then $F_1 \subseteq E'$. Let $E' \in \mathrm{CC}_{\mathfrak{p}}(E)$. Then $\sigma_i^{-1}(E') \in \mathrm{CC}_{\mathfrak{p}}(E) = \mathrm{CC}_{\mathfrak{p}}(F)$, so $V(\sigma_i^{-1}(E')) \neq \emptyset$ by assumption. Thus $V^{\sigma_i}(E') \neq \emptyset$ for all i, and therefore $W(E') \neq \emptyset$, as $F_1 \subseteq E'$. Since E is P$\mathfrak{S}$CC, $W(E) \neq \emptyset$, so in particular $W(F) \neq \emptyset$. Hence, since $F_0 \subseteq F$, it follows that $V(F) \neq \emptyset$, as claimed. □

Definition 9.3. If $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_e) \in \mathrm{Gal}(K)^e$, we denote by $\tilde{K}(\boldsymbol{\sigma})$ the fixed field of the group $\langle \sigma_1, \ldots, \sigma_e \rangle \leqslant \mathrm{Gal}(K)$ in $\tilde{K}$, and let $K^{\mathfrak{S}}(\boldsymbol{\sigma}) = K^{\mathfrak{S}} \cap \tilde{K}(\boldsymbol{\sigma})$. We say that a statement holds for *almost all* $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ if the set of those $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ for which it holds has measure 1 with respect to the unique Haar probability measure on $\mathrm{Gal}(K)^e$.

Proposition 9.4 (Geyer–Jarden). *Let $\mathfrak{S}$ be a finite set of local primes of a countable Hilbertian field K of characteristic zero, and let $e \geqslant 0$. Then, for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$, the field $K^{\mathfrak{S}}(\boldsymbol{\sigma})$ is* P$\mathfrak{S}$CC.

Proof. For an extension M of K, denote by $\mathcal{S}'_{\mathfrak{p}}(M)$ the set of orderings and *arbitrary* valuations on M lying over $\mathfrak{p} \in \mathfrak{S}$. By [18, Theorem A], for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$, the maximal Galois extension M of K inside $K^{\mathfrak{S}}(\boldsymbol{\sigma})$ satisfies a local–global principle with respect to $\mathcal{S}'_{\mathfrak{p}}(M)$. However, since M/K is totally $\mathfrak{S}$-adic by Lemma 9.1(1), and all extensions of $\mathfrak{p}$ to M are conjugate since M/K is Galois; in fact, $\mathcal{S}'_{\mathfrak{p}}(M) = \mathcal{S}_{\mathfrak{p}}(M)$. In other words, M is P$\mathfrak{S}$CC. Since $K \subseteq M \subseteq K^{\mathfrak{S}}(\boldsymbol{\sigma}) \subseteq K^{\mathfrak{S}}$, the claim follows from Lemma 9.2. □

Proposition 9.5 (Haran–Jarden–Pop). *Let $\mathfrak{S}$ be a finite set of local primes of a countable Hilbertian field K of characteristic zero, and let $e \geqslant 0$. Then, for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$, $\mathbf{Gal}_{\mathfrak{S}}(K^{\mathfrak{S}}(\boldsymbol{\sigma}))$ is isomorphic to the deficient reduct of the e-free semi-constant group pile of $(\mathrm{Gal}(K_{\mathfrak{p}}))_{\mathfrak{p} \in \mathfrak{S}}$ over perfect profinite spaces $(T_{\mathfrak{p}})_{\mathfrak{p} \in \mathfrak{S}}$.*

Proof. This is proven in [20]. Indeed, by [20, Proposition 12.3], for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$, the field $M = K^{\mathfrak{S}}(\boldsymbol{\sigma})$ satisfies condition (1) of §10 of that work. In the proof of [20, Proposition 11.2] it is proven that in this case $\mathbf{Gal}(M, \mathfrak{S}) := (G, \mathrm{Gal}(\tilde{K}(\boldsymbol{\sigma}))^G, \mathcal{G}_{\mathfrak{p}})_{\mathfrak{p} \in \mathfrak{S}}$, where $\mathbf{Gal}_{\mathfrak{S}}(M) = (G, \mathcal{G}_{\mathfrak{p}})_{\mathfrak{p} \in \mathfrak{S}}$, is a so-called 'Cantor group pile over $(\mathrm{Gal}(K_{\mathfrak{p}}))_{\mathfrak{p} \in \mathfrak{S}}$'.

[3] This proof corrects an inaccuracy in [18, Proof of Lemma 1.6 Part B].

By [20, Corollary 6.2] and [20, Proposition 6.3], every Cantor group pile over $(\mathrm{Gal}(K_\mathfrak{p}))_{\mathfrak{p}\in\mathfrak{S}}$ is isomorphic to the group pile $\mathbf{G}_T$ of [20, Proposition 5.3], which is exactly the e-free semi-constant group pile of $(\mathrm{Gal}(K_\mathfrak{p}))_{\mathfrak{p}\in\mathfrak{S}}$ over Cantor spaces $(T_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{S}}$. Since Cantor spaces are perfect and the deficient reduct of $\mathbf{Gal}(M,\mathfrak{S})$ is $\mathbf{Gal}_\mathfrak{S}(M)$, the claim follows. □

10. Axiomatization

We now present the axiomatization of the theory of almost all $K^\mathfrak{S}(\boldsymbol{\sigma})$. Several of the results in this section are generalizations of the corresponding results in the special case $\mathfrak{S}=\emptyset$ in [23].

Setting 10.1. *Let $\mathfrak{S}$ be a finite set of local primes of a countable Hilbertian field K of characteristic zero.*

Definition 10.2. For a set $S\subseteq\tilde{K}$, let $N_K(S)=\{f\in K[X]\colon f \text{ has no root in } S\}$.

Definition 10.3. Let the $\mathcal{L}_{\mathrm{ring}}(K)$-theory $T_{\mathrm{alg},\mathfrak{S}}$ consist of the following sentences.

(1) For each $f\in N_K(K^\mathfrak{S})$, the sentence
$$\neg(\exists x)(f(x)=0).$$

(2) For each $\mathfrak{p}\in\mathfrak{S}$ and each $f\in N_K(R_\mathfrak{p}(K^\mathfrak{S}))$, the sentence
$$\neg(\exists x)(\varphi_{\mathrm{R},\mathfrak{p}}(x)\wedge f(x)=0),$$
where $\varphi_{\mathrm{R},\mathfrak{p}}$ is the formula of Proposition 7.12.

Lemma 10.4. *A* P$\mathfrak{S}$CC *field $F\supseteq K$ is a model of $T_{\mathrm{alg},\mathfrak{S}}$ if and only if $F\cap\tilde{K}\subseteq K^\mathfrak{S}$ and $F/F\cap\tilde{K}$ is totally $\mathfrak{S}$-adic.*

Proof. Since F is P$\mathfrak{S}$CC, $\varphi_{\mathrm{R},\mathfrak{p}}(F)=R_\mathfrak{p}(F)$ for each $\mathfrak{p}\in\mathfrak{S}$ by Proposition 7.12. Let $L=F\cap\tilde{K}$. Since L/K is algebraic, L is $\mathfrak{S}$-SAP by Proposition 7.7.

Suppose that F satisfies $T_{\mathrm{alg},\mathfrak{S}}$. By (1), each polynomial $f\in K[X]$ that has a zero in L also has a zero in $K^\mathfrak{S}$, and hence, since $K^\mathfrak{S}$ is Galois over K, has all its zeros in $K^\mathfrak{S}$. Since L/K is algebraic, this implies that $L\subseteq K^\mathfrak{S}$. If $\mathfrak{p}\in\mathfrak{S}$, then $R_\mathfrak{p}(L)=R_\mathfrak{p}(K^\mathfrak{S})\cap L$ by Lemma 9.1. Since $R_\mathfrak{p}(K^\mathfrak{S})$ is $\mathrm{Gal}(K)$-invariant, (2) implies that $R_\mathfrak{p}(F)\cap\tilde{K}\subseteq R_\mathfrak{p}(K^\mathfrak{S})$. Therefore, $R_\mathfrak{p}(F)\cap L\subseteq R_\mathfrak{p}(K^\mathfrak{S})\cap L=R_\mathfrak{p}(L)$, so F/L is totally $\mathfrak{S}$-adic by Proposition 7.9.

