Extension of Hilbert's 1888 Theorem to Even Symmetric Forms

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To my parents

Rupa and Bharat Bhushan Goel

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Introduction and Overview

The question of deciding whether a real polynomial p positive semidefinite on $\mathbb{R}^n$ (i.e. $p \in \mathbb{R}[x_1, \ldots, x_n]$ and $p(\underline{x}) \geq 0 \ \forall \ \underline{x} \in \mathbb{R}^n$) can be written as a sum of squares of real polynomials (i.e. $p(\underline{x}) = \sum_i p_i(\underline{x})^2; p_i(\underline{x}) \in \mathbb{R}[x_1, \ldots, x_n]$) has many applications and has been studied extensively. Since a positive semidefinite (psd) polynomial always has even degree, it is sufficient to consider this question for even degree polynomials. Further upon homogenization, it is sufficient to study this question for forms, i.e. homogeneous polynomials (since the properties of being psd and sums of squares (sos) are preserved under homogenisation, see Lemma 1.36).

The most significant result in this direction was given by D. Hilbert [22] in 1888. His celebrated theorem states that a psd form is a sos if and only if $n = 2$ or $d = 1$ or $(n, 2d) = (3, 4)$, where n is the number of variables and $2d$ the degree of the form. Let $\mathcal{P}_{n,2d}$ and $\sum_{n,2d}$ denote the set of psd and sos $n-$ary $2d-$ic forms (i.e. forms of degree $2d$ in n variables, denoted by $\mathcal{F}_{n,2d}$) respectively. Hilbert made a careful study of quaternary quartics and ternary sextics, and demonstrated that $\Sigma_{3,6} \subsetneq \mathcal{P}_{3,6}$ and $\Sigma_{4,4} \subsetneq \mathcal{P}_{4,4}$. He in fact showed (see Proposition 1.49) that

$$\text{if } \Sigma_{4,4} \subsetneq \mathcal{P}_{4,4} \text{ and } \Sigma_{3,6} \subsetneq \mathcal{P}_{3,6}, \text{ then}$$
$$\Sigma_{n,2d} \subsetneq \mathcal{P}_{n,2d} \text{ for all } n \geq 3, 2d \geq 4 \text{ and } (n, 2d) \neq (3, 4). \tag{1}$$

In this thesis we will refer these two cases as the *basic cases*, since it is sufficient to produce psd not sos forms in these two crucial cases to get psd not sos forms in all remaining cases as in equation (1) above. In those two cases Hilbert described a method to produce examples of psd not sos forms, which was "elaborate and

unpractical" (see [7, p387]), so no explicit examples appeared in literature for next 80 years.
In 1967, T. S. Motzkin [29] presented a specific example $M(x,y,z) := z^6 + x^4y^2 + x^2y^4 - 3x^2y^2z^2$ of a ternary sextic form and showed (independently of Hilbert's method) that it is psd but not a sos. In 1969, R. M. Robinson [41] constructed examples of psd not sos ternary sextics as well as quaternary quartics, by drastically simplifying Hilbert's method (and independently of Motzkin). Further in 1974, more such examples were obtained by M.D. Choi and T.Y. Lam [5, 7, 8]. For instance they gave a quaternary quartic form $Q(x,y,z,w) = w^4 + x^2y^2 + y^2z^2 + z^2x^2 - 4xyzw$ and showed that it is psd not a sos .

In 1976, Choi and Lam [7] considered the question of deciding whether a psd form is a sos for the special case when the form considered is in addition symmetric (i.e. $n-$ary $2d-$ic form f such that $f^\sigma(x_1,\ldots,x_n) := f(x_{\sigma(1)},\ldots,x_{\sigma(n)}) = f(x_1,\ldots,x_n)\ \forall\ \sigma \in S_n$). They gave an answer to this question that classifies the pairs $(n,2d)$ for which a symmetric psd $n-$ary $2d-$ic form is a sos, namely,

$$S\mathcal{P}_{n,2d} = S\Sigma_{n,2d} \text{ iff } n = 2 \text{ or } d = 1 \text{ or } (n,2d) = (3,4).$$

Here $S\mathcal{P}_{n,2d}$ and $S\Sigma_{n,2d}$ denote the set of symmetric psd and symmetric sos $n-$ary $2d-$ic forms respectively. They demonstrated (see Proposition 3.3) that

$$\text{if } S\Sigma_{n,4} \subsetneq S\mathcal{P}_{n,4} \text{ for all } n \geq 4 \text{ and } S\Sigma_{3,6} \subsetneq S\mathcal{P}_{3,6}, \text{ then}$$

$$S\Sigma_{n,2d} \subsetneq S\mathcal{P}_{n,2d} \text{ for all } n \geq 3, 2d \geq 4 \text{ and } (n,2d) \neq (3,4). \tag{2}$$

They gave a specific example $f(x,y,z,w) = \sum x^2y^2 + \sum x^2yz - 2xyzw$ of a symmetric quaternary quartic form and showed (see proof of Proposition 3.4) that it is psd but not sos. Also Robinson [41] constructed the symmetric ternary sextic form $R(x,y,z) := x^6 + y^6 + z^6 - (x^4y^2 + y^4z^2 + z^4x^2 + x^2y^4 + y^2z^4 + z^2x^4) + 3x^2y^2z^2$ and showed that it is psd but not sos. So in view of (2) above, to get psd not sos symmetric $n-$ary $2d-$ic forms for all $n \geq 3, 2d \geq 4$ and $(n,2d) \neq (3,4)$, it remains to find psd not sos symmetric $n-$ary quartics for $n \geq 5$. We will do this in Section 3.1.2 of Chapter 3 (see Proposition 3.12 and Theorems 3.16, 3.17), i.e. we will construct explicit forms $f \in S\mathcal{P}_{n,4} \setminus S\Sigma_{n,4}$ for $n \geq 5$ thereby completing the answer to the question "when is a symmetric psd form a sos?".

In 1980, M.D. Choi, T.Y. Lam and B. Reznick [9] gave a test set (see Corollary 3.11 in Section 3.1.1) for symmetric quartics in $n \geq 4$ variables, where $\Omega \subseteq \mathbb{R}^n$ is a test set for a $n-$ary form f if f is psd iff $f(\underline{x}) \geq 0$ for all $\underline{x} \in \Omega$. This test set will play an important role in proving some of our results (e.g. Proposition 3.12) in Section 3.1.2. Test sets are particularly important since they allow us to determine directly whether a given form is psd or not just by checking its value at the points of the subset Ω of $\mathbb{R}^n$.

After this a lot of work was done to find test sets for even symmetric forms, specially even symmetric sextics by Choi, Lam, Reznick [10], even symmetric octics and ternary decics by W. R. Harris [20]; and their generalizations to test sets for symmetric and even symmetric polynomials of degree $2d$ in n variables by V. Timofte [49], D. Grimm [18], and C. Riener [39]. In 2003, Timofte [49] gave the following Half Degree Principle, which gives test sets for given symmetric and even symmetric polynomials:

- A symmetric real polynomial of degree $2d$ in n variables is non-negative (> 0 respectively) on $\mathbb{R}^n$ $\Leftrightarrow$ it is non-negative (> 0 respectively) on the subset $\Lambda_{n,k} := \{\underline{x} \in \mathbb{R}^n \mid$ number of distinct components in $\underline{x}$ is $\leq k\}$, where $k := \max\{2, d\}$. [If $d \geq 2$, then $\Lambda_{n,k} = \Lambda_{n,d}$].

- An even symmetric real polynomial of degree $2d \geq 4$ in n variables is non-negative (> 0 respectively) on $\mathbb{R}^n$ $\Leftrightarrow$ it is non-negative (> 0 respectively) on the subset $\Omega_{n,d/2} := \{\underline{x} \in \mathbb{R}^n_+ \mid$ number of distinct non-zero components in $\underline{x}$ is $\leq d/2\}$.

Timofte's half degree principle for symmetric polynomials was in fact a generalization of Corollary 3.11 from [9], since the work presented (for the ready reference of readers) in Section 3.1.1 from [9] was done much before [10] and [20].

We will consider the question of deciding whether a psd form is a sos for the special case when the form considered is in addition even symmetric (i.e. $n-$ary $2d-$ic symmetric form f such that in each term of $f(\underline{x})$ every variable has even degree, we will denote this by $S\mathcal{F}^e_{n,2d}$). Let $S\mathcal{P}^e_{n,2d}$ and $S\Sigma^e_{n,2d}$ denote the set of even symmetric psd and even symmetric sos $n-$ary $2d-$ic forms respectively. We are specifically interested in even symmetric forms because of the smaller dimension of the space of $S\mathcal{F}^e_{n,2d}$ and smaller test sets given by half degree principle. A partial known answer to this question is

- $S\mathcal{P}^e_{n,2d} = S\Sigma^e_{n,2d}$ if $n = 2, d = 1, (n, 2d) = (n, 4)_{n\geq 3}, (3, 8)$, and
- $S\mathcal{P}^e_{n,2d} \supsetneq S\Sigma^e_{n,2d}$ if $(n, 2d) = (n, 6)_{n\geq 3}, (3, 10), (4, 8)$.

For $n = 2, d = 1, (n, 2d) = (3, 4)$: $S\mathcal{P}^e_{n,2d} = S\Sigma^e_{n,2d}$ follows by Hilbert's Theorem; for $(n, 2d) = (n, 4)_{n\geq 4}$ we will give a proof in Proposition 4.1; Harris (in [20, 21]) proved that $S\mathcal{P}^e_{3,8} = S\Sigma^e_{3,8}$ and $S\mathcal{P}^e_{n,2d} \supsetneq S\Sigma^e_{n,2d}$ for $(n, 2d) = (3, 10), (4, 8)$; Choi, Lam and Reznick [10] proved that $S\mathcal{P}^e_{n,6} \supsetneq S\Sigma^e_{n,6}$ for $n \geq 3$.

For giving a further answer to this question, we will construct explicit forms $f \in S\mathcal{P}^e_{n,2d} \setminus S\Sigma^e_{n,2d}$ for the pairs $(n, 2d) = (3, 12)$, $(n, 8)_{n\geq 5}$ in Section 4.1 (see Propositions 4.9, 4.12, 4.15). We also give a Degree Jumping Principle (see Theorem 4.5) to find psd not sos even symmetric $n-$ary forms of degree $2d + 4r$ (for integer $r \geq 2$) and $2d + 2n$ from a given psd not sos even symmetric $n-$ary $2d-$ic form. We will then deduce that for the pairs $(n, 2d) = (n, 6)_{n\geq 3}, (n, 8)_{n\geq 4}, (3, 2d)_{d\geq 5}$, and $(n, 2d)_{n\geq 4, d\geq 7}$, the answer to this question is negative. This leads us to a version of Hilbert's 1888 Theorem for even symmetric forms (see Theorem 4.16), namely, $S\mathcal{P}^e_{n,2d} = S\Sigma^e_{n,2d}$ for $n = 2, d = 1, (n, 2d) = (n, 4)_{n\geq 3}, (3, 8)$, and $S\mathcal{P}^e_{n,2d} \supsetneq S\Sigma^e_{n,2d}$ for $(n, 2d) = (n, 6)_{n\geq 3}, (3, 2d)_{d\geq 5}, (n, 8)_{n\geq 4}, (n, 2d)_{n\geq 4, d\geq 7}$.

We will describe other results of this thesis by summarizing the chapters as follows:

In Chapter 1 we will provide most of the definitions and preliminary results which will be used in the rest of the chapters. We will give a characterization of symmetric forms via partitions of its degree. We then define the cones of psd and sos forms, give some of their properties, and explain in which cases they are equal and when there is a psd form that is not a sos. We close the chapter by defining Gram matrices, explaining their usage in obtaining a polynomial as a sos, and presenting the structure of Gram matrices for symmetric forms in the Hilbert cases where a psd form is always a sos and in the basic cases where a psd form might not always be a sos. This will be used later in Section 2.3 to present some Gram matrix tests for psdness of symmetric quadratic and ternary quartic forms.

In Chapter 2 we focus on the necessary and sufficient conditions for a form to be psd or a sos. We will start by surveying some known sufficient conditions on the coefficients of a form to be a sos by J. B. Lasserre [26], C. Fidalgo and A. Kovacec [13], and M. Ghasemi and M. Marshall [16, 17]; the last one will be used as one of the main tools later in Section 4.2 to find out when a psd even symmetric form that is a sos is in fact a sum of binomial squares (i.e. $f(\underline{x}) \in \mathcal{F}_{n,2d}$ such that f is a sum of squares of the form $(a\underline{x}^{\underline{\alpha}} - b\underline{x}^{\underline{\beta}})^2; \underline{\alpha}, \underline{\beta} \in \mathbb{N}^n$). In Section 2.2 we will recall test sets for psdness of symmetric quartics and even symmetric sextics by Choi, Lam, Reznick [10, 11]; of even symmetric octics and ternary decics by Harris [20]; and their generalizations to test sets for psdness of any symmetric and even symmetric polynomial by Timofte [49]. Further in Section 2.2.3 we deduce smaller test sets for even symmetric quartics and even symmetric ternary octics using Timofte's half degree principle. We will also give in Section 2.3 tests on the entries of a Gram matrix corresponding to a symmetric quadratic and ternary quartic form such that the form will be a sos. In the last Section 2.4 of this chapter, we will describe a filtration of intermediate cones between the sos and the psd cone and propose a generalization of Hilbert's theorem along the varieties containing the Veronese variety (Definition 2.30). It leads us to a reduced criterion for psdness and sosness of forms to psdness of quadratic forms on a subvariety of $\mathbb{R}^{N_0}$.

In Chapter 3, we will consider symmetric $n-$ary forms of degree $2d$ and revisit the question:

For what pairs $(n, 2d)$ will $S\mathcal{P}_{n,2d} \subseteq S\Sigma_{n,2d}$?, considered by Choi and Lam in [7]. We present our construction of explicit forms $p \in S\mathcal{P}_{n,4} \setminus S\Sigma_{n,4}$ for $n \geq 5$ in Section 3.1.2 (see Proposition 3.12 and Theorems 3.16, 3.17), thereby completing the answer of this question given by them. We will also present some results from [9] in Section 3.1.1 including Corollary 3.11 that gives test set for symmetric quartics in $n \geq 4$ variables, and will be used to prove Proposition 3.12.

In Chapter 4, we will consider even symmetric $n-$ary forms of degree $2d$ and investigate the question:

For what pairs $(n, 2d)$ will $S\mathcal{P}^e_{n,2d} \subseteq S^e\Sigma_{n,2d}$?

We will construct explicit forms $f \in S\mathcal{P}^e_{n,2d} \setminus S\Sigma^e_{n,2d}$ for the pairs $(n, 2d) = (3, 12)$, $(n, 8)_{n\geq5}$ in Section 4.1 (see Propositions 4.9, 4.12, 4.15) and give a Degree Jumping Principle (see Theorem 4.5). We will deduce that for $(n, 2d) = (n, 6)_{n\geq3}, (3, 2d)_{d\geq5}, (n, 8)_{n\geq4}, (n, 2d)_{n\geq4,d\geq7}$ an even symmetric psd form is not always a sos, and for the pairs $(n, 2d)$ with $n \geq 4, d = 5, 6$ we are still working on it. This leads to a version of Hilbert's 1888 Theorem for even symmetric forms (see Theorem 4.16). In Section 4.2 we will further work on the pairs $(n, 2d) = (n, 2), (2, 2d), (n, 4)_{n\geq3}, (3, 8)$ for which any psd even symmetric $n-$ary form f is a sos, and find out for which of these pairs f is in fact a sum of binomial squares (sobs). We will check this by applying a known sufficient condition (Theorem 2.5) on a form to be a sobs to special cases of even symmetric forms, and will see that:

- for the pairs $(n, 2d) = (n, 2), (2, 2d)_{d\leq3}, (n, 4)_{n\geq3}$ a psd (equivalently sos) even symmetric $n-$ary form of degree $2d$ is a sobs. This will follow from Proposition 4.18, Theorem 4.19 and Theorem 4.22 respectively, which are in fact much stronger results than "a psd even symmetric $n-$ary $2d-$ic form is a sobs for these pairs".
- a sos even symmetric binary form of degree $2d \geq 8$ is not necessarily a sobs (see Theorem 4.21).
- a sos even symmetric ternary octic form is not necessarily a sobs in gen-

eral (see Proposition 4.27), but we present some sufficient conditions under which an even symmetric ternary octic non-negative on just one point $(1,0,0)$ will be a sobs (see Proposition 4.26).

We will close this chapter by interpreting our results on even symmetric psd forms not being a sos (as in Theorem 4.16) in terms of preorderings (see Proposition 4.33), using the fact that to an even symmetric $n-$ary $2d-$ic psd form we can associate a symmetric $n-$ary $d-$ic form that is non-negative on $\mathbb{R}^n_+$. We will show how this interpretation relates to a result (Proposition 4.30) due to C. Scheiderer [45], strengthening degree bounds and in fact giving precise $(n, 2d)$ for which his result would hold for symmetric forms.

In Chapter 5, we will conclude our work and give some potential questions for future work.

Contents

Chapter 1

Preliminaries

In this Chapter we introduce some basic notations in real algebra dealing with the theory of sum of squares (sos) and positive semidefinite (psd) polynomials. We intent to survey and partially prove relevant facts; to prepare the reader for questions treated in the area of psd and sos polynomials in the later chapters. Suitable references are provided for the proofs that are omitted, to demonstrate the sequential advancements in the area.

Throughout we will denote the set of natural numbers $\{1, 2, 3, \dots\}$ by $\mathbb{N}$, the set of integers by $\mathbb{Z}$, the set of rationals by $\mathbb{Q}$, the set of reals by $\mathbb{R}$ and the set of complex numbers by $\mathbb{C}$. Also the set of non-negative integers will be denoted by $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ and the set of non-negative reals by $\mathbb{R}_+$. For a commutative ring A with 1, the ring of polynomials in n variables $x_1, \dots, x_n$ with coefficients in A will be denoted by $A[x_1, \dots, x_n]$ or $A[\underline{x}]$ for short.

1.1 Polynomials

Definition 1.1. For a **polynomial** $p \in \mathbb{R}[x_1, \dots, x_n]$, we write

$$p(\underline{x}) = \sum_{\underline{\alpha} \in \mathbb{Z}_+^n} a_{\underline{\alpha}}\, \underline{x}^{\underline{\alpha}}\ ;\ a_{\underline{\alpha}} \in \mathbb{R},$$

where $\underline{\alpha} = \{\alpha_1, \dots, \alpha_n\}$ is a n-tuple, $\underline{x}^{\underline{\alpha}} = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ is a **monomial** whose **degree** is $|\underline{\alpha}| = \sum_{k=1}^{n} \alpha_k$.

For $a_{\underline{\alpha}} \neq 0$, $a_{\underline{\alpha}}\ \underline{x}^{\underline{\alpha}}$ is called a **term** of $p(\underline{x})$. The **degree** of $p(\underline{x})$ is $\deg(p) = \max\{|\underline{\alpha}| : a_\alpha \neq 0 \text{ for some } \underline{\alpha} \in \mathbb{Z}_+^n\}$.

Definition 1.2. A polynomial p is said to be **irreducible** if it is non-constant and cannot be factored into a product of two or more non-constant polynomials.

Definition 1.3. A polynomial $p(\underline{x})$ is said to be **indefinite** if it takes both positive and negative values, i.e. less than 0 for some $\underline{x} \in \mathbb{R}^n$ and greater than 0 for others.

Definition 1.4. A field F is called **(formally) real** if -1 cannot be expressed as a sum of squares of elements of F. A **real closed field** F is a real field that has no non-trivial real algebraic extension $F_1 \supset F, F_1 \neq F$.

We recall the following result from [4], that we will use later in Chapter 3 for proving Lemma 3.2:

Theorem 1.5. *Let R be a real closed field and p an irreducible polynomial in $R[x_1, \dots, x_n]$. Then the following are equivalent:*

1. *$(p) = \mathcal{I}(Z(p))$, where $\mathcal{I}(A) = \{g \in R[\underline{x}] \mid g(\underline{a}) = 0 \ \ \forall\ \underline{a} \in A\}$ is the ideal of vanishing polynomials on $A \subseteq R^n$ and $Z(p) = \{\underline{x} \in R^n \mid p(\underline{x}) = 0\}$ is the zero set of p.*

2. *The sign of the polynomial p changes on R^n (i.e. $p(x)p(y) < 0$ for some $x, y \in R^n$).*

Proof. See [4, Theorem 4.5.1]. □

1.1.1 Homogeneous polynomials

Definition 1.6. A polynomial $p(\underline{x}) \in \mathbb{R}[\underline{x}]$ is called a **homogeneous polynomial** or **form** if all terms in p have the same degree.

Notation 1.7. $\mathcal{F}_{n,m} := \{f \in \mathbb{R}[x_1, \ldots, x_n] \mid f$ is a form, $\deg(f) = m\}$ denotes the set of all forms in n variables of degree m (called the "$n-$**ary** $m-$**ics** ") with real coefficients, for fixed $n, m \in \mathbb{N}$.
By convention $0 \in \mathcal{F}_{n,m}$.

Observation 1.8. By homogeneity:

$$f(\lambda \underline{x}) = \lambda^m (f(\underline{x})),\ \lambda \in \mathbb{R}, \tag{1.1}$$

for any $m-$ic form f (i.e. form of degree m).

Definition 1.9. The **homogenization** of a polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$ of degree m w.r.t. x_{n+1} is defined as

$$p_h(x_1, \ldots, x_n, x_{n+1}) := x_{n+1}^m\ p\left(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}\right).$$

Note that p_h is a homogeneous polynomial in $n+1$ variables $x_1, \ldots, x_n, x_{n+1}$ and of degree m, i.e. $p_h \in \mathcal{F}_{n+1,m}$.

We note the following known (see for example [43, p15]) fact:

Fact 1.10. Let $f(\underline{x}) \in \mathcal{F}_{n,m}$, then

$$\text{number of monomials of } f \text{ is} \leq N = N(n,m) := \binom{m+n-1}{n-1},$$

here $N = N(n,m)$ is the number of degree m monomials in n variables.

So the real vector space $\mathcal{F}_{n,m}$ is finite dimensional, in fact the following holds:

Proposition 1.11. The vector space $\mathcal{F}_{n,m}$ is isomorphic to $\mathbb{R}^N$.

Proof. For given $n, m \in \mathbb{N}$, consider

$$\alpha(n,m) := \{\underline{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \mid |\underline{\alpha}| = \alpha_1 + \ldots + \alpha_n = m\},$$

ordered lexicographically from left. This is a totally ordered set of cardinality $N = N(n,m) := \binom{m+n-1}{n-1}$. We enumerate it as $\alpha(n,m) = \{\underline{\alpha}(1), \ldots, \underline{\alpha}(N)\}$.

Now given a form $f = \sum_{\underline{\alpha}(i)\in\alpha(n,m)} a_{\underline{\alpha}(i)}\ \underline{x}^{\underline{\alpha}(i)} \in \mathcal{F}_{n,m}$, where $\underline{x}^{\underline{\alpha}} = x_1^{\alpha_1}\dots x_n^{\alpha_n}$ and $\{\underline{x}^{\underline{\alpha}(1)},\dots,\underline{x}^{\underline{\alpha}(N)}\}$ is the ordered monomial basis for $\mathcal{F}_{n,m}$. Consider the vector of coefficients (w.r.t. our ordered monomial basis) $\underline{a}_f := (a_1,\dots,a_N) \in \mathbb{R}^N$, where $a_i := a_{\underline{\alpha}(i)}$= coefficient of $\underline{x}^{\underline{\alpha}(i)}$ in f.

Fix the isomorphism

$$\begin{aligned}\lambda :\ \mathcal{F}_{n,m} &\longrightarrow \mathbb{R}^N\\ f &\longmapsto \underline{a}_f\end{aligned}$$

by identifying a form $f \in \mathcal{F}_{n,m}$ with the N-tuple of its coefficients in $\mathbb{R}^N$.

Thus we see that $\mathcal{F}_{n,m} \simeq \mathbb{R}^N$, and hence $\dim_{\mathbb{R}}\mathcal{F}_{n,m} = N$. □

1.1.2 Symmetric and even symmetric polynomials

Let S_n be the group of permutations of $\{1,\dots,n\}$, also called the **symmetric group** on n symbols.

Definition 1.12. A polynomial $p(\underline{x}) \in \mathbb{R}[x_1,\dots,x_n]$ is called **symmetric** iff it is unchanged by any permutation of its variables, i.e. iff

$$p^{\sigma}(x_1,\dots,x_n) := p(x_{\sigma(1)},\dots,x_{\sigma(n)}) = p(x_1,\dots,x_n),\ \forall\ \sigma \in S_n.$$

Similarly, a form $f(\underline{x}) \in \mathcal{F}_{n,m}$ is symmetric iff $f^{\sigma}(\underline{x}) = f(\underline{x})\ \forall\ \sigma \in S_n$.

In particular (see [20, definition 1.4]), let $f(\underline{x}) = \sum_{\underline{\alpha}=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n} a_{\underline{\alpha}} x_1^{\alpha_1}\dots x_n^{\alpha_n}$, then f is symmetric iff $a_{\underline{\alpha}}^{\sigma} := a_{(\alpha_{\sigma(1)},\dots,\alpha_{\sigma(n)})}$ is equal to $a_{\underline{\alpha}}$, $\forall\ \sigma \in S_n$.

Remark 1.13. We note some properties of a form as follows (which will be useful in the proof of Lemma 4.3 in Chapter 4):

1. Factors of a (non-constant) form are (non-constant) forms.
2. If $f \in \mathcal{F}_{n,m}$, then $f^{\sigma} \in \mathcal{F}_{n,m}$ for all $\sigma \in S_n$.
3. If $f \in \mathcal{F}_{n,m}$ is an irreducible form, then f^{σ} is also an irreducible form for all $\sigma \in S_n$.

Definition 1.14. (1) A form $f(\underline{x}) \in \mathcal{F}_{n,m}$ is **even** if in each term of $f(\underline{x})$ every variable has even degree.
(2) A form $f(\underline{x}) \in \mathcal{F}_{n,m}$ is **even symmetric** if it is both symmetric and even.

Fact 1.15. If $f \in \mathcal{F}_{n,m}$ is a symmetric form, then f is invariant under S_n, and if f is an even form, then f is invariant under the group $\mathbb{Z}_2^n = \{-1, 1\}^n$. Thus, even symmetric $n-$ary $m-$ic forms are invariant under the group $G = S_n \times \mathbb{Z}_2{}^n$.

Remark 1.16. If $f(x_1, \ldots, x_n)$ is an even symmetric form, then for all $\sigma \in S_n$:

$$g(x_1, \ldots, x_n) \mid f \;\Rightarrow\; g(x_{\sigma(1)}, \pm x_{\sigma(2)}, \ldots, \pm x_{\sigma(n)}) \mid f$$

We will use this property of an even symmetric form for proving Lemma 4.3 which will be crucial in the proof of our main Theorem 4.5 in Chapter 4.

Notation 1.17. Set

1. $\mathbb{R}[\underline{x}]^{S_n} := \{p \in \mathbb{R}[\underline{x}] \mid \sigma(p) = p \;\forall\; \sigma \in S_n\}$, the ring of symmetric polynomials in $\mathbb{R}[\underline{x}]$.

2. $S\mathcal{F}_{n,m} := \{f \in \mathcal{F}_{n,m} \mid \sigma(f) = f \;\forall\; \sigma \in S_n\}$, the set of symmetric forms in $\mathcal{F}_{n,m}$.

3. $S\mathcal{F}^e_{n,m}$:= the set of even symmetric forms in $\mathcal{F}_{n,m}$.

Note that $S\mathcal{F}^e_{n,m} \subsetneq S\mathcal{F}_{n,m} \subsetneq \mathcal{F}_{n,m}$.

1.1.3 Characterization of symmetric forms via partitions

In this section we will see how partitions of the degree m in $n-$parts characterize symmetric $n-$ary $m-$ic forms via their coefficients. We will use this coefficient characterization to

- compute a suitable basis of the subspace $S\mathcal{F}_{n,m}$ of symmetric forms of a given degree m in n number of variables (using Definition 1.18 and Fact 1.19 given below), and hence the dimension of $S\mathcal{F}_{n,m}$. We will see some selected examples in Section 1.1.4.

- study and characterize Gram matrices (see Definition 1.60) corresponding to a given $n-$ary $2d-$ ic symmetric form for $(n, 2d) = (n, 2), (2, 2d), (3, 4), (4, 4)$ and $(3, 6)$ in Section 1.4.1.

- give Gram matrix tests for psdness of symmetric quadratic and ternary quartic forms in Section 2.3.

Let f be a form of degree m in n variables. For every monomial of degree m in n variables associate a **partition** of m in $n-$parts as below:

if the monomial is $x_1^{d_1} \dots x_n^{d_n}$ with $d_1, \dots, d_n \in \mathbb{N} \cup \{0\}$, associate the partition $d_1 + \dots + d_n = m$ (from the exponents of the monomial).

Recall (from Proposition 1.11) that the monomials of degree m in n variables form a basis for the vector space $\mathcal{F}_{n,m}$ of all forms of degree m in n variables.
Also note that if two monomials associate to the same partition then there is a permutation σ of the variables which transforms one monomial into the other, and vice versa. So, we can define an equivalence relation on monomials of degree m in n variables as follows:

Definition 1.18. Two **monomials are equivalent** w.r.t. the symmetric group S_n iff they associate to the same partition of m.

The following fact follows from the above definition:

Fact 1.19. A form f of degree m in n variables is symmetric iff all equivalent monomials (appearing in the representation of f as a linear combination of monomials) appear with the same coefficient.

We observe the following:

Remark 1.20. There is no monomial basis for $S\mathcal{F}_{n,m}$.

Proof. Let f be a symmetric form of degree m in n variables. Consider the representation of f as a linear combination of monomials of degree m in n variables,

$$f = \lambda_1 M_1 + \dots + \lambda_N M_N, \text{ where } \lambda_i \in \mathbb{R}, N = N(n, m) := \binom{m+n-1}{n-1}.$$

Then all equivalent monomials appearing in this representation of f will appear with the same coefficient, since f is symmetric. So, f is a linear combination of symmetric forms $g_1, \ldots, g_{N_S}$; $g_r := \sum_j M_{r_j}, \forall\ 1 \leq r \leq N_S$, where j runs over all equivalent monomials in the equivalence class of a monomial.
So, there is no monomial basis for symmetric forms. □

Note that (as in the above proof) $\{g_1, \ldots, g_{N_S}\}$ spans $S\mathcal{F}_{n,m}$ and are linearly independent. So, $\{g_1, \ldots, g_{N_S}\}$ is a basis for $S\mathcal{F}_{n,m}$.

We need the following:

$$N_S = \dim(S\mathcal{F}_{n,m}) := \begin{cases} \text{the number of partitions of } m \text{ into at most } n\text{-parts} ;\, n < m \\ \text{the number of partitions of } m ;\, n \geq m. \end{cases}$$

Similarly for even symmetric forms (see [20, p205]), we have:

$$\dim(S\mathcal{F}^e_{n,m}) = \text{the number of partitions of } m \text{ into at most } n - \text{even parts.}$$

Using this fact we will compute a suitable basis of $S\mathcal{F}^e_{n,m}$ for some selected pairs (n, m) in Section 1.1.5.

1.1.4 Some examples of basis and dimension computations of a symmetric form via partitions of its degree

In this section we will compute a suitable basis and the dimension of $S\mathcal{F}_{n,m}$ for some selected n and m, i.e. the pairs $(n, m) = (n, 2), (2, 2d), (n, 4)_{n\geq 3}$ and $(3, 6)$.This will be used in characterizing Gram matrices corresponding to a given symmetric form in Section 1.4.1.

Example 1.21. Quadratic forms:
The partitions of $2d = 2$ are $2 + 0$ and $1 + 1$. So, $\left\{\sum_{i=1}^{n} x_i^2, \sum_{1\leq i<j\leq n} x_i x_j\right\}$ is a basis of symmetric quadratic forms, since it is linearly independent and generates $S\mathcal{F}_{n,2}$. So, $\dim(S\mathcal{F}_{n,2}) = 2$.

Example 1.22. Binary forms:
The partitions of $2d$ into at most $n = 2$ parts are $2d + 0, (2d - 1) + 1, \dots, (d + 1) + (d - 1), d + d$. So,

$$\left\{(x_1^{2d} + x_2^{2d}), (x_1^{2d-1}x_2 + x_1x_2^{2d-1}), \dots, (x_1^{d+1}x_2^{d-1} + x_1^{d-1}x_2^{d+1}), (x_1^d x_2^d)\right\}$$

is a basis of symmetric binary forms, since it is linearly independent and generates $S\mathcal{F}_{2,2d}$. So, $\dim(S\mathcal{F}_{2,2d}) = d + 1$.

Example 1.23. Ternary quartics:
The partitions of $2d = 4$ into at most $n = 3$ parts are $4+0$, $3+1$, $2+2$ and $2+1+1$. So,

$$\left\{\sum_{i=1}^{3} x_i^4,\ \sum_{1\le i\ne j\le 3} x_i^3x_j,\ \sum_{1\le i<j\le 3} x_i^2x_j^2,\ \sum_{\substack{1\le i\ne j\ne k\le 3\\ j<k}} x_i^2x_jx_k\right\}$$

is a basis of symmetric ternary quartics, since it is linearly independent and generates $S\mathcal{F}_{3,4}$. So, $\dim(S\mathcal{F}_{3,4}) = 4$.

Example 1.24. $n-$**ary quartics;** $n \ge 4$**:**
The partitions of $2d = 4$ are $4 + 0$, $3 + 1$, $2 + 2$, $2 + 1 + 1$ and $1 + 1 + 1 + 1$. So,

$$\left\{\sum_{i=1}^{n} x_i^4,\ \sum_{1\le i\ne j\le n} x_i^3x_j,\ \sum_{1\le i<j\le n} x_i^2x_j^2,\ \sum_{\substack{1\le i\ne j\ne k\le n\\ j<k}} x_i^2x_jx_k,\ \sum_{1\le i<j<k<l\le n} x_ix_jx_kx_l\right\}$$

is a basis of symmetric $n-$ary quartics for $n \ge 4$, since it is linearly independent and generates $S\mathcal{F}_{n,4}$. So, $\dim(S\mathcal{F}_{n,4}) = 5$ for $n \ge 4$.

Example 1.25. Ternary sextics:
The partitions of $2d = 6$ into at most $n = 3$ parts are $6 + 0$, $5 + 1$, $4 + 2$, $4 + 1 + 1$, $3 + 3$, $3 + 2 + 1$ and $2 + 2 + 2$. So,

$$\left\{\sum_{i=1}^{3} x_i^6,\ \sum_{1\le i\ne j\le 3} x_i^5x_j,\ \sum_{1\le i\ne j\le 3} x_i^4x_j^2,\ \sum_{\substack{1\le i\ne j\ne k\le 3\\ j<k}} x_i^4x_jx_k,\ \sum_{1\le i<j\le 3} x_i^3x_j^3,\ \sum_{1\le i\ne j\ne k\le 3} x_i^3x_j^2x_k,\ x_1^2x_2^2x_3^2\right\}$$

is a basis of symmetric ternary sextics, since it is linearly independent and generates $S\mathcal{F}_{3,6}$. So, $\dim(S\mathcal{F}_{3,6}) = 7$.

1.1.5 Some examples of basis computations of an even symmetric form via partitions of its degree

For fixed m, dim $(S\mathcal{F}^e_{n,m})$ is bounded as n goes to infinity and is substantially less than $N = N(n,m) := \binom{m+n-1}{n-1}$. Thus it is considerably easier to visualize the cones $S\mathcal{P}^e_{n,m}$ and $S\Sigma^e_{n,m}$ than $\mathcal{P}_{n,n}$ and $\sum_{n,m}$. For example, a general ternary sextic has $N(3,6) = 28$ coefficients, while the general even symmetric ternary sextic is

$$\alpha \sum_{i=1}^{3} x_i^6 + \beta \sum_{i \neq j} x_i^4 x_j^2 + \gamma x_1^2 x_2^2 x_3^2$$

(since partitions of 6 into at most 3 even parts are $6+0$, $4+2$ and $2+2+2$), which has only 3 coefficients.

We will compute, as below, a suitable basis of $S\mathcal{F}^e_{n,m}$ for the pairs $(n,m) = (n,2), (2,2d), (n,4)_{n\geq 3}, (n,6)_{n\geq 3}, (n,8)_{n\geq 3}$, and $(3,10)$. This will be used later in Section 2.2 and Chapter 4.

Example 1.26. Quadratic forms:
The only partition of $2d = 2$ into even parts is $2+0$. So, $\left\{ \sum_{i=1}^{n} x_i^2 \right\}$ is a basis of even symmetric quadratic forms, since it is linearly independent and generates $S\mathcal{F}^e_{n,2}$.

Example 1.27. Binary forms:
The partitions of $2d$ into at most $n = 2$ even parts are

$$\begin{cases} 2d+0, (2d-2)+2, \ldots, (d+1)+(d-1); \text{ if } d \text{ is odd, and} \\ 2d+0, (2d-2)+2, \ldots, (d+2)+(d-2), d+d; \text{ if } d \text{ is even.} \end{cases}$$

So,

$$\begin{cases} \left\{ (x_1^{2d} + x_2^{2d}), (x_1^{2d-2} x_2 + x_1 x_2^{2d-2}), \ldots, (x_1^{d+1} x_2^{d-1} + x_1^{d-1} x_2^{d+1}) \right\}; \text{ if } d \text{ is odd, and} \\ \left\{ (x_1^{2d} + x_2^{2d}), (x_1^{2d-2} x_2 + x_1 x_2^{2d-2}), \ldots, (x_1^{d+2} x_2^{d-2} + x_1^{d-2} x_2^{d+2}), x_1^d x_2^d \right\}; \text{ if } d \text{ is even} \end{cases}$$

is a basis of even symmetric binary forms, since it is linearly independent and generates $S\mathcal{F}^e_{2,2d}$.

