Universität Konstanz

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Konstanzer Schriften in Mathematik

Nr. 326, Februar 2014

ISSN 1430-3558

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Abstract

We consider various initial-value problems for ordinary integro-differential equations of first order that are characterized by convolution-terms, where all factors depend on the solutions of the equations. Applications of such problems are descriptions of certain glass-transition phenomena based on mode-coupling theory, for instance. We will prove results concerning well-posedness of such problems and the asymptotic behaviour of their solutions.

Keywords: integro-differential equations, well-posedness, asymptotic behaviour, glass-transition

1 Introduction

Mode-coupling theory of glass-transition lead to initial-value problems for ordinary integro-differential equation ([12]), i.e. problems of the kind

$$\phi(t) + \dot{\phi}(t) + \int_0^t F(\phi(t-s))\dot{\phi}(s)ds = 0 \ (t \in (0,\infty)), \quad \phi(0) = 1, \tag{1}$$

where $F : \mathbb{R} \to \mathbb{R}$ is a so-called kernel-function and $\Phi : [0,\infty) \to \mathbb{R}$ is a correlation-function. Especially the long-time limits of solutions (if they exist) are of physical interest, i.e. in case of $\lim_{t\to\infty} \Phi(t) = 0$, the considered undercooled liquid stays viscous and in case of $\lim_{t\to\infty} \Phi(t) \neq 0$, the liquid transitions into a glass. Physically relevant kernel-functions are of polynomial type, e.g. $F(x) = v_1 x + v_2 x^2$ $(v_1, v_2 \geq 0)$. Problem (1) is equivalent to the following integral equation

$$\Phi(t) = f(t) + \int_0^t g(\Phi(t-s))h(\Phi(s))ds, \tag{2}$$

with $f(t) = 1 - t$, $g(x) = 1 - x$ and $h(x) = 1 + F(x)$. Further glass-forming models work with more-parametric kernel-functions, i.e.

$$\phi(t) + \dot{\phi}(t) + \int_0^t F(\phi(t-s), t-s, s)\dot{\phi}(s)ds = 0 \ (t \in (0,\infty)), \quad \phi(0) = 1, \tag{3}$$

where $F : \mathbb{R} \times [0,\infty) \times [0,\infty) \to \mathbb{R}$ (see [6], [9], [15]), or with complex-valued equations ([10]).
The kernels of the convolutions-terms of all three equations (1)–(3) are depending on the solutions of the equations, i.e. they are given by functions $k = F(\Phi)$ resp. $k = F(\Phi,\cdot)$. This is the main difference to integral equations as studied extensively in literature (e.g. equations of Volterra-type, see [7], [8], [11] or [19]) and to mainly considered integro-differential equations from [1], [2], [3] and [4]. Until now, only two works are known to us that deal with integro-differential equations whose convolution terms are of similar type as in problems (1)–(3), namely [13] and [20]. In [13], well-posedness and asymptotic behaviour results have been proved for problem (1) under the restriction, that F is an absolutely monotone function. In [20], integro-differential equations of second order were studied, i.e. equations with semilinear structure which are essentially different from the equations in (1)–(3).
In this work, we aim to prove results for the problems (1)–(3) for a wider class of kernel-functions than introduced in [13]. In Chapter 2 we will extend the class of kernel-functions from absolutely monotone functions to monotonically increasing ones. In Chapter 3, we will present a class of kernel-functions that lead to ill-posed problems, i.e. we will prove under certain assumptions the existence of so called blow-up solutions, that are unbounded on a bounded interval of time. In Chapter 4, we will follow an ansatz from [20], to obtain results under smallness-conditions on the data. Problem (3) will be discussed in Chapter 5. In Chapter 6 we will present some comments on systems with real- and complex-valued equations. Examples and applications that use the results of Chapters 2–6 are subject of Chapter 7.
This work is based on the Ph.D. thesis [16]. The techniques of Chapter 4 can be extended to treat comparable problems of partial integro-differential equations of first order (see [17]).