Conversely, suppose that $L\subseteq K^\mathfrak{S}$ and F/L is totally $\mathfrak{S}$-adic. Since $L\subseteq K^\mathfrak{S}$, F satisfies (1). By Proposition 7.9, $R_\mathfrak{p}(F)\cap L=R_\mathfrak{p}(L)$. So, since $R_\mathfrak{p}(F)\cap L=R_\mathfrak{p}(F)\cap\tilde{K}$ and $R_\mathfrak{p}(L)=R_\mathfrak{p}(K^\mathfrak{S})\cap L$ by Lemma 9.1, F satisfies (2). □

Definition 10.5. Let the $\mathcal{L}_{\mathrm{ring}}(K)$-theory $T_{\mathrm{tot},\mathfrak{S},e}$ consist of the following axioms.

(0) The axioms for fields and the positive diagram of K; cf. [16, 7.3.1].

(1) The theory $T_{\mathrm{P}\mathfrak{S}\mathrm{CC}}$ (Proposition 7.11).

(2) The theory $T^{\mathrm{ring}}_{\mathrm{C},\mathfrak{S},e}$ (Definition 8.9).

(3) The theory $T_{\mathrm{alg},\mathfrak{S}}$ (Definition 10.3).

Lemma 10.6. *A field $F \supseteq K$ is a model of $T_{\mathrm{tot},\mathfrak{S},e}$ if and only if it satisfies the following conditions.*

(1) *F is* P$\mathfrak{S}$CC.

(2) *$\mathbf{Gal}_{\mathfrak{S}}(F)$ is an e-free C-pile.*

(3) *$F \cap \tilde{K} \subseteq K^{\mathfrak{S}}$ and $F/F \cap \tilde{K}$ is totally $\mathfrak{S}$-adic.*

Proof. See Proposition 7.11 for (1), Proposition 6.9 and Corollary 8.7 for (2), and Lemma 10.4 for (3). □

Lemma 10.7. *Let $K \subseteq L \subseteq E, F$ be fields such that the following conditions are satisfied.*

(1) *E and F are models of $T_{\mathrm{tot},\mathfrak{S},e}$.*

(2) *E/L and F/L are regular and totally $\mathfrak{S}$-adic.*

(3) *E is countable and F is $\aleph_1$-saturated.*

(4) *L is $\mathfrak{S}$-quasi-local.*

Then there exists an L-embedding $i : E \to F$ with $F/i(E)$ regular and totally $\mathfrak{S}$-adic.

Proof. By (1), E and F are P$\mathfrak{S}$CC (Lemma 10.6(1)), and so in particular are $\mathfrak{S}$-quasi-local (Proposition 7.7). Let $\mathbf{G} = \mathbf{Gal}_{\mathfrak{S}}(F)$, $\mathbf{B} = \mathbf{Gal}_{\mathfrak{S}}(E)$, and $\mathbf{A} = \mathbf{Gal}_{\mathfrak{S}}(L)$. Also by (1), $\mathbf{G}$ and $\mathbf{B}$ are e-free C-piles (Lemma 10.6(2)). By (2) and (4), the restriction maps $\mathrm{res}_{\tilde{F}/\tilde{L}} : \mathbf{G} \to \mathbf{A}$ and $\mathrm{res}_{\tilde{E}/\tilde{L}} : \mathbf{B} \to \mathbf{A}$ are rigid epimorphisms of group piles (Lemma 8.5). So $(\mathrm{res}_{\tilde{F}/\tilde{L}}, \mathrm{res}_{\tilde{E}/\tilde{L}})$ is a rigid e-generated deficient embedding problem for $\mathbf{G}$.

By (3), E is countable, so $\mathbf{B}$ has countable rank, and F is $\aleph_1$-saturated, so $\mathbf{Gal}_{\mathfrak{S}}(F)$ is $\aleph_1$-cosaturated by Corollary 8.8. Hence, by Proposition 6.10, there exists an epimorphism $\gamma : \mathbf{G} \to \mathbf{B}$ such that $\mathrm{res}_{\tilde{E}/\tilde{L}} \circ \gamma = \mathrm{res}_{\tilde{F}/\tilde{L}}$. By Proposition 7.15, this gives an L-embedding $i : E \to F$ such that $\gamma = \mathrm{res}_{\tilde{F}/\widetilde{i(E)}}$. Hence, since γ is an epimorphism of group piles, $F/i(E)$ is regular and totally $\mathfrak{S}$-adic by Lemma 8.5. □

The proof of the following proposition follows the proof of [16, 20.3.3].

Proposition 10.8. *Let $E, F \supseteq K$ be models of $T_{\mathrm{tot},\mathfrak{S},e}$ with $E \cap \tilde{K} \cong_K F \cap \tilde{K}$. Then $E \equiv_K F$; i.e., E and F are elementarily equivalent in $\mathcal{L}_{\mathrm{ring}}(K)$.*

Proof. Assume without loss of generality that $L := E \cap \tilde{K} = F \cap \tilde{K}$. Let E^*, F^* be $\aleph_1$-saturated elementary extensions of E (respectively, F). By Lemma 10.6, the fields E, F, E^*, F^* are P$\mathfrak{S}$CC, and in particular also $\mathfrak{S}$-SAP (Proposition 7.7), and the extensions E/L, F/L, E^*/L, and F^*/L are regular and totally $\mathfrak{S}$-adic.

By the downward Löwenheim–Skolem Theorem, there exists a countable elementary subfield E_0 of E^* that contains L. Then also E_0/L is regular and totally $\mathfrak{S}$-adic, and

E_0 is a model of $T_{\mathrm{tot},\mathfrak{S},e}$, and is in particular P$\mathfrak{S}$CC and $\mathfrak{S}$-quasi-local (Proposition 7.7). Since L/K is algebraic, L is $\mathfrak{S}$-quasi-local (Proposition 7.7). Therefore, by Lemma 10.7, there exists an L-embedding $\alpha_0 \colon E_0 \to F^*$ with $F^*/\alpha_0(E_0)$ regular and totally $\mathfrak{S}$-adic.

Identify E_0 with $\alpha_0(E_0)$. Let F_0 be a countable elementary subfield of F^* that contains E_0. Then F_0/E_0 is regular and totally $\mathfrak{S}$-adic, and F_0 is a model of $T_{\mathrm{tot},\mathfrak{S},e}$. Also E^*/E_0 is regular and totally $\mathfrak{S}$-adic by Proposition 7.14. Hence, by Lemma 10.7, there is an E_0-embedding $\beta_0 \colon F_0 \to E^*$ with $E^*/\beta_0(F_0)$ regular and totally $\mathfrak{S}$-adic.

Now iterate this process to construct a tower of countable fields $E_0 \subseteq F_0 \subseteq E_1 \subseteq F_1 \subseteq \cdots$ such that each E_i is an elementary subfield of E^* and each F_i is an elementary subfield of F^*. Then $M := \bigcup_{i\in\mathbb{N}} E_i = \bigcup_{i\in\mathbb{N}} F_i$ is an elementary subfield of both E^* and F^* (see, for example, [16, 7.4.1(b)]), so $E^* \equiv_M F^*$. In particular, $E^* \equiv_K F^*$, and hence $E \equiv_K F$. □

Lemma 10.9. *If F is a model of $T_{\mathrm{tot},\mathfrak{S},e}$, then $L := F \cap \tilde{K} \subseteq K^{\mathfrak{S}}$ and* $\mathrm{rk}(\mathrm{Gal}(K^{\mathfrak{S}}/L)) \leqslant e$.