Example 1.28. ***n*−ary quartics;** $n \geq 3$**:**
The only partitions of $2d = 4$ into even parts are $4 + 0$ and $2 + 2$. So,

$$\left\{ \sum_{i=1}^{n} x_i^4, \sum_{1 \leq i < j \leq n} x_i^2 x_j^2 \right\}$$

is a basis of even symmetric n−ary quartics for $n \geq 3$, since it is linearly independent and generates $S\mathcal{F}_{n,4}^e$.

Example 1.29. ***n*−ary sextics;** $n \geq 3$**:**
The partitions of $2d = 6$ into even parts are $6 + 0$, $4 + 2$ and $2 + 2 + 2$. So,

$$\left\{ \sum_{i=1}^{n} x_i^6, \sum_{1 \leq i \neq j \leq n} x_i^4 x_j^2, \sum_{1 \leq i < j < k \leq n} x_i^2 x_j^2 x_k^2 \right\}$$

is a basis of even symmetric n−ary sextics for $n \geq 3$, since it is linearly independent and generates $S\mathcal{F}_{n,6}^e$.

Example 1.30. ***n*−ary octics;** $n \geq 3$**:**
The partitions of $2d = 8$ into even parts are $8 + 0$, $6 + 2$, $4 + 4$, $4 + 2 + 2$ and $2 + 2 + 2 + 2$. So,

$$\left\{ \sum_{i=1}^{n} x_i^8, \sum_{1 \leq i \neq j \leq n} x_i^6 x_j^2, \sum_{1 \leq i < j \leq n} x_i^4 x_j^4, \sum_{\substack{1 \leq i \neq j \neq k \leq 3 \\ j < k}} x_i^4 x_j^2 x_k^2, \sum_{i < j < k < l} x_i^2 x_j^2 x_k^2 x_l^2 \right\}$$

(with the last element appearing only when $n \geq 4$) is a basis of even symmetric n−ary octics, since it is linearly independent and generates $S\mathcal{F}_{n,8}^e$.

Example 1.31. Ternary decics:
The partitions of $2d = 10$ into at most $n = 3$ even parts are $10 + 0$, $8 + 2$, $6 + 4$, $6 + 2 + 2$ and $4 + 4 + 2$. So,

$$\left\{ \sum_{i=1}^{3} x_i^{10}, \sum_{1 \leq i \neq j \leq 3} x_i^8 x_j^2, \sum_{1 \leq i \neq j \leq 3} x_i^6 x_j^4, \sum_{\substack{1 \leq i \neq j \neq k \leq 3 \\ j < k}} x_i^6 x_j^2 x_k^2, \sum_{\substack{1 \leq i \neq j \neq k \leq 3 \\ i < j}} x_i^4 x_j^4 x_k^2 \right\}$$

is a basis of even symmetric ternary decics, since it is linearly independent and generates $S\mathcal{F}_{3,10}^e$.

1.2 Positive semidefinite polynomials and sums of squares

Definition 1.32. (1) A polynomial $p(\underline{x}) \in \mathbb{R}[x_1, \ldots, x_n]$ is **positive semidefinite (psd)** if

$$p(\underline{x}) \geq 0 \ \forall \ \underline{x} \in \mathbb{R}^n.$$

We also say p is **non-negative** or simply $p \geq 0$.

(2) A homogeneous (respectively non-homogeneous) polynomial $p(\underline{x}) \in \mathbb{R}[x_1, \ldots, x_n]$ is **positive definite (pd)** if

$$p(\underline{x}) > 0 \ \forall \ \underline{x} \in \mathbb{R}^n \setminus \{0\} \text{ (respectively } \forall \ \underline{x} \in \mathbb{R}^n).$$

(3) A polynomial $p(\underline{x}) \in \mathbb{R}[x_1, \ldots, x_n]$ is a **sum of squares (sos)** if $\exists \ p_i(\underline{x}) \in \mathbb{R}[x_1, \ldots, x_n]$ s.t.

$$p(\underline{x}) = \sum_i p_i(\underline{x})^2.$$

(4) A polynomial $p(\underline{x}) \in \mathbb{R}[x_1, \ldots, x_n]$ is a **sum of binomial squares (sobs)**, if it is a sum of squares of the form

$$(a\underline{x}^{\underline{\alpha}} - b\underline{x}^{\underline{\beta}})^2, \text{ where } \underline{\alpha}, \underline{\beta} \in \mathbb{N}^n.$$

If $a = 0$ or $b = 0$, then $p(\underline{x})$ will be a sum of squares of monomials, considered as special cases of binomials.

Notation 1.33. Following Choi and Lam [7, 8] we adopt the following notation:

$\mathcal{P}_{n,m}$:= the set of all forms $f \in \mathcal{F}_{n,m}$ which are psd, and

$\sum_{n,m}$:= the set of all forms $f \in \mathcal{F}_{n,m}$ which are sos.

Remark 1.34. A psd form must have even degree, since if $f \in \mathcal{F}_{n,m}$ and m is odd then by homogeneity property (i.e. equation (1.1)) $f(-x) = (-1)^m f(x)$, which is ≥ 0 only when f is the zero form.

Clearly any polynomial which is a sos is non-negative on $\mathbb{R}^n$, but the converse is not always true. In 1888 Hilbert [22] solved the problem completely in the context of forms, we will see the details in Theorem 1.48. Before that lets see below some additional properties of psd and sos polynomials.

Lemma 1.35. (1) If a polynomial p is psd, then it must have even degree.

(2) If a polynomial $p(\underline{x})$ of degree m is sos, then m is even and any decomposition $p(\underline{x}) = \sum_i p_i(\underline{x})^2$, where $p_i(\underline{x}) \in \mathbb{R}[\underline{x}]$ satisfies $\deg(p_i(\underline{x})) \leq \frac{m}{2}\ \forall\ i$.

(3) Let p be a homogeneous polynomial of degree $2d$. If p is sos, then p is a sos of homogeneous polynomials (each of degree d).

Proof. See for example [27, p29]. □

Next we see in the lemma below that properties of being psd and sos are preserved under homogenisation. An equivalent statement to this lemma and its proof can be found in [28, p7].

Lemma 1.36. Let $p(\underline{x}) \in \mathbb{R}[x_1, \ldots, x_n]$ be a polynomial of degree m and (as in Definition 1.9) $p_h(x_1, \ldots, x_n, x_{n+1})$ its homogenisation w.r.t. x_{n+1}. Then

(1) $p \geq 0$ on $\mathbb{R}^n$ iff $p_h \geq 0$ on $\mathbb{R}^{n+1}$

(2) p is sos iff p_h is sos.

Proof. (1) If $p_h \geq 0$ on $\mathbb{R}^{n+1}$, then $p \geq 0$ on $\mathbb{R}^n$ follows from the fact that $p(\underline{x}) = p_h(\underline{x}, 1)$ for all $x \in \mathbb{R}^n$.
Conversely, if $p \geq 0$ on $\mathbb{R}^n$, then $\deg(p)$ is even, say $2d$, and

$$p_h(\underline{x}, x_{n+1}) = x_{n+1}^{2d} p_h\left(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}, 1\right) = x_{n+1}^{2d} p\left(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}\right) \geq 0$$

whenever $x_{n+1} \neq 0$. Thus $p_h \geq 0$ by continuity.

(2) If p_h is sos, then $p(\underline{x}) = p_h(\underline{x}, 1)$ is also sos.
Conversely, as in (1) an analogous argument shows that, if $p = \sum_j p_j^2$ with $p_j \in \mathbb{R}[\underline{x}]$, then $p_h = \sum_j p_{j_h}^2$, where p_{j_h} is the homogenisation of p_j. □

Observation 1.37. Since p and p_h are simultaneously psd or sos, so upon homogenization it is sufficient to study the question "does psd imply sos?" for forms.

Remark 1.38. Using Remark 1.34, from now on we will write the set of psd forms and the set of sos forms as $\mathcal{P}_{n,2d}$ and $\sum_{n,2d}$ respectively instead of $\mathcal{P}_{n,m}$ and $\sum_{n,m}$.

In 1888, Hilbert showed that a psd form is not in general a sum of squares of forms. Thus one led to study the cones $\mathcal{P}_{n,2d}$ and $\sum_{n,2d}$, and the relations between the two. We will see below that the sets $\mathcal{P}_{n,2d}$ and $\sum_{n,2d}$ are a special types of convex sets called cones, and present an interpretation of these two cones in terms of extremality.

1.2.1 The psd ($\mathcal{P}_{n,2d}$) and sos ($\sum_{n,2d}$) convex cones, and extremality

Definition 1.39. A subset C of $\mathbb{R}^n$ is **convex** if $\underline{a}, \underline{b} \in C \Rightarrow \lambda\underline{a} + (1-\lambda)\underline{b} \in C$; for $0 < \lambda < 1$.

Definition 1.40. The intersection of all convex sets containing a given subset $S \subseteq \mathbb{R}^n$ is called the **convex hull** of S.

Definition 1.41. A subset C of $\mathbb{R}^n$ is a **convex cone** if

$$\underline{x}, \underline{y} \in C \Rightarrow \underline{x} + \underline{y} \in C, \text{ and}$$
$$\underline{x} \in C, \lambda \in \mathbb{R}_+ \Rightarrow \lambda\underline{x} \in C$$

(i.e if it is closed under addition and under multiplication by non-negative scalar.)

Proposition 1.42. $\mathcal{P}_{n,2d}$ and $\sum_{n,2d}$ are closed convex cones.

Proof. See [37, p37]. □

Remark 1.43. $\sum_{n,2d}$ is a closed convex subcone of $\mathcal{P}_{n,2d}$.

By Proposition 1.11, $\mathcal{F}_{n,2d}$ can be identified with the vector space $\mathbb{R}^N$, where $N = \binom{2d+n-1}{n-1}$, and so $\mathcal{P}_{n,2d}$ and $\sum_{n,2d}$ can be viewed as closed convex cones in $\mathbb{R}^N$. Now lets look at extremality of the two cones.

Definition 1.44. Let C be any of the two cones $\mathcal{P}_{n,2d}$ or $\sum_{n,2d}$. A form $f \in C$ is said to be **extremal** in C if

$$f = f_1 + f_2; f_1, f_2 \in C \Rightarrow f_i = \lambda_i f; i = 1, 2 \text{ for } \lambda_i \in \mathbb{R}_+ \text{ s.t. } \lambda_1 + \lambda_2 = 1.$$

Notation 1.45. $\mathcal{E}(C)$ denotes the set of extremal elements in a cone C.

$\mathcal{E}(C)$ plays a major role in determining the structure and behavior of the cone C, since C is the convex hull of $\mathcal{E}(C)$ (see [7, p391]). In particular we have:

Lemma 1.46. Every $f \in \mathcal{P}_{n,2d}$ is a finite sum of forms in $\mathcal{E}(\mathcal{P}_{n,2d})$.

Proof. See [8, p1]. □

So the cone $\mathcal{P}_{n,2d}$ is completely determined when all its extremal elements are known. The following simple observation about extremal psd forms and their zeroes will motivate us to consider psd forms with non-trivial zeroes. It was central to the analysis done by Harris [20, 21] to prove that any psd even symmetric ternary octic is a sos, but there are psd not sos even symmetric quaternary octics and ternary decics; we will give more details of this in Chapter 4. Note that the homogeneity property (seen in Observation 1.8) allows us to look at forms projectively. Let $Z(f)$ be the set of projective real zeros of a form f.

Lemma 1.47. If f is extremal in $\mathcal{P}_{n,2d}$, then $Z(f) \neq \phi$.

Proof. See [20, p206]. □

1.2.2 Hilbert's 1888 Theorem for psd and sos cones

Clearly $\sum_{n,2d} \subseteq \mathcal{P}_{n,2d}$ (from Definition 1.32 and Notation 1.33). So it is natural to ask the following question:

$$(Q) : \text{For what pairs } (n, 2d) \text{ will } \mathcal{P}_{n,2d} \subseteq \Sigma_{n,2d}\ ? \tag{1.2}$$

In 1888, Hilbert [22] gave the following celebrated result that answers the above question (Q) completely and classifies the pairs $(n, 2d)$ for which the equality $\mathcal{P}_{n,2d} = \sum_{n,2d}$ holds:

Theorem 1.48. $\mathcal{P}_{n,2d} = \sum_{n,2d}$ iff $n = 2$ or $d = 1$ or $(n, 2d) = (3, 4)$.

The above answer to (Q) can be summarized by the following chart:

deg \ var	2	3	4	5	...
2	✓	✓	✓	✓	...
4	✓	✓	×	×	...
6	✓	×	×	×	...
8	✓	×	×	×	...
⋮	⋮	⋮	⋮	⋮	⋱

where, a tick (✓) denotes a positive answer to (Q), whereas a cross (×) denotes a negative answer to (Q).

Thus the question (Q) has an affirmative answer for binary forms, quadratic forms and ternary quartic forms, but a negative answer for all the other cases. In fact in the 3 cases where $\mathcal{P}_{n,2d} = \sum_{n,2d}$, the exact number of squares appearing in a sos representation are also known, as given below:

- $f \in \mathcal{P}_{2,2d}$ is a sum of squares of two binary forms of degree d.
- $f \in \mathcal{P}_{n,2}$ is a sum of squares of at most n linear forms.
- Any psd ternary quartic form is a sum of squares of ternary quadratics, and indeed three squares always suffice.

The arguments for the equality $\mathcal{P}_{n,2d} = \sum_{n,2d}$ for $n = 2$ and $d = 1$ are simple and were already known in the late 19th century. In the first case it follows from the factorization theory of binary forms and in the second case it follows from the diagonalization theorem of quadratic forms (see for example [24, Lecture 8]).

The statement $\mathcal{P}_{3,4} = \sum_{3,4}$ was originally proved by Hilbert [22] in 1888, moreover he showed that every psd ternary quartic is a sum of not more than three squares of quadratic forms. The central idea of Hilbert's proof is that one can associate to any ternary quartic a curve in the (complex) projective plane and then use the classically well-developed theory of algebraic curves.

Choi and Lam [8, p16] in 1977, gave an elementary proof of the equality of the two cones $\mathcal{P}_{3,4}$ and $\sum_{3,4}$, by exploiting extremal forms. They, however, did not show that only three quadratic forms suffice in such a sos representation.
A modern simplified version of Hilbert's proof due to Cassels, was given by Rajwade [34, p89] in 1993, this proof also shows that three squares suffice. In 2000, Rudin [42] and Swan [48] gave modern expositions of Hilbert's proof, which were more detailed than the original one. Also there are new modern proofs given by Powers, Scheiderer, Sottile and Reznick [32] in 2004, and by Pfister and Scheiderer [30] in 2012.

For proving the only if part of the Theorem 1.48, i.e. $\sum_{n,2d} \subsetneq \mathcal{P}_{n,2d}$ for all pairs $(n, 2d)$, $n \geq 3, 2d \geq 4$ and $(n, 2d) \neq (3, 4)$, Hilbert made a careful study of quaternary quartics and ternary sextics. He demonstrated that $\sum_{3,6} \subsetneq \mathcal{P}_{3,6}$ and $\sum_{4,4} \subsetneq \mathcal{P}_{4,4}$, and showed that for these two cases it is possible to construct psd not sos forms. In this thesis these two cases will be referred as the *basic cases*, since it is sufficient to produce psd not sos forms in these two cases to get psd not sos forms in all the following cases, i.e.

Proposition 1.49. If $\sum_{4,4} \subsetneq \mathcal{P}_{4,4}$ and $\sum_{3,6} \subsetneq \mathcal{P}_{3,6}$, then
$\sum_{n,2d} \subsetneq \mathcal{P}_{n,2d}$ for all $n \geq 3, 2d \geq 4$ and $(n, 2d) \neq (3, 4)$.

Proof. Trivially, $f \in \mathcal{P}_{n,2d} \setminus \sum_{n,2d} \Rightarrow f \in \mathcal{P}_{n+j,2d} \setminus \sum_{n+j,2d} \; \forall \; j \geq 0$
Moreover, we claim: $f \in \mathcal{P}_{n,2d} \setminus \sum_{n,2d} \Rightarrow x_1^{2i} f \in \mathcal{P}_{n,\, 2d+2i} \setminus \sum_{n,\, 2d+2i} \; \forall \; i \geq 0$
Indeed, assume for a contradiction that

$$x_1^2 f(x_1, \ldots, x_n) = \sum_{j=1}^{k} h_j^2(x_1, \ldots, x_n) \tag{1.3}$$

then L.H.S vanishes at $x_1 = 0$, so R.H.S also vanishes at $x_1 = 0$. It follows that $h_j(x_1, \ldots, x_n)$ vanishes at $x_1 = 0$ and so $x_1 \mid h_j \; \forall \; j$, so $x_1^2 \mid h_j^2 \; \forall \; j$. So, R.H.S of equation (1.3) is divisible by x_1^2.
Hence dividing both sides of equation (1.3) by x_1^2 we get a sos representation of f, a contradiction since $f \notin \sum_{n,2d}$. So, $x_1^{2i} f \in \mathcal{P}_{n,\, 2d+2i} \setminus \sum_{n,\, 2d+2i}$ for $i = 1$.
Proceeding similarly by induction on i, we will get

$$x_1^{2i} f \in \mathcal{P}_{n,\ 2d+2i} \setminus \Sigma_{n,\ 2d+2i} \ \forall\ i \geq 1.$$

□

We now give few examples of psd not sos forms in these two basic cases, from the literature:

1.2.3 Psd not sos ternary sextics and quaternary quartics

In the two basic cases Hilbert described a method to produce examples of psd not sos forms, which was elaborate and unpractical (as mentioned in [8, p387]), so no explicit examples appeared in literature for next 80 years.
In 1967 Motzkin [29] presented a specific example $M(x, y, z) := z^6 + x^4y^2 + x^2y^4 - 3x^2y^2z^2$ of a ternary sextic form and showed (independently of Hilberts method) that it is positive semidefinite but not a sum of squares.
Around the same time and independently of Motzkin, in 1973 R. M. Robinson [41] constructed examples of psd not sos ternary sextics as well as quaternary quartics based on the method described by Hilbert, but after drastically simplifying Hilbert's original ideas. For instance he showed that the form $R(x, y, z) := x^6 + y^6 + z^6 - (x^4y^2 + y^4z^2 + z^4x^2 + x^2y^4 + y^2z^4 + z^2x^4) + 3x^2y^2z^2 \in \mathcal{P}_{3,6} \setminus \sum_{3,6}$.
Further, in 1974, Choi and Lam [7, 8] discovered the two forms $Q(x, y, z, w) = w^4 + x^2y^2 + y^2z^2 + z^2x^2 - 4xyzw \in \mathcal{P}_{4,4} \setminus \sum_{4,4}$ and $S(x, y, z) = x^4y^2 + y^4z^2 + z^4x^2 - 3x^2y^2z^2 \in \mathcal{P}_{3,6} \setminus \sum_{3,6}$, using a slight variation of Motzkin's construction.

We note the following lemma which will be very useful in showing that a given form is psd:

Lemma 1.50. Arithmetic-geometric inequality:
For all $\alpha_i, x_i \geq 0$ with $\sum_{i=1}^{n} \alpha_i = 1$, $\alpha_1 x_1 + \ldots + \alpha_n x_n - x_1^{\alpha_1} \ldots x_n^{\alpha_n} \geq 0$,
and equality holds iff all the x_i are equal.

Proof. See for example [19, p17]. □

For proving that a given psd form is not a sos there are two methods from the 1970's. The first one compares coefficients (of the considered form to that of an assumed sos representation) by inspection of monomials involved, called term-inspection method and is in general simpler. This can be used to show that the forms $M(x, y, z)$, $Q(x, y, z, w)$ and $S(x, y, z)$ are not sos (see for example [8, p3]). The second one is Robinson's simplified version of Hilbert's method using zero sets, called Hilbert-Robinson method or zero-inspection method. This can be used to show that the forms $R(x, y, z)$ and $Q(x, y, z, w)$ are not sos (see [41, p 271], [34, p79] respectively), but its surprising that it fails to work for either of the two ternary sextics $M(x, y, z)$ and $S(x, y, z)$ (see for example [8, p5]).
Later in 1995, Choi, Lam and Reznick [11] developed a method for studying representations of a form as a sos, called the Gram matrix method. We will talk about it later in Section 1.4.

We will use the fact that the forms $R(x, y, z)$ and $Q(x, y, z, w)$ are psd and not sos, in the proof of Proposition 3.4 in Chapter 3, so we provide their proofs below.

Robinson's construction of a psd not sos real polynomial depends on the following lemma (from [41, p271]):

Lemma 1.51. A polynomial $P(x, y)$ of degree at most 3 which vanishes at eight of the nine points $(x, y) \in \{-1, 0, 1\} \times \{-1, 0, 1\}$ must also vanish at the ninth point.

Proof. Assign weights to the nine points as follows:

$$w(x, y) = \begin{cases} 1 \quad , \text{ if } x, y = \pm 1 \\ -2 \, , \text{ if } (x = \pm 1, y = 0) \text{ or } (x = 0, y = \pm 1) \\ 4 \quad , \text{ if } x, y = 0 \end{cases}$$

Define the weight of a monomial as:

$$w(x^k y^l) := \sum_{i=1}^{9} w(q_i) x^k y^l (q_i) \, , \ \text{ for } \ q_i \in \{-1, 0, 1\} \times \{-1, 0, 1\}$$

Define the weight of a polynomial $P(x, y) = \sum_{k,l} c_{k,l} \, x^k y^l$ as:

$$w(P) := \sum_{k,l} c_{k,l} \, w(x^k y^l)$$

One can see that:

$$w(P) := \sum_{k,l} c_{k,l}\, w(x^k y^l) = \sum_{k,l} c_{k,l} \sum_{i=1}^{9} w(q_i) x^k y^l(q_i)$$
$$= \sum_{i=1}^{9} w(q_i) \sum_{k,l} c_{k,l} x^k y^l(q_i) = \sum_{i=1}^{9} w(q_i) P(q_i) \tag{1.4}$$

Also, it is easy to see that $w(x^k y^l) = 0$ unless k and l are both strictly positive and even, by computing the monomial weights as follows

- if $k = 0, l \geq 0$: then we get
$$w(x^k y^l) = 1 + (-1)^l + 1 + (-1)^l + (-2) + (-2)(-1)^l = 0$$
- if $l = 0, k \geq 0$: then similarly we get $w(x^k y^l) = 0$, and
- if $k, l > 0$: then we get
$$w(x^k y^l) = 1 + (-1)^l + (-1)^k + (-1)^{k+l} = \begin{cases} 0\ , & \text{if either } k \text{ or } l \text{ is odd} \\ 4\ , & \text{otherwise.} \end{cases}$$

Thus, we see that $w(P) = 0$ if $\deg(P(x, y)) \leq 3$ and from equation (1.4) we get:

$$P(1,1) + P(1,-1) + P(-1,1) + P(-1,-1) + (-2)P(1,0)$$
$$+(-2)P(-1,0) + (-2)P(0,1) + (-2)P(0,-1) + 4P(0,0) = 0$$

Now it is clear that if $P(x, y)$ vanishes at any eight (of the nine) points, then it must also vanish at the ninth point □

The following can be found in [41, p272].

Theorem 1.52. The Robinson's form

$$R(x, y, z) := x^6 + y^6 + z^6 - (x^4y^2 + y^4z^2 + z^4x^2 + x^2y^4 + y^2z^4 + z^2x^4) + 3x^2y^2z^2$$
$$\in \mathcal{P}_{3,6} \setminus \Sigma_{3,6}.$$

Proof. Consider the polynomial

$$P(x, y) = (x^2 + y^2 - 1)(x^2 - y^2)^2 + (x^2 - 1)(y^2 - 1) \tag{1.5}$$

Note that $R(x, y, z) = P_h(x, y, z) = z^6 P(x/z, y/z)$.

By Lemma 1.36, it is enough to show that $P(x, y)$ is psd but not a sos.

Multiplying both sides of equation (1.5) by $(x^2 + y^2 - 1)$ and adding the result to equation (1.5) we get:

$$(x^2 + y^2)P(x, y) = x^2(x^2 - 1)^2 + y^2(y^2 - 1)^2 + (x^2 + y^2 - 1)^2(x^2 - y^2)^2 \qquad (1.6)$$

From equation (1.6) we see that $P(x, y) \geq 0$, i.e. $P(x, y)$ is psd.

Assume $P(x, y) = \sum_j P_j(x, y)^2$ s.t. $\deg(P_j) \leq 3 \ \forall \ j$.

By equation (1.5) it is easy to see that $P(0, 0) = 1$ and $P(x, y) = 0$ for all other eight points $(x, y) \in \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$, therefore every $P_j(x, y)$ must also vanish at these eight points. Hence by Lemma 1.51 each P_j must vanish at $(0, 0)$, thus $P(0, 0) = 0$, which is a contradiction. So, $P(x, y)$ cannot be a sos. □

The proof of the following proposition can be found in [7, p388] or [8, p3] or [34, p76].

Proposition 1.53. The form

$Q(x, y, z, w) := w^4 + x^2y^2 + y^2z^2 + z^2x^2 - 4xyzw \in \mathcal{P}_{4,4} \setminus \sum_{4,4}$.

Proof. The arithmetic-geometric inequality 1.50 clearly implies $Q \geq 0$.

Assume now that $Q = \sum_j q_j^2$, where $q_j \in \mathcal{F}_{4,2}$.

Now, forms in $\mathcal{F}_{4,2}$ can have only the following monomials:

$$x^2, y^2, z^2, w^2, xy, xz, xw, yz, yw, zw$$

If x^2 occurs in some of the q_j, then x^4 occurs in q_j^2 with positive coefficient and hence in $\sum_j q_j^2$ with positive coefficient too, but this is not the case. Similarly, since Q does not contain y^4 and z^4, we see that q_j does not contain y^2 and z^2.

The only way to write x^2w^2 as a product of allowed monomials is $x^2w^2 = (xw)^2$. Similarly for y^2w^2 and z^2w^2.

Thus each q_j involves only the monomials xy, xz, yz and w^2. But now there is no way to get the monomial $xyzw$ from $\sum_j q_j^2$, hence we get a contradiction. □

1.3 Positive semidefinite matrices

Definition 1.54. A **symmetric matrix** A is a square matrix that is equal to its transpose, i.e. $A = A^T$, i.e. if $A = (a_{ij}); a_{ij} \in \mathbb{R}$, then $(a_{ij}) = (a_{ji})$, for all indices i and j.

Proposition 1.55. For a real symmetric $n \times n$ matrix A, the following are equivalent:

(1) $x^T Ax \geq 0\ \forall x \in \mathbb{R}^n$.

(2) All eigenvalues of A are ≥ 0.

(3) $A = U^T U$ for some $n \times n$ matrix U.

(4) A is a non-negative linear combination of matrices of the form vv^T, $v \in \mathbb{R}^n$.

Proof. See [28, p2]. □

Definition 1.56. A square matrix A is **positive semidefinite (psd)** if A is real symmetric and the equivalent conditions of above proposition hold.
$A \succcurlyeq 0$ denotes that the matrix G is non-negative on $\mathbb{R}^n$.

Remark 1.57. In above proposition (1) ≡ (2) is equivalent (see [40, p25]) to the following:

(5) There is a unique lower triangular matrix $L \in M_{n\times n}(\mathbb{R})$ with $L_{jj} > 0$ for all j such that $LL^T = A$.

(6) All principal minors of A are non-negative.

Notation 1.58. We denote the vector space of all m by n matrices over $\mathbb{R}$ by $M_{m\times n}(\mathbb{R})$, and the set of all symmetric matrices in $M_{n\times n}(\mathbb{R})$ by $\mathrm{Sym}_n(\mathbb{R})$.

Remark 1.59. Let $A \in \mathrm{Sym}_n(\mathbb{R})$, then A is psd ⇔ the quadratic form $(x_1, \ldots, x_n)A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is a sos of linear forms in $\mathbb{R}[\underline{x}]$.

1.4 Gram matrices and sums of squares

Definition 1.60. Let $f \in \mathcal{F}_{n,m}$, then any $A \in \mathrm{Sym}_{N_0}(\mathbb{R})$ s.t. $\underline{v}A\underline{v}^T = f$ is called a **Gram matrix** of f, where $N_0 := \dim(\mathcal{F}_{n,d}) = \binom{d+n-1}{n-1}$ and $\underline{v}$:= vector (N_0-tuple) of a monomial basis for $\mathcal{F}_{n,d}$.

Such an A exists by following Proposition.

Proposition 1.61. Every polynomial $p \in \mathbb{R}[\underline{x}]$ of degree $2d$ can be written as $p = \underline{u}B\underline{u}^T$, where $\underline{u}$ is a vector of monomials of degree $\leq d$ and B a symmetric matrix. □

In 1995, Choi, Lam and Reznick [11] developed a general method for studying sums of squares in the polynomial ring $\mathbb{R}[\underline{x}]$, called the Gram matrix method. The gist of their method is that a sos representation $p = \sum h_i^2$ of a real polynomial corresponds to a real, symmetric psd matrix whose entries comes from the coefficients of h_i's. It is as explained in the proof of the Theorem 1.62 given below. Later, Powers and Wörmann [33] presented an algorithm to determine if a real polynomial is a sum of squares of polynomials and, in the affirmative case, to find an explicit representation. Their algorithm is based on the following theorem, that can be found in a slightly different form in [11, Theorem 2.4].

Theorem 1.62. A polynomial $p \in \mathbb{R}[\underline{x}]$ of degree $2d$ is a sum of squares iff there exists a vector $\underline{u}$ of monomials of degree $\leq d$ and a psd matrix B such that $p = \underline{u}^T B\underline{u}$.
Given such a matrix B of rank t, then we can construct polynomials $h_1, \ldots, h_t$ such that $p = \sum_{i=1}^{t} h_i^2$ and B is a Gram matrix of p associated to the h_i's.

Proof. ($\Rightarrow$) Assume that $p \in \mathbb{R}[\underline{x}]^2$, say $f = \sum_{i=1}^{t} h_i^2$, where each h_i has degree $\leq d$. Let $\underline{u} = \{\underline{x}^{\beta_k}, \ldots, \underline{x}^{\beta_N}\}$ be an enumeration of all possible monomials $\underline{x}^{\underline{\alpha}} = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ with $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ and $|\underline{\alpha}| := \alpha_1 + \ldots + \alpha_n \leq d$. Let A be the $N \times t$ matrix with $i-$th column as the coefficients of h_t. Then $p = \sum_{i=1}^{t} h_i^2$ can be written as $p = \underline{u}(A.A^T)\underline{u}^T$. Taking $B = A.A^T$, we are done since B is psd.

($\Leftarrow$) Conversely if $p = \underline{u}B\underline{u}^T$, where B is psd and rank of B is t. Then by Spectral theorem there exists a real matrix V and a real diagonal matrix $D = \operatorname{diag}(d_1, \ldots, d_t, 0, \ldots, 0)$ such that $B = V.D.V^T$ and $d_i \neq 0 \; \forall \; i$. So, $p = \underline{u}.V.D.V^T.\underline{u}^T$. Also each $d_i > 0 \; \forall \; i$, since B is psd. Suppose $V = (v_{i,j})$, then for $i = 1, \ldots, t$, set $h_i = \sqrt{d_i} \sum_{j=1}^{k} v_{j,i} \underline{x}^{\beta_i} \in \mathbb{R}[\underline{x}]$. It follows that $p = \sum_{i=1}^{t} h_i^2$. □

Note that by considering all expressions of p as a sos, we get a family of associated Gram matrices.

We will present some Gram matrix tests for psdness of symmetric quadratic and ternary quartic forms in Section 2.3. Before that we study here the special structure of Gram matrices of symmetric forms for some cases as follows:

1.4.1 Gram matrices of symmetric forms using coefficient characterization

Given a real symmetric $N_0 \times N_0$ matrix G, we aim at finding necessary and sufficient conditions on (the entries of) G s.t. $\underline{v}G\underline{v}^T$ is a symmetric form, where $\underline{v} := \{\underline{x}^{\beta(1)}, \ldots, \underline{x}^{\beta(N_0)}\}$ is a monomial basis for $\mathcal{F}_{n,d}$, $\underline{x}^{\underline{\beta}} = x_1^{\beta_1} \ldots x_n^{\beta_n}$ for $\underline{\beta} = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$ s.t. $|\underline{\beta}| = \beta_1 + \ldots + \beta_n = d$.

Using coefficient characterization of forms via partitions (from Section 1.1.3), we will present conditions on Gram matrices G giving symmetric $n-$ary $2d-$ic forms $\underline{v}G\underline{v}^T$ for the pairs $(n, 2d) = (n, 2), (2, 2d), (3, 4), (4, 4)$, and $(3, 6)$.

Remark 1.63. The values of N_0 for the pairs we will be considering in the coming Lemmas are as below:

1. For $(n, 2d) = (n, 2)$, $N_0 = N(n, 1) := \binom{1+n-1}{n-1} = n$.

2. For $(n, 2d) = (2, 2d)$, $N_0 = N(2, d) := \binom{d+2-1}{2-1} = d + 1$
3. For $(n, 2d) = (3, 4)$, $N_0 = N(3, 2) := \binom{2+3-1}{3-1} = 6$.
4. For $(n, 2d) = (4, 4)$, $N_0 = N(4, 2) := \binom{2+4-1}{4-1} = \binom{5}{3} = 10$.
5. For $(n, 2d) = (3, 6)$, $N_0 = N(3, 3) := \binom{3+3-1}{3-1} = \binom{5}{2} = 10$.

Lemma 1.64. If $f = a\Big(\sum_{i=1}^{n} x_i^2\Big) + 2b\Big(\sum_{1\le i\ne j\le n} x_i x_j\Big)$ is a quadratic form, and $G = (G_{ij})$ be a given real symmetric $n \times n$ matrix such that $f = \underline{v} G \underline{v}^T$; with $\underline{v} = (x_1, \ldots, x_n)$. Then

$$f \text{ is symmetric} \Leftrightarrow G_{ij} := \begin{cases} a \text{ if } \ i = j, \\ b \text{ if } \ i \ne j. \end{cases}$$

Lemma 1.65. If $f = b_0(x_1^{2d} + x_2^{2d}) + b_1(x_1^{2d-1}x_2 + x_1 x_2^{2d-1}) + \ldots + b_{d-1}(x_1^{d+1}x_2^{d-1} + x_1^{d-1}x_2^{d+1}) + b_d(x_1^d x_2^d)$ is a binary form, and $G = (G_{ij})$ be a given real symmetric $(d+1) \times (d+1)$ matrix s.t. $f(\underline{x}) = \underline{v} G \underline{v}^T$; with $\underline{v} := (x_1^d, x_1^{d-1}x_2, \ldots, , x_1 x_2^{d-1}, x_2^d)$. Then

$$f \text{ is symmetric} \Leftrightarrow G \text{ is defined by} \begin{cases} G_{11} = G_{d+1,d+1} = b_0 \ , \\ G_{12} = G_{d,d+1} = \dfrac{b_1}{2} \ , \\ 2G_{13} + G_{22} = 2G_{d-1,d+1} + G_{d,d} = b_2 \ , \\ G_{14} + G_{23} = G_{d-2,d+1} + G_{d-1,d} = \dfrac{b_3}{2} \ , \\ 2G_{15} + 2G_{24} + G_{33} = b_4 \ , \\ G_{16} + G_{25} + G_{34} = \dfrac{b_5}{2} \ , \\ 2G_{17} + 2G_{256} + 2G_{35} + G_{44} = b_6 \ , \\ \vdots \end{cases} .$$

Lemma 1.66. If

$f = \alpha\Big(\sum_{i=1}^{3} x_i^4\Big) + \beta\Big(\sum_{1\le i\ne j\le 3} x_i^3 x_j\Big) + \gamma\Big(\sum_{1\le i<j\le 3} x_i^2 x_j^2\Big) + \delta\Big(\sum_{1\le i\ne j\ne k\le 3} x_i^2 x_j x_k\Big)$ is a ternary

quartic form, and $G = (G_{ij})$ be a given real symmetric 6×6 matrix s.t. $f(\underline{x}) = \underline{v}G\underline{v}^T$; with $\underline{v} := (x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3)$. Then

$$f \text{ is symmetric} \Leftrightarrow G \text{ is defined by} \begin{cases} G_{11} = G_{22} = G_{33} = \alpha\ , \\ G_{14} = G_{15} = G_{24} = G_{26} = G_{35} = G_{36} = \dfrac{\beta}{2}\ , \\ 2G_{12} + G_{44} = 2G_{13} + G_{55} = 2G_{23} + G_{66} = \gamma\ , \\ G_{16} + G_{45} = G_{25} + G_{46} = G_{34} + G_{56} = \dfrac{\delta}{2}. \end{cases}$$

Lemma 1.67. If

$$f = \alpha\Big(\sum_{i=1}^{4} x_i^4\Big) + \beta\Big(\sum_{1\le i\ne j\le 4} x_i^3x_j\Big) + \gamma\Big(\sum_{1\le i<j\le 4} x_i^2x_j^2\Big) + \delta\Big(\sum_{1\le i\ne j\ne k\le 4} x_i^2x_jx_k\Big) + \epsilon x_1x_2x_3x_4$$

is a quaternary quartic form, and $G = (G_{ij})$ be a real symmetric 10×10 matrix s.t. $f = \underline{v}G\underline{v}^T$; with $\underline{v} := (x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4)$. Then f is symmetric $\Leftrightarrow G =$ is defined by

$$\begin{cases} G_{11} = G_{22} = G_{33} = G_{44} = \alpha\ , \\ G_{15} = G_{16} = G_{17} = G_{25} = G_{36} = G_{47} = G_{28} = G_{29} = G_{38} = G_{49} = G_{3,10} = G_{4,10} = \dfrac{\beta}{2}\ , \\ 2G_{12} + G_{55} = 2G_{13} + G_{66} = 2G_{14} + G_{77} = 2G_{23} + G_{88} = 2G_{24} + G_{99} = 2G_{34} + G_{10,10} = \gamma\ , \\ G_{18} + G_{56} = G_{19} + G_{57} = G_{1,10} + G_{67} = G_{26} + G_{58} = G_{27} + G_{59} = G_{35} + G_{68} = G_{45} + G_{79} \\ \qquad = G_{37} + G_{6,10} = G_{46} + G_{7,10} = G_{2,10} + G_{89} = G_{39} + G_{8,10} = G_{48} + G_{9,10} = \dfrac{\delta}{2}\ , \\ G_{5,10} + G_{69} + G_{78} = \dfrac{\epsilon}{2}. \end{cases}$$

Lemma 1.68. If

$$\begin{aligned} f = {} & \alpha_1\Big(\sum_{i=1}^{3} x_i^6\Big) + \alpha_2\Big(\sum_{1\le i\ne j\le 3} x_i^5x_j\Big) + \alpha_3\Big(\sum_{1\le i\ne j\le 3} x_i^4x_j^2\Big) + \alpha_4\Big(\sum_{1\le i\ne j\ne k\le 3} x_i^4x_jx_k\Big) \\ & +\alpha_5\Big(\sum_{1\le i<j\le 3} x_i^3x_j^3\Big) + \alpha_6\Big(\sum_{1\le i\ne j\ne k\le 3} x_i^3x_j^2x_k\Big) + \alpha_7\Big(x_1^2x_2^2x_3^2\Big) \end{aligned}$$

is a ternary sextic form, and $G = (G_{ij})$ be a given real symmetric 10×10 matrix s.t. $f = \underline{v}G\underline{v}^T$; with $\underline{v} := (x_1^3, x_2^3, x_3^3, x_1^2x_2, x_1^2x_3, x_1x_2^2, x_2^2x_3, x_1x_3^2, x_2x_3^2, x_1x_2x_3)$. Then f is symmetric $\Leftrightarrow G$ is defined by,

$$
\begin{cases}
G_{11} = G_{22} = G_{33} = \alpha_1 \ , \\
G_{14} = G_{15} = G_{26} = G_{27} = G_{38} = G_{39} = \dfrac{\alpha_2}{2} \ , \\
2G_{16} + G_{44} = 2G_{18} + G_{55} = 2G_{24} + G_{66} = 2G_{35} + G_{88} = 2G_{29} + G_{77} \\
\qquad = 2G_{37} + G_{99} = \alpha_3 \ , \\
G_{1,10} + G_{45} = G_{2,10} + G_{67} = G_{3,10} + G_{89} = \dfrac{\alpha_4}{2} \ , \\
G_{12} + G_{46} = G_{13} + G_{58} = G_{23} + G_{79} = \dfrac{\alpha_5}{2} \ , \\
G_{17} + G_{4,10} + G_{56} = G_{19} + G_{48} + G_{5,10} = G_{25} + G_{47} + G_{6,10} = G_{34} + G_{59} + G_{8,10} \\
\qquad = G_{28} + G_{69} + G_{7,10} = G_{36} + G_{78} + G_{9,10} = \dfrac{\alpha_6}{2} \ , \\
G_{10,10} + 2G_{49} + 2G_{57} + 2G_{68} = \alpha_7 .
\end{cases}
$$

Chapter 2

Tests for a form to be psd or sos

In this chapter we discuss necessary and sufficient conditions for a form to be psd or sos. We will present some known sufficient conditions on the coefficients of a form to be a sos [13, 16, 17, 26], from which Theorem 2.5 will be used as one of the main tools later in Section 4.2 to see when a psd even symmetric form that is a sos is in fact a sobs. In Section 2.2 we will recall some known test sets for psdness of symmetric quartics [9], even symmetric sextics [10], even symmetric octics, even symmetric ternary decics [20]; and their generalizations to test sets for psdness of any symmetric and even symmetric form [18, 39, 49]. Suitable references are provided for the proofs that are omitted, to indicate the chronological order of advancements in the area. Further we deduce smaller test sets for even symmetric quartics and even symmetric ternary octics using Timofte's Half degree principle (Theorem 2.16). We will also give in Section 2.3 tests on the entries of a Gram matrix corresponding to a symmetric quadratic and ternary quartic form so that the form is a sos. In the last Section 2.4 of this chapter, we will describe a filtration of intermediate cones between the sos and the psd cone and propose a generalization of Hilbert's theorem along the varieties containing the Veronese variety (see Definition 2.30). It leads us to a reduced criterion for psdness and sosness of forms to psdness of quadratic forms on a sub variety of $\mathbb{R}^{N_0}$.