2 Monotone kernel-functions

In this chapter we consider the following problem for an ordinary integro-differential equation

$$\phi(t) + \dot{\phi}(t) + \int_0^t F(\phi(t-s))\dot{\phi}(s)ds = 0 \ (t \in (0,\infty)), \quad \phi(0) = 1, \tag{4}$$

with a kernel-function $F : \mathbb{R} \to \mathbb{R}$. Problem (4) is equivalent to the following fixed-point problem

$$\phi(t) = 1 + \int_0^t F(\phi(s)) - \phi(s) - \phi(t-s)F(\phi(s))ds \ (t \in [0,\infty)). \tag{5}$$

Theorem 1. *Let* $F : \mathbb{R} \to \mathbb{R}$ *satisfy*

(i) $\exists x_0 < 1 : F(x_0) = \frac{x_0}{1-x_0}$,

(ii) $F|_{[x_0,1]}$ *is differentiable, monotonically increasing and locally Lipschitz-continous.*

Then problem (4) has a unique solution $\phi \in C^1([0,\infty),\mathbb{R})$, and ϕ is monotically decreasing with $x_0 \leq \phi(t) \leq 1$ for all $t \in [0,\infty)$.

Proof. We define $\tilde{F} : \mathbb{R} \to \mathbb{R}$ by

$$\tilde{F}(x) := \left\{ \begin{array}{ll} F(1), & x > 1 \\ F(x), & x_0 \leq x \leq 1 \\ F(x_0), & x < x_0 \end{array} \right\}.$$

Let $(F_n)_{n\in\mathbb{N}} \subseteq C^1(\mathbb{R},\mathbb{R})$ be a sequence of monotonically increasing functions such that

$$\sup_{x\in\mathbb{R}} |\tilde{F}(x) - F_n(x)| \overset{n\to\infty}{\longrightarrow} 0.$$

Due to the boundedness of F_n for all $n \in \mathbb{N}$, one can easily prove by using Banach fixed-point theorem that problem (5) with kernel-function F_n has a unique solution $\phi_n \in C^2([0,\infty),\mathbb{R})$ for alle $n \in \mathbb{N}$. Analogously we see that problem (5) with kernel-function $\tilde{F}$ has a unique solution $\phi \in C^1([0,\infty),\mathbb{R})$. Considering problem (4) with F_n instead of F, differentiation of the equation with respect to t leads to

$$\ddot{\phi}_n(t) \leq -(1 + F(1))\dot{\phi}_n(t),$$

for $t \in [0, t')$, where

$$t' := \left\{ \begin{array}{ll} \inf\{t > 0 : \dot{\phi}_n(t) = 0\}, & \{t > 0 : \dot{\phi}_n(t) = 0\} \neq \emptyset \\ \infty, & \{t > 0 : \dot{\phi}_n(t) = 0\} = \emptyset \end{array} \right\}.$$

Using Gronwall's inequality, we obtain for $t \in [0, t')$

$$\dot{\phi}_n(t) \leq -e^{-(1+F_n(1))t}.$$

If $\{t > 0 : \dot{\phi}_n(t) = 0\}$ was not empty, this inequality would lead to $\dot{\phi}_n(t') < 0$, which would contradict the assumption. It follows that ϕ_n is monotonically decreasing for all $n \in \mathbb{N}$. By using Gronwall's inequality one can easily prove that for all $N, \varepsilon > 0$ there exists a $k > 0$ only depending on N, ε and $\tilde{F}$ such that

$$\sup_{0\leq t\leq N} |\phi(t) - \phi_n(t)| \leq k \sup_{x\in\mathbb{R}} |\tilde{F} - F_n|.$$

It follows that ϕ is monotonically decreasing and we obtain with (4)

$$\phi(t) = -\dot{\phi}(t) - \int_0^t \tilde{F}(\phi(t-s))\dot{\phi}(s)ds \geq -F(x_0)(\phi(t) - 1),$$

i.e. $x_0 \leq \phi(t) \leq 1$ for all $t \in [0,\infty)$. We conclude $\tilde{F}(\phi(t)) = F(\phi(t))$ for all $t \in [0,\infty)$, so ϕ is a solution of (4) with kernel-function F. Uniqueness follows from Banach fixed-point theorem applied to (5). □

Corollary 2. *Let $x_0 < 1$ be the maximal fixed point of $F(x_0) = \frac{x_0}{1-x_0}$, then*

$$\phi(t) \overset{t\to\infty}{\longrightarrow} x_0.$$

Proof. From Theorem 1 we know that there exists a $g \geq x_0$ such that $\phi(t) \to g$ if $t \to \infty$. This implies the existence of a sequence $(t_n)_{n\in\mathbb{N}} \subseteq [0,\infty)$ with $t_n \to \infty$, $\phi(t_n) \to g$ and $\dot{\phi}(t_n) \to 0$ if $n \to \infty$. We have for every $0 \leq t_1 < t$

$$\begin{aligned}
&\left| \int_0^t F(\phi(t-s))\dot{\phi}(s)ds - F(g)(g-1) \right| \\
&\leq \left| \int_0^{t_1} F(\phi(t-s))\dot{\phi}(s)ds - F(g)(g-1) \right| + \left| \int_{t_1}^t F(\phi(t-s))\dot{\phi}(s)ds \right| \\
&=: I_1 + I_2.
\end{aligned}$$