Proof. Let $\mathbf{G} = \mathbf{Gal}_{\mathfrak{S}}(F)$, and $\mathbf{A} = \mathbf{Gal}_{\mathfrak{S}}(L)$. By Lemma 10.6(3), $L \subseteq K^{\mathfrak{S}}$ and F/L is totally $\mathfrak{S}$-adic. Since L/K is algebraic, L is $\mathfrak{S}$-quasi-local (Proposition 7.7), so the restriction $\mathbf{G} \to \mathbf{A}$ is an epimorphism of group piles by Lemma 8.5. By Lemma 10.6(2), $\mathbf{G}$ is an e-free C-pile. In particular, it is e-generated. Thus, by Lemma 3.16, $\mathbf{A}$ is also e-generated. By Lemma 9.1(2), $\mathrm{CC}_{\mathfrak{p}}(L) = \mathrm{CC}_{\mathfrak{p}}(K^{\mathfrak{S}})$, for all $\mathfrak{p} \in \mathfrak{S}$. Thus, $\mathbf{A}' = \mathbf{Gal}_{\mathfrak{S}}(K^{\mathfrak{S}})'$. But $\mathbf{Gal}_{\mathfrak{S}}(K^{\mathfrak{S}})$ is self-generated by the definition of $K^{\mathfrak{S}}$, so $\mathbf{A}' = \mathbf{Gal}_{\mathfrak{S}}(K^{\mathfrak{S}})$. Therefore, $\mathrm{Gal}(K^{\mathfrak{S}}/L) = A/A' = \bar{A}$ is generated by e elements. □

Definition 10.10. Let $T_{\mathrm{almost},\mathfrak{S},e}$ denote the set of all $\mathcal{L}_{\mathrm{ring}}(K)$-sentences that are true in almost all fields $K^{\mathfrak{S}}(\boldsymbol{\sigma})$, $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$.

The proof of the following result follows the proof of [16, 20.5.4].

Theorem 10.11. *The theory $T_{\mathrm{tot},\mathfrak{S},e}$ is an axiomatization of $T_{\mathrm{almost},\mathfrak{S},e}$; i.e., these two theories have the same models.*

Proof. First note that every model of $T_{\mathrm{almost},\mathfrak{S},e}$ is a field containing K. By Definition 10.5(0), the same holds for every model of $T_{\mathrm{tot},\mathfrak{S},e}$. Next observe that almost all $K^{\mathfrak{S}}(\boldsymbol{\sigma})$ satisfy $T_{\mathrm{tot},\mathfrak{S},e}$, as Lemma 10.6 shows. For almost all $\boldsymbol{\sigma}$, $K^{\mathfrak{S}}(\boldsymbol{\sigma})$ satisfies (1) by Proposition 9.4, (2) by Propositions 9.5 and 6.7, and (3) trivially. Thus, every model of $T_{\mathrm{almost},\mathfrak{S},e}$ is a model of $T_{\mathrm{tot},\mathfrak{S},e}$.

Conversely, let E be a model of $T_{\mathrm{tot},\mathfrak{S},e}$, and let $L = E \cap \tilde{K}$. If we can construct a model F of $T_{\mathrm{almost},\mathfrak{S},e}$ with $F \cap \tilde{K} \cong_K L$, then $E \equiv_K F$ by Proposition 10.8, so E is a model of $T_{\mathrm{almost},\mathfrak{S},e}$, and we are done. Lemma 10.9 implies that $L \subseteq K^{\mathfrak{S}}$ and there exist $\tau_1, \ldots, \tau_e \in \mathrm{Gal}(K^{\mathfrak{S}}/K)$ that generate $\mathrm{Gal}(K^{\mathfrak{S}}/L)$. Let $\mathcal{N}$ be the set of finite Galois extensions of K inside $K^{\mathfrak{S}}$. For each $N \in \mathcal{N}$, the set

$$\begin{aligned} \Sigma(N) &:= \{\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e \colon \mathrm{res}_{\tilde{K}/N}(\sigma_i) = \mathrm{res}_{\tilde{K}/N}(\tau_i),\ i = 1, \ldots, e\} \\ &\subseteq \{\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e \colon K^{\mathfrak{S}}(\boldsymbol{\sigma}) \cap N \cong_K L \cap N\} \end{aligned}$$

has positive Haar measure. If $N_1, \ldots, N_r \in \mathcal{N}$, then $N_1 \cdots N_r \in \mathcal{N}$ and $\Sigma(N_1) \cap \cdots \cap \Sigma(N_r) = \Sigma(N_1 \cdots N_r)$. Hence, by [16, 7.6.1], there exists an ultrafilter $\mathcal{D}$ on $\mathrm{Gal}(K)^e$ which contains each of the sets $\Sigma(N)$, $N \in \mathcal{N}$, and all sets of measure 1. Let

$$F = \prod_{\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e} K^{\mathfrak{S}}(\boldsymbol{\sigma})/\mathcal{D}$$

be the ultraproduct, and let $M = F \cap \tilde{K}$. Since $\mathcal{D}$ contains all sets of measure 1, and almost all $K^{\mathfrak{S}}(\boldsymbol{\sigma})$ are models of $T_{\mathrm{almost},\mathfrak{S},e}$, F is a model of $T_{\mathrm{almost},\mathfrak{S},e}$ by Łos' theorem. Furthermore, $M \subseteq K^{\mathfrak{S}}$, and $M \cap N \cong L \cap N$ for each $N \in \mathcal{N}$, since $\mathcal{D}$ contains $\Sigma(N)$. Therefore, $M \cong_K L$ (see, for example, [16, 20.6.3]), as claimed. □

11. Decidability

We now use the axiomatization of the previous section to prove the decidability of $T_{\mathrm{almost},\mathfrak{S},e}$. The method follows closely the proof of Jarden and Kiehne in [16, Ch. 20.6] for the special case $\mathfrak{S} = \emptyset$.

Definition 11.1. A set $X \subseteq \mathbb{N}^n$ is *recursive* if the characteristic function of X is a recursive function; cf. [16, Ch. 8.5]. If $\mathcal{L}$ is a countable language with a fixed embedding $\mathcal{L} \to \mathbb{N}$, then an $\mathcal{L}$-theory T is *decidable* (or recursive) if the set T, identified with a subset of $\mathbb{N}$ via a Gödel numbering, is recursive [16, Ch. 8.6].

A *presented field* is a countable field K together with an injection $\rho\colon K \to \mathbb{N}$ such that the images of the graphs of addition and multiplication are recursive. If $\mathcal{L}$ is a finite language containing $\mathcal{L}_{\mathrm{ring}}$, then the injection $\rho\colon K \to \mathbb{N}$ gives an injection of the set of $\mathcal{L}(K)$-formulas into $\mathbb{N}$, using a recursive pairing function $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$. We refer to this injection when we call an $\mathcal{L}(K)$-theory decidable.

If K is a presented field, one can inject the ring of polynomials $K[X]$ into $\mathbb{N}$, again using a recursive pairing function. We say that K has a *splitting algorithm* if the set of irreducible polynomials in $K[X]$ is a recursive subset of $K[X]$. In that case, one can recursively factor elements of $K[X]$ into irreducible factors.

Definition 11.2. A prime $\mathfrak{p}$ of a presented field $\rho\colon K \to \mathbb{N}$ is *recursive* if the set $\rho(\mathcal{O}_\mathfrak{p}) \subseteq \mathbb{N}$ is recursive.

Setting 11.3. *From now on, let $\mathfrak{S}$ be a finite set of recursive local primes of a presented countable Hilbertian field K of characteristic zero that has a splitting algorithm.*

Example 11.4. Every finite set of primes $\mathfrak{S}$ of a number field K satisfies Setting 11.3. Choose any standard representation $K = \mathbb{Q}^n \hookrightarrow \mathbb{N}^{2n} \hookrightarrow \mathbb{N}$ via recursive pairing functions. Then K is countable, Hilbertian [16, 13.4.2], and has a splitting algorithm [16, 19.1.3(b), 19.2.4], and every $\mathfrak{p} \in \mathfrak{S}$ is local and recursive. This last fact is well known, but for lack of reference we sketch a proof. Since $\mathcal{O}_\mathfrak{p}$ is existentially definable in K (see, for example, [34, p. 212] for the Archimedean case and [36, 4.2.4, 4.3.4] for the p-adic case), it is recursively enumerable. Since $K \smallsetminus \mathcal{O}_\mathfrak{p} = -\mathcal{O}_\mathfrak{p} \smallsetminus \{0\}$ in the Archimedean case and $K \smallsetminus \mathcal{O}_\mathfrak{p} = (\pi\mathcal{O}_\mathfrak{p})^{-1}$ in the p-adic case, where π is a uniformizer at $\mathfrak{p}$, $K \smallsetminus \mathcal{O}_\mathfrak{p}$ is also recursively enumerable, and so $\mathcal{O}_\mathfrak{p}$ is recursive.