2.1 Coefficient tests for sosness

In 1891, Hurwitz [23] gave an explicit representation of the form $f(x_1, \dots, x_{2d}) = \sum_{i=1}^{2d} x_i^{2d} - 2d \prod_{i=1}^{2d} x_i$ as a sum of squares using symmetric polynomials in $x_1, \dots, x_{2d}$. Note that here the degree of the form considered is same as the number of variables. Later, in 1987, Reznick [35, 36] gave the following result (in a slightly different form), that can be deduced from this representation:

Theorem 2.1. Suppose $f(\underline{x}) = \sum_{i=1}^{n} \alpha_i x_i^{2d} - 2d x_1^{\alpha_1} \dots x_n^{\alpha_n}$, where $\underline{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n, |\underline{\alpha}| = 2d$. Then f is a sobs.

Proof. See [19] or [15, p34]. □

2.1.1 Necessary and Sufficient conditions on diagonal minus tail forms

A diagonal minus tail form is defined (as in [13]) as:

Definition 2.2. A **diagonal minus tail (dmt)** form has the shape $f(x_1, \dots, x_n) = f(\underline{x}) = D(\underline{x}) - T(\underline{x})$, where the diagonal part $D(\underline{x})$, and tail $T(\underline{x})$ are defined by $D(\underline{x}) = \sum_{i=1}^{n} b_i x_i^{2d}$, and $T(\underline{x}) = \sum_{\underline{i} \in I} a_{\underline{i}} \underline{x}^{\underline{i}}$, with $d \in \mathbb{Z}_{\geq 1}, b_i, a_{\underline{i}} \geq 0$, $\underline{x}^{\underline{i}} = x_1^{i_1} \dots x_n^{i_n}$, and $I \subseteq \left\{ \underline{i} = (i_1, \dots, i_n) \in \mathbb{N}^n : 0 \leq i_1, \dots, i_n \leq 2d - 1, |\underline{i}| := \sum_{k=1}^{n} i_k = 2d \right\}$.

A dmt form is called **elementary** if its tail consists of at most one term.

In [13], Fidalgo and Kovacec gave necessary and sufficient conditions for an elementary dmt form to be psd or sos. The following result is a slightly different form of that and is a consequence of Theorem 2.1.

Theorem 2.3. For a form $E(\underline{x}) = \sum_{i=1}^{n} \beta_i x_i^{2d} - \mu \underline{x}^{\underline{\alpha}}$ such that $\underline{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n, |\underline{\alpha}| = 2d$, $\beta_i \in \mathbb{R}_+$ for $i = 1, \dots, n$, and $\mu \geq 0$ if all α_i are even, the following are equivalent:

1. E is psd
2. $\mu^{2d} \prod_{i=1}^{n} \alpha_i^{\alpha_i} \leq (2d)^{2d} \prod_{i=1}^{n} \beta_i^{\alpha_i}$
3. E is a sobs and hence a sos.

Proof. See [13, p631] or [15, p35]. □

Theorem 2.3 above shows that elementary psd dmt forms are sobs. In fact all psd dmt forms are sobs [13, p636].

2.1.2 Sufficient conditions on forms

In [17], Ghasemi and Marshall gave sufficient conditions on the coefficients of a real form to be a sos. We recall their main result as Theorem 2.5 below, that we will use as one of the main tools in Section 4.2 for showing when an even symmetric binary, even symmetric quartic and even symmetric ternary octic form is a sobs. They proved this result using Theorem 2.3.

Notation 2.4. Every polynomial $f \in \mathbb{R}[\underline{x}]$ can be written as $f(\underline{x}) = \sum_{\alpha \in \mathbb{N}^n} f_\alpha \underline{x}^\alpha$, for finitely many α. Assume now that f is non-constant and has even degree $2d$.

Let $\Omega_f := \{\alpha \in \mathbb{N}^n \mid f_\alpha \neq 0\} \setminus \{\underline{0}, 2d\epsilon_1, \ldots, 2d\epsilon_n\}$ where $f_0 := f_{\underline{0}}$, $f_{2d,i} := f_{2d\epsilon_i}$, and

$\epsilon_i := (\delta_{i1}, \ldots, \delta_{in})$ with $\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j. \end{cases}$

Let $\Delta_f := \{\alpha \in \Omega_f \mid f_\alpha \underline{x}^\alpha \text{ is not a square}\} = \{\alpha \in \Omega_f \mid f_\alpha < 0 \text{ or } \alpha \notin 2\mathbb{N}^n\}$.

Then $f(\underline{x}) = \sum_{i=1}^{n} f_{2d,i} x_i^{2d} + \sum_{\beta \in \Omega_f \setminus \Delta_f} f_\beta \underline{x}^\beta + \sum_{\alpha \in \Delta_f} f_\alpha \underline{x}^\alpha + f_0$.

So for a form $f \in \mathcal{F}_{n,2d}$,

$$f(\underline{x}) = \sum_{i=1}^{n} f_{2d,i} x_i^{2d} + \sum_{\beta \in \Omega_f \setminus \Delta_f} f_\beta \underline{x}^\beta + \sum_{\alpha \in \Delta_f} f_\alpha \underline{x}^\alpha.$$

Theorem 2.5. Suppose f is a form of degree $2d$. A sufficient condition for f to be sobs is that there exist real numbers $a_{\alpha,i}$ for $\alpha \in \Delta_f$, $i = 1, \dots, n$ such that

1. $\forall\ \alpha \in \Delta_f \quad (2d)^{2d} a_\alpha^\alpha = |f_\alpha|^{2d} \alpha^\alpha$.

2. $f_{2d,i} \geq \sum_{\alpha \in \Delta_f} a_{\alpha,i}$, $i = 1, \dots, n$.

(Note that $a_{\alpha,i} \geq 0$ and $a_{\alpha,i} = 0$ if and only if $\alpha_i = 0$.)

Proof. See [15, p36] or [17, p464]. □

We note the following corollaries to the above theorem that were already known before in [26] and [13] respectively:

Corollary 2.6. For any polynomial $f \in \mathbb{R}[\underline{x}]$ of degree $2d$, if

$$f_0 \geq \sum_{\alpha \in \Delta_f} |f_\alpha| \frac{2d - |\alpha|}{2d} \text{ and } f_{2d,i} \geq \sum_{\alpha \in \Delta_f} |f_\alpha| \frac{\alpha_i}{2d}, i = 1, \dots, n,$$

then f is a sum of squares.

Proof. See [15, p37] or [16, p346] or [17, p465]. □

Corollary 2.7. Suppose $f \in \mathbb{R}[\underline{x}]$ is a form of degree $2d$ and

$$\min_{i=1,\dots,n} f_{2d,i} \geq \frac{1}{2d} \sum_{\alpha \in \Delta_f} |f_\alpha| (\alpha^\alpha)^{\frac{1}{2d}}$$

Then f is a sobs.

Proof. See [15, p38]or [16, p347] or [17, p465]. □

2.2 Test sets for psdness of symmetric and even symmetric forms

Notation 2.8. Set

$S\mathcal{P}_{n,m} := \{f \in S\mathcal{F}_{n,m} \mid f$ is psd$\}= \{f \in \mathcal{P}_{n,m} \mid f$ is symmetric$\}$, and

$S\Sigma_{n,m} := \{f \in S\mathcal{F}_{n,m} \mid f$ is sos$\}= \{f \in \sum_{n,m} \mid f$ is symmetric$\}$, the set of symmetric psd forms and symmetric sos forms respectively.

$S\mathcal{P}^e_{n,m} := \{f \in S\mathcal{P}_{n,m} \mid f$ is even$\}= \{f \in S\mathcal{F}^e_{n,m} \mid f$ is psd$\}$, and

$S\Sigma^e_{n,m} := \{f \in S\Sigma_{n,m} \mid f$ is even$\}= \{f \in S\mathcal{F}^e_{n,m} \mid f$ is sos$\}$, the set of even symmetric psd forms and even symmetric sos forms respectively.
Here $S\mathcal{F}_{n,m}$ and $S\mathcal{F}^e_{n,m}$ are (as given in Definition 1.17) the set of symmetric and even symmetric forms respectively in $\mathcal{F}_{n,m}$.

Definition 2.9. Let $\mathcal{F}_n$ be a set of forms in n variables. We say that $\Omega \subseteq \mathbb{R}^n$ is a **test set** for $\mathcal{F}_n$ if $p \in \mathcal{F}_n$ is psd iff $p(\underline{x}) \geq 0$ for all $\underline{x} \in \Omega$.

Notation 2.10. Set
(1) $\Lambda_{n,k} := \{\underline{x} \in \mathbb{R}^n \mid x_i \in \{r_1, \ldots, r_k\}; r_i \neq r_j$ for $i \neq j\}$ denote the set of points in $\mathbb{R}^n$ with at most k distinct components.
(2) $\Omega_{n,k} := \{\underline{x} \in \mathbb{R}^n_+ \mid x_i \in \{0, r_1, \ldots, r_k\}; r_i \neq r_j$ for $i \neq j\}$ denote the set of points in $\mathbb{R}^n_+$ with at most k distinct non-zero components.

In 1980, Choi, Lam and Reznick, in their unpublished manuscript *Symmetric quartic forms* [9] gave the following test set for symmetric quartic forms. We will show the details of the proof in Section 3.1.1.

Theorem 2.11. The set $\Lambda_{n,2}$ is a test set for $S\mathcal{F}_{n,4}, n \geq 4$.

Proof. See Corollary 3.11. □

In Section 2.2.1 below, we will recall test sets for $S\mathcal{F}^e_{n,6}$ that came up from the work of Choi, Lam and Reznick in 1987, and test sets for $S\mathcal{F}^e_{n,8}$ and $S\mathcal{F}^e_{3,10}$, that came up from the work of Harris in 1999. Then in Section 2.2.2 a generalization of these known test sets to test sets for $S\mathcal{F}_{n,2d}$ and $S\mathcal{F}^e_{n,2d}$ for any $n; d \geq 2$ is presented, that was given by Timofte in 2003. Further in Section 2.2.3 we deduce smaller test sets for even symmetric quartics and even symmetric ternary octics using Timofte's Half degree principle.

2.2.1 Even symmetric sextics, octics and ternary decics

A general even symmetric $n-$ary sextic (as seen in Section 1.1.5) is:

$$f = \alpha \sum_{i=1}^{n} x_i^6 + \beta \sum_{i \neq j} x_i^4 x_j^2 + \gamma \sum_{i<j<k} x_i^2 x_j^2 x_k^2 \tag{2.1}$$

which can also be written in the form:

$$f = aM_6 + bM_4M_2 + cM_2^3, \tag{2.2}$$

where $M_r(\underline{x}) = \sum_{i=1}^{n} x_i^r$, and the above two representations of f are related by the equations

$\alpha = a + b + c, \beta = b + 3c, \gamma = 6c.$

Associating to a given sextic f, the auxiliary polynomial

$$f^*(t) = a + bt + ct^2, \tag{2.3}$$

we get the following necessary and sufficient conditions for an even symmetric sextic f to be sos or psd in terms of $f^*(t)$, as given in [10]:

Theorem 2.12. (i) $f \in S\mathcal{P}^e_{n,6}$ iff $f^*(t) \geq 0$ for $t \in \{1, 2, \ldots, n\}$;

(ii) $f \in S\Sigma^e_{n,6}$ iff $f^*(t) \geq 0$ for $t \in \{1\} \cap [2, n]$.

Proof. See [10, p568] and [10, p572] for (i) and (ii) respectively. □

Theorem 2.13. The set $\Omega_{n,2}$ is a test set for $S\mathcal{F}^e_{n,8}$, $n \geq 1$.

Proof. See [20, p21]. □

Theorem 2.14. The set $\Omega_{3,2}$ is a test set for $S\mathcal{F}^e_{3,10}$.

Proof. See [20, p215]. □

2.2.2 Half degree principle

In [49], Timofte gave the following reduced sets to check the psdness of a given symmetric and even symmetric polynomial:

Theorem 2.15. A symmetric real polynomial of degree $2d$ in n variables is non-negative (> 0 respectively) on $\mathbb{R}^n$ $\Leftrightarrow$ it is non-negative (> 0 respectively) on the subset $\Lambda_{n,k} := \{\underline{x} \in \mathbb{R}^n \mid$ number of distinct components in $\underline{x}$ is $\leq k\}$, where $k := \max\{2, d\}$. [If $d \geq 2$, then $\Lambda_{n,k} = \Lambda_{n,d}$].

Theorem 2.16. An even symmetric real polynomial of degree $2d \geq 4$ in n variables is non-negative (> 0 respectively) on $\mathbb{R}^n$ $\Leftrightarrow$ it is non-negative (> 0 respectively) on the subset $\Omega_{n,d/2} := \{\underline{x} \in \mathbb{R}^n_+ \mid$ number of distinct non-zero components in $\underline{x}$ is $\leq d/2\}$.

Riener in [39] gave another proofs of above two theorems.

As a corollary to the above two theorems, we get:

Corollary 2.17. (i) For a symmetric real polynomial f of degree $2d$ in n variables $\exists\, \underline{x} \in \mathbb{R}^n$ such that $f(\underline{x}) = 0 \Leftrightarrow \exists\, \underline{x} \in \Lambda_{n,k}$ s.t. $f(\underline{x}) = 0$, where $k := \max\{2, d\}$.

(ii) For an even symmetric real polynomial f of degree $2d$ in n variables $\exists\, \underline{x} \in \mathbb{R}^n$ s.t. $f(\underline{x}) = 0 \Leftrightarrow \exists\, \underline{x} \in \Omega_{n,d/2}$ such that $f(\underline{x}) = 0$.

Proof. (i) Suppose f is a symmetric real polynomial of degree $2d$ in n variables, then by Theorem 2.15:

(1) $f \geq 0$ on $\mathbb{R}^n \Leftrightarrow f \geq 0$ on $\Lambda_{n,k}$, and

(2) $f > 0$ on $\mathbb{R}^n \Leftrightarrow f > 0$ on $\Lambda_{n,k}$.

So, we have:

$$\exists\, \underline{x} \in \mathbb{R}^n \text{ s.t. } f(\underline{x}) = 0 \Leftrightarrow \neg(f > 0 \text{ on } \mathbb{R}^n) \underbrace{\Leftrightarrow}_{\text{by (2)}} \neg(f > 0 \text{ on } \Lambda_{n,k})$$
$$\underbrace{\Leftrightarrow}_{\text{since } f\geq 0 \text{ on } \Lambda_{n,k} \text{ by (1)}} \exists\, \underline{x} \in \Lambda_{n,k} \text{ s.t. } f(\underline{x}) = 0.$$

(ii) Similarly as in (i) using Theorem 2.16. □

Corollary 2.17 (i) was also worked out by Grimm in [18].

2.2.3 Even symmetric quartics and ternary octics

In the following two propositions we give smaller test sets for even symmetric quartics and even symmetric ternary octics respectively, using Theorem 2.16:

Proposition 2.18. *$f \in S\mathcal{F}^e_{n,4}$ is ≥ 0 on $\mathbb{R}^n \Leftrightarrow f \geq 0$ on $\{(x_1, \dots, x_n) \mid x_i \in \{0, 1\}\}$.*

Proof. Let $f = a \sum_{i=1}^{n} x_i^4 + b \sum_{i<j} x_i^2 x_j^2 \in S\mathcal{F}^e_{n,4}$, then applying Theorem 2.16 we get

$f \geq 0$ on $\mathbb{R}^n \Leftrightarrow f \geq 0$ on $\Omega_{n,1} = \{\underline{x} \in \mathbb{R}^n_+ \mid x_i \in \{0, y\}, y \in \mathbb{R}_+, y \neq 0\}$,

$$\Leftrightarrow \forall\, y \in \mathbb{R}_+ \begin{cases} f(\underline{x}) = ay^4 \geq 0, \text{for } \underline{x} \text{ with any one } x_i = y \text{ and all others } = 0; \\ f(\underline{x}) = (2a + b)y^4 \geq 0, \text{for } \underline{x} \text{ with any two } x_i, x_j = y \text{ and all others } = 0; \\ \vdots \\ f(\underline{x}) = \Big((n-1)a + \dfrac{(n-2)(n-1)}{2} b\Big) y^4 \geq 0, \text{for } \underline{x} \text{ with any one } x_i = 0 \text{ and} \\ \hfill \text{all others } = y; \\ f(y, \dots, y) = \Big(na + \dfrac{(n-1)n}{2} b\Big) y^4 \geq 0. \end{cases}$$

i.e.

$$f \geq 0 \text{ on } \mathbb{R}^n \Leftrightarrow \begin{cases} f(\underline{x}) = a \geq 0, \text{for } \underline{x} \text{ with any one } x_i = 1 \text{ and all others } = 0; \\ f(\underline{x}) = (2a+b) \geq 0, \text{for } \underline{x} \text{ with any two } x_i, x_j = 1 \text{ and all others } = 0; \\ \vdots \\ f(\underline{x}) = (n-1)a + \dfrac{(n-2)(n-1)}{2} b \geq 0, \text{for } \underline{x} \text{ with any one } x_i = 0 \text{ and} \\ \hfill \text{all others } = 1; \\ f(1,\dots,1) = na + \dfrac{(n-1)n}{2} b \geq 0. \end{cases}$$

i.e. $f \geq 0$ on $\mathbb{R}^n \Leftrightarrow f \geq 0$ on $\{(x_1,\dots,x_n) \mid x_i \in \{0,1\}\}$,

$$\Big(\text{equivalently} \Leftrightarrow ka + \frac{(k-1)k}{2} b \geq 0,\ \forall\ 1 \leq k \leq n.\Big)$$ □

Remark 2.19. Later in Section 4.2.3 we will get (by Theorem 4.23) even a much smaller and reduced test set than the one in above proposition (see Remark 4.25).

Proposition 2.20. $f \in S\mathcal{F}^e_{3,8}$ is ≥ 0 on $\mathbb{R}^3 \Leftrightarrow f \geq 0$ on $\{(1,0,0),(1,1,0),(1,1,1),(1,z,0),$ $(1,1,z) \mid z \neq 0,1; z \in \mathbb{R}_+\}$.

Proof. Let $f = a\sum_{i=1}^{3} x_i^8 + b\sum_{i\neq j} x_i^6x_j^2 + c\sum_{i<j} x_i^4x_j^4 + d\sum_{\substack{1\leq i\neq j\neq k\leq 3\\ j<k}} x_i^4x_j^2x_k^2 \in S\mathcal{F}^e_{3,8}$, then applying Theorem 2.16 we get

$$\begin{aligned} f \geq 0 \text{ on } \mathbb{R}^3 &\Leftrightarrow f \geq 0 \text{ on } A_2^+ := \{\underline{x} \in \mathbb{R}^3_+ \mid \text{ number of distinct non-zero components in } \underline{x} \text{ is} \leq 2\ \} \\ &\Leftrightarrow f \geq 0 \text{ on } A_2^+ := \{\underline{x} \in \mathbb{R}^3_+ \mid x_i \in \{0,x,y\}, x \neq y \in \mathbb{R}_+; x,y \neq 0\} \\ &\qquad = \{(x,0,0),(x,x,0),(x,x,x),(x,y,0),(x,x,y) \mid x \neq y \in \mathbb{R}_+; x,y \neq 0\} \\ &\qquad = \{(z,0,0),(z,z,0),(z,z,z),(1,z,0)_{z\neq 0,1},(1,1,z)_{z\neq 0,1} \mid z \in \mathbb{R}_+\} \end{aligned}$$

$$\Leftrightarrow \forall\, z \in \mathbb{R}_+ \begin{cases} f(z,0,0) = az^8 = z^8 f(1,0,0) \geq 0; \\ f(z,z,0) = (2a+2b+c)z^8 = z^8 f(1,1,0) \geq 0; \\ f(z,z,z) = 3(a+2b+c+d)z^8 = z^8 f(1,1,1) \geq 0; \\ f(1,z,0)_{z\neq 0,1} = az^8 + bz^6 + cz^4 + bz^2 + a \geq 0; \\ f(1,1,z)_{z\neq 0,1} = az^8 + 2bz^6 + (2c+d)z^4 + 2(b+d)z^2 + (2a+2b+c) \geq 0. \end{cases}$$

i.e. $f \geq 0$ on $\mathbb{R}^3 \Leftrightarrow f \geq 0$ on $\{(1,0,0),(1,1,0),(1,1,1),(1,z,0),(1,1,z) \mid z \neq 0,1;$
$z \in \mathbb{R}_+\}$.

$\square$

2.3 Gram matrix tests for symmetric forms

In this section we would like to exploit the connection between the coefficients of a given form $f \in \mathcal{F}_{n,2d}$ to the entries of a corresponding Gram matrix. For this consider the following map, called **Gram map**:

$$\mu : \mathrm{Sym}_{N_0}(\mathbb{R}) \longrightarrow \mathcal{F}_{n,2d}$$

$$A \longmapsto \underline{v} A \underline{v}^T$$

where $N_0 := N(n,d) = \dim(\mathcal{F}_{n,d}) = \binom{d+n-1}{n-1}$ and $\underline{v} := \{\underline{x}^{\beta(1)}, \ldots, \underline{x}^{\beta(N_0)}\}$ is a monomial basis for $\mathcal{F}_{n,d}$.
Note that μ is linear (clear) and surjective (by Proposition 1.61). But μ is not injective in general (as we will see in the case of ternary quartics in Section 2.3.2).

2.3.1 Quadratic forms

Theorem 2.21. There is a 1-1 correspondence between symmetric quadratic forms in n variables and symmetric matrices of the form $[a,b]_{n\times n} := \begin{bmatrix} a & b & \ldots & b \\ b & a & & \vdots \\ \vdots & & \ddots & b \\ b & \ldots & b & a \end{bmatrix}$,
where a and $2b$ are coefficients of x_i^2 and $x_i x_j$ respectively in the corresponding form.

Proof. Here $2d = 2$, $N_0 := \dim(\mathcal{F}_{n,1}) = \binom{1+n-1}{n-1} = n$.

Consider the map

$$\mu : \mathrm{Sym}_n(\mathbb{R}) \longrightarrow \mathcal{F}_{n,2}$$

$$A \longmapsto (x_1, \ldots, x_n)A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} := q_A$$

μ is well-defined: Given $A \in \mathrm{Sym}_n(\mathbb{R})$, computing $\underline{x}A\underline{x}^T$, we get a form $q_A \in \mathcal{F}_{n,2}$.

μ is surjective: Given $q \in \mathcal{F}_{n,2}$, we associate to it a symmetric $n \times n$ matrix A_q defined by:

$$(A_q)_{ij} := \begin{cases} \text{coefficient of } x_i x_j \text{ ; if } i = j, \\ \frac{1}{2}\text{coefficient of } x_i x_j \text{ ; if } i \neq j \end{cases}$$

s.t. $\mu(A_q) = q$.

Also, μ is a linear map and $\dim(\mathcal{F}_{n,2}) = \binom{2+n-1}{n-1} = \frac{n(n+1)}{2} = \dim(\mathrm{Sym}_n(\mathbb{R}))$.

So, (by Rank-Nullity theorem) $\dim(\ker\mu) = 0 \Rightarrow \mu$ is one to one.

Hence μ is an isomorphism.

Thus specifically for any $q \in \mathcal{F}_{n,2}$, we can find $A_q \in \mathrm{Sym}_n(\mathbb{R})$ s.t. $\mu^{-1}(q) = A_q$.

Now, consider the subspace

$\mathcal{GSF}_{(n,2)} = \mu^{-1}(\mathcal{SF}_{n,2}) := \{[a,b]_{n\times n} \mid a, b \in \mathbb{R}\}$ of $\mathrm{Sym}_n(\mathbb{R})$.

Then (the restriction map)

$$\mu\restriction_{\mathcal{GSF}_{(n,2)}} : \mathcal{GSF}_{(n,2)} \longrightarrow \mathcal{SF}_{n,2}, \text{ defined by}$$

$$[a,b]_{n\times n} \longmapsto \underline{x}\,[a,b]_{n\times n}\,\underline{x}^T = a\Big(\sum_{i=1}^{n} x_i^2\Big) + 2b\Big(\sum_{1\le i<j\le n} x_i x_j\Big)$$

is an isomorphism (since $\mu\restriction$ is well-defined, surjective linear map and $\dim(\mathcal{SF}_{n,2}) = 2 = \dim(\mathcal{GSF}_{(n,2)})$). □

Observation 2.22. In $(n,2)$ case it is more interesting to study rank n Gram matrices associated to a given sos representation of a $n-$ary quadratic form, since $f \in \mathcal{P}_{n,2}$ is a sos of at most n squares of linear forms.

In fact, we are done here since given a form $f = a\Big(\sum_{i=1}^{n} x_i^2\Big) + 2b\Big(\sum_{i<j} x_i x_j\Big) \in S\mathcal{P}_{n,2}$, there is a unique corresponding Gram matrix $[a,b]_{n\times n}$ of rank n.

The above theorem can be restated as:

Theorem 2.23. Let $q(x_1,\dots,x_n) = \sum_{i=1}^{n} a x_i^2 + \sum_{1\le i<j\le n} 2b x_i x_j$ be a symmetric quadratic form and $Q = \mu^{-1}(q) := [a,b]_{n\times n}$ be its associated matrix. Then
q is sos (equivalently psd) $\Leftrightarrow$
$$\begin{aligned} &a \ge 0,\\ &a^2 - b^2 \ge 0,\\ &a^3 + 2b^3 - 3ab^2 \ge 0,\\ &a^4 - 3b^4 - 6a^2b^2 + 8ab^3 \ge 0,\\ &\quad\vdots\\ &[a + (n-1)b](a-b)^{n-1} \ge 0. \end{aligned}$$

Proof. Since for $A \in \mathrm{Sym}_n(\mathbb{R})$, A is psd iff all principal minors of A are non-negative, so we are done using

$[a,b]_{n\times n}$ is psd $\Leftrightarrow |[a,b]_{k\times k}| = [a+(k-1)b](a-b)^{k-1} \ge 0\ \forall\ 1 \le k \le n$,

and Remark 1.59. □

2.3.2 Ternary quartics

Consider the map
$$\begin{aligned} \mu : \mathrm{Sym}_6(\mathbb{R}) &\longrightarrow \mathcal{F}_{3,4}\\ A &\longmapsto \underline{u}A\underline{u}^T := F_A\ ;\ \underline{u} := (x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3) \end{aligned}$$

Then, given $A = (a_{ij}) \in \mathrm{Sym}_6(\mathbb{R})$, from it we get a ternary quartic form
$$\begin{aligned} F_A := \underline{u}A\underline{u}^T = {}& a_{11}x_1^4 + a_{22}x_2^4 + a_{33}x_3^4 + 2a_{14}x_1^3x_2 + 2a_{15}x_1^3x_3 + 2a_{24}x_1x_2^3 + 2a_{26}x_2^3x_3 + \\ & 2a_{35}x_1x_3^3 + 2a_{36}x_2x_3^3 + (2a_{12}+a_{44})x_1^2x_2^2 + (2a_{13}+a_{55})x_1^2x_3^2 + (2a_{23} + \\ & a_{66})x_2^2x_3^2 + (2a_{16}+2a_{45})x_1^2x_2x_3 + (2a_{25}+2a_{46})x_1x_2^2x_3 + (2a_{34} + \\ & 2a_{56})x_1x_2x_3^2, \end{aligned}$$

but this ternary quartic form $F_A(\underline{x})$ is not necessarily symmetric. For $F_A(\underline{x})$ to be symmetric we need

$$a_{11} = a_{22} = a_{33} = \alpha$$
$$2a_{12} + a_{44} = 2a_{13} + a_{55} = 2a_{23} + a_{66} = \beta$$
$$2a_{14} = 2a_{15} = 2a_{24} = 2a_{26} = 2a_{35} = 2a_{36} = \gamma$$
$$2a_{16} + 2a_{45} = 2a_{25} + 2a_{46} = 2a_{34} + 2a_{56} = \delta.$$

Conversely, let $F(\underline{x}) = \sum_{i+j+k=4} \alpha_{i,j,k} x_1^i x_2^j x_3^k \in \mathcal{F}_{3,4}$, we associate to it by Gram matrix method (as in [31]) a symmetric 6×6 matrix

$$A_F := \begin{bmatrix} \alpha_{4,0,0} & a & b & \frac{1}{2}\alpha_{3,1,0} & \frac{1}{2}\alpha_{3,0,1} & d \\ a & \alpha_{0,4,0} & c & \frac{1}{2}\alpha_{1,3,0} & e & \frac{1}{2}\alpha_{0,3,1} \\ b & c & \alpha_{0,0,4} & f & \frac{1}{2}\alpha_{1,0,3} & \frac{1}{2}\alpha_{0,1,3} \\ \frac{1}{2}\alpha_{3,1,0} & \frac{1}{2}\alpha_{1,3,0} & f & \alpha_{2,2,0} - 2a & \frac{1}{2}\alpha_{2,1,1} - d & \frac{1}{2}\alpha_{1,2,1} - e \\ \frac{1}{2}\alpha_{3,0,1} & e & \frac{1}{2}\alpha_{1,0,3} & \frac{1}{2}\alpha_{2,1,1} - d & \alpha_{2,0,2} - 2b & \frac{1}{2}\alpha_{1,1,2} - f \\ d & \frac{1}{2}\alpha_{0,3,1} & \frac{1}{2}\alpha_{0,1,3} & \frac{1}{2}\alpha_{1,2,1} - e & \frac{1}{2}\alpha_{1,1,2} - f & \alpha_{0,2,2} - 2c \end{bmatrix},$$

where $\{a, b, c, d, e, f\}$ are parameters, s.t. $\mu(A_F) := \underline{u}\, A_F\, \underline{u}^T = F(\underline{x})$.

Remark 2.24. (1) If F is psd, then for some choice of the parameters $\{a, b, c, d, e, f\}$, this matrix will be psd and have rank 3.

(2) $\dim(\mathrm{Sym}_6(\mathbb{R})) = \dfrac{6(6+1)}{2} = 21 > 15 = \dbinom{4+3-1}{3-1} = \dim(\mathcal{F}_{3,4})$. So, μ is a well-defined, surjective linear map, but not $1-1$ (since $\dim(\ker\mu) = 21 - 15 = 6 \neq 0$).

(3) There is a family of Gram matrices corresponding to a given form $F \in \mathcal{F}_{3,4}$ for different choices of parameters $\{a, b, c, d, e, f\}$, and in fact corresponding to different sos representations of F.

Next we consider the subspace $\mathcal{GSF}_{(3,4)} = \mu^{-1}(S\mathcal{F}_{3,4})$ of $\mathrm{Sym}_6(\mathbb{R})$, call it **Gram of** $S\mathcal{F}_{3,4}$. Let

$$F(\underline{x}) = \alpha\Big(\sum_{i=1}^{3} x_i^4\Big) + \beta\Big(\sum_{1\le i\neq j\le 3} x_i^3 x_j\Big) + \gamma\Big(\sum_{1\le i<j\le 3} x_i^2 x_j^2\Big) + \delta\Big(\sum_{1\le i\neq j\neq k\le 3} x_i^2 x_j x_k\Big) \in S\mathcal{F}_{3,4},$$

then the associated matrix A_F (as defined above) becomes

$$\mathcal{G}[\alpha,\beta,\gamma,\delta,a,b,c,d,e,f] := \begin{bmatrix} \alpha & a & b & \frac{1}{2}\beta & \frac{1}{2}\beta & d \\ a & \alpha & c & \frac{1}{2}\beta & e & \frac{1}{2}\beta \\ b & c & \alpha & f & \frac{1}{2}\beta & \frac{1}{2}\beta \\ \frac{1}{2}\beta & \frac{1}{2}\beta & f & \gamma-2a & \frac{1}{2}\delta-d & \frac{1}{2}\delta-e \\ \frac{1}{2}\beta & e & \frac{1}{2}\beta & \frac{1}{2}\delta-d & \gamma-2b & \frac{1}{2}\delta-f \\ d & \frac{1}{2}\beta & \frac{1}{2}\beta & \frac{1}{2}\delta-e & \frac{1}{2}\delta-f & \gamma-2c \end{bmatrix}$$

Now, note that (the restriction map)

$$\mu\restriction_{\mathcal{GSF}_{(3,4)}} : \mathcal{GSF}_{(3,4)} \longrightarrow S\mathcal{F}_{3,4}, \text{ defined by}$$

$$\mathcal{G}[\alpha,\beta,\gamma,\delta,a,b,c,d,e,f] \longmapsto \underline{u}\,\mathcal{G}[\alpha,\beta,\gamma,\delta,a,b,c,d,e,f]\,\underline{u}^T$$

is also a well-defined, surjective linear map. But it is not 1-1 (as follows):

We analyze the elements in $\mu_{\restriction}^{-1}(S\mathcal{F}_{3,4})$:

$G \in \ker(\mu_{\restriction}) := \{G \in \mathcal{GSF}_{(3,4)} \mid \mu_{\restriction}(G) = 0 \text{ form }\} \Leftrightarrow \alpha = \beta = \gamma = \delta = 0$

$$\Leftrightarrow G = \begin{bmatrix} 0 & a & b & 0 & 0 & d \\ a & 0 & c & 0 & e & 0 \\ b & c & 0 & f & 0 & 0 \\ 0 & 0 & f & -2a & -d & -e \\ 0 & e & 0 & -d & -2b & -f \\ d & 0 & 0 & -e & -f & -2c \end{bmatrix}.$$

So, basis of $\ker(\mu_{\restriction})$ is
$$\left\{\begin{bmatrix}0&1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&-2&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix},\begin{bmatrix}0&0&1&0&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&-2&0\\0&0&0&0&0&0\end{bmatrix},\begin{bmatrix}0&0&0&0&0&0\\0&0&1&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&-2\end{bmatrix}\right.$$
$$\left.,\begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&-1&0\\0&0&0&-1&0&0\\1&0&0&0&0&0\end{bmatrix},\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&0\\0&0&0&0&0&-1\\0&1&0&0&0&0\\0&0&0&0&-1&0\end{bmatrix},\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&1&0&0\\0&0&1&0&0&0\\0&0&0&0&0&-1\\0&0&0&0&-1&0\end{bmatrix}\right\},$$

since these elements generate G and are linearly independent.