We have for fixed t_1

$$I_1 \overset{t\to\infty}{\longrightarrow} |F(g)||\phi(t_1) - g|$$

and

$$I_2 \leq C|\phi(t) - g| + |\phi(t_1) - g| \overset{t\to\infty}{\longrightarrow} C|\phi(t_1) - g|,$$

where $C := \sup_{t\in[0,\infty)} |F(\phi(t))|$. For $\varepsilon > 0$ arbitrary we choose t_1 large enough such that

$$|\phi(t_1) - g| < \min\left\{ \frac{\varepsilon}{2|F(g)|}, \frac{\varepsilon}{4C} \right\}$$

and it follows

$$\lim_{t\to\infty} I_1 + I_2 < \varepsilon.$$

Using (4) we obtain

$$F(g) = \frac{g}{1-g}$$

and from Theorem 1: $g = x_0$. □

We will now formulate a result concerning the rate of convergence in the special case of $x_0 = 0$. The case $x_0 \neq 0$ will be discussed later.

Theorem 3. *Assume $F \in C^1([0,1],\mathbb{R})$ is monotonically increasing and suppose*

(i) $F(x) < \frac{x}{1-x}$ *for* $x \in (0,1)$,

(ii) $F(0) = 0$,

(iii) $F'(0) < 1$.

Then there exists a constant $s_0 > 0$ such that

$$\lim_{t\to\infty} e^{s_0 t}\phi(t) = 0.$$

Proof. One easily proves that there is a constant $\varepsilon_0 \in (0,1)$ such that

$$F(x) \leq \frac{x}{1+\varepsilon_0 - x} =: G_{\varepsilon_0}(x) \ (x \in [0,1]).$$

G_{ε_0} is an absolute monotone function and fulfils $G'_{\varepsilon_0}(0) < 1$. It has been shown in [13] that there exist $x_0 > 0$ and $\varepsilon \in (0,1)$ such that for alle $n \in \mathbb{N}$ and $x > x_0$

$$\int_{x_0}^x t^n F(\phi(t))dt \leq \int_{x_0}^x t^n G_{\varepsilon_0}(\phi(t))dt \leq (1-\varepsilon)\int_{x_0}^x t^n \phi(t)dt. \tag{6}$$

Applying estimate (6) to the techniques of section 7 from [13] one proves

$$\int_0^\infty t^n \phi(t) dt < \infty$$

for all $n \in \mathbb{N}$ and finally the requested result. □

The restriction $F'(0) < 1$ in Theorem 3 implies $G'_{\varepsilon_0}(0) < 1$, which was needed for proving estimate (6). The question for rates of convergence in case of $F'(0) = 1$ is not answered yet. In the following theorem we will approach a certain class of functions with that property.

Theorem 4. *Let $F \in C^0([0,1], \mathbb{R})$ be differentiable and monotonically increasing with the following condition*

$$\exists c \in (0,1] \; \forall x \in [0,1] : 0 \leq F(x) \leq cx.$$

Then the solution ϕ of (4) with kernel-function F fulfils for alle $t \in [0, \infty)$

$$\phi(t) \leq c^{-\frac{1}{2}} t^{-\frac{1}{2}}.$$

Proof. Applying variation of constants formula to (4) leads to

$$\begin{aligned}
\phi(t) &= e^{-t} - e^{-t} \int_0^t e^s \int_0^s F(\phi(s-r)) \dot{\phi}(r) dr ds \\
&\leq e^{-t} - e^{-t} \int_0^t e^s \int_0^s c\phi(s-r)\dot{\phi}(r) dr ds \\
&= e^{-t} - e^{-t} \int_0^t e^s \left(\frac{d}{ds} c \int_0^s \phi(s-r)\phi(r) dr - c\phi(s) \right) ds \\
&\leq e^{-t} + e^{-t} \int_0^t e^s c \int_0^s \phi(s-r)\phi(r) dr ds - c \int_0^t \phi(t-s)\phi(s) ds + e^{-t} \int_0^t e^s \phi(s) ds.
\end{aligned}$$