Lemma 11.5. *The sets $N_K(K^{\mathfrak{S}})$ and $N_K(R_{\mathfrak{p}}(K^{\mathfrak{S}}))$, $\mathfrak{p} \in \mathfrak{S}$, are recursive.*

Proof. Let $f = \sum_{i=0}^{n} a_i X^i \in K[X]$ be given. We have to decide whether f has a root in $K^{\mathfrak{S}}$ (respectively, in $R_{\mathfrak{p}}(K^{\mathfrak{S}})$). Using the splitting algorithm, we can assume without loss of generality that f is irreducible. Since $K^{\mathfrak{S}} = \bigcap_{\mathfrak{q}\in\mathfrak{S}} \bigcap \mathrm{CC}_{\mathfrak{q}}(K)$ and $R_{\mathfrak{p}}(K^{\mathfrak{S}}) = K^{\mathfrak{S}} \cap \bigcap_{K'\in \mathrm{CC}_{\mathfrak{p}}(K)} R_{\mathfrak{p}}(K')$ by Lemma 9.1(3), and all elements of $\mathrm{CC}_{\mathfrak{p}}(K)$ are K-conjugate, it suffices to decide whether f has all roots in $K_{\mathfrak{q}}$ (respectively, in $R_{\mathfrak{q}}(K_{\mathfrak{q}})$) for all $\mathfrak{q} \in \mathfrak{S}$.

By [14, Proposition 8.2] we can compute a universal $\mathcal{L}_{\mathrm{ring},\mathfrak{q}}$-formula φ_1 such that f has all roots in $K_{\mathfrak{q}}$ (respectively, in $R_{\mathfrak{q}}(K_{\mathfrak{q}})$) iff $(K, \mathcal{O}_{\mathfrak{q}}) \models \varphi_1(\mathbf{a})$. By applying the same result to the negation, we get a universal $\mathcal{L}_{\mathrm{ring},\mathfrak{q}}$-formula φ_2 such that f has all roots in $K_{\mathfrak{q}}$ (respectively, in $R_{\mathfrak{q}}(K_{\mathfrak{q}})$) iff $(K, \mathcal{O}_{\mathfrak{q}}) \not\models \varphi_2(\mathbf{a})$. Therefore, since $\mathcal{O}_{\mathfrak{q}}$ is recursive, the set of such f is both recursively enumerable and co-recursively enumerable, and hence recursive. □

Lemma 11.6. *The theory $T_{\mathrm{tot},\mathfrak{S},e}$ is recursive.*

Proof. We check that the different sets of axioms in Definition 10.5 are recursive.

(0). Since K is a presented field, the positive diagram of K is recursive.

(1). In order to check that $T_{\mathrm{P}\mathfrak{S}\mathrm{CC}}$ is recursive, we have to look into the definition [14, Definition 9.1]. Part (1) of these axioms is recursive, because [14, Definition 6.5] (1), (3)–(5) consist of finitely many sentences, and (2) is recursive because $\mathcal{O}_{\mathfrak{p}}$ is. Part (2) of [14, Definition 9.1] is recursive because the map $\eta_n \mapsto (\hat{\eta}_n)_{\mathfrak{p},\forall}$ is recursive by [14, Proposition 8.4].

(2). The theory $T^{\mathrm{ring}}_{\mathrm{C},\mathfrak{S},e}$ is recursive since $T^{\mathrm{co}}_{\mathrm{C},\mathfrak{S},e}$ is obviously recursive and the map $\varphi \mapsto \varphi_{\mathrm{ring}}$ of Corollary 8.7 is recursive.

(3). Since $N_K(K^{\mathfrak{S}})$ and $N_K(R_{\mathfrak{p}}(K^{\mathfrak{S}}))$ are recursive by Lemma 11.5, the theory $T_{\mathrm{alg},\mathfrak{S}}$ is recursive. □

Definition 11.7. The set of *test sentences* is the smallest set of $\mathcal{L}_{\mathrm{ring}}(K)$-sentences that contains all sentences of the form $(\exists x)(f(x) = 0)$, where $f \in K[X]$ is a polynomial that completely decomposes over $K^{\mathfrak{S}}$, and is closed under negations, conjunctions, and disjunctions.

Lemma 11.8. *Let $E, F \supseteq K$ be models of $T_{\mathrm{tot},\mathfrak{S},e}$. Then $E \equiv_K F$ if and only if E and F satisfy the same test sentences.*

Proof. Trivially, if E and F are elementarily equivalent over K, then they satisfy the same test sentences. Conversely, assume that E and F satisfy the same test sentences, and let $E_0 = E \cap \tilde{K}$ and $F_0 = F \cap \tilde{K}$. By Lemma 10.6(3), $E_0 \subseteq K^{\mathfrak{S}}$ and $F_0 \subseteq K^{\mathfrak{S}}$. Let $f \in K[X]$ be an irreducible polynomial. If f does not completely decompose over $K^{\mathfrak{S}}$, then it has no root in $K^{\mathfrak{S}}$, so it has no root in E_0 and it has no root in F_0. If f completely decomposes over $K^{\mathfrak{S}}$, then $(\exists x)(f(x) = 0)$ is a test sentence. Hence, f has a root in E_0 if and only if it has a root in F_0. Therefore, $E_0 \cong_K F_0$; see, for example, [16, 20.6.3]. By Proposition 10.8, $E \equiv_K F$. □

Lemma 11.9. *The set of test sentences is recursive.*

Proof. Given a polynomial $f \in K[X]$, one can decide whether $(\exists x)(f(x) = 0)$ is a test sentence because $N_K(K^{\mathfrak{S}})$ is recursive by Lemma 11.5. Induction on the structure of formulas then shows that the set of test sentences is recursive. □

Definition 11.10. For each $\mathcal{L}_{\text{ring}}(K)$-sentence θ, let $\Sigma_{\mathfrak{S},e}(\theta) = \{\boldsymbol{\sigma} \in \text{Gal}(K)^e : K^{\mathfrak{S}}(\boldsymbol{\sigma}) \models \theta\}$. We denote by μ the unique Haar probability measure on $\text{Gal}(K)^e$.

Lemma 11.11. *Let λ be a test sentence. Then $\Sigma_{\mathfrak{S},e}(\lambda)$ is open–closed in $\text{Gal}(K)^e$ and $\mu(\Sigma_{\mathfrak{S},e}(\lambda)) \in \mathbb{Q}$. The map $\lambda \mapsto \mu(\Sigma_{\mathfrak{S},e}(\lambda))$ from test sentences to $\mathbb{Q}$ is recursive.*

Proof. Let $f_1, \ldots, f_n \in K[X]$ be the polynomials occurring in λ. Their splitting field L_λ is a finite Galois extension of K inside $K^{\mathfrak{S}}$. Let L/K be a Galois extension with $L_\lambda \subseteq L \subseteq K^{\mathfrak{S}}$ (this is needed for the induction). Then $K^{\mathfrak{S}}(\boldsymbol{\sigma}) \cap L = L(\text{res}_{\tilde{K}/L}(\boldsymbol{\sigma}))$ for each $\boldsymbol{\sigma} \in \text{Gal}(K)^e$. Let

$$\Sigma_{L,\lambda} = \{\boldsymbol{\tau} \in \text{Gal}(L/K)^e : L(\boldsymbol{\tau}) \models \lambda\}.$$