$\Rightarrow \dim(\ker \mu_{\restriction}) = 6$.

Also, as shown in Example 1.23, we have $\dim(S\mathcal{F}_{3,4}) = 4$

So, $\dim(\mathcal{GSF}_{3,4}) = 6 + 4 = 10$ (using Rank-Nullity Theorem).

Remark 2.25. Since $\ker(\mu_{\restriction})$ is a subspace of $\mathcal{GSF}_{(3,4)}$, so any element in $\mu_{\restriction}^{-1}(S\mathcal{F}_{3,4})$ is of the form B_o+G, where $B_o := \mathcal{G}[\alpha,\beta,\gamma,\delta] =$
$$\begin{bmatrix}\alpha&0&0&\frac{\beta}{2}&\frac{\beta}{2}&0\\0&\alpha&0&\frac{\beta}{2}&0&\frac{\beta}{2}\\0&0&\alpha&0&\frac{\beta}{2}&\frac{\beta}{2}\\\frac{\beta}{2}&\frac{\beta}{2}&0&\gamma&\frac{\delta}{2}&\frac{\delta}{2}\\\frac{\beta}{2}&0&\frac{\beta}{2}&\frac{\delta}{2}&\gamma&\frac{\delta}{2}\\0&\frac{\beta}{2}&\frac{\beta}{2}&\frac{\delta}{2}&\frac{\delta}{2}&\gamma\end{bmatrix} \in \mu_{\restriction}^{-1}(S\mathcal{F}_{(3,4)})$$
is fixed.

Note that B_o has rank 6.

Theorem 2.26. There is a 1-1 correspondence between symmetric ternary quartic forms
$$F = \alpha\Big(\sum_{i=1}^{3} x_i^4\Big) + \beta\Bigg(\sum_{1\le i\ne j\le 3} x_i^3x_j\Bigg) + \gamma\Bigg(\sum_{1\le i<j\le 3} x_i^2x_j^2\Bigg) + \delta\Bigg(\sum_{1\le i\ne j\ne k\le 3} x_i^2x_jx_k\Bigg),$$

and symmetric matrices of the form $\mathcal{G}[\alpha,\beta,\gamma,\delta] := \begin{bmatrix} \alpha & 0 & 0 & \frac{\beta}{2} & \frac{\beta}{2} & 0 \\ 0 & \alpha & 0 & \frac{\beta}{2} & 0 & \frac{\beta}{2} \\ 0 & 0 & \alpha & 0 & \frac{\beta}{2} & \frac{\beta}{2} \\ \frac{\beta}{2} & \frac{\beta}{2} & 0 & \gamma & \frac{\delta}{2} & \frac{\delta}{2} \\ \frac{\beta}{2} & 0 & \frac{\beta}{2} & \frac{\delta}{2} & \gamma & \frac{\delta}{2} \\ 0 & \frac{\beta}{2} & \frac{\beta}{2} & \frac{\delta}{2} & \frac{\delta}{2} & \gamma \end{bmatrix}$.

So we get the following sufficient condition for a symmetric ternary form to be a sos:

Theorem 2.27. Let F be as in above theorem and $\mathcal{G}[\alpha,\beta,\gamma,\delta]$ (as defined above) be its associated matrix s.t. $F = \underline{u}\, \mathcal{G}[\alpha,\beta,\gamma,\delta]\, \underline{u}^T$; $\underline{u} = (x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3)$.
If $\mathcal{G}[\alpha,\beta,\gamma,\delta]$ is psd, then $F = \underline{u}\, \mathcal{G}[\alpha,\beta,\gamma,\delta]\, \underline{u}^T$ is sos.

Proof. Using Theorem 1.62. □

Corollary 2.28. Let $F \in S\mathcal{F}_{3,4}$, be as above in theorem.
If $\alpha \geq 0$,
$\alpha^3\gamma - \frac{1}{2}\alpha^2\beta^2 \geq 0$,
$\alpha^3\gamma^2 - \frac{1}{4}\alpha^3\delta^2 - \alpha^2\beta^2\gamma + \frac{1}{4}\alpha^2\beta^2\delta + \frac{3}{16}\alpha\beta^4 \geq 0$,
$\alpha^3\gamma^3 - \frac{3}{4}\alpha^3\gamma\delta^2 + \frac{1}{4}\alpha^3\delta^3 + \frac{3}{4}\alpha^2\beta^2\gamma\delta - \frac{3}{2}\alpha^2\beta^2\gamma^2 - \frac{3}{16}\alpha\beta^4\delta + \frac{9}{16}\alpha\beta^4\gamma - \frac{1}{16}\beta^6 \geq 0$,
then F is sos.

2.29. Converse to Theorem 2.27 is not true in general, i.e.
$f[\alpha,\beta,\gamma,\delta] \in S\mathcal{F}_{3,4}$ sos may not imply that $\mathcal{G}[\alpha,\beta,\gamma,\delta]$ is psd (non-negative on $\mathbb{R}^{N_0}$; $N_0 = N(3,2) = 6$).
For example: If $\alpha = 1, \beta = 0 = \delta, \gamma = -1$, then $\mathcal{G}[\alpha,\beta,\gamma,\delta]$ is not psd, but
$f = \sum_{i=1}^{3} x_i^4 - \sum_{1\leq i<j\leq 3} x_i^2x_j^2 = (x_1^2 - x_2^2 - x_3^2)^2 + \sum_{1\leq i<j\leq 3} x_i^2x_j^2$ is a sos.
So, the following question occur (noting the fact that ternary quartic psd forms are sos):

Does $\exists\ V \subseteq \mathbb{R}^6$ s.t. $f[\alpha,\beta,\gamma,\delta] = \underline{u}\, \mathcal{G}[\alpha,\beta,\gamma,\delta]\, \underline{u}^T \in S\mathcal{F}_{3,4}$ and F psd implies $\mathcal{G}[\alpha,\beta,\gamma,\delta] \succcurlyeq 0$ on V ? (2.4)

For answering this, we first give the following definition:

Definition 2.30. The map

$$\nu_d : \mathbb{R}^n \longrightarrow \mathbb{R}^{N_0}, \text{defined by}$$

$$(x_1, \ldots, x_n) \longmapsto (\ldots, \underline{x}^{\beta(i)}, \ldots),$$

where $N_0 = N(n,d) := \binom{d+n-1}{n-1}$ and $\underline{x}^{\beta(i)}$ ranges over all monomials of degree d in $x_1, \ldots, x_n$; is the **Veronese map** of degree d. The image $\nu_d(\mathbb{R}^n)$ of the Veronese map is an algebraic variety, often called a **Veronese variety**.

Since, $f \in \mathcal{F}_{n,2d}$ is psd on $\mathbb{R}^n \Leftrightarrow \forall\, A \in \mu^{-1}(f) : A \geqslant 0$ on $\nu_d(\mathbb{R}^n)$. So, the answer to the above question in equation (2.4) is yes for $V = \nu_2(\mathbb{R}^3)$.
We want to see if something even bigger than $V = \nu_2(\mathbb{R}^3)$ satisfies equation (2.4), i.e.
Does $\exists\, V \subseteq \mathbb{R}^6, V \supsetneq \nu_2(\mathbb{R}^3)$ s.t. $f[\alpha, \beta, \gamma, \delta] = \underline{u}\ \mathcal{G}[\alpha, \beta, \gamma, \delta]\ \underline{u}^T \in S\mathcal{F}_{3,4}$ and F psd $\Rightarrow \mathcal{G}[\alpha, \beta, \gamma, \delta] \geqslant 0$ on V ?

In fact for any n, d, we will consider the following more general question:
Does $\exists\, V \subseteq \mathbb{R}^{N_0}, V \supsetneq \nu_d(\mathbb{R}^n)$ s.t. $f \in \mathcal{P}_{n,2d} \Rightarrow \exists\, \mathcal{G} \in \mu^{-1}(f)$ such that $\underline{v}\ \mathcal{G}\underline{v}^T$ is non-negative for all $\underline{v} \in V$? (2.5)

We will answer this in the next section below.

2.4 Intermediate cones between sos and psd cones

We know that for any n, d; the sos cone $\sum_{n,2d}$ is contained in the psd cone $\mathcal{P}_{n,2d}$, and

1. $f \in \mathcal{F}_{n,2d}$ is sos $\Leftrightarrow \exists$ some $G \in \mu^{-1}(f)$ that is non-negative on $\mathbb{R}^{N_0}$,
2. $f \in \mathcal{F}_{n,2d}$ is psd on $\mathbb{R}^n \Leftrightarrow \forall\, G \in \mu^{-1}(f) : G$ is non-negative on $\nu_d(\mathbb{R}^n)$,
3. $f \in \mathcal{F}_{n,2d}$ is psd on $K \subseteq \mathbb{R}^n \Leftrightarrow \forall\, G \in \mu^{-1}(f) : G$ is non-negative on $\nu_d(K)$.

So given a form $f \in \mathcal{F}_{n,2d}$, it is interesting to find a non negativity domain bigger than $\nu_d(\mathbb{R}^n)$, for a Gram matrix corresponding to f.

Definition 2.31. Let $\mathbb{K}$ be a given basic closed semi algebraic set s.t. $\nu_d(\mathbb{R}^n) \subseteq \mathbb{K} \subseteq \mathbb{R}^{N_0}$, define

$$C_{\mathbb{K}}^{\exists} := \{f \in \mathcal{F}_{n,2d} \mid \exists\, G \succcurlyeq 0 \text{ on } \mathbb{K}, \text{ for some Gram matrix } G \in \mu^{-1}(f)\}, \text{ and}$$

$$C_{\mathbb{K}}^{\forall} := \{f \in \mathcal{F}_{n,2d} \mid G \succcurlyeq 0 \text{ on } \mathbb{K}, \forall \text{ Gram matrices } G \in \mu^{-1}(f)\}, \text{ where}$$

$G \succcurlyeq 0$ on $\mathbb{K}$ denotes that G is non-negative on $\mathbb{K}$.

Remark 2.32. By above definition $C_{\mathbb{K}}^{\forall} \subseteq C_{\mathbb{K}}^{\exists}$, and trivially $C_{\mathbb{K}}^{\exists}$, $C_{\mathbb{K}}^{\forall}$ are cones in $\mathcal{F}_{n,2d}$.

Lemma 2.33. If $\mathbb{K} = \nu_d(\mathbb{R}^n)$, then $C_{\mathbb{K}}^{\forall} = \mathcal{P}_{n,2d} = C_{\mathbb{K}}^{\exists}$.

Proof. $f \in C_{\nu_d(\mathbb{R}^n)}^{\exists} \Rightarrow \exists\, G \in \mu^{-1}(f)$ s.t. $G \succcurlyeq 0$ on $\nu_d(\mathbb{R}^n)$

$\Rightarrow \exists\, G \in \mu^{-1}(f)$ s.t. $\underline{u}G\underline{u}^T \geq 0\ \forall\ \underline{u} \in \nu_d(\mathbb{R}^n) \subseteq \mathbb{R}^{N_0}$,

i.e. $\underline{w}(\underline{x})G\underline{w}^T(\underline{x}) \geq 0\ \forall\ \underline{x} \in \mathbb{R}^n$, where $\underline{w}$ is a vector of monomials of degree d in $x_1, \ldots, x_n$,

equivalently $f(x_1, \ldots, x_n) \geq 0\ \forall\ \underline{x} \in \mathbb{R}^n$.

$\Rightarrow \forall\ G \in \mu^{-1}(f), G \succcurlyeq 0$ on $\nu_d(\mathbb{R}^n)$.

$\Rightarrow f \in C_{\nu_d(\mathbb{R}^n)}^{\forall} = \mathcal{P}_{n,2d}$

Thus $C_{\nu_d(\mathbb{R}^n)}^{\exists} \subseteq C_{\nu_d(\mathbb{R}^n)}^{\forall}$. □

For $\mathbb{K}$ s.t. $\nu_d(\mathbb{R}^n) \subseteq \mathbb{K} \subseteq \mathbb{R}^{N_0}$, we will compute $C_{\mathbb{K}}^{\forall}$ depending on whether μ is injective or not. For this we will first give some definitions and lemmas as below.

For the purpose of Lemma 2.35, we give the following definition of Veronese map in the projective space, similar to the Definition 2.30 of Veronese map in the affine space:

Definition 2.34. Let k be a field and K a fixed algebraically closed superior field of k. Let $d \geq 1$ and $m_0, \ldots, m_N$ be monomials of degree d in $n+1$ variables $x_0, \ldots, x_n$, where $N + 1 = \binom{n+d}{n}$. Then for $M \subset k[x_0, \ldots, x_n]$:

$$\mathcal{V}_+(M) := \{x \in \mathbb{P}^n(K) \mid \forall\, f \in M, f(x) = 0\}$$

is called **variety** of M. Note that $\mathcal{V}_+(m_0, \ldots, m_N) = \phi$.
Also the map

$$\nu_d : \mathbb{P}^n \longrightarrow \mathbb{P}^N, \text{defined by}$$

$$\underline{x} = (x_0, x_1, \ldots, x_n) \longmapsto (m_0(\underline{x}) : \ldots : m_N(\underline{x})),$$

is a morphism, called the d^{th} **Veronese-map**.

Lemma 2.35. $\nu_d(\mathbb{P}^n)$ is a closed subvariety of $\mathbb{P}^n$, and $\nu_d : \mathbb{P}^n \to \nu_d(\mathbb{P}^n)$ is an isomorphism of k-varieties.

Proof. Let $\mathcal{J} := \{\alpha \in \mathbb{Z}_+^n \mid |\alpha| = d\}$, then $\mathbb{P}^N$ has homogeneous coordinates z_α ($\alpha \in \mathcal{J}$) and the veronese map ν_d is given by $\nu_d(x) = (x^\alpha)_{\alpha \in \mathcal{J}}$ (homogeneous tupel).
Let $\mathcal{Z} := \mathcal{Z}_{\alpha\beta\gamma\delta}$ be the zeroset of all quadratic polynomials $z_\alpha z_\beta - z_\gamma z_\delta$ with $\alpha, \beta, \gamma, \delta \in \mathcal{J}$ and $\alpha + \beta = \gamma + \delta$.

Then clearly $\nu_d(\mathbb{P}^n) \subseteq \mathcal{Z}_{\alpha\beta\gamma\delta}$.

Also $\nu_d : \mathbb{P}^n \longrightarrow \mathcal{Z}$ is a morphism (of varieties)
$(x \longmapsto (x^\alpha)_{\alpha \in \mathcal{J}})$.

<u>Conversely</u> (i.e. for $\Phi : \mathcal{Z} \longrightarrow \mathbb{P}^n$),
for every $\alpha \in \mathcal{J}$ and $i \in \{0, \ldots, n\}$ with $\alpha_i \geq 1$, define the morphisms

$$\Phi_{\alpha,i} : \mathcal{Z} \cap D_+(z_\alpha) \longrightarrow D_+(X_i) \subseteq \mathbb{P}^n$$

by

$$\Phi_{\alpha,i}(z) = (z_{\alpha - e_i + e_0} : \ldots : z_{\alpha - e_i + e_n}),$$

where, for $f \in k[x_0, \ldots, x_n]$, $D_+(f) := \{x \in \mathbb{P}^n(K) \mid f(x) \neq 0\}$ and
$e_i = (0, \ldots, 0, \underbrace{1}_{(i^{th})}, 0, \ldots, 0)$.

Now if $z \in \mathcal{Z}$ with $z_\alpha z_\beta \neq 0$ and s.t. $\alpha_i \geq 1, \beta_i \geq 1$, then $\Phi_{\alpha,i}(z) = \Phi_{\beta,j}(z)$
because (by definition of $\mathcal{Z}$):

$$z_{\alpha - e_i + e_k}\, z_{\beta - e_j + e_l} = z_{\alpha - e_i + e_l}\, z_{\beta - e_j + e_k} \ \forall\, k, l = 0, 1, \ldots, n.$$

So gluing (covering up) the $\Phi_{\alpha,i}$'s to a morphism $\Phi : \mathcal{Z} \longrightarrow \mathbb{P}^n$,
it follows clearly (from the construction) that $\Phi\ o\ \nu_d = Id_{\mathbb{P}^n}$
[since for $x \in \mathbb{P}^n$ with $x_i \neq 0$, take $\alpha = de_i$, then $\nu_d(x) \in D_+(z_\alpha)$ and
$\Phi\ o\ \nu_d(x) = \Phi_{\alpha,i}(\nu_d(x)) = (x_0 x_i^{d-1} : \ldots : x_n x_i^{d-1}) = x$],
and conversely $\nu_d\ o\ \Phi(x) = Id_{\mathcal{Z}}$ also holds:
since for $\alpha \in \mathcal{J}$ with $\alpha_i \geq 1$ ($i \in \{0, \ldots, n\}$) and $z \in \mathcal{Z} \cap D_+(z_\alpha)$:
$\nu_d\ o\ \Phi(z) = \nu_d(\Phi_{\alpha,i}(z)) = (z^{\beta_0}_{\alpha-e_i+e_0} : \ldots : z^{\beta_n}_{\alpha-e_i+e_n})_{\beta\in\mathcal{J}}$, and this point in $\mathcal{Z}$ is same as z (i.e. $= (z_\beta)_{\beta\in\mathcal{J}}$). Indeed it means that $\forall\, \beta, \gamma \in \mathcal{J}$ the equality

$$z_\beta\, z^{\gamma_0}_{\alpha-e_i+e_0} : \ldots : z^{\gamma_n}_{\alpha-e_i+e_n} = z_\gamma z^{\beta_0}_{\alpha-e_i+e_0} : \ldots : z^{\beta_n}_{\alpha-e_i+e_n} \tag{2.6}$$

holds.
Now to prove equation (2.6), observe that the sum of the tuple of subscripts (with multiplicity) on both sides is same:

$$\beta + \sum_{j=0}^{n} \gamma_j(\alpha - e_i + e_j) = \beta + \gamma + d(\alpha - e_i) = \gamma + \sum_{j=0}^{n} \beta_j(\alpha - e_i + e_j).$$

Thus to prove $\nu_d\ o\ \Phi(x) = Id_{\mathcal{Z}}$, it's enough to show that:
if $\zeta_1, \ldots, \zeta_r,\ \eta_1, \ldots, \eta_r \in \mathcal{J}$ with $\sum_{j=1}^{r} \zeta_j = \sum_{j=1}^{r} \eta_j$ (in $\mathbb{Z}_+^{n+1}$), then
$z_{\zeta_1} \ldots z_{\zeta_r} = z_{\eta_1} \ldots z_{\eta_r}\ \forall z \in \mathcal{Z}$.
Hence $\nu_d : \mathbb{P}^n \longrightarrow \mathcal{Z}_{\alpha\beta\gamma\delta}$ is an isomorphism [since a morphism $f : X \longrightarrow Y$ is an isomophism if $\exists$ a morphism $g : Y \longrightarrow X$ s.t $f\ o\ g = Id_Y$ and $g\ o\ f = Id_X$]. Also $\nu_d(\mathbb{P}^n) \subseteq \mathcal{Z}_{\alpha\beta\gamma\delta}$. So we get $\nu_d(\mathbb{P}^n) = \mathcal{Z}_{\alpha\beta\gamma\delta}$. □

The above proof is taken from [44].

Definition 2.36. For $(0 \neq)\ G \in \mathrm{Sym}_{N_0}(\mathbb{R})$ such that $\mu(G) \in \mathcal{F}_{n,2d}$, consider $q_G \in \mathbb{R}[x_1, \ldots, x_{N_0}]$ defined by :

$$q_G(u_1, \ldots, u_{N_0}) := (u_1, \ldots, u_{N_0})\ G \begin{pmatrix} u_1 \\ \vdots \\ u_{N_0} \end{pmatrix},$$

then q_G is a **quadratic form associated to** G, in the N_0 variables.

Lemma 2.37. $\nu_d(\mathbb{R}^n) = \bigcap_{G \in \ker\mu} \mathcal{Z}_{q_G}$, where $\mathcal{Z}_{q_G} := \{\underline{u} \in \mathbb{R}^{N_0} \mid q_G(\underline{u}) = 0\}$.

Proof. "$\subseteq$" Clearly $\nu_d(\mathbb{R}^n) \subseteq \bigcap_{G \in \ker\mu} \mathcal{Z}_{q_G}$, since $G \in \ker\mu$ if and only if q_G vanishes on $\nu_d(\mathbb{R}^n)$, by definition.

"$\supseteq$" For this, we will prove that

$$\nu_d(\mathbb{R}^n) = \mathcal{Z}'_{\alpha\beta\gamma\delta} \supseteq \bigcap_{G \in \ker\mu} \mathcal{Z}_{q_G}, \tag{2.7}$$

where $\mathcal{Z}'_{\alpha\beta\gamma\delta}$:= Zero locus of all quadratic forms $z_\alpha z_\beta - z_\gamma z_\delta$ with $\alpha, \beta, \gamma, \delta \in \mathcal{J} := \{\alpha \in \mathbb{Z}^n_+ \mid |\alpha| = d\}$ such that $\alpha + \beta = \gamma + \delta$.

Claim 1: $\nu_d(\mathbb{R}^n) = \mathcal{Z}'_{\alpha\beta\gamma\delta}$

Proof of Claim 1: In Lemma 2.35 above, we showed that $\nu_d(\mathbb{P}^n) = \mathcal{Z}_{\alpha\beta\gamma\delta}$. By working in affine space we get $\nu_d(\mathbb{R}^n) = \mathcal{Z}'_{\alpha\beta\gamma\delta}$.

Claim 2: $\mathcal{Z}'_{\alpha\beta\gamma\delta} \supseteq \bigcap_{G \in \ker\mu} \mathcal{Z}_{q_G}$

Proof of Claim 2: We know that

$$\mathcal{Z}'_{\alpha\beta\gamma\delta} = \mathcal{V}_+(I)\ ;\ \text{where } I = \{z_\alpha z_\beta - z_\gamma z_\delta \mid \alpha, \beta, \gamma, \delta \in \mathcal{J} \text{ such that } \alpha + \beta = \gamma + \delta\}$$

and

$$\bigcap_{G \in \ker\mu} \mathcal{Z}_{q_G} = \mathcal{V}_+(J)\ ;\ \text{where } J = \{q_G \in \mathcal{F}_{N_0,2} \mid G \in \ker\mu\}$$

So, to prove $\mathcal{Z}'_{\alpha\beta\gamma\delta} \supseteq \bigcap_{G \in \ker\mu} \mathcal{Z}_{q_G}$, its enough to prove $I \subset J$

i.e. for $z_\alpha z_\beta - z_\gamma z_\delta \in I$, we need to find a $G \in \ker\mu$ s.t.

$$\begin{aligned} z_\alpha z_\beta - z_\gamma z_\delta &= (u_1, \ldots, u_{N_0})\, G \begin{pmatrix} u_1 \\ \vdots \\ u_{N_0} \end{pmatrix} \\ &= q_G \in \mathcal{F}_{N_0,2}. \end{aligned}$$

Observe that

$$z_\alpha z_\beta - z_\gamma z_\delta = \underline{z} G \underline{z}^T\ ;\ \text{where } \underline{z} \in \{\underline{x}^{\underline{\alpha}} \mid \underline{\alpha} \in \mathbb{Z}^n_+ \text{ s.t. } |\alpha| = d\} \text{ and}$$

$$G := G_{(\alpha\beta\gamma\delta)} = (g_{\epsilon\eta})_{\epsilon,\eta\,\in\,\mathcal{J}}\ ; g_{\epsilon\eta} = \begin{cases} \frac{1}{2} & , \text{ if } (\epsilon = \alpha) \wedge (\eta = \beta) \wedge (\alpha \neq \beta) \\ -\frac{1}{2} & , \text{ if } (\epsilon = \gamma) \wedge (\eta = \delta) \wedge (\gamma \neq \delta) \\ 1 & , \text{ if } \epsilon = \alpha = \beta = \eta \\ -1 & , \text{ if } \epsilon = \gamma = \delta = \eta \\ 0 & , \text{ otherwise} \end{cases}$$

Also,

$$\mu(G) = \underline{u}G\underline{u}^T = x^\alpha x^\beta - x^\gamma x^\delta = x^{\alpha+\beta} - x^{\gamma+\delta} = 0$$
$$\Rightarrow\ G \in \ker\mu$$

So we have $z_\alpha z_\beta - z_\gamma z_\delta = q_G$, for $G = G_{(\alpha\beta\gamma\delta)} \in \ker\mu$. □

Remark 2.38. For $d = 1$; $N_0 = n$ and μ is injective (as seen in proof of Theorem 2.21). Also we have

$$\nu_1 : \mathbb{R}^n \longrightarrow \mathbb{R}^n;$$
$$(x_1, \ldots, x_n) \longmapsto \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

which implies ν_1 = identity map. This implies $\nu_1(\mathbb{R}^n) = \mathbb{R}^n$.
Indeed Lemma 2.37 shows in particular that μ is injective iff $\nu_d(\mathbb{R}^n) = \mathbb{R}^{N_0}$, since

$$\nu_d(\mathbb{R}^n) = \bigcap_{G \in \ker\mu} \mathcal{Z}_{q_G} = \bigcap_{G \in \{0\}} \mathcal{Z}_{q_G} = \mathcal{Z}_{q_0} = \mathbb{R}^{N_0}.$$

Also $\nu_d(\mathbb{R}^n) = \mathbb{R}^{N_0}$ iff ν_d is surjective iff $N_0 = n$ iff $d = 1$.
This is because the Veronese map ν_d is always injective, so it is an isomorphism from $\mathbb{R}^n$ to $\mathbb{R}^{N_0}$, so necessarily $N_0 = n$, and moreover
$N_0 = \binom{d+n-1}{n-1} = n \Rightarrow d = 1$, since $\dfrac{(d+n-1)!}{d!(n-1)!} > n$ if $d > 1$.
So, injectivity is only in case of quadratic forms i.e. for $f \in \mathcal{F}_{n,2}$.

For any fixed n, d, and for $\mathbb{K}$ s.t. $\nu_d(\mathbb{R}^n) \subseteq \mathbb{K} \subseteq \mathbb{R}^{N_0}$, the following proposition gives a complete description of $C^\forall_\mathbb{K}$ depending on whether μ is injective or not.

Proposition 2.39. Let $\mathbb{K} \subseteq \mathbb{R}^{N_0}$ be a semi algebraic set. Assume $\nu_d(\mathbb{R}^n) \subseteq \mathbb{K}$,

1. if μ is injective, then $\mathbb{K} = \nu_d(\mathbb{R}^n) = \mathbb{R}^{N_0}$ and $C^\forall_\mathbb{K} = C^\exists_\mathbb{K}$,

2. if $\mathbb{K} \supsetneq \nu_d(\mathbb{R}^n)$ and μ is not injective, then $C_{\mathbb{K}}^{\forall} = \phi$.

Proof. 1. By above remark, μ is injective iff $\nu_d(\mathbb{R}^n) = \mathbb{R}^{N_0}$. So, for $\mathbb{K}$ s.t. $\nu_d(\mathbb{R}^n) \subseteq \mathbb{K} \subseteq \mathbb{R}^{N_0}$, we fall back into the case $\mathbb{K} = \nu_d(\mathbb{R}^n) = \mathbb{R}^{N_0}$. Hence as proved in Lemma 2.33, we get $C_{\mathbb{K}}^{\forall} = C_{\mathbb{K}}^{\exists}$.

2. We need to show that: $\forall\, f \in \mathcal{F}_{n,2d}, \exists\, G \in \mu^{-1}(f)$ and $\exists\, \underline{u} \in \mathbb{K}$ s.t. $\underline{u}G\underline{u}^T < 0$.

Given $f \in \mathcal{F}_{n,2d}$, it is enough to show that :

$$\exists\, G_0 \in \ker(\mu) \text{ s.t } G_0 \neq 0 \text{ on } \mathbb{K}. \tag{2.8}$$

Indeed, since $\ker(\mu) \subseteq \mu^{-1}(f)$, so any element $\mathcal{G} \in \mu^{-1}(f)$ has the form $G + G_0$, for some $G \in \mu^{-1}(f)$, and $u(G + G_0)u^T = uGu^T + uG_0u^T$. Say $uG_0u^T = r_0$. Then considering $(G + sG_0) \in \mu^{-1}(f)$; for $s \in \mathbb{R}$ large enough

s.t. $s \to \begin{cases} +\infty, \text{if } r_0 < 0 \\ -\infty, \text{if } r_0 > 0, \end{cases}$ we get $u(G + sG_0)u^T < 0$.

So it remains to show (2.8), i.e. $\exists\, G_0 \in \ker(\mu)$ and $\exists\, \underline{u}_0 \in \mathbb{K}$ s.t. $\underline{u}_0 G_0 \underline{u}_0^T \neq 0$.

Take a basis $G_1, \ldots, G_l$ of $\ker(\mu)$, and assume for a contradiction that such G_0 does not exist,

that means $\forall\, G \in \ker(\mu)$ and $\forall\, \underline{u} \in \mathbb{K}\ :\ \underline{u}G\underline{u}^T = 0$,

equivalently, $\forall\, \underline{u} \in \mathbb{K}$ and $\forall\, j = 1, \ldots, l\ :\ q_{G_j}(\underline{u}) = \underline{u}\, G_j\, \underline{u}^T = 0$,

that means $\mathbb{K} \subseteq \left\{\underline{u} \in \mathbb{R}^{N_0} \mid \forall\, j : q_{G_j}(\underline{u}) = 0 \right\} = \bigcap_{j=1}^{l} \mathcal{Z}_{q_{G_j}} = V_{1,\ldots,l}$, where $\bigcap_{j=1}^{l} \mathcal{Z}_{q_{G_j}}$ is a variety in $\mathbb{R}^{N_0}$. This is a contradiction since $\mathbb{K} \supsetneq \nu_d(\mathbb{R}^n)$ (as given) and $\nu_d(\mathbb{R}^n) = \bigcap_{G \in \ker\mu} \mathcal{Z}_{q_G}$ by Lemma 2.37.

□

Remark 2.40. We have following remarks to the above proposition:

1. If $\mathbb{K} = \mathbb{R}^{N_0}$, then $C_{\mathbb{K}}^{\exists} = \sum_{n,2d}$, and $C_{\mathbb{K}}^{\forall}$ is as given in above proposition.

2. For $(n, 2d) = (n, 2), (2, 2d), (3, 4)$: $\mathcal{P}_{n,2d} = \sum_{n,2d}$ (by Theorem 1.48), so in these three cases we have:

   $C^{\exists}_{\mathbb{R}^{N_0}} = \sum_{n,2d} = C^{\exists}_{\nu_d(\mathbb{R}^n)} = C^{\forall}_{\nu_d(\mathbb{R}^n)} = \mathcal{P}_{n,2d}$, but this is not always $= C^{\forall}_{\mathbb{R}^{N_0}}$.

Observation 2.41. Proposition 2.39 implies that $C^{\forall}_{\mathbb{K}}$ is not interesting for us since its is either empty or equal to $C^{\exists}_{\mathbb{K}}$. That means from now on the interesting case is $C^{\exists}_{\mathbb{K}}$. For simplicity we will denote $C^{\exists}_{\mathbb{K}}$ by $C_{\mathbb{K}}$ from now on.

It is interesting to see how the cone $C_{\mathbb{K}}$ varies in between the sos and psd cones for different semi algebraic sets $\mathbb{K}$ s.t. $\nu_d(\mathbb{R}^n) \subseteq \mathbb{K} \subseteq \mathbb{R}^{N_0}$, i.e.

$$\begin{array}{ccccc} \nu_d(\mathbb{R}^n) & \subseteq & \mathbb{K} & \subseteq & \mathbb{R}^{N_0} \\ \mathcal{P}_{n,2d} & \supseteq & C_{\mathbb{K}} & \supseteq & \sum_{n,2d}. \end{array}$$

This is equivalent to seeing the maximal and minimal domain of non negativity for a Gram matrix corresponding to a form $f \in C_{\mathbb{K}}$. We will explain more about non negativity of a Gram matrix on a variety in Section 2.4.1. For now we present a viewpoint of intermediate cones between sos and psd cones, as below:

Remark 2.42. Trivially (by Definition 2.31) $C_{\mathbb{K}_1} \supseteq C_{\mathbb{K}_2}$, for distinct semi algebraic sets $\mathbb{K}_1, \mathbb{K}_2$ s.t. $\nu_d(\mathbb{R}^n) \subseteq \mathbb{K}_1 \subseteq \mathbb{K}_2 \subseteq \mathbb{R}^{N_0}$.

Lemma 2.43. If $\mathbb{K}_1, \mathbb{K}_2$ are distinct semi algebraic sets such that $\nu_d(\mathbb{R}^n) \subseteq \mathbb{K}_1, \mathbb{K}_2 \subseteq \mathbb{R}^{N_0}$, and $(n, 2d) = (2, 2d)$ or $(n, 2)$ or $(3, 4)$, then $C_{\mathbb{K}_1} = C_{\mathbb{K}_2}$.

Proof. For given $\mathbb{K}_1, \mathbb{K}_2$, we have:

$$\mathcal{P}_{n,2d} = C_{\nu_d(\mathbb{R}^n)} \supseteq C_{\mathbb{K}_1}, C_{\mathbb{K}_2} \supseteq C_{\mathbb{R}^{N_0}} = \Sigma_{n,2d}$$

Also we know that $\mathcal{P}_{n,2d} = \sum_{n,2d}$ for $(n, 2d) = (n, 2), (2, 2d), (3, 4)$, by Theorem 1.48. So we get $C_{\mathbb{K}_1} = C_{\mathbb{K}_2}$. □

The above lemma shows that for distinct semi algebraic sets $\mathbb{K}_1, \mathbb{K}_2$, the two cones $C_{\mathbb{K}_1}$ and $C_{\mathbb{K}_2}$ are equal when $(n, 2d) = (2, 2d)$ or $(n, 2)$ or $(3, 4)$. We proceed further to look what happens in all other cases, i.e when $(n, 2d) \neq (2, 2d), (n, 2)$, and $(3, 4)$.

For semi algebraic sets $\mathbb{K}_1, \mathbb{K}_2$ such that $\nu_d(\mathbb{R}^n) \subseteq \mathbb{K}_1 \subsetneq \mathbb{K}_2 \subseteq \mathbb{R}^{N_0}$, does $C_{\mathbb{K}_1} \supsetneq C_{\mathbb{K}_2}$ when $(n, 2d) \neq (2, 2d), (n, 2), (3, 4)$?
Equivalently, we

$$\text{look for conditions on such } \mathbb{K}_1 \text{ and } \mathbb{K}_2 \text{ such that } C_{\mathbb{K}_1} \supsetneq C_{\mathbb{K}_2} \text{ when } \mathbb{K}_1 \subsetneq \mathbb{K}_2 \text{ and } (n, 2d) \neq (2, 2d), (n, 2), (3, 4). \tag{2.9}$$

Observation 2.44. For $\mathbb{K}_1 = \nu_d(\mathbb{R}^n)$ and $\mathbb{K}_2 = \mathbb{R}^{N_0}$, we have : $C_{\mathbb{K}_1} = \mathcal{P}_{n,2d}$, $C_{\mathbb{K}_2} = \sum_{n,2d}$.
So, by Theorem 1.48 we get $C_{\mathbb{K}_1} \supsetneq C_{\mathbb{K}_2}$ when $(n, 2d) \neq (2, 2d), (n, 2), (3, 4)$.

Thus the problem in equation (2.9) above is reduced to looking at the following question:

Question 2.45. Let $\mathbb{K}_1, \mathbb{K}_2$ be semi algebraic sets such that $\nu_d(\mathbb{R}^n) \subseteq \mathbb{K}_1 \subsetneq \mathbb{K}_2 \subseteq \mathbb{R}^{N_0}$ and either $\mathbb{K}_1 \neq \nu_d(\mathbb{R}^n)$ or $\mathbb{K}_2 \neq \mathbb{R}^{N_0}$ or both. Find conditions on $\mathbb{K}_1$ and $\mathbb{K}_2$ such that $C_{\mathbb{K}_1} \supsetneq C_{\mathbb{K}_2}$ when $(n, 2d) \neq (2, 2d), (n, 2), (3, 4)$.

2.4.1 Reducing psdness and sosness to non negativity of quadratic forms on a variety defined by finitely many quadratic forms

In this Section, we consider the quadratic form associated (as in Definition 2.36) to a Gram matrix corresponding to a given form. We will show, using Lemma 2.37, how the psdness of a form is reduced to checking non negativity of a corresponding quadratic form on a variety defined by finitely many quadratic forms (see Problem 2.47). We will also give an easier approach to find sufficient conditions for a form to be sos (see Question 2.48).

For a form $f \in \mathcal{F}_{n,2d}$, let $G \in \mu^{-1}(f)$ be a corresponding Gram matrix. The domain of non negativity of G is the following subset of $\mathbb{R}^{N_0}$:

$$K_G := \{(u_1, \ldots, u_{N_0}) \in \mathbb{R}^{N_0} \mid (u_1, \ldots, u_{N_0})G(u_1, \ldots, u_{N_0})^T \geq 0\}.$$

Considering the quadratic form $q_G(u_1,\ldots,u_{N_0}) = (u_1,\ldots,u_{N_0})G(u_1,\ldots,u_{N_0})^T$ associated to G (as defined in Definition 2.36), we see that K_G is just the domain of non negativity of the quadratic form q_G. So we can work either with G or q_G. Thus we have the following:

Observation 2.46. A $n-$ary $2d-$ic form is psd (i.e. non-negative on $\mathbb{R}^n$) iff any of its Gram matrices G is non-negative on the Veronese variety $v_d(\mathbb{R}^n)$ iff the corresponding quadratic form q_G is non-negative on the Veronese variety $v_d(\mathbb{R}^n)$.
Also we proved (in Lemma 2.37) that $v_d(\mathbb{R}^n)$ is the intersection of finitely many quadratic forms (i.e. hyper surfaces). So the problem of finding conditions on the coefficients of a form to be psd can be reformulated as:

Problem 2.47. Given a quadratic form q in r variables, give sufficient (and necessary) conditions so that q is non-negative on the variety $\mathcal{V}$ defined by $q_1 = ... = q_s = 0$, for q_i quadratic forms, i.e. on $\mathcal{V} := \{\underline{x} \in \mathbb{R}^r \mid q_1(\underline{x}) = ... = q_s(\underline{x}) = 0\}$.