By using Gronwall's inequality we obtain

$$e^t c \int_0^t \phi(t-s)\phi(s) ds \leq e^t$$

and due to the monotonicity of ϕ it follows

$$\phi(t) \leq c^{-\frac{1}{2}} t^{-\frac{1}{2}}$$

for all $t \in [0, \infty)$. □

Remark 5. *(i) We consider problem (4) with $F(x) = x$ ($x \in \mathbb{R}$) and we assume that there are $k, \delta > 0$ such that for all $t \in [0, \infty)$*

$$\phi(t) \leq k \frac{1}{(1+t)^{\frac{1}{2}+\delta}} =: h(t).$$

It then follows

$$\phi(t) \geq 1 - \int_0^t h(t-s)h(s)ds.$$

It has been shown in [14] that there exists a constant $k_1 > 0$ *such that*

$$\int_0^t h(t-s)h(s)ds \leq k_1 \frac{1}{(1+t)^{2\delta}} \stackrel{t\to\infty}{\longrightarrow} 0,$$

so we have $\phi(t) \to c$ $(t \to \infty)$ *for a* $c \geq 1$*, that contradicts the result of Theorem 4. Due to this example, the rate of convergence in Theorem 4 is optimal.*

(ii) *The results of Theorems 3 and 4 can be generalized to the case of a maximal fixed point* $g \neq 0$ *of* $F(g) = \frac{g}{1-g}$*. This can be done in a similar way as presented in [13] by defining*

$$\tilde{F}(x) := \left[F((1-g)x+g) - F(g)\right](1-g)$$

and

$$\tilde{\phi}(t) := \frac{\phi((1-g)t) - g}{1-g}.$$

Then one has

$$\tilde{\phi}(t) + \dot{\tilde{\phi}}(t) + \int_0^t \tilde{F}(\tilde{\phi}(t-s))\dot{\tilde{\phi}}(s)ds = 0 \;\; (t \in (0,\infty)), \qquad \tilde{\phi}(0) = 1. \tag{7}$$

Applying Theorems 3 and 4 to the problem (7) one abtains similar results for the general case.

(iii) *The results of this chapter can easily be extended to more general (not necessary physically relevant) cases with initial conditions* $\phi(0) \neq 1$ *and inhomogeneous right-hand sides* $f : [0,\infty) \to \mathbb{R}$ *that fulfil* $\bar{f} := \lim_{t\to\infty} f(t) < \infty$ *and* $f(0) < \phi(0)$[1]*. The fixed-point equation from Theorem 1 then proceeds to*

$$F(x) = \frac{x - \bar{f}}{\phi(0) - x}.$$

To prove results concerning rates of convergence of the solutions as seen in Theorems 3 and 4, it will be necessary to call for additional decay rates of f*.*

3 Blow-up solutions

In the previous chapter we discussed the existence of global solutions of the problem (4) under certain restrictions on the kernel-function F. It is a natural

[1] This condition is necessary to obtain monotonically decreasing solutions.

question whether one can always expect global solutions or whether there are kernel-functions such that related solutions are unbounded on a bounded intervall $[0,T)$ for a $T>0$, i.e. they only exist on $[0,T)$ and produce a so called blow-up at time T. In this chapter we will prove the existence of such blow-up solutions under certain conditions on the kernel-function. We start quoting a version of a lemma from [8] for volterra-integral equations.

Lemma 6. *Let $g \in C^1(\mathbb{R},\mathbb{R})$ be monotonically increasing with $g(x)>0$ if $x>0$, $k \in C^1((0,\infty),\mathbb{R})$ be nonnegative and monotonically increasing with*

$$K(x) = \int_0^x k(s)ds > 0 \text{ if } x > 0$$

and assume $f \in C^0([0,\infty),\mathbb{R})$ is nonnegative and monotonically increasing. Furthermore, let g satisfy

$$\limsup_{x\to\infty} \frac{x}{g(x)} < \infty$$

and

$$\int_0^\infty \frac{g'(s)}{g(s)} K^{-1}\left(\frac{s}{g(s)}\right) ds < \infty.$$

If $u:[0,T)\to\mathbb{R}$ is a solution of the following volterra-integral equation

$$u(t) = f(t) + \int_0^t k(t-s)g(u(s))ds$$

with maximal interval of existence $[0,T)$ such that $u(t)>0$ for all $t\in(0,T)$, then $T<\infty$ and $u(t)\to\infty$ if $t\to T$.