We claim that $\Sigma_{\mathfrak{S},e}(\lambda) = \{\boldsymbol{\sigma} \in \text{Gal}(K)^e : \text{res}_{\tilde{K}/L}(\boldsymbol{\sigma}) \in \Sigma_{L,\lambda}\}$. Indeed, if λ is of the form $(\exists x)(f(x) = 0)$, where $f \in K[X]$ completely decomposes over $K^{\mathfrak{S}}$, then

$$\Sigma_{L,\lambda} = \{\boldsymbol{\tau} \in \text{Gal}(L/K)^e : f \text{ has a zero in } L(\boldsymbol{\tau})\}.$$

Since L contains all roots of f, $K^{\mathfrak{S}}(\boldsymbol{\sigma}) \models \lambda$ if and only if $K^{\mathfrak{S}}(\boldsymbol{\sigma}) \cap L \models \lambda$, so the claim is true in that case. Induction on the structure of λ shows that the claim holds for all test sentences λ. Thus, $\Sigma_{\mathfrak{S},e}(\lambda)$ is open–closed, and in particular is measurable. Furthermore,

$$\mu(\Sigma_{\mathfrak{S},e}(\lambda)) = \frac{|\Sigma_{L_\lambda,\lambda}|}{[L_\lambda : K]^e} \in \mathbb{Q}$$

is computable since K has a splitting algorithm; see, for example, [16, 19.3.2]. □

Theorem 11.12. *Under Setting 11.3, the following hold.*

(1) *For every $\mathcal{L}_{\text{ring}}(K)$-sentence θ, $\Sigma_{\mathfrak{S},e}(\theta)$ is μ-measurable, and $\mu(\Sigma_{\mathfrak{S},e}(\theta)) \in \mathbb{Q}$.*

(2) *The map $\theta \mapsto \mu(\Sigma_{\mathfrak{S},e}(\theta))$ from $\mathcal{L}_{\text{ring}}(K)$-sentences to $\mathbb{Q}$ is recursive.*

In particular, the theory $T_{\text{almost},\mathfrak{S},e}$ of almost all fields $K^{\mathfrak{S}}(\boldsymbol{\sigma})$, $\boldsymbol{\sigma} \in \text{Gal}(K)^e$, is decidable.

Proof. By Theorem 10.11, $T_{\text{tot},\mathfrak{S},e} \models T_{\text{almost},\mathfrak{S},e}$ and $T_{\text{almost},\mathfrak{S},e} \models T_{\text{tot},\mathfrak{S},e}$. By Lemma 11.8 and [16, 7.8.2], for every $\mathcal{L}_{\text{ring}}(K)$-sentence θ there exists a test sentence λ such that the sentence $\theta \leftrightarrow \lambda$ is in $T_{\text{almost},\mathfrak{S},e}$. In particular, $\Sigma_{\mathfrak{S},e}(\theta)$ and $\Sigma_{\mathfrak{S},e}(\lambda)$ differ only by a zero set. Lemma 11.11 implies that $\Sigma_{\mathfrak{S},e}(\lambda)$ is μ-measurable and $\mu(\Sigma_{\mathfrak{S},e}(\lambda)) \in \mathbb{Q}$, so $\Sigma_{\mathfrak{S},e}(\theta)$ is also μ-measurable and $\mu(\Sigma_{\mathfrak{S},e}(\theta)) = \mu(\Sigma_{\mathfrak{S},e}(\lambda)) \in \mathbb{Q}$.

Since $T_{\text{tot},\mathfrak{S},e} \models T_{\text{almost},\mathfrak{S},e}$, we have $T_{\text{tot},\mathfrak{S},e} \models \theta \leftrightarrow \lambda$. The set of test sentences is recursive by Lemma 11.9. By Lemma 11.6, the theory $T_{\text{tot},\mathfrak{S},e}$ is recursive, so the set of consequences of $T_{\mathrm{tot},\mathfrak{S},e}$ is recursively enumerable. Therefore, there is a recursive map $\theta \mapsto \lambda_\theta$ from $\mathcal{L}_{\mathrm{ring}}(K)$-sentences to test sentences such that $\theta \leftrightarrow \lambda_\theta$ is in $T_{\mathrm{almost},\mathfrak{S},e}$ for every θ. In particular, $\mu(\Sigma_{\mathfrak{S},e}(\theta)) = \mu(\Sigma_{\mathfrak{S},e}(\lambda_\theta))$. Since the map $\lambda \mapsto \mu(\Sigma_{\mathfrak{S},e}(\lambda))$ from test sentences to $\mathbb{Q}$ is also recursive by Lemma 11.11, so is the composition $\theta \mapsto \lambda_\theta \mapsto \mu(\Sigma_{\mathfrak{S},e}(\lambda_\theta)) = \mu(\Sigma_{\mathfrak{S},e}(\theta))$. □

Proof of Theorem 1.1. By applying Theorem 11.12 to Example 11.4, we finally deduce Theorem 1.1 given in § 1.

Remark 11.13. Note that the assumption that the primes in $\mathfrak{S}$ are recursive is *necessary*. Indeed, $R_{\mathfrak{p}}(K^{\mathfrak{S}})$ is K-definable in $K^{\mathfrak{S}}$ for each $\mathfrak{p} \in \mathfrak{S}$; cf. Proposition 7.12. An element $x \in K$ lies in $\mathcal{O}_{\mathfrak{p}}$ if and only if $x \in R_{\mathfrak{p}}(K^{\mathfrak{S}})$, so the decidability of the complete $\mathcal{L}_{\mathrm{ring}}(K)$-theory of $K^{\mathfrak{S}}$ implies that $\mathcal{O}_{\mathfrak{p}}$ is recursive. On the other hand, we do not know whether the assumption that K has a splitting algorithm is necessary.

Remark 11.14. The theorem does certainly not hold anymore if we allow $\mathfrak{S}$ to be an arbitrary (possibly infinite) set of recursive local primes of K. In fact, although there exist trivial examples of Hilbertian fields K with an infinite set of local primes $\mathfrak{S}$ such that $K^{\mathfrak{S}}$ is decidable, we do not know any infinite set of primes $\mathfrak{S}$ of $K = \mathbb{Q}$ for which the theorem holds. Moreover, [22, Example 10.4] gives an example of an infinite set of primes $\mathfrak{S}$ of $\mathbb{Q}$ *that has Dirichlet density zero*, but $\mathbb{Q}^{\mathfrak{S}} = \mathbb{Q}$, and hence $T_{\mathrm{almost},\mathfrak{S},e} = \mathrm{Th}(\mathbb{Q})$ is undecidable.

Remark 11.15. With the machinery developed here and some additional work one can show that Theorem 11.12 holds with $K^{\mathfrak{S}}(\boldsymbol{\sigma})$ replaced by the maximal Galois extension $K^{\mathfrak{S}}[\boldsymbol{\sigma}]$ of K inside $K^{\mathfrak{S}}(\boldsymbol{\sigma})$. We refer the interested reader to [13, Ch. 5], where this is shown for number fields K.

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A. Real closed fields

We recall the notion of real closed fields and quote some well-known results from [33]; see also [27].

Let K be a field. A *positive cone* of K is a semi-ring $P \subseteq K$ such that $P \cup (-P) = K$ and $P \cap (-P) = \{0\}$. An *ordering* of K is a total order $\leqslant$ on K such that $\{x \in K : x \geqslant 0\}$ is a positive cone. The map that assigns to an ordering the corresponding positive cone induces a natural bijection between the orderings of K and the positive cones of K. An *ordered field* is a field K together with an ordering. An ordering $\leqslant$ of K is *Archimedean* if for every $x \in K$ there exists $y \in \mathbb{N} \subseteq K$ with $x < y$. A *pre-positive cone* of K is a semi-ring $P \subseteq K$ such that $K^2 \subseteq P$ and $-1 \notin P$.