This also relates to the work of Blekherman, Smith, and Velasco [3] in which they proved that every real quadratic form that is non-negative on a real non-degenerate variety X is a sos of linear forms if and only if X is a variety of minimal degree, where a non-degenerate variety $X \subseteq \mathbb{CP}^{n-1}$ is called a **variety of minimal degree** iff $\deg(X) = \mathrm{Codim}(X) + 1$.

Since sos is a special case of psdness the following question also naturally arises:

Question 2.48. Let S be a subset of $\mathbb{R}^{N_0}$ and q a quadratic form non-negative on S. When can we find another quadratic form q' s.t. $q = q'$ on S and q' is psd?

When S is the Veronese variety $v_d(\mathbb{R}^n)$, a solution to the above question would analyze the situation when a psd form $f \in \mathcal{F}_{n,2d}$ has a Gram matrix G necessarily non-negative on $v_d(\mathbb{R}^n)$ and another Gram matrix G' which is psd, revealing that f is a sos.

We close this chapter by pointing out the fact that if we have a solution to Problem 2.47 and an answer to Question 2.48, we will get very nice explicit sufficient conditions for a $n-$ary $2d-$ic form to be psd and to be sos for any n and d. This will solve one of the most challenging questions regarding tests for a form to be psd and sos, giving more insights to future work.

Chapter 3

Symmetric forms

The inclusion $S\Sigma_{n,2d} \subseteq S\mathcal{P}_{n,2d}$ holds for all n, d, since every sos form is clearly psd. In this chapter we will investigate the converse, i.e. we will revisit the following question considered by Choi and Lam in [7]:

$$Q(S) : \text{For what pairs } (n, 2d) \text{ will } S\mathcal{P}_{n,2d} \subseteq S\Sigma_{n,2d} \text{ ?} \tag{3.1}$$

We will construct explicit forms $f \in S\mathcal{P}_{n,4} \setminus S\Sigma_{n,4}$ for $n \geq 5$ in Section 3.1.2 (see Theorems 3.16, 3.17). This will be our main contribution towards completion of the answer (Theorem 3.1 below) to $Q(S)$ given in [7].

$Q(S)$ is an extension of Hilbert's original question (Q), mentioned before as equation (1.2), to the special case when the form considered is in addition symmetric.

The following theorem answers $Q(S)$ completely and classifies the pairs $(n, 2d)$ for which the equality $S\mathcal{P}_{n,2d} = S\Sigma_{n,2d}$ holds.

Theorem 3.1. $S\mathcal{P}_{n,2d} = S\Sigma_{n,2d}$ iff $n = 2, 2d = 2, (n, 2d) = (3, 4)$.

In other words the answer to $Q(S)$ is given by the same chart which answers (Q), i.e.

deg \ var	2	3	4	5	...
2	✓	✓	✓	✓	...
4	✓	✓	×	×	...
6	✓	×	×	×	...
8	✓	×	×	×	...
⋮	⋮	⋮	⋮	⋮	⋱

where, a tick (✓) denotes a positive answer to (Q), whereas a cross (×) denotes a negative answer to $Q(S)$.

By Hilbert's Theorem 1.48 we know that a psd $n-$ary form of degree $2d$ is sos if $n = 2, 2d = 2$, and $(n, 2d) = (3, 4)$. So this must hold in particular for symmetric forms also.

Conversely for proving $S\mathcal{P}_{n,2d} \subseteq S\Sigma_{n,2d}$ only if $n = 2, 2d = 2$, and $(n, 2d) = (3, 4)$, i.e. $S\Sigma_{n,2d} \subsetneq S\mathcal{P}_{n,2d}$ if $n \geq 3, 2d \geq 4$ and $(n, 2d) \neq (3, 4)$, it is enough to find symmetric forms $f \in \mathcal{P}_{n,2d} \setminus \Sigma_{n,2d}$ for all pairs $(n, 4)$ with $n \geq 4$, and for the pair $(3, 6)$. Since once we have found such f's, we can construct symmetric $n-$ary forms of higher degree, as shown in Proposition 3.3 below. Note that the reduction strategy used in the proof of Proposition 3.3 is similar to the one used by Hilbert [22] to get psd not sos $n-$ary $2d-$ic forms for $n \geq 3, 2d \geq 4; (n, 2d) \neq (3, 4)$ from psd not sos ternary sextics and quaternary quartics, as seen in Proposition 1.49.

In [7], the authors mentioned that "the construction of $f_{n,4} \in S\mathcal{P}_{n,4} \setminus S\Sigma_{n,4}$ ($n \geq 4$) requires considerable effort, so we shall not go into the full details here. Suffice it to record the special form $f_{4,4} = \sum x^2y^2 + \sum x^2yz - 2xyzw$. Here the two summations denote the full symmetric sums (w.r.t. the variables x, y, z, w); hence the summation lengths are respectively 6 and 12". Without any claim and as per a conversation with Bruce Reznick, this statement was probably based on the beginnings of their work on the unpublished manuscript [9]. In 1980, Choi, Lam and Reznick [9] described the extremal psd forms and the extremal sos forms in the cone of $n-$ary symmetric quartics for $n \geq 4$. They also provided a method for determining whether a particular symmetric quartic p belongs to $S\mathcal{P}_{n,4}$ or $S\Sigma_{n,4}$. In Section 3.1.1 we will present some of their results (that will be used in proving a result in Section 3.1.2) for the ready reference of the readers.

For completing the proof of Theorem 3.1 we first need the following lemma:

Lemma 3.2. Let $f \in \mathcal{F}_{n,2d}$ be a psd not sos form and p an irreducible indefinite form of degree r in $\mathbb{R}[x_1, \ldots, x_n]$. Then $p^2 f \in \mathcal{F}_{n,2d+2r}$ will also be a psd not sos form.

Proof. Clearly $p^2 f$ is psd. If $p^2 f = \sum_k h_k^2$, then for every real tuple $\underline{a}$ with $p(\underline{a}) = 0$, it follows that $(p^2 f)(\underline{a}) = 0$.
$\Rightarrow h_k^2(\underline{a}) = 0 \ \forall\ k$ (since h_k^2 is psd),
and so on the real variety $p = 0$, we have $h_k = 0$ as well.
So by Theorem 1.5: $h_k = p\ g_k$, which gives $f = \sum_k g_k^2$, a contradiction. □

Proposition 3.3. If $S\Sigma_{n,4} \subsetneq S\mathcal{P}_{n,4}$ for all $n \geq 4$ and $S\Sigma_{3,6} \subsetneq S\mathcal{P}_{3,6}$, then $S\Sigma_{n,2d} \subsetneq S\mathcal{P}_{n,2d}$ for all $n \geq 3, d \geq 2$ and $(n, 2d) \neq (3, 4)$.

Proof. If we have forms $f \in S\mathcal{P}_{n,2d} \setminus S\Sigma_{n,2d}$ for all pairs $(n, 4)$ with $n \geq 4$, and for the pair $(3, 6)$. Then we can construct symmetric $n-$ary forms of higher degree by taking $(x_1 + \ldots + x_n)^{2i} f$, which can be seen to be in $S\mathcal{P}_{n,2d+2i} \setminus S\Sigma_{n,2d+2i}\ \forall\ i\ \geq 0$, by Lemma 3.2 taking $p = (x_1 + \ldots + x_n)$; i times. □

Proposition 3.4. $S\Sigma_{n,4} \subsetneq S\mathcal{P}_{n,4}$ for all $n \geq 4$ and $S\Sigma_{3,6} \subsetneq S\mathcal{P}_{3,6}$.

Proof. 1. For $f \in S\mathcal{P}_{3,6} \setminus S\Sigma_{3,6}$, consider the ternary sextic constructed by Robinson in [41]:

$R(x, y, z) := x^6 + y^6 + z^6 - (x^4y^2 + y^4z^2 + z^4x^2 + x^2y^4 + y^2z^4 + z^2x^4) + 3x^2y^2z^2.$

This is a symmetric ternary sextic form and we have already shown in Theorem 1.52 that it is psd and not sos.

2. For $f \in S\mathcal{P}_{n,4} \setminus S\Sigma_{n,4}$ with $n \geq 4$:

   (a) First we consider for $n = 4$ the special symmetric quaternary quartic $F = \sum x^2y^2 + \sum x^2yz - 2xyzw$ (from [7]), where the two summations denote the full symmetric sums (w.r.t. the variables x, y, z, w); hence the summation lengths are respectively 6 and 12. This form F

turns out to be congruent (under the invertible linear transformation $(w, x, y, z) \mapsto (w + x + y + z, w + x - y - z, w - x + y - z, w - x - y + z)$ of the variables) to the form $Q(x, y, z, w) = w^4 + x^2y^2 + y^2z^2 + z^2x^2 - 4xyzw$, which $\in \mathcal{P}_{4,4} \setminus \Sigma_{4,4}$, as already shown in Proposition 1.53.

(b) Next we need to find $f \in S\mathcal{P}_{n,4} \setminus S\Sigma_{n,4}$ for $n \geq 5$. This will be done in Section 3.1.2 as Theorems 3.16, 3.17, where we will present our construction of such explicit forms.

□

3.1 $n-$ary quartics for $n \geq 4$

We will consider two useful bases as given below, for representing symmetric quartics with at least 4 variables. Any symmetric quartic form $f(x_1, \ldots, x_n)$ (for $n \geq 4$) can be written uniquely as follows in (3.2) and (3.3):

$$f(x_1, \ldots, x_n) = \alpha \sum x_i^4 + \beta \sum x_i^3 x_j + \gamma \sum x_i^2 x_j^2 + \delta \sum x_i^2 x_j x_k + \epsilon \sum x_i x_j x_k x_l; \tag{3.2}$$

$$f(x_1, \ldots, x_n) = aM_4 + bM_1M_3 + cM_2^2 + dM_1^2M_2 + eM_1^4, \tag{3.3}$$

where each summation in (3.2) is taken symmetrically over distinct terms with the given shape, and $M_r = \sum_{j=1}^{n} x_j^r$ is the $r-$th Newton function in (3.3). If (3.2) and (3.3) hold for f, then we write the following quintuples which can be used as a shorthand representation of the form they represent :

$$(f) = (\alpha, \beta, \gamma, \delta, \epsilon); \quad [f] = [a, b, c, d, e]. \tag{3.4}$$

Note that (f) and $[f]$ are related to each other by a matrix M (given below) as

$$(f)^T = M[f]^T; [f]^T = M^{-1}(f)^T, \text{where} \tag{3.5}$$

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 2 & 2 & 6 \\ 0 & 0 & 0 & 2 & 12 \\ 0 & 0 & 0 & 0 & 24 \end{pmatrix} \text{is independent of } n, \text{ and } M^{-1} = \frac{1}{24}\begin{pmatrix} 24 & -24 & -12 & 24 & -6 \\ 0 & 24 & 0 & -24 & 8 \\ 0 & 0 & 12 & -12 & 3 \\ 0 & 0 & 0 & 12 & -6 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The representation (f) is more useful for questions dealing with sos while $[f]$ is more useful for psd questions.
Both $S\mathcal{P}_{n,4}$ and $S\Sigma_{n,4}$ can be viewed as closed convex cones in $\mathbb{R}^5$ by identifying a given form f with (f) or $[f]$.

3.1.1 Zeroes and Test set of symmetric quartics

In this section we shall use $\underline{x} = (x_1, \ldots, x_n)$ to denote a general $n-$tuple and $\underline{y} = (y_1, \ldots, y_n)$ to denote a specific $n-$tuple. As a companion definition to the $r-$th Newton function M_r, let $N_r = \sum_{j=1}^{n} y_j^r$. We will present some results from [9] including Corollary 3.11 that gives test set for symmetric quartics in $n \geq 4$ variables. This corollary is the main result of this section and will be used to prove Proposition 3.12 in Section 3.1.2.
Timofte's half degree principle [49] (seen as Theorems 2.15, 2.16) gave test sets for symmetric and even symmetric $n-$ary forms of degree $2d$, as generalization of test set for even symmetric $n-$ary sextics given by Choi, Lam and Reznick in [10], and test set for even symmetric $n-$ary octics given by Harris in [20]. His results for symmetric forms were in fact a generalization of Corollary 3.11 given below from [9], since the work we present below from [9] was done much before [10] and [20].

Theorem 3.5. Suppose $n \geq 4$, $f \in S\mathcal{P}_{n,4}$ and $\underline{y} = (y_1, \ldots, y_n) \in Z(f)$, where $\underline{y}$ has at least three distinct coordinates y_i, y_j, y_k. Then f is quadratic in M_2 and M_1^2, i.e. $[f] = [0, 0, c, d, e]$.

For proving the above theorem we need a lemma, as given below.

Lemma 3.6. Suppose $g(x_1, x_2, x_3) \in \mathcal{P}_{3,4}$ and $(y_i, y_j, 1) \in Z(g)$ for $1 \le i \ne j \le 3$ with $y_1 < y_2 < y_3$. Then g is a perfect square and $(u, v, 1) \in Z(g)$ for all (u, v) on the non-degenerate ellipse determined by (y_i, y_j).

Proof. Since $g \in \mathcal{P}_{3,4}$, $g = \sum h_k^2$ (by Theorem 1.48) and $(y_i, y_j, 1) \in Z(h_k)$ for each k; $1 \le i \ne j \le 3$ with $y_1 < y_2 < y_3$.
Suppose $(y_i, y_j, 1) \in Z(h)$ and $h(x_1, x_2, x_3) = \sum_{i \le j} \alpha_{ij} x_i x_j$, then (evaluating h at its zeroes and solving simultaneously) it is easy to see that

$$\alpha_{11} = \alpha_{12} = \alpha_{22},$$

$$\alpha_{13} = \alpha_{23} = -(y_1 + y_2 + y_3)\alpha_{11}, \text{ and}$$

$$\alpha_{33} = (y_1 y_2 + y_1 y_3 + y_2 y_3)\alpha_{11}.$$

Hence

$$g(x_1, x_2, x_3) = \lambda\Big[x_1^2 + x_2^2 + [(y_1 + y_2 + y_3)x_3 - (x_1 + x_2)]^2 - (y_1^2 + y_2^2 + y_3^2)x_3^2\Big]^2, \text{ and}$$

$(u, v, 1) \in Z(g)$ if (u, v) lies on the ellipse $u^2+v^2+(y_1+y_2+y_3-(u+v))^2 = y_1^2+y_2^2+y_3^2$. This ellipse is not degenerate because its discriminant is non-zero (or since no three of the the points (y_i, y_j) for $i \ne j$ are collinear). □

3.7. *Proof of Theorem 3.5.*
Suppose $[f] = [a, b, c, d, e]$, then $f(\underline{x}) = aM_4 + bM_1M_3 + cM_2^2 + dM_1^2M_2 + eM_1^4$, and since $\dfrac{\partial M_r}{\partial x_j} = r x_j^{r-1}$ we have

$$\frac{\partial f}{\partial x_j}(x_1, \dots, x_n) = 4ax_j^3 + 3bM_1x_j^2 + (4cM_2 + 2dM_1^2)x_j + (bM_3 + 2dM_1M_2 + 4eM_1^3)$$

Now suppose $f(\underline{y}) = 0$. Since f is psd, $\dfrac{\partial f}{\partial x_j}(\underline{y}) = 0$ for all j. So by above equation the following holds for all j:

$$0 = 4ay_j^3 + 3bN_1y_j^2 + (4cN_2 + 2dN_1^2)y_j + (bN_3 + 2dN_1N_2 + 4eN_1^3) \tag{3.6}$$

- If $\underline{y}$ has four or more distinct coordinates, then the cubic in equation (3.6) above has at least four solutions, hence each coefficient vanishes. Therefore

$a = 0$ and $bN_1 = 0$, we will show that $b = 0$. For if for a contradiction $b \neq 0$, then $N_1 = 0$. So by equation (3.6): $4cN_2 + 2dN_1^2 = 0$, which implies $cN_2 = 0$, so $c = 0$. But if $a = c = 0$, then $(1, 1, -2, 0, \ldots, 0) \in Z(p)$, and for this choice of $\underline{y}$ we get $N_1 = 0, N_2 = 6, N_3 = -6$. So by equation (3.6): $0 = -6b$, i.e. $b = 0$. So $a = b = 0$ and f is a quadratic in M_2 and M_1^2.

- Now suppose $\underline{y}$ has exactly three distinct coordinates and assume, without loss of generality, that $y_1 < y_2 < y_3$. Let

$$g(x_1, x_2, x_3) = f(x_1, x_2, (y_1 + y_2 + y_3)x_3 - (x_1 + x_2), y_4x_3, \ldots, y_nx_3).$$

g need not be symmetric, but $g \in \mathcal{P}_{n,4}$ and $(y_i, y_j, 1) \in Z(g)$ for $1 \leq i \neq j \leq 3$. By Lemma 3.6 (just proved above), $(u, v, 1) \in Z(g)$ for all (u, v) on a non-degenerate ellipse, hence $(u, v, y_1 + y_2 + y_3 - (u + v), y_4, \ldots, y_n) \in Z(f)$. By avoiding a finite set of lines, we can find (u, v) so that $u, v, y_1+y_2+y_3-(u+v)$ are mutually distinct and distinct from y_4. Thus f has a zero with at least four distinct components and as in the above case, f is a quadratic in M_2 and M_1^2. □

The above theorem suggests the following definition:

Definition 3.8. For $n \geq 4$, $f \in S\mathcal{P}_{n,4}$ is a **dull form** if it is quadratic in M_2 and M_1^2, i.e. if $[f] = [0, 0, c, d, e]$.

The above description of a dull form is motivated by Proposition 3.10 given below. We first note the following simple lemma on linear functions, which will be useful in proving Proposition 3.10:

Lemma 3.9. Suppose $at + b \geq 0$ for all t with $c \leq t \leq d$, where $c < d$. Then (a, b) is a non-negative linear combination of $(1, -c)$ and $(-1, d)$.

Proof. For $t = \lambda c + (1 - \lambda)d$, $at + b = \lambda(ac + b) + (1 - \lambda)(ad + b)$, hence it is necessary and sufficient that $ac + b \geq 0$ and $ad + b \geq 0$.
Any $(a, b) = \lambda(1, -c)+\mu(-1, d)$ for some λ, μ. Evaluation of $at+b$ at the end points gives $\lambda, \mu \geq 0$, so the combination is a non-negative linear combination. □

Proposition 3.10. Suppose f is a psd dull form. Then f is a non-negative linear combination of $(\lambda M_2 - M_1^2)^2, 0 \le \lambda \le n$, and $M_1^2(nM_2 - M_1^2)$ and is a sum of squares. Further, a dull form f is psd iff $f(x_1, \dots, x_n) \ge 0$ for every $\underline{x}$ with at most two distinct coordinates.

Proof. For any $\underline{y}$, $N_1^2 = \lambda N_2$ with $0 \le \lambda \le n$ by the Cauchy-Schwarz inequality. If $[f] = [0, 0, c, d, e]$, then $f(\underline{y}) = N_2^2(c + d\lambda + c\lambda^2)$, so f is psd iff the following holds:

$$\hat{f}(\lambda) = c + d\lambda + e\lambda^2 \ge 0 \text{ for } 0 \le \lambda \le n.$$

Let α be the minimum of $\hat{f}$ on $[0, n]$. Then we can write $f(\underline{x}) = \alpha M_2^2 + q(\underline{x})$, where q is psd and $\hat{q}(\lambda_0) = 0$ for some $\lambda_0 \in [0, n]$. (Note that $M_2^2 = \frac{4}{n^2}\Big[\Big(\frac{n}{2}M_2 - M_1^2\Big)^2 + M_1^2(nM_2 - M_1^2)\Big]$ satisfies the conclusion.)
If $0 < \lambda_0 < n$, then $\hat{q}'(\lambda_0) = 0$ also holds so that $\hat{q}(\lambda) = e(\lambda - \lambda_0)^2$ and $q(\underline{x}) = e(\lambda_0 M_2 - M_1^2)^2$.
If $\lambda_0 = 0$, then $\hat{q}(\lambda) = \lambda(e\lambda + d)$ and $(e\lambda + d) \ge 0$ for $\lambda \in [0, n]$. By Lemma 3.9 (just proved above), (e, d) is a non-negative linear combination of $(1, 0)$ and $(-1, n)$, so $\hat{q}$ is a non-negative linear combination λ^2 and $\lambda(n - \lambda)$ and q is a non-negative linear combination of M_1^4 and $M_1^2(nM_2 - M_1^2)$.
Similarly if $\lambda_0 = n$, q is a non-negative linear combination of $M_1^2(nM_2 - M_1^2)$ and $(nM_2 - M_1^2)^2$. Since $nM_2 - M_1^2 = \sum_{i<j}(x_i - x_j)^2$, a dull psd quartic f is a sos.
Finally, for $\underline{y}_t = (t, 1, \dots, 1); -(n-1) \le t \le 1$, λ increases from 0 to n. Thus, if $f(\underline{x}) \ge 0$ for all $\underline{x}$ with at most two distinct components, then $f\big(\underline{y}_t\big) \ge 0$ for $0 \le \lambda \le n$ and f is psd. □

Theorem 3.5 and Proposition 3.10 have the following remarkable corollary which allows us to determine directly whether a given symmetric quartic is psd or not only by checking its value at $\underline{x}$ with two distinct components. This corollary is the main result of this section and will be used to prove Proposition 3.12 in Section 3.1.2.

Corollary 3.11. A symmetric $n-$ary quartic f is psd iff $f(\underline{x}) \ge 0$ for every $\underline{x} \in \mathbb{R}^n$ with at most two distinct coordinates (if $n \ge 4$), i.e. $\Lambda_{n,2} = \{\underline{x} \in \mathbb{R}^n \mid x_i \in \{r, s\}; r \ne s\}$ is a test set for symmetric $n-$ary quartics.

Proof. "⇒" Clearly, $f \in S\mathcal{F}_{n,4}$ psd implies $f(\underline{x}) \geq 0$ for every $\underline{x} \in \Lambda_{n,2} \subset \mathbb{R}^n$.
"⇐" Suppose $f(\underline{x}) \geq 0$ for every $\underline{x} \in \mathbb{R}^n$ with at most two distinct coordinates but f is not psd. Let $f(\underline{x}) \geq -\lambda; \lambda > 0$ for the unit sphere $\sum x_i^2 = 1$ with $f(\underline{y}) = -\lambda, \sum y_i^2 = 1$. Let $q(\underline{x}) = f(\underline{x}) + \lambda M_2^2$, then $q(\underline{x}) \geq 0$ for the unit sphere and so q is psd by homogeneity and $q(\underline{y}) = 0$. By hypothesis, $\underline{y}$ has more than two distinct components, so by Theorem 3.5 q is dull and so $f = q - \lambda M_2^2$ is also dull. By Proposition 3.10, f would then be psd, a contradiction to supposition. □

(Since the form considered above is symmetric, so after a suitable parametrization, we need to check only psdness of a finite set of binary quartic forms. Some details of this parametrization were given in [9], but we will not discuss them here.)

3.1.2 Psd not sos symmetric $n-$ary quartics for $n \geq 5$

In this section, we will construct explicit forms $f \in S\mathcal{P}_{n,4} \setminus S\Sigma_{n,4}$ for $n \geq 5$, in Proposition 3.12 and Theorems 3.16, 3.17. This will complete the proof of Theorem 3.1 and hence the answer to $Q(S)$.

Consider the following symmetric quartic $L_n(\underline{x})$ in $n \geq 4$ variables, which will be central to our discussion in this section, and is defined by

$$L_n(x_1, \ldots, x_n) := m(n-m)\sum_{i<j}(x_i - x_j)^4 - \Big(\sum_{i<j}(x_i - x_j)^2\Big)^2,$$

where $m = \left[\frac{n}{2}\right]$.

We will see in Proposition 3.12 that the symmetric quartic $L_n(\underline{x})$ defined above is psd for all n. Then we will prove that L_n for odd n is not a sos (see Theorem 3.16), which will finish half of the work that we intend to present in this section since we can then take $f := L_n \in S\mathcal{P}_{n,4} \setminus S\Sigma_{n,4}$ for $n = 2m + 1 \geq 5$. In contrast we will show in Proposition 3.13 that L_n for even n is a sos and therefore L_{2m} will no more be a candidate for psd not sos symmetric $n-$ary quartic for even $n = 2m \geq 6$. So, we will define another psd symmetric quartic C_{2m} in $2m$ (≥ 4)

number of variables and prove that it is not sos (see Theorem 3.17). This will serve our purpose of $f \in S\mathcal{P}_{n,4} \setminus S\Sigma_{n,4}$ for even $n = 2m \geq 6$.

For $n = 5$, $L_n(\underline{x})$ has been discussed by Anneli and Peter Lax in [25]. Let $A_n(\underline{x})$ be defined as below

$$A_n(\underline{x}) := \sum_{i=1}^{n} \prod_{j \neq i} (x_i - x_j),$$

which appeared on the 1971 International Mathematical Olympiad for high school students. The problem was to determine those n for which $A_n(\underline{x})$ is psd, and the answer is $n = 3$ or $n = 5$. Lax and Lax [25, p72] showed that $A_5(\underline{x})$, a psd symmetric quartic in five variables, is not a sos. By direct computation (and using a shorthand representation of a symmetric quartic form from equation (3.4)), we get $(A_5) = (1, -1, 0, 1, 3)$, so $A_5 = \frac{1}{8} L_5$. In general, L_n for odd n is not a sos (see Theorem 3.16).

Proposition 3.12. *L_n is psd for all n.*

Proof. In view of Corollary 3.11, it is enough to prove that $L_n \geq 0$ on the test set $\Lambda_{n,2} = \{(\underbrace{r, \ldots, r}_{k}, \underbrace{s, \ldots, s}_{n-k}) \mid r \neq s \in \mathbb{R}; 0 \leq k \leq n\}$.
Now for $\underline{x} \in \Lambda_{n,2}$

$$x_i - x_j = \begin{cases} \pm(r - s) \neq 0, & \text{for } k(n-k) \text{ terms,} \\ 0 & \text{, otherwise} \end{cases}$$

so L_n takes the value

$L_n(\underline{x}) = m(n-m)k(n-k)(r-s)^4 - [k(n-k)(r-s)^2]^2$

$= k(n-k)(r-s)^4[m(n-m) - k(n-k)]$

$= k(n-k)(r-s)^4[(m-k)(n-m-k)]$

≥ 0 (since there is no integer between m and $n - m$). □

Proposition 3.13. *If n is even, then L_n is sos.*

Proof. If n is even, $n = 2m$, then

$L_{2m}(\underline{x}) = m^2 \sum_{i<j} (x_i - x_j)^4 - \Big(\sum_{i<j} (x_i - x_j)^2 \Big)^2$

$$= (m^2-1)\sum_{i<j}(x_i-x_j)^4 - 2\sum_{(i,j)\neq(i',j')}(x_i-x_j)^2(x_{i'}-x_{j'})^2$$

$$= m^2\sum_{i<j}(x_i^2-x_j^2)^2 - 2m\sum_{i<j}(x_i+x_j)(x_i-x_j)^2(x_1+\ldots+x_{2m}) + \sum_{i<j}(x_i-x_j)^2(x_1+\ldots+x_{2m})^2$$

$$= \sum_{i<j}(x_i-x_j)^2\Big(-(x_1+\ldots+x_{2m}) + m(x_i+x_j)\Big)^2, \text{ which is clearly a sos.} \qquad \square$$

For proving that L_n for odd n is not a sos, we first give the following definition and thereafter prove a lemma:

Definition 3.14. A subset $S \subseteq \mathbb{R}^n$ of the form $S = \{\underline{x} \mid x_i \in \{0,1\}\ \forall\ i = 1,\ldots,n\}$ is said to be a **0/1 set** and a point $\underline{x} \in S$ is called a **0/1 point**.

Lemma 3.15. If $h(x_1,\ldots,x_n) = \sum_{i=1}^{n} a_i x_i^2 + \sum_{i<j} a_{ij}x_ix_j$ is a quadratic form that vanishes on all 0/1 points with m or $(m+1)$ 1's, where $m = \left[\frac{n}{2}\right] \geq 2$, i.e. $h(\underline{x}) = 0$ for all $\underline{x}$ with m or $(m+1)$ 1's and $\begin{cases}(m+1) \text{ or } m \text{ 0's (respectively) for odd } n = 2m+1;\\ m \text{ or } (m-1) \text{ 0's (respectively) for even } n = 2m.\end{cases}$
Then h is identically zero.

Proof. Fix distinct i, j, k and let S such that $|S| = m-1$, be a set of indices not containing i, j, k. Then $h = 0$ on $\underline{x}$, where the 1's on $\underline{x}$ occur precisely on $S \cup \{i\}$, $S \cup \{i,k\}$, $S \cup \{j\}$, $S \cup \{j,k\}$. So we have:

$$\text{on } S \cup \{i\}: \quad 0 = \sum_{l\in S} a_l + a_i + \sum_{l<l'\in S} a_{ll'} + \sum_{l\in S} a_{il}$$

$$\text{on } S \cup \{i,k\}: \quad 0 = \sum_{l\in S} a_l + a_i + a_k + \sum_{l<l'\in S} a_{ll'} + \sum_{l\in S} a_{il} + \sum_{l\in S} a_{kl} + a_{ik}$$

Subtracting above two equations gives:

$$a_k + \sum_{l\in S} a_{kl} + a_{ik} = 0 \tag{3.7}$$

Doing the same with $S \cup \{j\}$ and $S \cup \{j,k\}$ gives:

$$a_k + \sum_{l\in S} a_{kl} + a_{jk} = 0 \tag{3.8}$$

Thus $a_{ik} = a_{jk}$ (from equations (3.7), (3.8)).
Since i, j, k are arbitrary, $a_{ik} = a_{jk} = a_{jl}$ for any $l \neq i, j, k$. So all the coefficients of $x_i x_j$ (for $i \neq j$) in h are equal, say $a_{ij} = u; i \neq j$.
It follows from equation (3.7) that $a_k + mu = 0$. So $a_k = -mu \; \forall \; k$, which gives:

$$h(x_1, \ldots, x_n) = u\big(- m \sum_{i=1}^{n} x_i^2 + \sum_{i<j} x_i x_j \big)$$

But then $h(\underbrace{1, \ldots, 1}_{m}, 0, \ldots, 0) = 0$ gives

$$0 = u\Big(- m(m) + \frac{m(m-1)}{2} \Big),$$

which implies $u = 0$, which implies $h = 0$. □

Theorem 3.16. If $n \geq 5$ is odd, then L_n is not a sos.

Proof. Fix odd $n \geq 5, n = 2m + 1$. Then

$$L_{2m+1} = m(m+1) \sum_{i<j} (x_i - x_j)^4 - \Big(\sum_{i<j} (x_i - x_j)^2 \Big)^2.$$

If $L_{2m+1} = \sum_t h_t^2$, then $L_{2m+1}(\underline{x}) = 0 \Rightarrow$ each $h_t(\underline{x}) = 0$, for any $\underline{x} \in \mathbb{R}^n$.
In particular, $L_{2m+1}(\underline{x}) = 0$ when $\underline{x}$ has m or $(m+1)$ 1's and $(m+1)$ or m 0's. So, $h_t(\underline{x}) = 0$ for $\underline{x}$ with m or $(m+1)$ 1's and $(m+1)$ or m 0's respectively.
Write

$$h_t(\underline{x}) = \sum_{i=1}^{n} a_i x_i^2 + \sum_{i<j}^{n} a_{ij} x_i x_j$$

($a_{ij} = a_{ji}$ if needed).
Then by Lemma 3.15 above, we get $h_t = 0$. Hence L_{2m+1} is not a sos. □

Next we construct $f \in S\mathcal{P}_{n,4} \setminus S\Sigma_{n,4}$ for n even, $n \geq 4$

For $m \geq 2$, consider the symmetric quartic $L_{2m+1}(x_1, \ldots, x_{2m}, x_{2m+1})$ and substitute one of the variables among $x_1, \ldots, x_{2m}, x_{2m+1}$ equal to 0, w.l.o.g. say $x_{2m+1} = 0$. Then we get a symmetric quartic $C_{2m}(\underline{x})$ in $2m$ number of variables for $m \geq 2$, defined by

$$C_{2m}(x_1, \ldots, x_{2m}) := L_{2m+1}(x_1, \ldots, x_{2m}, 0).$$

Trivially $C_{2m}(x_1, \ldots, x_{2m}) \in S\mathcal{P}_{2m,4}$, in addition we prove that it is not a sos as follows:

Theorem 3.17. For $m \geq 2$, $C_{2m}(x_1, \ldots, x_{2m})$ is not a sos.

Proof. If $C_{2m} = \sum_t h_t^2$, then $C_{2m}(\underline{x}) = 0 \Rightarrow$ each $h_t(\underline{x}) = 0$, for any $\underline{x} \in \mathbb{R}^n$.
In particular, $C_{2m}(\underline{x}) = 0$ when $\underline{x}$ has m or $(m+1)$ 1's and m or $(m-1)$ 0's. So, $h_t(\underline{x}) = 0$ for $\underline{x}$ with m or $(m+1)$ 1's and m or $(m-1)$ 0's respectively.
Write

$$h_t(\underline{x}) = \sum_{i=1}^{n} a_i x_i^2 + \sum_{i<j}^{n} a_{ij} x_i x_j$$

($a_{ij} = a_{ji}$ if needed).
Then by Lemma 3.15 above, we get $h_t = 0$. Hence, C_{2m} is not a sos. □

From Proposition 3.12 and Theorems 3.16, 3.17, it follows that:

Corollary 3.18.

1. For all $m \geq 2$, $L_{2m+1} \in S\mathcal{P}_{2m+1,4} \setminus S\Sigma_{2m+1,4}$, and
2. For all $m \geq 3$, $C_{2m} \in S\mathcal{P}_{2m,4} \setminus S\Sigma_{2m,4}$.

This finishes the construction of explicit psd not sos symmetric $n-$ary quartic forms for all $n \geq 5$, thereby completing the proof of Theorem 3.1 and hence the answer to $Q(S)$.

In Chapter 4 we will actually prove a stronger version (see Theorems 4.12, 4.15) of the above Corollary 3.18 by showing that $L_{2m+1}(x_1^2, ..., x_{2m+1}^2)$ for $m \geq 2$ and $C_{2m}(x_1^2, \ldots, x_{2m}^2)$ for $m \geq 3$ are not sos, from which it follows that $L_{2m+1}(\underline{x})$ for $m \geq 2$ and $C_{2m}(\underline{x})$ for $m \geq 3$ are not sos (because if $L_{2m+1}(\underline{x})$ or $C_{2m}(\underline{x})$ or both were sos, then by substituting $x_i \rightarrow x_i^2\ \forall\ i = 1, \ldots, n$ they still have to be sos, but it is not the case).

Chapter 4

Even symmetric forms

The inclusion $S\Sigma^e_{n,2d} \subseteq S\mathcal{P}^e_{n,2d}$ holds for all n, d, since every sos form is clearly psd. In this chapter we will investigate the converse, i.e. the following question:

$$Q(S^e) : \text{For what pairs } (n, 2d) \text{ will } S\mathcal{P}^e_{n,2d} \subseteq S\Sigma^e_{n,2d} \text{ ?} \tag{4.1}$$

In Section 4.1, we will construct explicit forms $f \in S\mathcal{P}^e_{n,2d} \setminus S\Sigma^e_{n,2d}$ for the pairs $(n, 2d) = (3, 12)$, $(n, 8)_{n\geq 5}$ (see Propositions 4.9, 4.12, 4.15) and give a degree jumping principle (see Theorem 4.5) to find psd not sos even symmetric $n-$ary forms of degree $2d + 8, 2d + 12, 2d + 16, \ldots$ etc. and $2d + 2n$ from given psd not sos even symmetric $n-$ary $2d-$ic form. We will deduce that for the pairs $(n, 2d) = (n, 6)_{n\geq 3}, (n, 8)_{n\geq 4}, (3, 2d)_{d\geq 5}$, and $(n, 2d)_{n\geq 4, d\geq 7}$, the answer to $Q(S^e)$ is negative.

$Q(S^e)$ is an extension of Hilbert's original question (Q), mentioned before as equation (1.2), to the special case when the form considered is in addition even symmetric.

The following proposition gives a partial answer to $Q(S^e)$ based on the results already known in the literature:

Proposition 4.1. 1. $S\mathcal{P}^e_{n,2d} = S\Sigma^e_{n,2d}$ if $n = 2, d = 1, (n, 2d) = (n, 4)_{n\geq 3}, (3, 8)$.

2. $S\mathcal{P}^e_{n,2d} \supsetneq S\Sigma^e_{n,2d}$ for $(n, 2d) = (n, 6)_{n\geq 3}, (3, 10), (4, 8)$.

For part 1 of the above proposition, we give a proof for $n = 2, d = 1, (n, 2d) = (n, 4)_{n\geq3}$ and an idea of the proof for $(n, 2d) = (3, 8)$, as below. For part 2, we give explicit examples of forms $f \in S\mathcal{P}^e_{n,2d} \setminus S\Sigma^e_{n,2d}$ for the pairs $(n, 2d) = (n, 6)_{n\geq3}, (3, 10)$ and $(4, 8)$ as below.

1. $S\mathcal{P}^e_{n,2d} = S\Sigma^e_{n,2d}$ if $n = 2, d = 1, (n, 2d) = (n, 4)_{n\geq3}, (3, 8)$.

   (a) For $n = 2, d = 1, (n, 2d) = (3, 4)$:

   From Hilbert's Theorem 1.48 we know that a psd $n-$ary form of degree $2d$ is a sos if $n = 2, 2d = 2$, and $(n, 2d) = (3, 4)$. So this must hold in particular for even symmetric forms also.