Theorem 7. *Let $F \in C^1((-\infty,1],\mathbb{R})$ be monotonically increasing with $F(x) < -1$ for $x \in (-\infty,1]$ and let there exist $x_0 \in (-\infty,1]$ and $\varepsilon > 0$ such that for $x \in (-\infty, x_0)$*

$$F(x) \le \varepsilon x - (\varepsilon+1). \tag{8}$$

Furthermore, let there exist $\delta>0$ such that

$$\int_{-\infty}^{-\delta} \frac{F'(x)\sqrt{-x}}{(-F(x))^{\frac{3}{2}}} dx < \infty. \tag{9}$$

Then there is a $T>0$, so that problem (4) has a unique solution $\phi:[0,T)\to\mathbb{R}$ that satisfies $\phi(t)\to-\infty$ if $t\to T$, i.e. there is no global solution for (4).

Proof. We assume that problem (4) has a global solution $\phi:[0,\infty)\to\mathbb{R}$ and we aim to produce a contradiction with the help of Lemma 6. We define for $t,x\in[0,\infty)$

$$k(t) := 1-\phi(t),\ f(t) := t,\ u(t) := 1-\phi(t) \text{ and } g(x) := -1-F(1-x).$$

With this, u is a global solution of the following volterra-integral equation

$$u(t) = f(t) + \int_0^t k(t-s)g(u(s))ds.$$

It is easy to see that (8) implies $\frac{x}{g(x)} \leq \frac{1}{\varepsilon}$ for $x \geq 1 - x_0$, i.e.

$$\limsup_{x\to\infty} \frac{x}{g(x)} < \infty.$$

Due to $F(x) < -1$ ($x \in (-\infty, 1]$), one has by using (4): $\dot{\phi}(t) \leq -1$, i.e. $\phi(t) \leq 1 - t$ ($t \in [0, \infty)$). It follows $k(t) \geq t$ and from that $K(x) \geq \frac{1}{2}x$, so we obtain for $x \in [0, \infty)$

$$K^{-1}(x) \leq \sqrt{2x}.$$

It follows

$$\int_0^\infty \frac{g'(s)}{g(s)} K^{-1}\left(\frac{s}{g(s)}\right) ds \leq \sqrt{2} \int_{-\infty}^1 \frac{F'(x)\sqrt{1-x}}{(-1-F(x))^{\frac{3}{2}}} dx.$$

The integral on the right-hand side is bounded because of (9) and

$$\int_{-\infty}^{-\delta_1} \frac{F'(x)\sqrt{1-x}}{(-1-F(x))^{\frac{3}{2}}} dx \leq \frac{1}{2} \int_{-\infty}^{-\delta_1} \frac{F'(x)\sqrt{-x}}{(-F(x))^{\frac{3}{2}}} dx,$$

where $\delta_1 \geq \delta$ was chosen suitable. Using Lemma 6, we obtain a contradiction of the assumption from the beginning. This finshes the proof. □

4 Kernels under smallness-conditions

In this chapter, we aim at results for well-posedness und asymptotic behaviour of solutions of (4) without using monotonicity conditions on the kernel-functions. This will be done by regarding the convolution-integral term in (4) as a small perturbation of the linear equation, so that the exponential decaying solution of the linear part will dominate. First of all, we consider the following related linear problem

$$\begin{gathered} \phi(t) + \dot{\phi}(t) + \int_0^t m(t-s)\dot{\phi}(s) ds = 0 \ (t \in (0, \infty)), \\ \phi(0) = 1, \end{gathered} \tag{10}$$

where $m : [0, \infty) \to \mathbb{R}$.

Theorem 8. *Let $m \in C^1([0, \infty), \mathbb{R})$ satisfy $m(0) > -1$, $\lim_{t\to\infty} m(t) = 0$ and*

$$|m'(t)| \leq k e^{-c_1 t}$$

for all $t \in [0, \infty)$, where $k, c_1 > 0$ such that $c(c_1 - c) > k$ with $c := 1 + m(0)$. Then problem (10) has a unique solution $\phi \in C^1([0, \infty), \mathbb{R})$ that satisfies

$$\left|\dot{\phi}(t)\right| \leq e^{\frac{k - c(c_1 - c)}{c_1 - c} t} \text{ and } |\phi(t)| \leq \frac{-(c_1 - c)}{k - c(c_1 - c)} e^{\frac{k - c(c_1 - c)}{c_1 - c} t}.$$