Lemma A.1. *Each pre-positive cone of K is the intersection of the positive cones of K containing it. In particular, each pre-positive cone of K is contained in a positive cone of K.*

Proof. See [33, 1.6]. □

A field is *real closed* if it has an ordering but each proper algebraic extension has no ordering. A real closed field K has a unique ordering, given by the positive cone K^2, [33, 3.2]. A real closed field F is a *real closure* of an ordered field K if F is an algebraic extension of K and the unique ordering of F extends the ordering of K. Any ordered field K has a real closure, which is unique up to K-isomorphism [33, 3.10]. If L is a finite extension of an ordered field K, then the extensions of the ordering of K to L bijectively correspond to the K-embeddings of L into a fixed real closure of K [33, 3.12].

Lemma A.2. *A field which is algebraically closed in a real closed field is real closed.*

Proof. See [33, 3.13]. □

Proposition A.3 (Artin–Schreier). *A field K is real closed if and only if* $\mathrm{Gal}(K) \cong \mathbb{Z}/2\mathbb{Z}$.

Proof. This follows from [28, VI.9.3] and [33, 3.3]. □

Proposition A.4. *Let V be a smooth absolutely irreducible variety over a real closed field K. If $V(K) \neq \emptyset$, then $V(K)$ is Zariski-dense in V.*

Proof. This follows for example from [33, 7.10]. □

Proposition A.5 (Tarski). *The $\mathcal{L}_{\mathrm{ring}}$-theory of real closed ordered fields is model complete.*

Proof. See [29, 3.3.15, 3.3.16] for model completeness in the language of ordered rings. Since a real closed field carries a unique ordering, this immediately implies model completeness in the language of rings. □

B. p-adically closed fields

We recall the notion of p-adically closed fields and quote some well-known results from [32] and some properties of the absolute Galois group of a p-adically closed field.

A valuation v on a field K of characteristic zero with residue field of characteristic $p > 0$ and corresponding valuation ring $\mathcal{O}$ is a *p-valuation of p-rank $d \in \mathbb{N}$* if $\dim_{\mathbb{F}_p}\mathcal{O}/p\mathcal{O} = d$. We also say that the valued field (K, v) is a *p-valued* field.

The residue field $\bar{K}_v$ of a p-valued field (K, v) is finite, and the value group $v(K^\times)$ is discrete and $v(p) \in \mathbb{Z}$. If $e = v(p)$ and $f = [\bar{K}_v : \mathbb{F}_p]$, then $d = ef$ [32, p. 15]. We call (p, e, f) the *type* of (K, v). Thus, if two p-valued fields have the same type, then they have the same p-rank. If L/K is an extension of p-valued fields, then L and K have the same p-rank if and only if they have the same type. In that case, this type is also the type of each intermediate extension of L/K.

A p-valued field is *p-adically closed* if it has no proper p-valued algebraic extension of the same p-rank. Every p-adically closed valued field (K, v) has a unique p-valuation [32, 6.15]. We therefore also call K p-adically closed. A *p-adic closure* of a p-valued field (K, v) is an algebraic extension of (K, v) which is p-adically closed of the same p-rank as (K, v). A p-valued field (K, v) is p-adically closed if and only if it is Henselian and the value group $v(K^\times)$ is a $\mathbb{Z}$-group [32, 3.1]. Here, an ordered abelian group Γ is a *$\mathbb{Z}$-group* if it is discrete and $(\Gamma : n\Gamma) = n$ for each $n \in \mathbb{N}$. Any p-valued field (K, v) has a p-adic closure. A p-adic closure of (K, v) is unique up to K-isomorphism if and only if $v(K^\times)$ is a $\mathbb{Z}$-group [32, 3.2].

Lemma B.1. *If a field is algebraically closed in a p-adically closed field K, then it is p-adically closed of the same p-rank as K.*

Proof. See [32, 3.4]. □

Proposition B.2. *Let V be a smooth absolutely irreducible variety over a p-adically closed field K. If $V(K) \neq \emptyset$, then $V(K)$ is Zariski-dense in V.*

Proof. This follows for example from [32, 7.8]. □

Proposition B.3. *The $\mathcal{L}_{\text{ring}}$-theory of p-adically closed fields of p-rank d is model complete.*

Proof. See [32, 5.1, 5.2, 5.6, 5.4] for model completeness in the language of valued fields. Since a p-adically closed field carries a unique p-valuation, this immediately implies model completeness in the language of rings. □

A *p-adic field* is the completion of a p-valued number field, i.e., a finite extension of the field of p-adic numbers $\mathbb{Q}_p$. A p-adic field F is p-adically closed of p-rank $[F : \mathbb{Q}_p]$, [32, p. 21].

Lemma B.4. *Every p-adically closed field is elementarily equivalent to a p-adic field.*

Proof. Let E be p-adically closed, and let $K = \tilde{\mathbb{Q}} \cap E$. By Lemma B.1, K is p-adically closed of the same p-rank as E; hence it contains a field $K_0 \cong \tilde{\mathbb{Q}} \cap \mathbb{Q}_p$. Then $[K : K_0] < \infty$, cf. [32, 2.9], so $F := K\mathbb{Q}_p$ is a p-adic field. Since K is algebraically closed in F, K and F have the same p-rank by Lemma B.1. Therefore, $E \equiv K \equiv F$ by model completeness (Proposition B.3). □

Proposition B.5. *Let K be p-adically closed. Then* $\mathrm{Gal}(K)$ *is finitely generated and torsion-free.*

Proof. Let K be a p-adically closed field. By Lemma B.4, K is elementarily equivalent to a p-adic field K_0. By [30, 7.4.1, 7.1.8(i)], $\mathrm{Gal}(K_0)$ is finitely generated and $\mathrm{cd}_l(\mathrm{Gal}(K)) = 2$

for every l, so in particular it is torsion free. The facts that $\mathrm{Gal}(K_0)$ is finitely generated and $K \equiv K_0$ imply that $\mathrm{Gal}(K) \cong \mathrm{Gal}(K_0)$, cf. [16, 20.4.6], and hence $\mathrm{Gal}(K)$ is also finitely generated and torsion free. □

Proposition B.6 (Neukirch–Pop–Efrat–Koenigsmann). *Let K be p-adically closed, and let L be a field. If $\mathrm{Gal}(K) \cong \mathrm{Gal}(L)$, then L is p-adically closed of the same type as K.*

Proof. By Lemma B.4, K is elementarily equivalent to a p-adic field K_0, and $\mathrm{Gal}(K) \cong \mathrm{Gal}(K_0)$ by Proposition B.5 and [16, 20.4.6]. By [26, Theorem 4.1], if $\mathrm{Gal}(K_0) \cong \mathrm{Gal}(L)$, then L is p-adically closed. But $\mathrm{Gal}(L)$ determines the type of L; see, for example, [24, Lemma 1]. □

C. Model theory of absolute Galois groups

We now review how to translate statements about the inverse system $S(\mathrm{Gal}(F))$ of the absolute Galois group of a field into statements about the field itself. Such a translation is claimed already in the unpublished [6], but without proof. In fact, the reader might be tempted to assume that one can prove [6, Lemma 17] by giving an honest interpretation of the ω-sorted structure $S(\mathrm{Gal}(F))^\omega$ in F, but experts think that this is actually impossible. On the other hand, the abstract proof presented in [4] is complete, but does not allow one to conclude that the translation is recursive. For this section, we fix $\mathfrak{S} = \emptyset$ and let $\mathcal{L}_{\mathrm{co}} = \mathcal{L}_{\mathrm{co},\emptyset}$.

Definition C.1. As explained for example in [6, §1], [4, (5.10)], and [1, Proposition 5.1], one can encode finite extensions of F of degree n by n^3-tuples $\mathbf{a} \in F^{n^3}$. We denote the extension defined by $\mathbf{a} \in F^{n^3}$ by $F_{\mathbf{a}}$. If $\mathbf{a}_1 \in F^{n_1^3}$ and $\mathbf{a}_2 \in F^{n_2^3}$ encode extensions of degree n_1 (respectively, n_2) then an F-embedding of $F_{\mathbf{a}_1}$ into $F_{\mathbf{a}_2}$ is encoded by an n_1n_2-tuple $\mathbf{b} \in F^{n_1 n_2}$.