   (b) For $(n, 2d) = (n, 4); n \geq 4$:

   The vector space of even symmetric $n-$ary quartics is two-dimensional (as seen in Section 1.1.5). One basis is $\left\{\sum_{j=1}^{n} x_j^4, \sum_{j<k} x_j^2x_k^2\right\}$. We will consider a more useful basis $\{f, g\}$, where $f = \sum_{j<k}(x_j^2 - x_k^2)^2$, $g = \sum_{j<k} x_j^2x_k^2$. Clearly, f and g are sos.

   If p is any even symmetric quartic, then there exist uniquely determined reals a and b such that $p = af + bg$.

   Let $u = (1, 0, .., 0)$ and $v = (1, ..., 1)$. Then

   $f(u) = n - 1, g(u) = 0, f(v) = 0, g(v) = n(n - 1)/2$.

   If p is psd, then $0 \leq p(u) = (n - 1)a$ and $0 \leq p(v) = n(n - 1)b/2$.

   Thus $a, b \geq 0$ and hence p is sos.

   (c) For $(n, 2d) = (3, 8)$:

   The vector space of even symmetric ternary octics is four-dimensional (as seen in Section 1.1.5). Any even symmetric ternary octic form $p(x, y, z)$ can be written uniquely as:

$$p(x, y, z) := [\alpha, \beta, \gamma, \delta] = \alpha \sum^{3} x^8 + \beta \sum^{6} x^6y^2 + \gamma \sum^{3} x^4y^4 + \delta \sum^{3} x^4y^2z^2,$$

where each summation $\sum^{k}$ denotes the sum taken over all the k permutations of variables which yield distinct expressions (for example $\sum^{3} x^8 = x^8 + y^8 + z^8$).

Harris showed (in [20, p221]) that any psd even symmetric ternary octic form is sos, by demonstrating that the following elements and families of elements in $S\Sigma^e_{3,8}$ comprise all the extremal forms of $S\mathcal{P}^e_{n,8}$:

(i) $A(x,y,z) = [0,1,-2,0] = \sum^{3}(x^2-y^2)^2y^2$;

(ii) $B(x,y,z) = [0,0,1,-1] = \frac{1}{2}\sum^{3}(x^2-y^2)^2z^4$;

(iii) $C(x,y,z) = [0,0,0,1] = \sum^{3} x^4y^2z^2$;

(iv) $D_b(x,y,z) = [1, 2b, b^2+2, 2b^2+2b]$

$= (x^4+y^4+z^4+b(x^2y^2+x^2z^2+y^2z^2))^2$, for $b < -1$;

(v) $E_u(x,y,z) = [1, -(u+1), u^2+2u, -u^2+1]$

$= \frac{1}{6}\sum^{3}(2x^4-y^4-z^4-(u+1)(x^2y^2+x^2z^2-2y^2z^2))^2$, for $u \geq 0$,

and then using the fact that any element of $S\mathcal{P}^e_{3,8}$ is a finite nonnegative linear combination of its extremal elements (i.e. A, B, C, D_t, E_u).

2. To show: $S\mathcal{P}^e_{n,2d} \supsetneq S\Sigma^e_{n,2d}$ for $(n,2d) = (n,6)_{n\geq 3}, (3,10), (4,8)$.

Proof. We divide this into subparts depending on the order of their appearance in the literature:

(a) <u>For $(n,2d) = (n,6); n \geq 3$:</u>

For $n \geq 3$; $t \in \mathbb{Z}, 2 \leq t \leq n-1$, Choi , Lam and Reznick (in [10, p572]) gave the following psd not sos even symmetric $n-$ary sextic forms:

$$f_t(x_1,\dots,x_n) := (t^2-t)\sum_{i=1}^{n} x_i^6 - 2(t-1)\sum_{i\neq j} x_i^4x_j^2 + 6\sum_{i<j<k} x_i^2x_j^2x_k^2.$$

For example, for $t = 2, n = 3$:

$$\begin{aligned} f_2(x,y,z) &= 2(x^6+y^6+z^6-(x^4y^2+y^4z^2+z^4x^2+x^2y^4+y^2z^4+z^2x^4)+3x^2y^2z^2) \\ &= 2R(x,y,z) \in S\mathcal{P}^e_{3,6} \setminus S\Sigma^e_{3,6}, \end{aligned}$$

where $R(x,y,z)$ is the Robinson's even symmetric ternary sextic.

Thus the forms $f_2,\dots,f_{n-1}$ were actually generalizations of the Robinson's form to the case of any arbitrary number (≥ 3) of variables.

(b) <u>For $(n, 2d) = (3, 10)$:</u>

For $0 \leq u \leq 2$, $u \neq 1$, the following even symmetric ternary decic forms are psd but not sos (see [20, p239]):

$$N_u = \sum^{3} x^{10} - (u+1)\sum^{6} x^8y^2 + u\sum^{6} x^6y^4 + (u+1)^2\sum^{3} x^6y^2z^2 - u(u+2)\sum^{3} x^4y^4z^2,$$

where each summation $\sum^{k}$ denotes the sum taken over all the k permutations of variables which yield distinct expressions (for example $\sum^{3} x^6y^2z^2 = x^6y^2z^2 + x^2y^6z^2 + x^2y^2z^6$).

(c) <u>For $(n, 2d) = (4, 8)$</u>

For $0 \leq v \leq 5$, $v \neq 1$, the following even symmetric quaternary octic forms are psd but not sos (see [21, p81]):

$$E_v = \sum^{4} x^8 - \frac{2}{3}(v+1)\sum^{12} x^6y^2 + \frac{2}{3}(2v-1)\sum^{6} x^4y^4 + (\frac{1}{3}v^2+1)\sum^{12} x^4y^2z^2 - 4(v^2+1)x^2y^2z^2w^2,$$

where each summation $\sum^{k}$ denotes the sum taken over all the k permutations of variables which yield distinct expressions (for example $\sum^{6} x^4y^4 = x^4y^4 + x^4z^4 + x^4w^4 + y^4z^4 + y^4w^4 + z^4w^4$).

(For more examples of psd not sos even symmetric ternary decics and quaternary octics see [20] or [21].) □

The above partially known answer to $Q(S^e)$ given in Proposition 4.1 can be summarized by the following chart:

deg \ var	2	3	4	5	...
2	✓	✓	✓	✓	...
4	✓	✓	✓	✓	...
6	✓	×	×	×	...
8	✓	✓	×	?	?
10	✓	×	?	?	?
12	✓	?	?	?	?
⋮	⋮	?	?	?	?

where, a tick (✓) denotes a positive answer to $Q(S^e)$, a cross (×) denotes a negative answer to $Q(S^e)$, and a (?) denotes an unknown answer to $Q(S^e)$.

To get a complete answer to $Q(S^e)$ it is interesting to look at the following remaining cases in which it is not known whether $S\mathcal{P}^e_{n,2d} \subseteq S\Sigma^e_{n,2d}$ or not:

1. $(3, 2d)$ for $d \geq 6$,

2. $(n, 8)$ for $n \geq 5$, and

3. $(n, 2d)$ for $n \geq 4, d \geq 5$.

In the coming section we will deal with these cases one by one and prove some results that will answer $Q(S^e)$ in more detail, and will bring us very close to a complete answer of $Q(S^e)$.

4.1 Version of Hilbert's 1888 theorem

In this section we will construct explicit forms $f \in S\mathcal{P}^e_{n,2d} \setminus S\Sigma^e_{n,2d}$ for the pairs $(n, 2d) = (3, 12)$, $(n, 8)_{n \geq 5}$ and show that for the pairs $(n, 2d) = (3, 2d)_{d \geq 6}$ and $(n, 2d)_{n \geq 4, d \geq 7}$, the answer to $Q(S^e)$ is negative. For the pairs $(n, 2d)$ for $n \geq 4$ and $d = 5, 6$ we are still working on getting an answer to $Q(S^e)$.

Our answer to $Q(S^e)$ can be summarized by the following chart, giving a version of Hilbert's 1888 Theorem (i.e. Theorem 1.48) for even symmetric forms (see Theorem 4.16):

deg \ var	2	3	4	5	6	...
2	✓	✓	✓	✓	✓	...
4	✓	✓	✓	✓	✓	...
6	✓	×	×	×	×	...
8	✓	✓	×	×	×	...
10	✓	×	?	?	?	?
12	✓	×	?	?	?	?
14	✓	×	×	×	×	...
⋮	⋮	⋮	⋮	⋮	⋮	⋱

where, a tick (✓) denotes a positive answer to $Q(S^e)$, a cross (×)\(×) denotes a negative known\our answer respectively to $Q(S^e)$, and a (?) denotes an unknown answer to $Q(S^e)$.

We first need the following lemmas for proving that $S\mathcal{P}^e_{n,2d} \supsetneq S\Sigma^e_{n,2d}$ for the pairs $(n, 2d) = (3, 2d)_{d \geq 6}$, $(n, 8)_{n \geq 5}$, and $(n, 2d)_{n \geq 4, d \geq 7}$:

Lemma 4.2. For $\theta \in \left(\frac{1}{3}, 1\right)$, define

$$\begin{aligned} F_\theta &= \frac{1}{1-\theta}((x^4 + y^4 + z^4) - \theta(x^2 + y^2 + z^2)^2) \\ &= (x^4 + y^4 + z^4) - \frac{2\theta}{1-\theta}(x^2y^2 + x^2z^2 + y^2z^2). \end{aligned}$$

Then F_θ is irreducible if and only if $\theta \neq \frac{1}{2}$.

Proof. See [20, p214]. □

Lemma 4.3. If $2t = 4, 6$, and $n \geq 3$, then

$$h_t(x_1, \ldots, x_n) := \sum_{i=1}^{n} x_i^{2t} - 10 \sum_{i \neq j} x_i^{2t-2} x_j^2$$

is an indefinite irreducible even symmetric $n-$ary form of degree $2t$.

Proof. For $t = 2, 3$; h_t is clearly indefinite (since $h_t(1, 0, \ldots, 0) = 1$ and $h_t(1, \ldots, 1) << 0$). Also h_2, h_3 are irreducible as below:

1. We have $h_2 = \sum_{i=1}^{n} x_i^4 - 10 \sum_{i \neq j} x_i^2 x_j^2$. Suppose for a contradiction h_2 is reducible, and let $h_2 = fg$; where f, g are non-constant forms (by Remark 1.13) such that $\deg(f) \geq 1$, $\deg(g) \geq 1$, $\deg(f) + \deg(g) = 4$.

   Setting $x_4 = x_5 = \ldots = x_n = 0$, we get:

   $$h_2(x_1, x_2, x_3, 0, \ldots, 0) = h_2(x_1, x_2, x_3)$$
   $$= f(x_1, x_2, x_3, 0, \ldots, 0)\, g(x_1, x_2, x_3, 0, \ldots, 0)$$

   $\Rightarrow h_2$ has a factorization as a product of two (or more) non-constant forms, which is a contradiction, since $h_2(x, y, z) = x^4 + y^4 + z^4 - 20(x^2y^2 + x^2z^2 + y^2z^2) = F_{\frac{10}{11}}(x, y, z)$ is irreducible by above lemma with $\theta = \frac{10}{11}$.

2. We have $h_3 = \sum_{i=1}^{n} x_i^6 - 10 \sum_{i \neq j} x_i^4 x_j^2$. Suppose for a contradiction h_3 is reducible, and let $h_3 = fg$; where f, g are non-constant forms (by Remark 1.13) such that $\deg(f) \geq 1$, $\deg(g) \geq 1$, $\deg(f) + \deg(g) = 6$.

   Setting $x_4 = x_5 = \ldots = x_n = 0$, we get:

   $$h_3(x_1, x_2, x_3, 0, \ldots, 0) = h_3(x_1, x_2, x_3)$$
   $$= f(x_1, x_2, x_3, 0, \ldots, 0)\, g(x_1, x_2, x_3, 0, \ldots, 0)$$

$\Rightarrow h_3$ has a factorization as a product of two (or more) non-constant forms, which is not possible since $h_3(x,y,z) = x^6 + y^6 + z^6 - 10(x^4y^2 + x^4z^2 + x^2y^4 + y^4z^2 + x^2z^4 + y^2z^4)$ cannot have a linear, irreducible quadratic or irreducible cubic factor as shown below in (a), (b) and (c) respectively. For simplicity say $h_3 = h$.

(a) <u>h cannot have a linear factor:</u>

Suppose $l(x,y,z) = ax + by + cz \mid h$, then

$a \neq 0, b \neq 0, c \neq 0$

(because if $a = 0$, i.e. if

$$h(x,y,z) = (by + cz)q(x,y,z);\ \deg(q) = 5, \tag{4.2}$$

then there will be no x^6 term on the R.H.S. of equation (4.2) and hence in h, which is a contradiction. Similarly $b \neq 0, c \neq 0$).

So by Remark 1.16, we have:

$$l(x,y,z) = ax + by + cz \mid h$$
$$l(x,y,-z) = ax + by - cz \mid h$$
$$l(x,-y,z) = ax - by + cz \mid h$$
$$l(x,-y,-z) = ax - by - cz \mid h.$$

Now

i. <u>if $|a| \neq |b|$</u> then again by Remark 1.16:

$$l(y,x,z) = bx + ay + cz \mid h$$
$$l(y,x,-z) = bx + ay - cz \mid h$$
$$l(-y,x,z) = bx - ay + cz \mid h$$
$$l(-y,x,-z) = bx - ay - cz \mid h.$$

So we have 8 distinct linear factors of h, namely $(ax+by+cz), (ax+by-cz), (ax-by+cz), (ax-by-cz), (bx+ay+cz), (bx+ay-cz), (bx-ay+cz)$, and $(bx-ay-cz)$, which is a contradiction since $\deg(h) = 6$.

ii. <u>if $|a| = |b|$</u> then $|a| = |b| = |c|$. So, $(x+y+z), (x+y-z), (x-y+z)$, and $(x-y-z)$ are linear factors of h.

Let $T(x,y,z) := (x+y+z)(x+y-z)(x-y+z)(x-y-z)$.
If
$$h(x,y,z) = T(x,y,z)q(x,y,z),$$
for some quadratic form $q(x,y,z)$, then $T(1,2,1) = 0 \Rightarrow h(1,2,1) = 0$, which gives a contradiction since $h(1,2,1) = -354 \neq 0$.

Hence h cannot have a linear factor.

(b) h cannot have an irreducible quadratic factor:

Suppose $q(x,y,z) = a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + a_{12}xy + a_{13}xz + a_{23}yz \mid h$, then either of the following holds:

i. $a_{ij} = 0$ for all $i < j$, or

ii. one of $a_{ij} \neq 0$ for $i < j$, or

iii. two of $a_{ij} \neq 0$ for $i < j$, or

iv. $a_{ij} \neq 0$ for all $i < j$.

i. if $a_{12} = a_{13} = a_{23} = 0$ then $q(x,y,z) = a_{11}x^2 + a_{22}y^2 + a_{33}z^3$, and either of the following holds:

- if a_{11}, a_{22}, a_{33} are all distinct then by Remark 1.16:

$$\begin{aligned} q_1 &:= q(x,y,z) = a_{11}x^2 + a_{22}y^2 + a_{33}z^2 \\ q_2 &:= q(x,z,y) = a_{11}x^2 + a_{33}y^2 + a_{22}z^2 \\ q_3 &:= q(y,x,z) = a_{22}x^2 + a_{11}y^2 + a_{33}z^2 \\ q_4 &:= q(z,x,y) = a_{22}x^2 + a_{33}y^2 + a_{11}z^2 \\ q_5 &:= q(y,z,x) = a_{33}x^2 + a_{11}y^2 + a_{22}z^2 \\ q_6 &:= q(z,y,x) = a_{33}x^2 + a_{22}y^2 + a_{11}z^2 \end{aligned}$$

are 6 irreducible quadratic factors of h. If at least 4 among them are pairwise non scalar multiples, then their product divides h implying $\deg(h) \geq 8$, which is a contradiction since $\deg(h) = 6$. Otherwise, if for some $i,j,k,l \in \{1,\ldots,6\}$, q_i is a non-zero scalar multiple of q_j and q_k is a non-zero scalar multiple of q_l, say w.l.o.g. $q_1 = \lambda q_6$, then we get:

$$\lambda a_{22} = a_{22} \tag{4.3}$$
$$\lambda a_{11} = a_{33}, \lambda a_{33} = a_{11}; \tag{4.4}$$

equation (4.3) implies $\lambda = 1$ or $a_{22} = 0$; but $\lambda = 1$ implies $a_{11} = a_{33}$ (by equation (4.4)), which is not possible (since a_{11}, a_{33} must be distinct), so $a_{22} = 0$. Now, equation (4.4) implies $\lambda = \pm 1$ or $a_{11} = a_{33} = 0$; so only possibility is $\lambda = -1$ (since $\lambda \neq 1$ and a_{11}, a_{33} must be distinct), so $a_{33} = -a_{11}$. Thus we have $a_{22} = 0, a_{33} = -a_{11}$, but then $p(x,y,z) = {a_{11}}^6(x^2 - z^2)(x^2 - y^2)(y^2 - z^2) = q_1q_2q_3 = -q_4q_5q_6$ does not divide h, since $p(1,1,0) = 0$ but $h(1,1,0) = -18 \neq 0$.

- if two of a_{11}, a_{22}, a_{33} are same w.l.o.g. let $a_{11} = r$; $a_{22} = a_{33} = s$ for $r \neq s$, then $(rx^2 + sy^2 + sz^2), (sx^2 + ry^2 + sz^2)$, and $(sx^2 + sy^2 + rz^2)$ are 3 distinct quadratic factors of h.
  So, h is a scalar multiple of
  $$p = (rx^2 + sy^2 + sz^2)(sx^2 + ry^2 + sz^2)(sx^2 + sy^2 + rz^2),$$
  say $h = \lambda p$, then comparing the coefficients of λp with those of $h = \sum_{i=1}^{3} x_i^6 - 10\sum_{i\neq j} x_i^4 x_j^2$ we get an insolvable system of equations in r, s:
  $$\lambda rs^2 = 1; \lambda(rs^2 + sr^2 + s^3) = -10; \lambda(r^3 + 2s^3 + 3rs^2) = 0.$$
  So this case doesn't arise.
- if $a_{11} = a_{22} = a_{33}$ then $q = x^2 + y^2 + z^2$
  Replacing z^2 by $-(x^2 + y^2)$, we get
  $h = (x^2 + y^2)(11(x^4 + y^4) + 7x^2y^2)$, which is positive definite, a contradiction, so nothing to prove.

ii. if one of $a_{ij} \neq 0$ for $i < j$ suppose w.l.o.g. $a_{12} \neq 0$ and $a_{13} = a_{23} = 0$, then by Remark 1.16:
$$q(x,y,z) = a_{11}x^2 + a_{22}y^2 + a_{33}z^3 + a_{12}xy,$$
$$q(x,-y,z) = a_{11}x^2 + a_{22}y^2 + a_{33}z^3 - a_{12}xy,$$
$$q(z,y,x) = a_{33}x^2 + a_{22}y^2 + a_{11}z^3 + a_{12}yz, \text{ and}$$
$$q(z,-y,x) = a_{33}x^2 + a_{22}y^2 + a_{11}z^3 - a_{12}yz$$

are 4 distinct quadratic factors of h. This would imply $\deg(h) \geq 8$, which is a contradiction since $\deg(h) = 6$.

iii. <u>if two of $a_{ij} \neq 0$ for $i < j$</u> suppose w.l.o.g. $a_{12}, a_{13} \neq 0$, then $q(x, y, z)$ and (by Remark 1.16) $q(x, -y, z), q(x, y, -z), q(x, -y, -z)$ are 4 distinct quadratic factors of h. This would imply $\deg(h) \geq 8$, which is a contradiction since $\deg(h) = 6$.

iv. <u>if $a_{ij} \neq 0$ for all $i < j$</u> similarly as in case (iii) above.

Hence h cannot have an irreducible quadratic factor.

(c) <u>h cannot have an irreducible cubic factor:</u>

Suppose $c(x, y, z) = \alpha x^3 + \beta y^3 + \gamma z^3 + c_1(x, y, z) \mid h$, where $c_1(x, y, z) := a_{210}x^2y + a_{201}x^2z + a_{120}xy^2 + a_{021}y^2z + a_{102}xz^2 + a_{012}yz^2 + a_{111}xyz$.

Then $\alpha, \beta, \gamma \neq 0$

$\Big($because if

$$h = c(x, y, z)(\alpha' x^3 + \beta' y^3 + \gamma' z^3 + c_1'(x, y, z)), \tag{4.5}$$

where $c_1'(x, y, z)$ is a cubic form defined like $c_1(x, y, z)$, then expanding the R.H.S. of equation (4.5) and comparing its coefficients with those of $h = \sum_{i=1}^3 x_i^6 - 10 \sum_{i \neq j} x_i^4 x_j^2$ we get $\alpha\alpha' = 1, \beta\beta' = 1, \gamma\gamma' = 1\Big)$.

So, $c(x, y, z)$ and (by Remark 1.16)

$$c(x, -y, z) = \alpha x^3 - \beta y^3 + \gamma z^3 + c_1(x, -y, z),$$
$$c(x, y, -z) = \alpha x^3 + \beta y^3 - \gamma z^3 + c_1(x, y, -z),$$
$$c(x, -y, -z) = \alpha x^3 - \beta y^3 - \gamma z^3 + c_1(x, -y, -z),$$

are 4 distinct cubic factors of h. This would imply $\deg(h) \geq 12$, which is a contradiction since $\deg(h) = 6$.

□

Lemma 4.4. If $r \in \mathbb{Z}_+; r \geq 2$, then there exists $a, b \in \mathbb{Z}_+$ such that $r = 2a + 3b$.

Proof. If r is even, then take $a = \dfrac{r}{2}, b = 0$.

If r is odd, then $r \geq 3$. So, $r - 3$ is even. So, $r = 2a + 3b; a \in \mathbb{Z}_+, b = 1$. □

Now we present as follows in Theorem 4.5, two amazing techniques that can be used to find psd not sos even symmetric forms of degree $2d + 8, 2d + 12, 2d + 16, \ldots$ etc. and $2d + 2n$ from a given psd not sos even symmetric form of degree $2d$. We will call it a Degree Jumping Principle, since the degree of the newly constructed form will be

= (degree of the previous form) + jump of degree $4r$ (for integer $r \geq 2$) and $2n$ respectively.

Theorem 4.5. Degree Jumping Principle:
If $f \in S\mathcal{P}^e_{n,2d} \setminus S\Sigma^e_{n,2d}$, $(n \geq 3)$, then

1. for any integer $r \geq 2$, the form $fh_2^{2a}h_3^{2b} \in S\mathcal{P}^e_{n,2d+4r} \setminus S\Sigma^e_{n,2d+4r}$, where $r = 2a + 3b$; $a, b \in \mathbb{Z}_+$.

2. $(x_1 \ldots x_n)^2 f \in S\mathcal{P}^e_{n,2d+2n} \setminus S\Sigma^e_{n,2d+2n}$.

Proof. 1. By above lemma, for $r \in \mathbb{Z}_+; r \geq 2$, $\exists$ non-negative $a, b \in \mathbb{Z}$ such that $r = 2a + 3b$.

Consider the form
$$fh_2^{2a}h_3^{2b}$$
of degree $2d + 4r$ in n variables, where $h_t(\underline{x}) := \sum_{i=1}^{n} x_i^{2t} - 10\sum_{i \neq j} x_i^{2t-2}x_j^2$ for $t = 2, 3$ as defined in Lemma 4.3.

Since $fh_2^{2a}h_3^{2b}$ is a product of even symmetric forms, it is even and symmetric. Also it is a product of psd forms so it is still psd. Thus we have $fh_2^{2a}h_3^{2b} \in S\mathcal{P}^e_{n,2d+4r}$.

Since h_2 and h_3 are indefinite and irreducible forms by Lemma 4.3, so taking $p = h_2$ or h_3 in Lemma 3.2 we get $fh_2^2 \in S\mathcal{P}^e_{n,2d+8} \setminus S\Sigma^e_{n,2d+8}$ and $fh_3^2 \in S\mathcal{P}^e_{n,2d+12} \setminus S\Sigma^e_{n,2d+12}$. Repeating this argument we get $fh_2^{2a}h_3^{2b} \notin S\Sigma^e_{n,2d+4r}$.

2. Taking $p = x_i$ in turn for each $1 \leq i \leq n$, we are done by Lemma 3.2.

$\square$

Proposition 4.6. If we can find psd not sos even symmetric $n-$ary $2d-$ic forms for the following pairs:

1. $(n, 2d) = (n, 8)$ for $n \geq 5$, and
2. $(n, 2d)$ for $n \geq 4, d = 5, 6$.

then the complete answer to $Q(S^e)$ will be:

$$S\mathcal{P}^e_{n,2d} \subseteq S\Sigma^e_{n,2d} \text{ if and only if } n = 2, d = 1, (n, 2d) = (n, 4)_{n\geq 3}, (3, 8).$$

Proof. We proved in Proposition 4.1 that

$$S\mathcal{P}^e_{n,2d} \subseteq S\Sigma^e_{n,2d} \text{ if } n = 2, d = 1, (n, 2d) = (n, 4)_{n\geq 3}, (3, 8).$$

For all the remaining cases we break the discussion into the following two cases:

1. <u>For $n = 3$:</u>
   (a) for $2d = 6$: there exists $f \in S\mathcal{P}^e_{3,6} \setminus S\Sigma^e_{3,6}$, as seen in Proposition 4.1
   (b) for $2d = 8$: $S\mathcal{P}^e_{3,8} = S\Sigma^e_{3,8}$
   (c) for $2d = 10$: there exists $f \in S\mathcal{P}^e_{3,10} \setminus S\Sigma^e_{3,10}$, as seen in Proposition 4.1
   (d) for $2d \geq 12$: using Theorem 4.5 (part 1 and 2 respectively):
      - $p \in S\mathcal{P}^e_{3,6} \setminus S\Sigma^e_{3,6} \Rightarrow \exists\, p' \in S\mathcal{P}^e_{3,14} \setminus S\Sigma^e_{3,14}$, and
      - $$\begin{array}{rccc} & f \in S\mathcal{P}^e_{3,6}\setminus S\Sigma^e_{3,6}; & g \in S\mathcal{P}^e_{3,10}\setminus S\Sigma^e_{3,10}; & h \in S\mathcal{P}^e_{3,14}\setminus S\Sigma^e_{3,14} \\ \text{(jump of deg 6)} & \Downarrow & \Downarrow & \Downarrow \\ \exists & f_1 \in S\mathcal{P}^e_{3,12}\setminus S\Sigma^e_{3,12}; & g_1 \in S\mathcal{P}^e_{3,16}\setminus S\Sigma^e_{3,16}; & h_1 \in S\mathcal{P}^e_{3,20}\setminus S\Sigma^e_{3,20} \\ \text{(jump of deg 6)} & \Downarrow & \Downarrow & \Downarrow \\ \exists & f_2 \in S\mathcal{P}^e_{3,18}\setminus S\Sigma^e_{3,18}; & g_2 \in S\mathcal{P}^e_{3,22}\setminus S\Sigma^e_{3,22}; & h_2 \in S\mathcal{P}^e_{3,26}\setminus S\Sigma^e_{3,26} \\ & \vdots & \vdots & \vdots \end{array}$$

2. <u>For $n \geq 4$:</u>
   (a) for $2d = 6$: there exists $f \in S\mathcal{P}^e_{n,6} \setminus S\Sigma^e_{n,6}$, as seen in Proposition 4.1
   (b) for $2d = 8$:

- for $n = 4$, there exists $f \in S\mathcal{P}^e_{4,8} \setminus S\Sigma^e_{4,8}$, as seen in Proposition 4.1
- for $n \geq 5$, there exists $f \in S\mathcal{P}^e_{n,8} \setminus S\Sigma^e_{n,8}$ by hypothesis (we will do this in Section 4.1.2)

(c) for $2d = 10$: there exists $f \in S\mathcal{P}^e_{n,10} \setminus S\Sigma^e_{n,10}$ by hypothesis

(d) for $2d = 12$: there exists $f \in S\mathcal{P}^e_{n,12} \setminus S\Sigma^e_{n,12}$ by hypothesis

(e) for $2d \geq 14$: using (part 1 of) Theorem 4.5:

$$\begin{array}{lcc} & f \in S\mathcal{P}^e_{n,6} \setminus S\Sigma^e_{n,6}; & g \in S\mathcal{P}^e_{n,8} \setminus S\Sigma^e_{n,8} \\ \text{(jump of degree 8)} & \Downarrow & \Downarrow \\ & \exists\, f_1 \in S\mathcal{P}^e_{n,14} \setminus S\Sigma^e_{n,14}; & g_1 \in S\mathcal{P}^e_{n,16} \setminus S\Sigma^e_{n,16} \\ \text{(jump of degree 12)} & \Downarrow & \Downarrow \\ & \exists\, f_2 \in S\mathcal{P}^e_{n,18} \setminus S\Sigma^e_{n,18}; & g_2 \in S\mathcal{P}^e_{n,20} \setminus S\Sigma^e_{n,20} \\ & \vdots & \vdots \end{array}$$

□

So we get the following as an immediate corollary to Proposition:

Corollary 4.7. $S\mathcal{P}^e_{n,2d} \supsetneq S\Sigma^e_{n,2d}$ for $(n, 2d) = (3, 2d)_{d\geq 6}, (n, 2d)_{n\geq 4, d\geq 7}$.

In addition to the degree jumping techniques presented above in Theorem 4.5 we also note the following result that can be used to find psd not sos even symmetric ternary forms of degree $6, 10, 14, 18\ldots$ etc.:

Lemma 4.8. If μ is odd integer,

$$H_\mu(x, y, z) := h_\mu(x^2, y^2, z^2) \in S\mathcal{P}^e_{3,2\mu+4} \setminus S\Sigma^e_{3,2\mu+4},$$

where $h(x, y, z) = x^\mu(x - y)(x - z) + y^\mu(y - z)(y - x) + z^\mu(z - x)(z - y)$.

Proof. See [8, p6]. □

4.1.1 Psd not sos even symmetric ternary dodecics

We give some explicit examples of psd not sos even symmetric ternary dodecics as below.

Proposition 4.9. The forms $R_1 := R(xy, xz, yz)$ and $R_2 := (xyz)^2 R(x, y, z) \in S\mathcal{P}^e_{3,12} \setminus S\Sigma^e_{3,12}$, where $R(x, y, z) = x^6 + y^6 + z^6 - (x^4y^2 + y^4z^2 + z^4x^2 + x^2y^4 + y^2z^4 + z^2x^4) + 3x^2y^2z^2$ is the psd not sos Robinson's form.

Proof. (i) Clearly $R_1 \in S\mathcal{P}^e_{3,12}$. Suppose $R_1(x, y, z) = \sum f_k^2(x, y, z)$, then

$$R_1(xy, xz, yz) = \sum f_k^2(xy, xz, yz)$$

Let $g_k(x, y, z) := f_k(xy, xz, yz)$, then

$R_1(xy, xz, yz) = R(x^2yz, xy^2z, xyz^2) = (xyz)^6 R(x, y, z) = \sum g_k^2(x, y, z)$
$\Rightarrow x^3y^3z^3 \mid g_k$
$\Rightarrow g_k = x^3y^3z^3 h_k(x, y, z)$
$\Rightarrow R = \sum h_k^2(x, y, z)$, a contradiction.

(ii) $R_2 \in S\mathcal{P}^e_{3,12} \setminus S\Sigma^e_{3,12}$ follows directly from (part 2 of) Theorem 4.5. □

Proposition 4.10. The form $H(x, y, z) := h(x^2, y^2, z^2) \in S\mathcal{P}^e_{3,12} \backslash S\Sigma^e_{3,12}$, for $h(x, y, z) = \sum^3 x^6 - \sum^6 x^5y + \sum^3 x^4yz$, where each summation $\sum^k$ denotes the sum taken over all the k permutations of variables which yield distinct expressions.

Proof. See [8, p6]. □

4.1.2 Psd not sos even symmetric $n-$ary octics for $n \geq 5$

For finding psd not sos even symmetric $n-$ary octics for $n \geq 5$ we first note the following lemma which will be particularly useful in proving the main results of this section.

Lemma 4.11. Suppose $p = \sum_{i=1}^{r} h_i^2$ is an even sos form. Then we may write $p = \sum_{j=1}^{s} q_j^2$, where each form q_j^2 is even. In particular, $q_j(\underline{x}) = \sum c_j(\alpha)\underline{x}^\alpha$, where the sum is taken over α's in one congruence class mod 2 componentwise.

Proof. See [10, Theorem 4.1]. □

Proposition 4.12. The form

$$B(x_1, \dots, x_5) := A_5(x_1^2, \dots, x_5^2) \in S\mathcal{P}_{5,8}^e \setminus S\Sigma_{5,8}^e,$$

where $A_5(\underline{x}) := \sum_{i=1}^{5} \prod_{j \neq i} (x_i - x_j)$ is a symmetric psd not sos 5−ary quartic form.

($A_5 = \frac{1}{8} L_5$ was discussed already in Section 3.1.2).

Proof. Clearly B is psd, since A_5 is psd. Suppose $B = \sum h_t^2$. Since B is an even form, by Lemma 4.11 each h_t^2 is even. So h_t can be only of the following 3 types:

1. $h_t = e_{ijkl} x_i x_j x_k x_l$, or
2. $h_t = x_i x_j \Big(d_1 x_i^2 + d_2 x_j^2 + \sum_{k \neq i,j} d_k x_k^2 \Big)$, or
3. $h_t = \sum_{i=1}^{5} c_i x_i^4 + \sum_{i<j} c_{ij} x_i^2 x_j^2$.

Since $\Lambda_{5,2} = \{\underline{x} \in \mathbb{R}^5 \mid \exists\, r \neq s \in \mathbb{R} : x_i \in \{r, s\}\}$ is a test set for symmetric quartics (by Corollary 3.11), so $A_5 \geq 0$ on $\Lambda_{5,2}$. In particular, $A_5(\underline{x}) = 0$ on the points $\underline{x} \in \{(r, r, r, s, s)$ and its permutations, $(r, r, r, r, r); r \neq s \in \mathbb{R}\}$. Therefore $B = 0$ on such points $\underline{x}$ and hence for $r \neq s \in \mathbb{R}$, each $h_t = 0$ on the points (r, r, r, s, s), their permutations and (r, r, r, r, r). Now

1. if $h_t = e_{ijkl} x_i x_j x_k x_l$, then

   $h_t(1, 1, 1, 1, 1) = 0 \Rightarrow e_{ijkl} = 0$, which implies $h_t = 0$.

2. if $h_t = x_i x_j\Big(d_1 x_i^2 + d_2 x_j^2 + \sum_{k\neq i,j} d_k x_k^2\Big)$

   W.l.o.g. take $h_t = x_1 x_2\Big(d_1 x_1^2 + d_2 x_2^2 + \sum_{k=3}^{5} d_k x_k^2\Big)$, then

$$h_t(1,1,0,0,0) = 0 \Rightarrow d_1 + d_2 = 0 \tag{4.6}$$

$$h_t(1,1,1,0,0) = 0 \Rightarrow d_1 + d_2 + d_3 = 0 \tag{4.7}$$

$$h_t(1,1,0,1,0) = 0 \Rightarrow d_1 + d_2 + d_4 = 0 \tag{4.8}$$

$$h_t(1,1,0,0,1) = 0 \Rightarrow d_1 + d_2 + d_5 = 0 \tag{4.9}$$

$$h_t(1,2,1,1,2) = 0 \Rightarrow 2(d_1 + 4d_2 + d_3 + d_4 + 4d_5) = 0 \tag{4.10}$$

   Using equation (4.6) in equations (4.7), (4.8) and (4.9), we get $d_3 = d_4 = d_5 = 0$. Using this in equation (4.10), we get $2(d_1 + 4d_2) = 0$, and then again using equation (4.6), we get $d_1 = d_2 = 0$. So $h_t = 0$.

3. if $h_t = \sum_{i=1}^{5} c_i x_i^4 + \sum_{i<j} c_{ij} x_i^2 x_j^2$, then

   $h_t(1,1,0,0,0) = 0 \Rightarrow c_1 + c_2 + c_{12} = 0.$

   Similarly at $\underline{x}$ with $x_i = x_j = 1; x_k = 0$ for $k \neq i, j; i, j, k \in \{1,\ldots,5\}$:

$$h_t(\underline{x}) = 0 \Rightarrow c_i + c_j + c_{ij} = 0 \Rightarrow c_{ij} = -(c_i + c_j) \tag{4.11}$$

   Evaluating h_t at $\underline{x}$ with $x_i = x_j = x_k = 1; x_l = 0$ for $l \neq i, j, k;\ i, j, k, l \in \{1,\ldots,5\}$:

$$h_t(\underline{x}) = 0 \Rightarrow c_i + c_j + c_k + c_{ij} + c_{ik} + c_{jk} = 0$$

$$\underbrace{\Rightarrow}_{\text{(equation (4.11))}} \quad c_i + c_j + c_k - (c_i + c_j) - (c_i + c_k) - (c_j + c_k) = 0$$

$$\Rightarrow c_i + c_j + c_k = 0; i, j, k \in \{1,\ldots,5\} \text{ distinct} \tag{4.12}$$

   Similarly,

$$c_i + c_j + c_l = 0; i, j, k, l \in \{1,\ldots,5\} \text{ distinct} \tag{4.13}$$

   Equations (4.12) and (4.13) $\Rightarrow c_l = c_k; l, k \in \{1,\ldots,5\}$ distinct, i.e. $c_l = c_k$ for arbitrary l, k. This $\Rightarrow c_1 = c_2 = \ldots = c_5$

$$\underbrace{\Rightarrow}_{\text{(equation (4.12))}} \quad 3c_i = 0 \Rightarrow c_i = 0 \;\forall\; i$$

$$\underbrace{\Rightarrow}_{\text{(equation (4.11))}} \quad c_{ij} = 0 \;\forall\; i < j$$

So each $h_t = 0$.

□

For finding psd not sos even symmetric $n-$ary octics for $n \geq 6$ we first prove two lemmas, as below, that will be used in proving Theorem 4.15 following immediately after these:

Lemma 4.13. For $n \geq 6$, if $h(x_1, \ldots, x_n) = x_1 x_2 \Big(d_1 x_1^2 + d_2 x_2^2 + \sum_{k=3}^{n} d_k x_k^2 \Big)$ is a quartic form that vanishes on all 0/1 points with m or $(m+1)$ 1's, where $m = \left[\frac{n}{2}\right]$, i.e. $h(\underline{x}) = 0$ for all $\underline{x}$ with m or $(m+1)$ 1's and $\begin{cases} (m+1) \text{ or } m \text{ 0's (resp.) for odd } n = 2m+1; \\ m \text{ or } (m-1) \text{ 0's (resp.) for even } n = 2m. \end{cases}$
Then $h = d_1 x_1 x_2 (x_1^2 - x_2^2)$.

Proof. Pick $r \in \{3, \ldots, n\}$ and choose $A \subseteq \{3, \ldots, n\}$ such that $|A| = m - 2$ and $r \notin A$. Then $|A \cup \{r\}| = m - 1$. So, $h = 0$ on $\underline{x}$, where the 1's on $\underline{x}$ occur precisely on $A \cup \{1, 2\}$, $A \cup \{1, 2, r\}$. So we have:

$$\text{on } A \cup \{1,2\}: \quad d_1 + d_2 + \sum_{k \in A} d_k = 0 \tag{4.14}$$

$$\text{on } A \cup \{1,2,r\}: \; d_1 + d_2 + \sum_{k \in A} d_k + d_r = 0$$

Subtracting above two equations gives: $d_r = 0$. Since r was arbitrary, we can arrange this for any $r \in \{3, \ldots, n\}$. So, $d_r = 0$ for all $r \in \{3, \ldots, n\}$. This with equation (4.14) implies $d_1 + d_2 = 0$, i.e. $d_2 = -d_1$.
Thus we get, $h = d_1 x_1 x_2 (x_1^2 - x_2^2)$. □

Lemma 4.14. For $n \geq 6$, if $h(x_1, \ldots, x_n) = \sum_{i=1}^{n} c_i x_i^4 + \sum_{i<j} c_{ij} x_i^2 x_j^2$ is an even quartic form that vanishes on all 0/1 points with m or $(m+1)$ 1's, where $m = \left[\frac{n}{2}\right]$, i.e. $h(\underline{x}) = 0$ for all $\underline{x}$ with m or $(m+1)$ 1's and $\begin{cases} (m+1) \text{ or } m \text{ 0's (resp.) for odd } n = 2m+1; \\ m \text{ or } (m-1) \text{ 0's (resp.) for even } n = 2m. \end{cases}$
Then h is identically zero.

Proof. Pick $r \in \{1, \ldots, n\}$ and choose distinct $A, B \subseteq \{1, \ldots, n\}$ such that $r \notin A, B$, and $|A| = |B| = m$. Then $|A \cup \{r\}| = |B \cup \{r\}| = m + 1$. So, $h = 0$ on $\underline{x}$, where the 1's on $\underline{x}$ occur precisely on A, $A \cup \{r\}$, B, $B \cup \{r\}$. So we have:

on A : $\quad \sum_{i \in A} c_i + \sum_{i<j \in A} c_{ij} = 0$

on $A \cup \{r\}$: $\sum_{i \in A} c_i + c_r + \sum_{i<j \in A} c_{ij} + \sum_{i \in A} c_{ir} = 0$

Subtracting above two equations gives:

$$c_r + \sum_{i \in A} c_{ir} = 0 \tag{4.15}$$

Doing the same with B and $B \cup \{r\}$ gives:

$$c_r + \sum_{i \in B} c_{ir} = 0 \tag{4.16}$$

Pick $s, t \in \{1, \ldots, n\}$ such that r, s, t are distinct. Let $S \subseteq \{1, \ldots, n\}$ such that $|S| = m - 1$ be a set not containing r, s, t. Let $A = S \cup \{s\}$ and $B = S \cup \{t\}$, then equations (4.15), (4.16) implies

$$c_r + \sum_{i \in S} c_{ir} + c_{sr} = 0$$

$$c_r + \sum_{i \in S} c_{ir} + c_{tr} = 0$$

Subtracting above two equations gives $c_{sr} = c_{tr}$. Since r, s, t were arbitrary, this holds for all r, s, t distinct. So all the coefficients of $x_i^2 x_j^2$ (for $i \neq j$) in h are the same, say $c_{ij} = u; i \neq j$.

From equation (4.15) it follows that $c_r + mu = 0$. So $c_r = -mu \; \forall \; r$, which gives:

$$h(x_1, \dots, x_n) = u\big(- m \sum_{i=1}^{n} x_i^4 + \sum_{i<j} x_i^2 x_j^2 \big)$$

But then $h(\underbrace{1, \dots, 1}_{m}, 0, \dots, 0) = 0$ gives

$$0 = u\Big(- m(m) + \frac{(m)(m-1)}{2}\Big),$$

which implies $u = 0$, which implies $h = 0$. □

Theorem 4.15. For $m \geq 3$,

1. $M_{2m+1} := L_{2m+1}(x_1^2, \dots, x_{2m+1}^2) \in S\mathcal{P}^e_{2m+1,8} \setminus S\Sigma^e_{2m+1,8}$, and
2. $D_{2m} := C_{2m}(x_1^2, \dots, x_{2m}^2) \in S\mathcal{P}^e_{2m,8} \setminus S\Sigma^e_{2m,8}$,

where $L_{2m+1} = m(m+1) \sum_{i<j} (x_i - x_j)^4 - \Big(\sum_{i<j} (x_i - x_j)^2 \Big)^2$ is a symmetric psd not sos $(2m+1)-$ary quartic form and $C_{2m} := L_{2m+1}(x_1, \dots, x_{2m}, 0)$ is a symmetric psd not sos $2m-$ary quartic form (already discussed in Section 3.1.2).

Proof. Clearly both M_{2m+1} and D_{2m} are psd, since L_{2m+1} and C_{2m} are psd.
Suppose $M_{2m+1} = \sum_t h_t^2$, respectively $D_{2m} = \sum_{t'} g_{t'}^2$. Then since M_{2m+1} and D_{2m} are even forms, by Lemma 4.11 each h_t^2 and $g_{t'}^2$ is even. So h_t respectively $g_{t'}$ can be only of the following 3 types:

(i) $h_t = e_{ijkl} x_i x_j x_k x_l$; $g_{t'} = e'_{ijkl} x_i x_j x_k x_l$, or

(ii) $h_t = x_i x_j \Big(d_i x_i^2 + d_j x_j^2 + \sum_{k \neq i,j} d_k x_k^2 \Big)$; $g_{t'} = x_i x_j \Big(d'_i x_i^2 + d'_j x_j^2 + \sum_{k \neq i,j} d'_k x_k^2 \Big)$, or

(iii) $h_t = \sum_{i=1}^{2m+1} c_i x_i^4 + \sum_{i<j} c_{ij} x_i^2 x_j^2$; $g_{t'} = \sum_{i=1}^{2m} c'_i x_i^4 + \sum_{i<j} c'_{ij} x_i^2 x_j^2$.

We will show below that the h_t's, $g_{t'}$'s of type (i) and (iii) will be identically zero, and the h_t's, $g_{t'}$'s of type (ii) will give a contradiction. Hence proving that M_{2m+1} and D_{2m} are not sos.

For $\underline{x} \in \Lambda_{2m+1,2} = \{(\underbrace{r,\ldots,r}_{k}, \underbrace{s,\ldots,s}_{2m+1-k}) \mid r \neq s \in \mathbb{R};\ 1 \leq k \leq 2m+1\}$

$$x_i - x_j = \begin{cases} \pm(r-s) \neq 0, & \text{for } k(2m+1-k) \text{ terms,} \\ 0 & \text{, otherwise} \end{cases}$$

so $L_{2m+1}(\underline{x}) = k(2m+1-k)(r-s)^4[m(m+1) - k(2m+1-k)]$, which is $= 0$ when $k = 2m+1, m, m+1$. In particular $L_{2m+1}(\underline{x}) = 0$ at $\underline{x} = (1,\ldots,1)$ and at $\underline{x}$ with m or $(m+1)$ 1's and $(m+1)$ or m 0's (respectively). Therefore

1. $M_{2m+1}(\underline{x}) = 0$ on such points $\underline{x}$ and hence each $h_t(\underline{x}) = 0$ on such points $\underline{x}$. Now

   - if $h_t = e_{ijkl}x_ix_jx_kx_l$, then
     $h_t(1,\ldots,1) = 0 \Rightarrow e_{ijkl} = 0$, which implies $h_t = 0$.
   - if $h_t = x_ix_j\Big(d_ix_i^2 + d_jx_j^2 + \sum_{k\neq i,j} d_kx_k^2\Big)$, then by Lemma 4.13, we get $h_t = d_ix_ix_j(x_i^2 - x_j^2)$.
   - if $h_t = \sum_{i=1}^{2m+1} c_ix_i^4 + \sum_{i<j} c_{ij}x_i^2x_j^2$, then by Lemma 4.14, we get $h_t = 0$.

   Hence the only possibility sos decomposition of M_{2m+1} is

   $$M_{2m+1}(\underline{x}) = \sum_{i\neq j} \Big(d_ix_ix_j(x_i^2 - x_j^2)\Big)^2,$$

   which is not possible since at $\underline{x} = (1,0,\ldots,0)$ each term on R.H.S. is $= 0$, but $M_{2m+1}(\underline{x}) \neq 0$. Hence M_{2m+1} is not a sos.

2. $C_{2m}(x_1,\ldots,x_{2m}) = 0$ at $\underline{x}$ with m or $(m+1)$ 1's and m or $(m-1)$ 0's (respectively). Therefore $D_{2m}(\underline{x}) = 0$ on such points $\underline{x}$ and hence each $g_{t'}(\underline{x}) = 0$ on such points $\underline{x}$. Now

   - if $g_{t'} = e'_{ijkl}x_ix_jx_kx_l$, then
     $g_{t'}(\underline{x}) = 0$ at $\underline{x}$ with m or $(m+1)$ 1's including 1's at the indices i, j, k, l
     $\Rightarrow e_{ijkl} = 0$, which implies $g_{t'} = 0$.

- if $g_{t'} = x_i x_j\Big(d'_i x_i^2 + d'_j x_j^2 + \sum_{k\neq i,j} d'_k x_k^2\Big)$, then by Lemma 4.13, we get $g_{t'} = d'_i x_i x_j (x_i^2 - x_j^2)$.
- if $g_{t'} = \sum_{i=1}^{2m} c'_i x_i^4 + \sum_{i<j} c'_{ij} x_i^2 x_j^2$, then by Lemma 4.14, we get $g_{t'} = 0$.

Hence the only possibility sos decomposition of D_{2m} is

$$D_{2m}(\underline{x}) = \sum_{i\neq j} \Big(d'_i x_i x_j (x_i^2 - x_j^2)\Big)^2,$$

which is not possible since at $\underline{x} = (1, 0, \ldots, 0)$ each term on R.H.S. is $= 0$, but $D_{2m}(\underline{x}) \neq 0$. Hence D_{2m} is not a sos.

□

After constructing psd not sos even symmetric ternary do-decics and $n-$ary octics for $n \geq 5$, we have the following answer to question $Q(S^e)$:

Theorem 4.16. Version of Hilbert's 1888 Theorem for even symmetric forms: $S\mathcal{P}^e_{n,2d} = S\Sigma^e_{n,2d}$ for $n = 2, d = 1, (n, 2d) = (n, 4)_{n\geq3}, (3, 8)$, and $S\mathcal{P}^e_{n,2d} \supsetneq S\Sigma^e_{n,2d}$ for $(n, 2d) = (n, 6)_{n\geq3}, (3, 2d)_{d\geq5}, (n, 8)_{n\geq4}, (n, 2d)_{n\geq4, d\geq7}$.

Proof. Follows from Proposition 4.1, Corollary 4.7, Proposition 4.12, and Theorem 4.15. □

Observation 4.17. By substituting $x_i \to x_i^2;\ \forall\ i = 1, \ldots, n$, in a psd not sos (symmetric) $n-$ary $2d-$ic form we get an even psd (symmetric) $n-$ary $4d-$ic form. But this technique doesn't necessarily result in a non sos form, as shown below:

- Note the following theorem from [38]:

  If p is positive definite such that $Z(p) = 0$, then $\exists$ computable N s.t. $(\sum x_i^2)^N p$ is sos.

- The Robinson form $R(x, y, z) = x^6 + y^6 + z^6 - (x^4y^2 + y^4z^2 + z^4x^2 + x^2y^4 + y^2z^4 + z^2x^4) + 3x^2y^2z^2$ is a psd not sos ternary sextic, but the form $R(x^2, y^2, z^2)$ is a sos ternary dodecic, since there exists a Gram matrix corresponding to f with positive eigen values. We noticed this by using SeDuMi [47].

- The Motzkin form $M(x, y, z) = z^6 + x^4y^2 + x^2y^4 - 3x^2y^2z^2$ is a psd not sos ternary sextic, but the form $M(x^2, y^2, z^2)$ is a sos ternary dodecic since $M(x^2, y^2, z^2) = 2(x^3y^3 - xyz^4)^2 + (x^4y^2 - x^2y^4)^2 + (z^6 - x^2y^2z^2)^2$ (as given in [36, p446]) .

- It is interesting to note the following conjecture by B. Reznick [36]:

  If $p(x_1, \ldots, x_n)$ is a psd not sos form, then $\exists\, k \in \mathbb{Z}$ s.t. $p(x_1^k, \ldots, x_n^k)$ is sos.

  He showed later that this conjecture is true for agiforms (i.e. forms arising from monomial substitution into the arithmetic-geometric inequality (see Lemma 1.50) for example the Motzkin form $M(x, y, z), S(x, y, z), Q(x, y, z)$ (as given in Section 1.2.3), but false for the Horn form $H(x_1, \ldots, x_5) = (x_1^2 + \ldots + x_5^2)^2 - 4(x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_4^2 + x_4^2x_5^2 + x_5^2x_1^2)$, which is a psd not sos even quartic form (see [7, p396]). He proved that $\{k \in \mathbb{Z} \mid H(x_1^k, \ldots, x_5^k)$ is a sos$\} = \phi$, in particular $H(x_1^2, \ldots, x_5^2)$ is not a sos (as mentioned in [36, p462]).

4.2 As sum of binomial squares

We have seen in Proposition 4.1 that for the pairs $(n, 2d) = (n, 2), (2, 2d), (n, 4)_{n\geq 3}$ and $(3, 8)$ any psd even symmetric $n-$ary form f of degree $2d$ is a sos. In this section we move further to find out for which of these pairs $(n, 2d)$, f is in fact a sobs. We will check this by applying Theorem 2.5, i.e. sufficient condition on a form to be a sobs, to special cases of even symmetric forms. In particular, we will prove that after applying it on even symmetric psd quartics the form considered comes out to be a sobs, which is a stronger result than the result “even symmetric psd $n-$ary quartics are sos” due to Choi, Lam and Reznick.

4.2.1. Even symmetric sos quadratic forms

Proposition 4.18. $f \in S\mathcal{F}^e_{n,2}$ is a sobs iff $f(1, 0, \ldots, 0) \geq 0$.

Proof. Trivially $f \in S\mathcal{F}^e_{n,2}$ sobs implies $f(1, 0, \ldots, 0) \geq 0$.

Conversely, if $f \in S\mathcal{F}^e_{n,2}$, then $f = a\sum_{i=1}^{n} x_i^2, a \in \mathbb{R}$. So, $f(1, 0, \ldots, 0) = a \geq 0$ implies f is a sum of monomial squares and hence a sobs. □

4.2.2. Even symmetric sos binary forms

Theorem 4.19. For $d = 2, 3 : f \in S\mathcal{F}^e_{2,2d}$ is a sobs iff $f(1, 0) \geq 0$, $f(1, 1) \geq 0$.

Proof. Trivially $f \in S\mathcal{F}^e_{2,2d}$ sobs implies $f(1, 0) \geq 0$, $f(1, 1) \geq 0$.

Conversely, to prove that $f \in S\mathcal{F}^e_{2,2d}$ s.t. $f(1, 0) \geq 0$ and $f(1, 1) \geq 0$ is a sobs, it is enough to check the two sufficient conditions in Theorem 2.5. We consider the two cases $d = 2$ and $d = 3$ as below:

1. Let $f = a(x_1^4 + x_2^4) + bx_1^2x_2^2 \in S\mathcal{F}^e_{2,4}$ s.t. $f(1, 0) = a \geq 0$, $f(1, 1) \geq 0$.

- if $b \geq 0$, then clearly f is a sobs.
- if $b < 0$, then following the Notation 2.4: $\Delta_f = \{(2, 2)\}$

  Note that for $\alpha \in \Delta_f : f_\alpha = b$.

  For $\alpha \in \Delta_f; i = 1, 2$ taking $a_{\alpha,i} = \alpha_i \frac{|b|}{4} \in \mathbb{R}_+$:

  <u>Condition 1.</u> $\forall\ \alpha \in \Delta_f$:

  L.H.S.= $(2d)^{2d} a_\alpha{}^\alpha = 4^4 a_{\alpha,1}^{\alpha_1} a_{\alpha,2}^{\alpha_2} = 4^4 \alpha_1{}^{\alpha_1} \alpha_2{}^{\alpha_2} \left(\frac{|b|}{4}\right)^{\alpha_1+\alpha_2} = |b|^4 \alpha^\alpha$= R.H.S.

  <u>Condition 2.</u> for $i = 1, 2$:

  R.H.S.=$\sum_{\alpha\in\Delta_f} a_{\alpha,i} = \sum_{\alpha\in\Delta_f} \alpha_i \frac{|b|}{4} = 2\frac{|b|}{4} = \frac{(-b)}{2}$, is $\leq a = f_{2d,i} = L.H.S. \Leftrightarrow$ $2a + b \geq 0$, which is true since $f(1, 1) = 2a + b \geq 0$ as supposed.

Hence f is a sobs.

2. Let $f = a(x_1^6 + x_2^6) + b(x_1^4x_2^2 + x_1^2x_2^4) \in S\mathcal{F}^e_{2,6}$ s.t $f(1,0) = a \geq 0$, $f(1,1) \geq 0$.

- if $b \geq 0$, then clearly f is a sobs.
- if $b < 0$, then following the Notation 2.4: $\Delta_f = \{(4,2),(2,4)\}$

  Note that for $\alpha \in \Delta_f : f_\alpha = b$.

  For $\alpha \in \Delta_f$; $i = 1,2$ taking $a_{\alpha,i} = \alpha_i \dfrac{|b|}{6} \in \mathbb{R}_+$:

  Condition 1. $\forall\ \alpha \in \Delta_f$:

  L.H.S.= $(2d)^{2d} a_\alpha{}^\alpha = 6^6 a_{\alpha,1}^{\alpha_1}\, a_{\alpha,2}^{\alpha_2} = 6^6 \alpha_1{}^{\alpha_1}\alpha_2{}^{\alpha_2}\left(\dfrac{|b|}{6}\right)^{\alpha_1+\alpha_2} = |b|^6\alpha^\alpha$= R.H.S.

  Condition 2. for $i = 1,2$:

  R.H.S.=$\displaystyle\sum_{\alpha\in\Delta_f} a_{\alpha,i} = \sum_{\alpha\in\Delta_f} \alpha_i\frac{|b|}{6} = (4+2)\frac{|b|}{6} = -2$, is $\leq a = f_{2d,i} = L.H.S.$
  $\Leftrightarrow a + b \geq 0$, which is true since $f(1,1) = 2(a+b) \geq 0$ as supposed.

  Hence f is a sobs.

□

For $d = 3$, the above theorem gives a simpler version of the following result from [10] for $n = 2$:

Theorem 4.20. Let f be an even symmetric sextic. The following statements are equivalent:

1. f is sobs.
2. There exists $c_i \geq 0$ such that $f = c_0 f_0 + c_1 f_1 + c_2 F$, where

   $f_0 = (n-2)\displaystyle\sum_{i\neq j} x_i^4x_j^2 - 6\sum_{i<j<k} x_i^2x_j^2x_k^2$, $f_1 = 6\displaystyle\sum_{i<j<k} x_i^2x_j^2x_k^2$, and

   $F = (n-1)\displaystyle\sum_{i=1}^{n} x_i^6 - \sum_{i\neq j} x_i^4x_j^2.$
3. $f^*(1) \geq 0, f^*(n) \geq 0$ and $(n-1)f^*(2) - (n-2)f^*(1) \geq 0$, where $f^*(t) = a + bt + ct^2$ is as in equation (2.3).

Proof. See [10, p573]. □

Proposition 4.21. For $d \geq 4$, there exists $f \in S\Sigma^e_{2,2d}$ that is not a sobs.

Proof. For integer $r \geq 2$, consider

- $f = (x^2 - y^2)^{2r} \in S\Sigma^e_{2,4r}$

  Suppose f is a sobs, i.e. $f = \sum_k h_k^2$, where h_k's are binomials of degree $2r$.

  $\Rightarrow (x^2 - y^2)^r \mid h_k(x, y)$ for each k

  $\Rightarrow h_k = \lambda_k(x^2 - y^2)^r; \lambda_r \in \mathbb{R}$, which is a contradiction since $\lambda_k(x^2 - y^2)^r$ is not a binomial.

  Thus we have even symmetric sos binary form f of degree $8, 12, 16, \ldots$ that are not sobs.

- $f = (x^2 + y^2)(x^2 - y^2)^{2r} \in S\Sigma^e_{2,4r+2}$

  Suppose f is a sobs, i.e. $f = \sum_k g_k^2$, where g_k's are binomials of degree $2r + 1$.

  $\Rightarrow (x^2 - y^2)^r \mid g_k(x, y)$ for each k

  $\Rightarrow g_k = (\alpha_k x + \beta_k y)(x^2 - y^2)^r; \alpha_k, \beta_k \in \mathbb{R}$, which is a contradiction since $(\alpha_k x + \beta_k y)(x^2 - y^2)^r$ is not a binomial.

  Thus we have even symmetric sos binary form f of degree $10, 14, 18, \ldots$ that are not sobs.

□

4.2.3. Even symmetric sos quartic forms

In the following two results we give necessary and sufficient conditions for an even symmetric quartic form to be a sobs:

Theorem 4.22. For $n \geq 3$: $f \in S\mathcal{F}^e_{n,4}$ is sobs iff $f(1, 0, \ldots, 0) \geq 0$, $f(1, \ldots, 1) \geq 0$.

Proof. Trivially $f \in S\mathcal{F}^e_{n,4}$ sobs implies $f(1,0,\dots,0) \geq 0$, $f(1,\dots,1) \geq 0$.

Conversely, to prove that f s.t. $f(1,0,\dots,0) \geq 0$ and $f(1,\dots,1) \geq 0$ is a sobs, it is enough to check the two sufficient conditions in Theorem 2.5.

Let $f = a\sum_{i=1}^{n} x_i^4 + b\sum_{i<j} x_i^2 x_j^2$; $a, b \in \mathbb{R}$.

- If $b \geq 0$, then $f(1,0,\dots,0) = a \geq 0$ trivially implies f is a sobs.
- If $b < 0$, then following the Notation 2.4:

  $\Omega = \{\alpha \in \mathbb{N}^n \mid |\alpha| = 4, f_\alpha \neq 0\} \setminus \{4\epsilon_1,\dots,4\epsilon_n\}$, where $\epsilon_i := (\delta_{i1},\dots,\delta_{in})$ with

  $$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j. \end{cases}$$

  $\Delta = \{\alpha \in \Omega \mid f_\alpha < 0 \text{ or } \alpha \notin 2\mathbb{N}^n\}$

  $= \{\alpha \in \Omega \mid$ exactly 2 of the α_i's $= 2$ and all other α_i's$= 0\}$.

  Note that for $\alpha \in \Delta : f_\alpha = b$.

  For $\alpha \in \Delta$; $i = 1,\dots,n$ taking $a_{\alpha,i} = \alpha_i \dfrac{|b|}{4} \in \mathbb{R}_+$:

  <u>Condition 1.</u> $\forall\ \alpha \in \Delta$:

  $$\text{L.H.S.} = (2d)^{2d} a_\alpha{}^\alpha = 4^4 a_{\alpha,1}^{\alpha_1}\dots a_{\alpha,n}^{\alpha_n} = 4^4 \alpha_1{}^{\alpha_1}\dots\alpha_n{}^{\alpha_n}\left(\frac{|b|}{4}\right)^{\alpha_1+\dots+\alpha_n}$$

  $$= |b|^4\alpha^\alpha = \text{R.H.S.}$$

  <u>Condition 2.</u> for $i = 1,\dots,n$:

  $$\text{R.H.S.} = \sum_{\alpha\in\Delta} a_{\alpha,i} = \sum_{\alpha\in\Delta}\alpha_i\frac{|b|}{4} = \sum_{j=1}^{n-1} 2\frac{|b|}{4} = (n-1)\frac{(-b)}{2}, \text{ is} \leq a = f_{2d,i} = L.H.S.$$

  $\Leftrightarrow 2a + (n-1)b \geq 0$, which is true since $f(1,\dots,1) = an + \dfrac{bn(n-1)}{2} \geq 0$ as given.

  Hence f is a sobs.

$\square$

So we have proved the following:

Theorem 4.23. For $f = a\sum_{i=1}^{n} x_i^4 + b\sum_{i<j} x_i^2x_j^2 \in S\mathcal{F}^e_{n,4}$. The following are equivalent:

1. f is psd
2. $f \geq 0$ on $\{(x_1,\ldots,x_n) \mid x_i \in \{0,1\}\}$,
3. $2ak + (k-1)kb \geq 0 \ \forall\ 1 \leq k \leq n$
4. $f(1,0,\ldots,0) = a \geq 0$ and $f(1,\ldots,1) = an + \dfrac{bn(n-1)}{2} \geq 0$
5. f is sobs and hence sos.

Proof. $(5) \Rightarrow (1) \underbrace{\Leftrightarrow}_{\text{(by Proposition 2.18)}} (2) \Leftrightarrow (3) \Rightarrow (4) \underbrace{\Leftrightarrow}_{\text{(by Theorem 4.22)}} (5)$ □

As a corollary to Theorem 4.23 (above), we have proved the following result due to Choi, Lam and Reznick:

Corollary 4.24. $S\mathcal{P}^e_{n,4} = S\Sigma^e_{n,4}$.

Remark 4.25. From Proposition 2.18 and Theorem 4.23 we observe that for even symmetric $n-$ary quartics we have got a smaller test set for psdness than Timofte, i.e. we got
$f \in S\mathcal{F}^e_{n,4}$ is ≥ 0 on $\mathbb{R}^n \Leftrightarrow f \geq 0$ on $\{(1,0,\ldots,0),(1,\ldots,1)\}$.

4.2.4. Even symmetric sos ternary octic forms

In the following proposition we give sufficient conditions for an even symmetric ternary octic form to be a sobs:

Proposition 4.26. If $f = a\sum_{i=1}^{3} x_i^8 + b\sum_{i\neq j} x_i^6x_j^2 + c\sum_{i<j} x_i^4x_j^4 + d\sum_{\substack{1\leq i\neq j\neq k\leq 3\\ j<k}} x_i^4x_j^2x_k^2 \in S\mathcal{F}^e_{3,8}$ such that $f(1,0,0) \geq 0$, then f is a sobs if

1. $b < 0$ and $c, d \geq 0 \Rightarrow a + 2b \geq 0$

2. $c < 0$ and $b, d \geq 0 \Rightarrow a + c \geq 0$

3. $d < 0$ and $b, c \geq 0 \Rightarrow a + d \geq 0$

4. $b, c < 0$ and $d \geq 0 \Rightarrow a + 2b + c \geq 0$

5. $b, d < 0$ and $c \geq 0 \Rightarrow a + 2b + d \geq 0$

6. $c, d < 0$ and $b \geq 0 \Rightarrow a + c + d \geq 0$

7. $b, c, d < 0 \Rightarrow a + 2b + c + d \geq 0$.

Proof. We have $f(1,0,0) = a \geq 0$.

- If $b, c, d \geq 0$, then $f(1,0,0) = a \geq 0$ trivially implies f is a sobs.
- If any 1 or 2 or all of b, c, d is/are < 0, then following the Notation 2.4:

  $\Omega = \{\alpha \in \mathbb{N}^3 \mid |\alpha| = 8, f_\alpha \neq 0\} \setminus \{(8,0,0), (0,8,0), (0,0,8)\}$,

  $\Delta = \{\alpha \in \Omega \mid f_\alpha \underline{x}^\alpha$ is not a square$\}$=$\{\alpha \in \Omega \mid f_\alpha < 0$ or $\alpha \notin 2\mathbb{N}^n\}$, and

  $$f(\underline{x}) = \sum_{i=1}^{n} f_{2d,i} x_i^{2d} + \sum_{\beta \in \Omega \setminus \Delta} f_\beta \underline{x}^\beta + \sum_{\alpha \in \Delta} f_\alpha \underline{x}^\alpha.$$

  To prove that f is a sobs, it is enough to check the two sufficient conditions in Theorem 2.5. For $\alpha \in \Delta$; $i = 1, 2, 3$ taking $a_{\alpha,i} = \alpha_i \dfrac{|f_\alpha|}{8} \in \mathbb{R}_+$:

  Condition 1. $\forall\ \alpha \in \Delta$:

  $$\text{L.H.S.} = (2d)^{2d} a_\alpha{}^\alpha = 8^8 a_{\alpha_1}^{\alpha_1} a_{\alpha_2}^{\alpha_2} a_{\alpha_3}^{\alpha_3} = 8^8 \alpha_1{}^{\alpha_1} \alpha_2{}^{\alpha_2} \alpha_3{}^{\alpha_3} \left(\frac{|f_\alpha|}{8}\right)^{\alpha_1+\alpha_2+\alpha_3}$$
  $$= |f_\alpha|^8 \alpha^\alpha = \text{R.H.S.}$$

  Condition 2. To prove: $(a =) f_{2d,i} \geq \sum_{\alpha \in \Delta} a_{\alpha,i}$, $i = 1, 2, 3$. We will consider the following 7 subcases:

  (i) If $b < 0$ and $c, d \geq 0$:

  $\Delta = \Delta_b := \{(6,2,0), (6,0,2), (2,6,0), (0,6,2), (2,0,6), (0,2,6)\}$, and

$c, d \geq 0$ implies $c \sum_{i<j} x_i^4 x_j^4 + d \sum_{i \neq j \neq k} x_i^4 x_j^2 x_k^2 \in \sum_{\beta \in \Omega \setminus \Delta} f_\beta \underline{x}^\beta$.

For $i = 1, 2, 3$:

R.H.S.=$\sum_{\alpha \in \Delta} a_{\alpha,i} = \sum_{\alpha \in \Delta} \alpha_i \frac{|f_\alpha|}{8} = (2 + 2 + 6 + 6)\frac{|b|}{8} = 2(-b)$, is $\leq a = f_{2d,i}$

$= L.H.S. \Leftrightarrow a + 2b \geq 0$, which is given.

Hence f is a sobs.

(ii) <u>If $c < 0$ and $b, d \geq 0$</u>:

$\Delta = \Delta_c := \{(4, 4, 0), (4, 0, 4), (0, 4, 4)\}$, and

$b, d \geq 0$ implies $b \sum_{i \neq j} x_i^6 x_j^2 + d \sum_{i \neq j \neq k} x_i^4 x_j^2 x_k^2 \in \sum_{\beta \in \Omega \setminus \Delta} f_\beta \underline{x}^\beta$.

For $i = 1, 2, 3$:

R.H.S.=$\sum_{\alpha \in \Delta} a_{\alpha,i} = \sum_{\alpha \in \Delta} \alpha_i \frac{|f_\alpha|}{8} = (4 + 4)\frac{|c|}{8} = (-c)$, is $\leq a = f_{2d,i} = L.H.S.$

$\Leftrightarrow a + c \geq 0$, which is given.

Hence f is a sobs.

(iii) <u>If $d < 0$ and $b, c \geq 0$</u>:

$\Delta = \Delta_d := \{(4, 2, 2), (2, 4, 2), (2, 2, 4)\}$, and

$b, c \geq 0$ implies $b \sum_{i \neq j} x_i^6 x_j^2 + c \sum_{i<j} x_i^4 x_j^4 \in \sum_{\beta \in \Omega \setminus \Delta} f_\beta \underline{x}^\beta$.

For $i = 1, 2, 3$:

R.H.S.=$\sum_{\alpha \in \Delta} a_{\alpha,i} = \sum_{\alpha \in \Delta} \alpha_i \frac{|f_\alpha|}{8} = (2 + 2 + 4)\frac{|d|}{8} = (-d)$, is $\leq a = f_{2d,i} = L.H.S.$

$\Leftrightarrow a + d \geq 0$, which is given.

Hence f is a sobs.

(iv) <u>If $b, c < 0$ and $d \geq 0$</u>:

$\Delta = \Delta_b \cup \Delta_c$ and $d \geq 0$ implies $d \sum_{i \neq j \neq k} x_i^4 x_j^2 x_k^2 \in \sum_{\beta \in \Omega \setminus \Delta} f_\beta \underline{x}^\beta$.

For $i = 1, 2, 3$:

R.H.S.=$\sum_{\alpha\in\Delta} a_{\alpha,i} = \sum_{\alpha\in\Delta_b} \alpha_i \frac{|b|}{8} + \sum_{\alpha\in\Delta_c} \alpha_i \frac{|c|}{8} = 2|b| + |c| = -2b - c$, is $\leq a = f_{2d,i}$

$= L.H.S. \Leftrightarrow a + 2b + c \geq 0$, which is given.

Hence f is a sobs.

(v) <u>If $b, d < 0$ and $c \geq 0$</u>:

$\Delta = \Delta_b \cup \Delta_d$ and $c \geq 0$ implies $c \sum_{i<j} x_i^4 x_j^4 \in \sum_{\beta\in\Omega\setminus\Delta} f_\beta \underline{x}^\beta$.

For $i = 1, 2, 3$:

R.H.S.=$\sum_{\alpha\in\Delta} a_{\alpha,i} = \sum_{\alpha\in\Delta_b} \alpha_i \frac{|b|}{8} + \sum_{\alpha\in\Delta_d} \alpha_i \frac{|d|}{8} = 2|b| + |d| = -2b - d$, is $\leq a = f_{2d,i}$

$= L.H.S. \Leftrightarrow a + 2b + d \geq 0$, which is given.

Hence f is a sobs.

(vi) <u>If $c, d < 0$ and $b \geq 0$</u>:

$\Delta = \Delta_c \cup \Delta_d$ and $b \geq 0$ implies $b \sum_{i\neq j} x_i^6 x_j^2 \in \sum_{\beta\in\Omega\setminus\Delta} f_\beta \underline{x}^\beta$.

For $i = 1, 2, 3$:

R.H.S.=$\sum_{\alpha\in\Delta} a_{\alpha,i} = \sum_{\alpha\in\Delta_c} \alpha_i \frac{|c|}{8} + \sum_{\alpha\in\Delta_d} \alpha_i \frac{|d|}{8} = |c| + |d| = -c - d$, is $\leq a = f_{2d,i}$

$= L.H.S. \Leftrightarrow a + c + d \geq 0$, which is given.

Hence f is a sobs.

(vii) <u>If $b, c, d < 0$</u>:

$\Delta = \Delta_b \cup \Delta_c \cup \Delta_d$.

For $i = 1, 2, 3$:

R.H.S.=$\sum_{\alpha\in\Delta_b} \alpha_i \frac{|b|}{8} + \sum_{\alpha\in\Delta_c} \alpha_i \frac{|c|}{8} + \sum_{\alpha\in\Delta_d} \alpha_i \frac{|d|}{8} = -2b - c - d$, is $\leq a = f_{2d,i}$

$= L.H.S. \Leftrightarrow a + 2b + c + d \geq 0$, which is given.

Hence f is a sobs.

□

Proposition 4.27. Let $f \in S\Sigma^e_{3,8}$ such that it vanishes on $(1,1,z); z \in \mathbb{R}_+$, then f is not a sobs.

Proof. Let $(1,1,k),(1,1,l)$ be two distinct zeroes of f.

If $f = \sum_i h_i^2$, where h_i's are binomials, then each h_i vanishes on $(1,1,z); z \in \mathbb{R}_+$. Suppose $h_i = (ex^{a_1}y^{a_2}z^{a_3} - dx^{b_1}y^{b_2}z^{b_3})$, where $a_j, b_j \in \mathbb{Z}_+$ for $j = 1,2,3$ such that each $a_j + b_j$ is even and $a_1 + a_2 + a_3 = b_1 + b_2 + b_3 = 4$.

- If $a_j = b_j$ for each j, then $h_i(1,1,1) = e - d = 0$ implies $h_i = 0$.
- If at least one of $a_j \neq b_j$, say w.l.o.g. $a_3 \neq b_3$, then

  $$h_i(1,1,k) = ek^{a_3} - dk^{b_3} = 0;$$
  $$h_i(1,1,l) = el^{a_3} - dl^{b_3} = 0$$

  implies $e = d = 0$ unless $|k| = |l|$. But $k \neq l \in \mathbb{Z}_+$, so we get $e = d = 0$, which implies $h_i = 0$.

Hence f is not a sobs. □

Summarizing our above results, we have proved the following:

- An even symmetric $n-$ary form of degree $2d$ non-negative on just two points $(1,0,\ldots)$ and $(1,\ldots,1)$ is a sobs for the pairs $(n,2d) = (2,2d)$ with $d \leq 3$, and $(n,2d) = (n,4)$ with $n \geq 3$. This follows from Theorem 4.19 and Theorem 4.22 respectively proved above, which are in fact much stronger results than ‶a psd even symmetric $n-$ary $2d-$ic form is sobs for these two pairs″.
- A sos even symmetric binary form of degree $2d \geq 8$ is not necessarily a sobs, as shown above in Theorem 4.21.
- A sos even symmetric ternary octic is not necessarily a sobs in general, but we present some sufficient conditions under which an even symmetric ternary octic non-negative on just one point $(1,0,0)$ will be a sobs. This was shown in Proposition 4.26.

4.3 Reduction of psdness to preordering

In this Section, we will interpret our results on even symmetric psd not sos forms presented in Theorem 4.16 in terms of preorderings, see Proposition 4.33 below. We will also show how they strengthen a known result (Proposition 4.30 below) due to Scheiderer [45] giving precise degrees and number of variables for which it holds in the case of symmetric forms.