Definition C.2. An *admissible* sequence of length $\lambda \leqslant \omega$ and degrees $(n_i)_{i<\lambda}$ is a sequence $\mathbf{a} = \mathbf{a}_1\mathbf{a}_2 \ldots$ of elements of F such that, for $i < \lambda$, the tuple $\mathbf{a}_i$ encodes

(1) a finite Galois extension $F_{\mathbf{a},i}$ of F of degree at most n_i,
(2) an automorphism $\sigma_{\mathbf{a},i} \in \mathrm{Gal}(F_{\mathbf{a},i}/F)$,
(3) a finite extension $F^*_{\mathbf{a},i}$ of F,
(4) an embedding $\epsilon_{\mathbf{a},i} : F_{\mathbf{a},i} \to F^*_{\mathbf{a},i}$, and,
(5) for each $j = 0, \ldots, i$, an embedding $\epsilon_{\mathbf{a},j,i} : F^*_{\mathbf{a},j} \to F^*_{\mathbf{a},i}$,

such that $F^*_{\mathbf{a},i}$ is the compositum of all $\epsilon_{\mathbf{a},j,i}(\epsilon_{\mathbf{a},j}(F_{\mathbf{a},j}))$, $j \leqslant i$, and, for $k \leqslant j \leqslant i$, $\epsilon_{\mathbf{a},i,i} = \mathrm{id}_{F^*_{\mathbf{a},i}}$ and $\epsilon_{\mathbf{a},j,i} \circ \epsilon_{\mathbf{a},k,j} = \epsilon_{\mathbf{a},k,i}$.

We intentionally do not keep track of the size of the tuples in order to avoid an overload of notation. We will also handle finite and infinite sequences rather sloppily.

Lemma C.3. *For each $\lambda < \omega$ and $\mathbf{n} = (n_i)_{i<\lambda}$ there is an $\mathcal{L}_{\mathrm{ring}}$-formula $\alpha_{\lambda,\mathbf{n}}$ such that, for a tuple $\mathbf{a}$ of elements of F, $F \models \alpha_{\lambda,\mathbf{n}}(\mathbf{a})$ iff $\mathbf{a}$ is admissible of length λ and degrees $\mathbf{n}$.*

Proof. This is clear; see also [6, §1]. □

Definition C.4. Every $x \in S(\mathrm{Gal}(F))$ is of the form gN for some open normal subgroup $N \trianglelefteq \mathrm{Gal}(F)$, and we denote by F_x the finite extension of F that is the fixed field of N, and by $\sigma_x \in \mathrm{Gal}(F_x/F)$ the automorphism corresponding to x under the natural isomorphism $\mathrm{Gal}(F)/N \cong \mathrm{Gal}(F_x/F)$ induced by restriction.

Definition C.5. A sequence $\mathbf{x} = (x_i)_{i<\lambda}$ of elements of $S(\mathrm{Gal}(F))$ and an admissible sequence $\mathbf{a}$ of length λ are *compatible* if there exist F-isomorphisms $\phi_i : F_{x_0} \cdots F_{x_i} \to F^*_{\mathbf{a},i}$ with $\phi_i(F_{x_i}) = \epsilon_{\mathbf{a},i}(F_{\mathbf{a},i})$ – so ϕ_i induces an isomorphism $\tilde{\phi}_i : F_{x_i} \to F_{\mathbf{a},i}$ – and $\tilde{\phi}_i \circ \sigma_{x_i} = \sigma_{\mathbf{a},i} \circ \tilde{\phi}_i$ for all $i < \lambda$, such that, for $j \leqslant i$, $\phi_i|_{F_{x_0}\cdots F_{x_j}} = \epsilon_{\mathbf{a},j,i} \circ \phi_j$.

Lemma C.6. *Let* $\mathbf{x}$ *and* $\mathbf{a}$ *be compatible sequences of length* $\lambda < \omega$.

(a) *For every* $x_\lambda \in S(\mathrm{Gal}(F))$ *there exists* $\mathbf{a}'$ *such that* $\mathbf{aa}'$ *is admissible of length* $\lambda+1$ *and* $(x_0, \dots, x_\lambda)$ *and* $\mathbf{aa}'$ *are compatible.*

(b) *For every* $\mathbf{a}'$ *such that* $\mathbf{aa}'$ *is admissible of length* $\lambda+1$ *there exists* $x_\lambda \in S(\mathrm{Gal}(F))$ *such that* $(x_0, \dots, x_\lambda)$ *and* $\mathbf{aa}'$ *are compatible.*

Proof. Let $\phi_i : F_{x_0} \cdots F_{x_i} \to F^*_{\mathbf{a},i}$ for $i < \lambda$ be as in Definition C.5.

(a). Choose $\mathbf{a}'$ such that there are isomorphisms $\phi' : F_{x_\lambda} \to F_{\mathbf{aa}',\lambda}$ and $\phi_\lambda : F_{x_0} \cdots F_{x_\lambda} \to F^*_{\mathbf{aa}',\lambda}$, and $\epsilon_{\mathbf{aa}',\lambda} = \phi_\lambda \circ \phi'^{-1}$, $\epsilon_{\mathbf{aa}',i,\lambda} = \phi_\lambda \circ \phi_i^{-1}$, $\sigma_{\mathbf{a},\lambda} = \phi' \circ \sigma_{x_\lambda} \circ \phi'^{-1}$.

(b). Extend the isomorphism $\phi_{\lambda-1}^{-1} : F^*_{\mathbf{a},\lambda-1} \to F_{x_0} \cdots F_{x_{\lambda-1}}$ to an embedding $\psi : F^*_{\mathbf{aa}',\lambda} \to (F_{x_0} \cdots F_{x_\lambda})\tilde{}$ with $\psi \circ \epsilon_{\mathbf{aa}',\lambda-1,\lambda} = \phi_{\lambda-1}^{-1}$, and choose x_λ such that $F_{x_\lambda} = \psi(\epsilon_{\mathbf{aa}',\lambda}(F_{\mathbf{aa}',\lambda}))$, and σ_{x_λ} corresponds to $\sigma_{\mathbf{aa}',\lambda}$ under $\psi \circ \epsilon_{\mathbf{aa}',\lambda}$. □

Definition C.7. We assign to each bounded $\mathcal{L}_{\mathrm{co}}$-formula $\varphi(v_0, \dots, v_{\lambda-1})$ and tuple $\mathbf{n} = (n_i)_{i<\omega}$ an $\mathcal{L}_{\mathrm{ring}}$-formula $\varphi_{\mathrm{ring},\mathbf{n}}(\mathbf{u})$, where $\mathbf{u} = \mathbf{u}_0\mathbf{u}_1 \dots$, as follows.

(1) If φ is $v_i \leqslant v_j$, then $\varphi_{\mathrm{ring},\mathbf{n}}$ expresses that $\alpha_{k+1,\mathbf{n}}(\mathbf{u})$, where $k = \max\{i, j\}$, and that $\epsilon_{\mathbf{u},i,k}(\epsilon_{\mathbf{u},i}(F_{\mathbf{u},i})) \supseteq \epsilon_{\mathbf{u},j,k}(\epsilon_{\mathbf{u},j}(F_{\mathbf{u},j}))$, so that $\epsilon_{\mathbf{u}}$ induces an embedding $F_{\mathbf{u},j} \to F_{\mathbf{u},i}$.

(2) If φ is $v_i \sqsubseteq v_j$, then $\varphi_{\mathrm{ring},\mathbf{n}}$ expresses that $\alpha_{k+1,\mathbf{n}}(\mathbf{u})$, where $k = \max\{i, j\}$, and that $\epsilon_{\mathbf{u}}$ induces an embedding $F_{\mathbf{u},j} \to F_{\mathbf{u},i}$, and, under this embedding, $\sigma_{\mathbf{u},i}|_{F_{\mathbf{u},j}} = \sigma_{\mathbf{u},j}$.