Definition 4.28. A subset $M \subseteq \mathbb{R}[x_1, \ldots, x_n]$ is a **quadratic module** if $M + M \subseteq M, a^2 M \subseteq M \ \forall\ a \in \mathbb{R}[\underline{x}], 1 \in M$.
A **preordering** T in $\mathbb{R}[\underline{x}]$ is a quadratic module with $TT \subseteq T$.

Definition 4.29. Let $S = \{g_1, \ldots, g_s\} \subseteq \mathbb{R}[\underline{x}]$, then

$$T_S := \Big\{ \sum_{e_1,\ldots,e_s \in \{0,1\}} \sigma_{\underline{e}}\, g_1^{e_1} \ldots g_s^{e_s} \mid \sigma_{\underline{e}} \in \Sigma\mathbb{R}[\underline{x}]^2, \underline{e} = (e_1, \ldots, e_s) \Big\}$$

is a **preordering** generated by $g_1, \ldots, g_s$.

Note that $p \in T_S$ implies p is non-negative on $K_S := \{\underline{x} \in \mathbb{R}^n \mid g_i(\underline{x}) \geq 0 \ \forall\ i = 1, \ldots, s\}$, but not conversely. In fact we have the following result due to Scheiderer [45]:

Proposition 4.30. Let S be a finite subset of $\mathbb{R}[\underline{x}]$, such that $\dim(K_S) \geq 3$. Then $\exists\ g \in \mathbb{R}[\underline{x}]$ s.t. $g \geq 0$ on $\mathbb{R}^n$ but $g \notin T_S$.

Proof. See [45, p1064]. □

In the setting of non-negative variables, the suitable analogue of a preordering is:

$$T_{S'} := \Big\{ \sum_{e_1,\ldots,e_s \in \{0,1\}} \sigma_{\underline{e}}\, x_1^{e_1} \ldots x_n^{e_n} \mid \sigma_{\underline{e}} \in \Sigma\mathbb{R}[\underline{x}]^2, \underline{e} = (e_1, \ldots, e_s) \Big\},$$

where $S' = \{x_1 \geq 0, \ldots, x_n \geq 0\}$.

Note that $p \in T_{S'}$ implies p is non-negative for $x_1, \ldots, x_n \geq 0$, but not conversely. So it is interesting to find out when the converse holds, i.e. for which particular degrees and number of variables p non-negative for $x_1, \dots, x_n \geq 0$ implies $p \in T_{S'}$. We will do this in the case of symmetric forms, see Proposition 4.33 below.

Clearly, p is non-negative for $x_1, \dots, x_n \geq 0$ if and only if $p(x_1^2, \dots, x_n^2)$ is non-negative everywhere. The relevance of $T_{S'}$ is explained by the following result due to Frenkel and Horvath [14], which will play a central role in our interpretation of even symmetric psd forms in terms of preorderings.

Lemma 4.31. Let $p \in \mathbb{R}[x_1, \dots, x_n]$. Then $p \in T_{S'}$ if and only if $p(x_1^2, \dots, x_n^2) \in \sum \mathbb{R}[x_1, \dots, x_n]^2$.

Proof. See [14, Lemma 1]. □

Remark 4.32. Let $f(x_1, \dots, x_n)$ be an even symmetric form of degree $2d$, assume f is psd, and let $g(y_1, \dots, y_n)$ be such that $g(x_1^2, \dots, x_n^2) = f(x_1, \dots, x_n)$. Then g is a $n-$ary symmetric form of degree d and $g \geq 0$ on $K_{S'}$, where $S' := \{y_1 \geq 0, \dots, y_n \geq 0\}$, i.e. g is non-negative on the first quadrant.
By above lemma, $f \in \Sigma\mathbb{R}[\underline{x}]^2$ if and only if $g \in T_{S'}$.
So, finding out pairs $(n, 2d)$ for which an even symmetric $n-$ary $2d-$ic psd form f is a sos is equivalent to finding the pairs $(n, 2d)$ for which the corresponding symmetric $n-$ary $d-$ic form g non-negative on the first quadrant belongs to $T_{S'}$.

In Theorem 4.16 we saw the pairs $(n, 2d)$ for which an even symmetric $n-$ary $2d-$ic psd form is a sos and the pairs $(n, 2d)$ for which there exists an even symmetric $n-$ary $2d-$ic psd form that is not a sos. We interpret these results in terms of preorderings as follows:

Proposition 4.33. For $n \geq 3$, let $S' = \{x_1 \geq 0, \dots, x_n \geq 0\}$ be a finite subset of $\mathbb{R}[\underline{x}]$, then $\dim(K_{S'}) \geq 3$ and we have:

1. for $n \geq 3$, $\forall\ g \in S\mathcal{F}_{n,2}$: $g \geq 0$ on $\mathbb{R}_+^n$ implies $g \in T_{S'}$.
2. $\forall\ g \in S\mathcal{F}_{3,4}$: $g \geq 0$ on $\mathbb{R}_+^3$ implies $g \in T_{S'}$.
3. for $n \geq 3$, $\exists\ g \in S\mathcal{F}_{n,3}$ s.t. $g \geq 0$ on $\mathbb{R}_+^n$ but $g \notin T_{S'}$.

4. for $d \geq 5$, $\exists\ g \in S\mathcal{F}_{3,d}$ s.t. $g \geq 0$ on $\mathbb{R}^n_+$ but $g \notin T_{S'}$.

5. for $n \geq 4$, $\exists\ g \in S\mathcal{F}_{n,4}$ s.t. $g \geq 0$ on $\mathbb{R}^n_+$ but $g \notin T_{S'}$.

6. for $n \geq 4, d \geq 7$; $\exists\ g \in S\mathcal{F}_{n,d}$ s.t. $g \geq 0$ on $\mathbb{R}^n_+$ but $g \notin T_{S'}$.

Proof. Since $K_{S'} = \mathbb{R}^n_+$ and $n \geq 3$, we have clearly $\dim(K_{S'}) \geq 3$.
In Section 4.1 (see Theorem 4.16) we showed that $S\mathcal{P}^e_{n,2d} = S\Sigma^e_{n,2d}$ for $(n, 2d) = (n, 4)_{n\geq 3}, (3, 8)$, and $S\mathcal{P}^e_{n,2d} \supsetneq S\Sigma^e_{n,2d}$ for $(n, 2d) = (n, 6)_{n\geq 3}, (3, 2d)_{d\geq 5}, (n, 8)_{n\geq 4}$ and $(n, 2d)_{n\geq 4, d\geq 7}$. So using Remark 4.32 we are done. □

Observation 4.34. Above proposition strengthens the Proposition 4.30, since it holds in addition for symmetric forms and gives us precise degrees and number of variables for which the statement in Proposition 4.30 is true or not in case of symmetric forms.

Chapter 5

Concluding remarks and future work

Concluding this work, we are presenting some potential questions which we consider for immediate further research.

We investigated the following question in Chapter 4:

$$Q(S^e) : \text{For what pairs } (n, 2d) \text{ will } \ S\mathcal{P}^e_{n,2d} \subseteq S\Sigma^e_{n,2d} \ ? \tag{5.1}$$

and concluded (in Theorem 4.16) that the answer to this is

- yes for $n = 2, d = 1, (n, 2d) = (n, 4)_{n\geq 3}, (3, 8)$, and
- no for $(n, 2d) = (n, 6)_{n\geq 3}, (3, 2d)_{d\geq 5}, (n, 8)_{n\geq 4}, (n, 2d)_{n\geq 4, d\geq 7}$.

For the pairs $(n, 2d)$ for $n \geq 4$ and $d = 5, 6$, the answer to $Q(S^e)$ is still unknown. To get a complete answer to $Q(S^e)$ and noting the fact that the answer is negative for all the pairs $(n, 2d) = (3, 2d)_{d\geq 5}, (n, 8)_{n\geq 4}, (n, 2d)_{n\geq 4, d\geq 7}$, we find it natural to ask the following question:

$$\text{Does } \ S\mathcal{P}^e_{n,2d} \supsetneq S\Sigma^e_{n,2d} \text{ for } n \geq 4; d = 5, 6 \ ? \tag{5.2}$$

For the pair $(n, 2d) = (4, 10)$, we got the following example of an even symmetric psd not sos quaternary decic, that was found by Amir Ali Ahmadi during our discussion with him in April 2014 at MFO, Oberwolfach:

Example 5.1. The even symmetric quaternary decic

$$f = 4\sum_i x_i^{10} - 23\sum_{i\neq j} x_i^8x_j^2 + 27\sum_{i\neq j} x_i^6x_j^4 - 177\sum_{i\neq j\neq k} x_i^6x_j^2x_k^2 + 3541\sum_{i\neq j\neq k} x_i^4x_j^4x_k^2$$

$$-10102\sum_{i\neq j\neq k\neq l} x_i^4x_j^2x_k^2x_l^2,$$

is positive definite but not a sos. The process of producing a formal certificate is explained by Ahmadi and Parrilo in [1, 2].

We constructed (in Section 4.1) even symmetric psd not sos n−ary $2d$−ic forms for $(n, 2d) = (3, 2d)_{d\geq 6}, (n, 2d)_{n\geq 4, d\geq 7}$ from even symmetric psd not sos n−ary $2d$−ic forms for $(n, 2d) = (n, 6)_{n\geq 3}, (3, 10), (n, 8)_{n\geq 4}$; using Lemma 3.2 and the indefinite irreducible forms h_2, h_3 (in Lemma 4.3). We see below, that a similar method will not work to construct even symmetric psd not sos n−ary $2d$−ic forms for $n \geq 4, d = 5, 6$ from given $f \in S\mathcal{P}^e_{n,6} \setminus S\Sigma^e_{n,6}$, $g \in S\mathcal{P}^e_{n,8} \setminus S\Sigma^e_{n,8}$ for $n \geq 4$, and we discuss some other strategies for such constructions:

1. since the only symmetric quadratic whose square is even symmetric is $p = \sum_{i=1}^n x_i^2$, which is not indefinite; and there is no symmetric cubic whose square is an even form, so we cannot use Lemma 3.2 for such constructions.

2. clearly $p^2 f, pg \in S\mathcal{P}^e_{n,10}$ and $p^2 g \in S\mathcal{P}^e_{n,12}$, but it needs to be checked if $p^2 f, pg$ and $p^2 g$ are sos or not.

3. recall f_t's for $t \in \mathbb{Z}, 2 \leq t \leq n-1$ given (as in [10]) by:

$$f_t(x_1,\dots,x_n) := (t^2-t)\sum_{i=1}^n x_i^6 - 2(t-1)\sum_{i\neq j} x_i^4x_j^2 + 6\sum_{i\neq j\neq k} x_i^2x_j^2x_k^2 \in S\mathcal{P}^e_{n,6}\setminus S\Sigma^e_{n,6},$$

   which were generalizations of the Robinson's form $R(x, y, z) = \frac{1}{2} f_2(x, y, z)$. Clearly $f_t(x_1^2, \dots, x_n^2) \in S\mathcal{P}^e_{n,12}$, but it needs to be checked if it a sos or not. Note that the technique of substituting x_i to x_i^2 in a form might also result in a sos, for instance $R(x^2, y^2, z^2)$ is a sos (as seen in Observation 4.17).

Zussamenfassung auf Deutsch

Die Frage, ob ein reelles positiv semidefinites Polynom p (d.h. $p \in \mathbb{R}[x_1, \ldots, x_n]$ und $p(\underline{x}) \geq 0 \ \forall \ \underline{x} \in \mathbb{R}^n$) eine Quadratsumme (d.h. $p(\underline{x}) = \sum_i p_i(\underline{x})^2; p_i(\underline{x}) \in \mathbb{R}[x_1, \ldots, x_n]$) ist, hat viele Anwendungen und wurde extensiv studiert. Weil ein positiv semidefinites (psd) Polynom immer geraden Grad hat, reicht es aus, diese Frage für Polynome vom geradem Grad zu betrachten. Desweiteren reicht es aus, diese Frage für Formen, d.h. homogene Polynome zu betrachten (weil die Eigenschaften psd und eine Summe von Quadraten (sos) zu sein, unter Homogenisierung erhalten bleiben, siehe Lemma 1.36).

Das wichtigste Ergebnis in diese Richtung ist von D. Hilbert [22] aus dem Jahr 1888. Sein behrühmter Satz besagt, dass eine psd Form sos ist, genau dann wenn $n = 2$ oder $d = 1$ oder $(n, 2d) = (3, 4)$, wobei n die Anzahl den Variablen und $2d$ der Grad der Form ist. Sei $\mathcal{P}_{n,2d}$ und $\sum_{n,2d}$ die Menge der psd beziehungsweise sos $n-$äre $2d-$iken Formen (d.h. Formen vom Grad $2d$ in n Variablen, bezeichnet mit $\mathcal{F}_{n,2d}$). Hilbert hat quartische Quadriken und ternäre Sextiken genau untersucht. Er hat bewiesen, dass $\Sigma_{3,6} \subsetneq \mathcal{P}_{3,6}$ und $\Sigma_{4,4} \subsetneq \mathcal{P}_{4,4}$. Tatsächlich hat er gezeigt (siehe Proposition 1.49), dass

$$\text{wenn } \Sigma_{4,4} \subsetneq \mathcal{P}_{4,4} \text{ und } \Sigma_{3,6} \subsetneq \mathcal{P}_{3,6} \text{ gilt, dann gilt}$$

$$\Sigma_{n,2d} \subsetneq \mathcal{P}_{n,2d} \text{ für alle } n \geq 3, 2d \geq 4 \text{ und } (n, 2d) \neq (3, 4). \tag{1}$$

In dieser Arbeit werden wir diese beiden Fälle als Ausgangslage heranziehen, da es ausreicht, psd nicht sos Formen in diesen beiden ausschlaggebenden Fällen zu produzieren, um psd nicht sos Formen für alle restlichen Fälle von Gleichung (1) zu erhlaten. Für diese zwei Fälle beschrieb Hilbert eine Methode zur Konstruktion

von psd nicht sos Formen, die "aufwändig und unpraktisch" (siehe [7, p387]) war, so dass in den nächsten 80 Jahren keine expliziten Beispiele in der Literatur erschienen.
1967 präsentierte T. S. Motzkin [29] ein geignetes Beispiel $M(x,y,z) := z^6 + x^4y^2 + x^2y^4 - 3x^2y^2z^2$ einer ternären Sextik und zeigte (unabhängig von Hilbert's Methode), dass sie psd und nicht sos ist. 1969 konstruierte R. M. Robinson [41] durch drastische Vereinfachung von Hilbert's Methode (und unabhängig von Motzkin) Beispiele sowohl von psd nicht sos ternären Sextiken als auch von quartischen Quadriken. Ferner wurden 1974 weitere Beispiele von M.D. Choi und T.Y. Lam [5, 7, 8] entdeckt. Sie erhielten zum Beispiel eine quartische Quadrik $Q(x,y,z,w) = w^4 + x^2y^2 + y^2z^2 + z^2x^2 - 4xyzw$ und zeigten, dass sie psd nicht sos ist.

1976 befassten sich Choi und Lam [7] mit der Frage, ob eine psd Form im speziellen Fall dass die betrachtete Form symmetrisch ist (d.h. $n-$äre $2d-$ike Form f so dass $f^{\sigma}(x_1,\ldots,x_n) := f(x_{\sigma(1)},\ldots,x_{\sigma(n)}) = f(x_1,\ldots,x_n)\ \forall\ \sigma \in S_n$), sos ist. Sie gaben eine Antwort auf diese Frage, die die Paare $(n,2d)$, für welche eine symmetrische psd $n-$äre $2d-$ike Form sos ist, vollständig klassifizierte, nämlich

$$S\mathcal{P}_{n,2d} = S\Sigma_{n,2d} \text{ genau dann wenn } n = 2 \text{ oder } d = 1 \text{ oder } (n,2d) = (3,4).$$

Hierbei bezeichnet $S\mathcal{P}_{n,2d}$ und $S\Sigma_{n,2d}$ die Menge der symmetrischen psd beziwhungsweise symmetrischen sos $n-$ären $2d-$iken Formen. Sie zeigten (siehe Proposition 3.3), dass

$$\text{wenn } S\Sigma_{n,4} \subsetneq S\mathcal{P}_{n,4} \text{ für alle } n \geq 4 \text{ und } S\Sigma_{3,6} \subsetneq S\mathcal{P}_{3,6} \text{ gilt, dann gilt}$$

$$S\Sigma_{n,2d} \subsetneq S\mathcal{P}_{n,2d} \text{ für alle } n \geq 3, 2d \geq 4 \text{ und } (n,2d) \neq (3,4). \qquad (2)$$

Sie gaben ein geeignetes Beispiel einer symmetrischen quartischen Quadrik $f(x,y,z,w) = \sum x^2y^2 + \sum x^2yz - 2xyzw$ an und zeigten (siehe Beweis von Proposition 3.4), dass sie psd aber nicht sos ist. Ebenso konstruierte Robinson [41] die symmetrische ternäre Sextik $R(x,y,z) := x^6 + y^6 + z^6 - (x^4y^2 + y^4z^2 + z^4x^2 + x^2y^4 + y^2z^4 + z^2x^4) + 3x^2y^2z^2$ und zeigte, dass sie psd aber nicht sos ist. Um für alle $n \geq 3, 2d \geq 4$ und $(n, 2d) \neq (3, 4)$ symmetrische psd nicht sos $n-$äre $2d-$iken zu erhalten, genügt es also im Hinblick auf (2) symmetrische psd nicht sos $n-$äre Quadriken für $n \geq 5$ zu finden. Wir werden dies im Abschnitt 3.1.2 von Kapitel 3 (siehe Proposition 3.12 und Theoreme 3.16, 3.17) machen, d.h. wir werden explizite Formen $f \in S\mathcal{P}_{n,4} \setminus S\Sigma_{n,4}$ für $n \geq 5$ konstrurieren und dadurch die Antwort auf die Frage "Wann ist eine symmetrische psd Form sos?"vervollständigen.

1980 gaben M.D. Choi, T.Y. Lam und B. Reznick [9] eine Testmenge (siehe Korollar 3.11 in Abschnitt 3.1.1) für symmetrische Quartiken in $n \geq 4$ Variablen an. Eine Menge $\Omega \subseteq \mathbb{R}^n$ wird dabei Testmenge für eine $n-$äre Form f genannt, wenn f genau dann psd ist, wenn $f(\underline{x}) \geq 0$ für alle $\underline{x} \in \Omega$. Diese Testmenge wird eine wichtige Rolle in den Beweisen einiger unserer Resultate (z.B. Proposition 3.12) in Abschnitt 3.1.2 spielen. Testmengen sind insbesondere so wichtig, da sie es uns erlauben einfach festzustellen, ob eine gegebene Form psd ist oder nicht, indem nur ihre Werte in den Punkten der Teilmenge Ω von $\mathbb{R}^n$ getestet werden.

Im Anschluss daran wurde viel getan, um Testmengen für gerade symmetrische Formen zu finden, insbesondere für gerade symmetrische Sextiken durch Choi, Lam, Reznick [10], für gerade symmetrische Oktiken und ternäre Deziken durch W. R. Harris [20]; und ihre Verallgemeinerungen auf Testmengen für symmetrische und gerade symmetrische Polynome von Grad $2d$ in n Variablen durch V. Timofte [49], D. Grimm [18], und C. Riener [39]. 2003 präsentierte Timofte [49] das folgende Halber Grad Prinzip, welches Testmengen für gegebene symmetrische und gerade symmetrische Polynome liefert:

- Ein symmetrisches reelles Polynom vom Grad $2d$ in n Variablen ist nichtnegativ (beziehungsweise > 0) auf $\mathbb{R}^n \Leftrightarrow$ es ist nichtnegativ (beziehungsweise > 0) auf der Teilmenge $\Lambda_{n,k} := \{\underline{x} \in \mathbb{R}^n \mid$ die Anzahl der verschiedenen Komponenten von $\underline{x}$ ist $\leq k$ $\}$, wobei $k := \max\{2, d\}$. [Wenn $d \geq 2$, dann $\Lambda_{n,k} = \Lambda_{n,d}$].

- Ein gerades symmetrisches Polynom vom Grad $2d \geq 4$ in n Variablen ist nichtnegativ (beziehungsweise > 0) auf $\mathbb{R}^n \Leftrightarrow$ es ist nichtnegativ (beziehungsweise > 0) auf der Teilmenge $\Omega_{n,d/2} := \{\underline{x} \in \mathbb{R}^n_+ \mid$ die Anzahl der verschiedenen Komponenten ungleich Null von $\underline{x}$ ist $\leq d/2$ $\}$.

Timofte's Halber Grad Prinzip für symmetrische Polynome war eigentlich eine Verallgemeinerung von Korollar 3.11 von [9], da die in Abschnitt 3.1.1 präsentierte Arbeit von [9] viel früher entstand als [10] und [20].

Wir wollen die Frage, ob eine psd Form sos ist oder nicht, für den speziellen Fall, dass sie zusätzlich gerade symmetrisch ist, betrachten (d.h. eine $n-$äre $2d-$ike symmetrische Form f, so dass in jedem Term von $f(\underline{x})$ jede Variable geraden Grad hat, wir bezeichnen diese Menge mit $S\mathcal{F}^e_{n,2d}$). Wir nennen $S\mathcal{P}^e_{n,2d}$ und $S\Sigma^e_{n,2d}$ die Menge der geraden symmetrischen psd beziehungsweise die Menge der geraden symmetrischen sos $n-$ären $2d-$iken Formen. Wir sind insbesondere an geraden symmetrischen Formen interessiert wegen der kleineren Dimension von $S\mathcal{F}^e_{n,2d}$ und kleineren, durch das Halber Grad Prinzip gegebenen, Testmengen. Eine teilweise bekannte Antwort auf diese Frage ist

- $S\mathcal{P}^e_{n,2d} = S\Sigma^e_{n,2d}$ wenn $n = 2, d = 1, (n, 2d) = (n, 4)_{n\geq 3}, (3, 8)$, und
- $S\mathcal{P}^e_{n,2d} \supsetneq S\Sigma^e_{n,2d}$ wenn $(n, 2d) = (n, 6)_{n\geq 3}, (3, 10), (4, 8)$.

Für $n = 2, d = 1, (n, 2d) = (3, 4)$: $S\mathcal{P}^e_{n,2d} = S\Sigma^e_{n,2d}$ folgt durch Hilbert's Theorem; für $(n, 2d) = (n, 4)_{n\geq 4}$ liefern wir einen Beweis in Proposition 4.1; Harris bewies (in [20, 21]), dass $S\mathcal{P}^e_{3,8} = S\Sigma^e_{3,8}$ und $S\mathcal{P}^e_{n,2d} \supsetneq S\Sigma^e_{n,2d}$ für $(n, 2d) = (3, 10), (4, 8)$ gilt; Choi, Lam und Reznick [10] zeigten, dass $S\mathcal{P}^e_{n,6} \supsetneq S\Sigma^e_{n,6}$ für $n \geq 3$ gilt.

Um eine zusätzliche Antwort auf diese Frage zu geben, werden wir in Abschnitt 4.1 (siehe Propositionen 4.9, 4.12, 4.15) explizite Formen $f \in S\mathcal{P}^e_{n,2d} \setminus S\Sigma^e_{n,2d}$ für die Paare $(n, 2d) = (3, 12), (n, 8)_{n\geq 5}$ konstrurieren. Wir liefern au*ß*erdem ein Grad-Sprung Prinzip (siehe Theorem 4.5) um aus gegebenen psd nicht sos geraden symmetrischen $n-$ären $2d-$iken Formen neue psd nicht sos gerade symmetrische $n-$äre Formen vom Grad $2d+4r$ (für ganze Zahlen $r \geq 2$) und $2d+2n$ zu erhalten. Wir werden dann folgern, dass für die Paare $(n, 2d) = (n, 6)_{n\geq 3}, (n, 8)_{n\geq 4}$, $(3, 2d)_{d\geq 5}$, und $(n, 2d)_{n\geq 4, d\geq 7}$, die Antwort auf die Frage negativ ist. Dies führt uns zu einer Version von Hilbert's 1888 Theorem für gerade symmetrische Formen (siehe Theorem 4.16), nämlich, $S\mathcal{P}^e_{n,2d} = S\Sigma^e_{n,2d}$ für $n = 2, d = 1, (n, 2d) = (n, 4)_{n\geq 3}, (3, 8)$, und $S\mathcal{P}^e_{n,2d} \supsetneq S\Sigma^e_{n,2d}$ für $(n, 2d) = (n, 6)_{n\geq 3}, (3, 2d)_{d\geq 5}, (n, 8)_{n\geq 4}, (n, 2d)_{n\geq 4, d\geq 7}$.

Wir erhalten andere Resultate dieser Arbeit durch folgenden zusammengefassten Verlauf der Kapitel:

In Kapitel 1 werden wir die meisten Definitionen und vorausgehenden Resultate, welche in den restlichen Kapiteln benutzt werden, bereitstellen. Wir liefern eine Charakterisierung von symmetrischen Formen durch eine Aufteilung nach ihrem Grad. Danach definieren wir die Kegel der psd und sos Formen, geben einige ihrer Eigenschaften an, und erklären in welchen Fällen sie gleich sind und wenn es eine psd Form gibt, die nicht sos ist. Wir schlieen das Kapitel ab, indem wir Gram-Matrizen definieren, ihren Nutzen zur Gewinnung von sos Polynomen erklären, und indem wir die Strukturen von Gram-Matrizen für symmetrische Formen in den Hilbert-Fällen, in denen eine psd Form immer sos ist, und in den Grundfällen, in denen eine psd Form nicht notwendigerweise sos ist, präsentieren. Dies alles wird später in Abschnitt 2.3 benutzt, um einige Gram-Matrix Tests zur positiven Definitheit von symmetrischen quadratischen und ternären Quartiken zu präsentieren.

In Kapitel 2 konzentrieren wir uns auf die notwendigen und hinreichenden Bedingungen an eine Form um psd oder sos zu sein. Wir starten mit der genaueren Betrachtung einiger bekannten hinreichenden Bedingungen an die Koeffizienten einer Form um sos zu sein, die von J. B. Lasserre [26], C. Fidalgo und A. Kovacec [13], und M. Ghasemi und M. Marshall [16, 17] stammen; die Letzteren werden später als eine der Hauptwerkzeuge in Abschnitt 4.2 benutzt, um herauszufinden, wann eine psd gerade symmetrische Form, die sos ist, sogar eine Summe von binomischen Quadraten (i.e. $f(\underline{x}) \in \mathcal{F}_{n,m}$ so dass f eine Summe von Quadraten der Form $(a\underline{x}^{\underline{\alpha}} - b\underline{x}^{\underline{\beta}})^2; \underline{\alpha}, \underline{\beta} \in \mathbb{N}^n$ ist). In Abschnitt 2.2 erinnern wir uns an Testmengen für positve Definitheit von symmetrischen Quartiken und geraden symmetrischen Sextiken von Choi, Lam, Reznick [10, 11]; an Testmengen von geraden symmetrischen Oktiken und ternären Deziken von [20]; und an ihre

Verallgemeinerungen auf Testmengen für positive Definitheit von beliebigen symmetrischen und geraden symmetrischen Polynomen von Timofte [49]. Desweiteren werden wir in Abschnitt 2.2.3 unter Benutzung von Timofte‘s Halber Grad Prinzip kleinere Testmengen für gerade symmetrische Quartiken und gerade symmetrische ternäre Oktiken herleiten. Au*β*rdem liefern wir in Abschnitt 2.3 Tests an die Einträge einer, zu einer symmetrischen quadratischen und ternären Quartik gehörenden, Gram-Matrix, so dass die Form sos sein wird. Im letzten Abschnitt 2.4 des Kapitels werden wir eine Filtration von, zwischen dem sos- und dem psd-Kegel liegenden, Kegeln beschreiben und eine Verallgemeinerung von Hilbert‘s Theorem entlang den Varietäten, die die Veronese Varietät (Definition 2.30) enthalten, aufstellen. Das führt zu einem Kriterium für die psd- und die sos-Eigenschaft einer Form, das sich auf ein Kriterium für die psd- und die sos-Eigenschaft von quadratischen Formen auf einer Untervarietät von $\mathbb{R}^{N_0}$ reduziert.

In Kapitel 3 betrachten wir symmetrische $n-$äre Formen vom Grad $2d$ und greifen die Frage auf:

Für welche Paare $(n, 2d)$ ist $S\mathcal{P}_{n,2d} \subseteq S\Sigma_{n,2d}$?,

die von Choi und Lam in [7] betrachtet wurde. Wir präsentieren unsere Konstruktion von expliziten Formen $p \in S\mathcal{P}_{n,4} \setminus S\Sigma_{n,4}$ für $n \geq 5$ in Abschnitt 3.1.2 (siehe Proposition 3.12 und Theoreme 3.16, 3.17), und vervollständigen dadurch die Antwort auf diese Frage. Wir präsentieren au*β*erdem einige Resultate von [9] in Abschnitt 3.1.1 inklusive Korollar 3.11, das Testmengen für symmetrische Quartiken in $n \geq 4$ Variablen liefert, und für den Beweis von Proposition 3.12 benutzt wird.

In Kapitel 4 betrachten wir gerade symmetrische $n-$äre Formen vom Grad $2d$ und untersuchen die Frage:

Für welche Paare $(n, 2d)$ ist $S\mathcal{P}^e_{n,2d} \subseteq S^e\Sigma_{n,2d}$?

Wir konstruieren explizite Formen $f \in S\mathcal{P}^e_{n,2d} \setminus S\Sigma^e_{n,2d}$ für die Paare $(n, 2d) = (3, 12)$, $(n, 8)_{n\geq 5}$ in Abschnitt 4.1 (siehe Propositionen 4.9, 4.12, 4.15) und geben ein Grad-Sprung Prinzip an (siehe Theorem 4.5). Damit leiten wir ab, dass für

$(n, 2d) = (n, 6)_{n\geq3}, (3, 2d)_{d\geq5}, (n, 8)_{n\geq4}, (n, 2d)_{n\geq4,d\geq7}$ eine gerade symmetrische psd Form nicht immer sos ist. An Resultaten für die Paare $(n, 2d)$ mit $n \geq 4; d = 5, 6$ arbeiten wir im Moment noch. Dies führt uns insgesamt zu einer Version von Hilbert‘s 1888 Theorem für gerade symmetrische Formen (siehe Theorem 4.16). In Abschnitt 4.2 arbeiten wir ferner an den Paaren $(n, 2d) = (n, 2), (2, 2d), (n, 4)_{n\geq3}$, $(3, 8)$ für welche jede psd gerade symmetrische $n-$äre Form f sos ist und stellen fest, für welche von diesen Paaren f sogar eine Summe von binomischen Quadraten (sobs) ist. Wir prüfen dies, indem wir eine bekannte hinreichende Bedingung (Theorem 2.5) für die sobs-Eigenschaft einer Form auf den Spezialfall von geraden symmetrischen Formen anwenden und werden sehen, dass gilt:

- Für die Paare $(n, 2d) = (n, 2), (2, 2d)_{d\leq3}, (n, 4)_{n\geq3}$ ist eine psd (oder äquivalent sos) gerade symmetrische $n-$äre Form vom Grad $2d$ sobs. Dies folgt aus Proposition 4.18, Theorem 4.19 und Theorem 4.22, welche tatsächlich viel stärker sind als ‶eine psd gerade symmetrische $n-$äre $2d-$ike Form ist sobs für diese Paare″.

- eine sos gerade symmetrische binäre Form vom Grad $2d \geq 8$ ist nicht notwendigerweise sobs (siehe Theorem 4.21).

- eine sos gerade symmetrische ternäre Oktik ist im Allgemeinen nicht sobs (siehe Proposition 4.27), aber wir präsentieren einige hinreichende Bedingungen unter welchen eine gerade symmetrische ternäre Oktik, die nichtnegativ auf dem Punkt $(1, 0, 0)$ ist, sobs sein wird (siehe Proposition 4.26).

Wir schlie*ß*en das Kapitel ab, indem wir unsere Resultate über gerade symmetrische psd Formen, die nicht sos sind (wie in Theorem 4.16), im Sinne von Präordnungen (siehe Proposition 4.33) interpretiren. Dabei benutzen wir die Tatsache, dass wir zu einer geraden symmetrischen $n-$ären $2d-$iken psd Form eine symmetrische $n-$äre $d-$ike Form, die nichtnegativ auf $\mathbb{R}^n_+$ist, assoziieren können. Wir werden zeigen, wie diese Interpretation zu einem Resultat (Proposition 4.30) von C. Scheiderer [45] in Verbindung steht, wir werden Gradschranken verschärfen und wir geben exakte Paare $(n, 2d)$ an, für welche sein Resultat für symmetrische Formen gilt.

In Kapitel 5 konkludieren wir unsere Arbeit und geben einige Fragen für zukünftige Arbeit an.

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List of Notations

$\mathcal{I}(A) := \{g \in R[\underline{x}] \mid g(\underline{a}) = 0 \ \ \forall\ \underline{a} \in A\}$, ideal of vanishing polynomials on $A \subseteq R^n$.

$Z(p) := \{\underline{x} \in R^n \mid p(\underline{x}) = 0\}$, the zero set of p.

$p_h :=$ homogenisation of a polynomial p w.r.t. a new variable x_{n+1}.

$\mathcal{F}_{n,m} := \{f \in \mathbb{R}[x_1, \ldots, x_n] \mid f$ is a form, $\deg(f) = m;\ m \in \mathbb{N}\}$, set of all forms in n variables of degree m (called "$n-$ary $m-$ics") with real coefficients.

$N(n,m) := \binom{m+n-1}{n-1}$, number of degree m monomials in n variables.

$\mathbb{R}[\underline{x}]^{S_n} := \{p \in \mathbb{R}[x_1, \ldots, x_n] \mid \sigma(p) = p\ \forall\ \sigma \in S_n\}$, the ring of symmetric polynomials in $\mathbb{R}[x_1, \ldots, x_n]$.

$S\mathcal{F}_{n,m} := \{f \in \mathcal{F}_{n,m} \mid \sigma(f) = f\ \forall\ \sigma \in S_n\}$, set of symmetric forms in $\mathcal{F}_{n,m}$.

$S\mathcal{F}^e_{n,m} := \{f \in S\mathcal{F}_{n,m} \mid$ Every variable in each term of $f(\underline{x})$ has even degree$\}$, set of even symmetric forms in $\mathcal{F}_{n,m}$.

$\mathcal{P}_{n,m} := \{f \in \mathcal{F}_{n,m} \mid f(\underline{x}) \geq 0\ \forall \underline{x} \in \mathbb{R}^n\}$, set of positive semidefinite forms in $\mathcal{F}_{n,m}$.

$\sum_{n,m} := \left\{f \in \mathcal{F}_{n,m} \mid \exists\ p_i \in \mathbb{R}[\underline{x}] \text{ s.t. } f(\underline{x}) = \sum_i p_i(\underline{x})^2\right\}$, set of all forms in $\mathcal{F}_{n,m}$ which are sum of squares.

$\mathcal{E}(C)$:= set of extremal elements in a cone C.

$M_{m\times n}(\mathbb{R})$:= the vector space of all m by n matrices over $\mathbb{R}$.

$\mathrm{Sym}_n(\mathbb{R})$:= the set of all symmetric matrices in $M_{n\times n}(\mathbb{R})$.

$G \succcurlyeq 0$ on $\mathbb{K}$ denotes that the matrix G is non-negative on the set $\mathbb{K}$.

$S\mathcal{P}_{n,m} := \{f \in \mathcal{F}_{n,m} \mid f \text{ is psd}\}$, set of symmetric psd forms in $\mathcal{F}_{n,m}$.

$S\Sigma_{n,m} := \{f \in \mathcal{F}_{n,m} \mid f \text{ is psd}\}$, set of symmetric sos forms in $\mathcal{F}_{n,m}$.

$S\mathcal{P}^e_{n,m} := \{f \in S\mathcal{F}^e_{n,m} \mid f \text{ is psd }\}$, set of even symmetric psd forms in $\mathcal{F}_{n,m}$.

$S\Sigma^e_{n,m} := \{f \in S\mathcal{F}^e_{n,m} \mid f \text{ is sos }\}$, set of even symmetric sos forms in $\mathcal{F}_{n,m}$.

$\Lambda_{n,k} := \{\underline{x} \in \mathbb{R}^n \mid x_i \in \{r_1, \ldots, r_k\}; r_i \neq r_j \text{ for } i \neq j\}$, set of points in $\mathbb{R}^n$ with at most k distinct components.

$\Omega_{n,k} := \{\underline{x} \in \mathbb{R}^n_+ \mid x_i \in \{0, r_1, \ldots, r_k\}; r_i \neq r_j \text{ for } i \neq j\}$, set of points in $\mathbb{R}^n_+$ with at most k distinct non-zero components.

$C_{\mathbb{K}} = C^{\exists}_{\mathbb{K}} := \{f \in \mathcal{F}_{n,2d} \mid \exists\, G \succcurlyeq 0 \text{ on } \mathbb{K}, \text{ for some Gram matrix } G \in \mu^{-1}(f)\}$, where $\mathbb{K}$ is a basic closed semi algebraic set s.t. $\nu_d(\mathbb{R}^n) \subseteq \mathbb{K} \subseteq \mathbb{R}^{N_0}$ and $N_0 = N(n,d)$.

$C^{\forall}_{\mathbb{K}} := \{f \in \mathcal{F}_{n,2d} \mid G \succcurlyeq 0 \text{ on } \mathbb{K}, \forall \text{ Gram matrices } G \in \mu^{-1}(f)\}$, where $\mathbb{K}$ is a basic closed semi algebraic set s.t. $\nu_d(\mathbb{R}^n) \subseteq \mathbb{K} \subseteq \mathbb{R}^{N_0}$ and $N_0 = N(n,d)$.

$\mathcal{V}_+(M) := \{x \in \mathbb{P}^n(K) \mid \forall\, f \in M, f(x) = 0\}$ is the variety of $M \subset K[x_0, \ldots, x_n]$.

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