(3) If φ is $P(v_{i_1}, v_{i_2}, v_{i_3})$, then $\varphi_{\mathrm{ring},\mathbf{n}}$ expresses that $\alpha_{k+1,\mathbf{n}}(\mathbf{u})$, where $k = \max\{i_1, i_2, i_3\}$, and that the three embeddings $\epsilon_{\mathbf{u},i_\nu,k} \circ \epsilon_{\mathbf{u},i_\nu}$, $\nu = 1, 2, 3$, induce isomorphisms between the $F_{\mathbf{u},i_\nu}$, and, under these isomorphisms, $\sigma_{\mathbf{u},i_1} \circ \sigma_{\mathbf{u},i_2} = \sigma_{\mathbf{u},i_3}$.

(4) If φ is $v_i = v_j$, then $\varphi_{\mathrm{ring},\mathbf{n}}$ expresses that $\alpha_{k+1,\mathbf{n}}(\mathbf{u})$, where $k = \max\{i, j\}$, and that $\epsilon_{\mathbf{u}}$ induces an isomorphism $\phi : F_{\mathbf{u},i} \to F_{\mathbf{u},j}$, and $\phi \circ \sigma_{\mathbf{u},i} = \sigma_{\mathbf{u},j} \circ \phi$.

(5) If φ is $v_i \in G_n$, then $\varphi_{\mathrm{ring},\mathbf{n}}$ expresses that $\alpha_{i+1,\mathbf{n}}(\mathbf{u})$ and that $[F_{\mathbf{u},i} : F] \leqslant n$.

(6) If φ is of the form $\psi \wedge \eta$, then $\varphi_{\mathrm{ring},\mathbf{n}}$ is $\psi_{\mathrm{ring},\mathbf{n}} \wedge \eta_{\mathrm{ring},\mathbf{n}}$, and if φ is of the form $\neg\psi$, then $\varphi_{\mathrm{ring},\mathbf{n}}$ is $\neg\psi_{\mathrm{ring},\mathbf{n}}$.

(7) If φ is of the form $(\exists v_i \in G_n)(\psi)$ and $\lambda \geqslant i$ is minimal such that the free variables in ψ are among $v_0, \dots, v_\lambda$, then, after renumbering the variables, we can assume

that $i = \lambda$, and $\varphi_{\text{ring},\mathbf{n}}(\mathbf{u})$ is

$$(\exists \mathbf{u}')(\alpha_{\lambda+1,\mathbf{n}'}(\mathbf{u}\mathbf{u}') \wedge \psi_{\text{ring},\mathbf{n}'}(\mathbf{u}_0 \ldots \mathbf{u}_{\lambda-1}\mathbf{u}')),$$

where $\mathbf{n}' := (n_0, \ldots, n_{\lambda-1}, n)$.

If Σ is a ranked set of coformulas in the variables $(v_i)_{i<\lambda}$, we let $\mathbf{n} := \mathbf{n}_\Sigma := (n_i)_{i<\lambda}$ with n_i minimal such that the formula $G_{n_i}(v_i)$ is in Σ, and let Σ_{ring} consist of all $\alpha_{i,\mathbf{n}}$, for $i < \lambda$, and all $\varphi_{\text{ring},\mathbf{n}}$, for $\varphi \in \Sigma$.

Lemma C.8. *Let* $\mathbf{x}$, $\mathbf{a}$ *be compatible sequences of length* $\lambda \leqslant \omega$ *and degrees* $\mathbf{n}$, *and let* $\varphi(v_0, \ldots, v_{\lambda-1})$ *be a bounded* $\mathcal{L}_{\text{co}}$*-formula. Then* $S(\text{Gal}(F)) \models \varphi(\mathbf{x})$ *if and only if* $F \models \varphi_{\text{ring},\mathbf{n}}(\mathbf{a})$.

Proof. This is proven by case distinction according to Definition C.7. In cases (1)–(5), the claim follows immediately from the compatibility. In case (6), the claim is trivial by induction. In case (7), the claim follows by induction and Lemma C.6. □

Proposition C.9. *Let* Σ *be a ranked set of coformulas in the variables* $v_0, v_1, \ldots$. *Then* Σ *is cosatisfied in* $\text{Gal}(F)$ *if and only if* Σ_{ring} *is satisfied in* F.

Proof. Let $\mathbf{n} = \mathbf{n}_\Sigma$ as in Definition C.7.

If Σ is cosatisfied in $\text{Gal}(F)$ by $\mathbf{x} = (x_0, x_1, \ldots)$, then $[F_{x_i} : F] \leqslant n_i$ for all i. By applying Lemma C.6 iteratively we get an admissible sequence $\mathbf{a}$ of degrees $\mathbf{n}$ such that $\mathbf{x}$ and $\mathbf{a}$ are compatible. In particular, $F \models \alpha_{i,\mathbf{n}}(\mathbf{a})$ for every i. For every $\varphi \in \Sigma$, $S(\text{Gal}(F)) \models \varphi(\mathbf{x})$ implies that $F \models \varphi_{\text{ring},\mathbf{n}}(\mathbf{a})$ by Lemma C.8, so Σ_{ring} is satisfied in F.

Conversely, if Σ_{ring} is satisfied by a sequence $\mathbf{a}$, then, since Σ_{ring} contains all $\alpha_{i,\mathbf{n}}$, $\mathbf{a}$ is admissible of degrees $\mathbf{n}$. Lemma C.6 gives a sequence $\mathbf{x}$ in $S(\text{Gal}(F))$ such that $\mathbf{x}$ and $\mathbf{a}$ are compatible. For every $\varphi \in \Sigma$, $F \models \varphi_{\text{ring},\mathbf{n}}(\mathbf{a})$ implies that $S(\text{Gal}(F)) \models \varphi(\mathbf{x})$ by Lemma C.8, so Σ is satisfied in F. □

Corollary C.10. *There is a recursive map* $\varphi \mapsto \varphi_{\text{ring}}$ *from bounded* $\mathcal{L}_{\text{co}}$*-sentences to* $\mathcal{L}_{\text{ring}}$*-sentences such that, for any field* F, $S(\text{Gal}(F)) \models \varphi$ *if and only if* $F \models \varphi_{\text{ring}}$.

Proof. This is the special case $\Sigma = \{\varphi\}$ of Proposition C.9. □

Corollary C.11. *If* F *is* $\aleph_1$*-saturated, then* $\text{Gal}(F)$ *is* $\aleph_1$*-cosaturated.*

Proof. Let Σ be a countable ranked set of coformulas in the variables $v_0, v_1, \ldots$ with parameters in $S(\text{Gal}(F))$ for which every finite subset is cosatisfied in $\text{Gal}(F)$, and let $\mathbf{n} = \mathbf{n}_\Sigma$. For a finite subset

$$\Sigma_0' = \{(\varphi_1)_{\text{ring},\mathbf{n}}, \ldots, (\varphi_m)_{\text{ring},\mathbf{n}}, \alpha_{i_1,\mathbf{n}}, \ldots, \alpha_{i_l,\mathbf{n}}\} \subseteq \Sigma_{\text{ring}}$$

choose $\lambda \geqslant \max\{i_1, \ldots, i_l\}$ such that the free variables of the φ_i are among $v_0, \ldots, v_\lambda$, and let

$$\Sigma_0 := \{\varphi_1, \ldots, \varphi_m, G_{n_0}(v_0), \ldots, G_{n_\lambda}(v_\lambda)\}.$$

Since $(\varphi_i)_{\mathrm{ring},\mathbf{n}} = (\varphi_i)_{\mathrm{ring},\mathbf{n}_{\Sigma_0}}$, we see that $\Sigma_0' \subseteq (\Sigma_0)_{\mathrm{ring}}$, so, since Σ_0 is cosatisfied in $\mathrm{Gal}(F)$, Proposition C.9 gives that $(\Sigma_0)_{\mathrm{ring}}$, and hence Σ_0' is satisfied in F. Therefore, since F is $\aleph_1$-saturated, Σ_{ring} is satisfied in F, and hence Σ is cosatisfied in $\mathrm{Gal}(F)$, again by Proposition C.9. □

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