Proof. Problem (10) is equivalent to the following fixed-point equation

$$\phi(t) = 1 + \int_0^t m(s) - \phi(s) - m(s)\phi(t-s)ds. \tag{11}$$

By using Banach fixed-point theorem, it is easy to prove that (11) has a unique solution $\phi \in C^1([0,\infty),\mathbb{R})$. Differentiation of (10) with respect to t and variation of constants lead to

$$e^{ct}\dot{\phi}(t) = -1 - \int_0^t \int_r^t e^{cs} m'(s-r)ds\dot{\phi}(r)dr.$$

Using the conditions on m, we obtain

$$e^{ct}|\dot{\phi}(t)| \leq 1 + \frac{k}{c_1 - c}\int_0^t e^{cr}|\dot{\phi}(r)|dr.$$

By using Gronwall's inequality, one has

$$|\dot{\phi}(t)| \leq e^{\frac{k-c(c_1-c)}{c_1-c}t}$$

and it follows the existence of a $g \in \mathbb{R}$ such that

$$|\phi(t) - g| \leq \frac{-(c_1-c)}{k - c(c_1-c)} e^{\frac{k-c(c_1-c)}{c_1-c}t}.$$

By using similar techniques as presented in the proof of Corollary 2, it is easy to see that

$$\lim_{t\to\infty}\left|\int_0^t m(t-s)\dot{\phi}(s)ds\right| = 0$$

and it follows $g = 0$. □

We will now discuss the nonlinear problem (10). Assume $F \in C^1(\mathbb{R},\mathbb{R})$ with $F(1) > -1$ and let $c := 1 + F(1)$ and $v_2, \beta > 0$, $c_1 > c$ constants chosen arbitrary.

(i) Let $k > 0$ such that $c(c_1 - c) > k \geq (c-1)(c_1 - c)$,

(ii) let $\alpha > 0$ satisfy $(\alpha+1)\frac{k-c(c_1-c)}{c_1-c} \leq -c_1$,

(iii) let $v_1 > 0$ fulfil $v_1\left(\frac{-(c_1-c)}{k-c(c_1-c)}\right)^{\alpha} \leq k$,

(iv) let $a > 0$ such that $a \geq v_2\left(\frac{c_1-c}{k-c(c_1-c)}\right)^{\beta}$ and

(v) let $b > 0$ satisfy $b \leq -\beta\frac{k-c(c_1-c)}{c_1-c}$.

In addition to that, suppose that F satisfies the smallness conditions

$$|F(x)| \leq v_2|x|^{\beta} \text{ und } |F'(x)| \leq v_1|x|^{\alpha} \tag{12}$$

for $x \in \mathbb{R}$. We define $X := \{f \in C^1([0,\infty),\mathbb{R}) | f, f' \text{ are bounded}\}$ together with the norm $\|f\|_X := \max\{\|f\|_\infty, \|f'\|_\infty\}$ and the following subset of X

$$C := \left\{ f \in X \,\middle|\, f(0) = 1, \forall t \in [0,\infty) : \begin{array}{l} |f(t)| \leq \frac{-(c_1-c)}{k-c(c_1-c)} e^{\frac{k-c(c_1-c)}{c_1-c}t}, \\ |f'(t)| \leq e^{\frac{k-c(c_1-c)}{c_1-c}t} \end{array} \right\}.$$

$C \subseteq X$ is bounded, closed, convex and due to (i) not empty. We define

$$T : C \to C, \ v \mapsto Tv := u_v,$$

where u_v is the solution of the linear problem (11) with kernel-function $m := F \circ v$. Due to the conditions (i)–(v) and Theorem 8 we easily see that T is well-defined. By using Schauder fixed-point theorem, we obtain a fixed-point $\phi \in C$ of T that is a solution of (4) with kernel-function F. Due to the equivalence of (4) and (5), Banach fixed-point arguments on (5) lead to the uniqueness of the solution ϕ of (4) in X. Altogether we have proved the following

Theorem 9. *Assume $F \in C^1(\mathbb{R},\mathbb{R})$ with $F(1) > -1$ and $F(0) = 0$. Furthermore let $c := 1 + F(1)$ and $c_1 > c$.*

(i) Let $k > 0$ such that $c(c_1 - c) > k \geq (c-1)(c_1 - c)$,

(ii) let $\alpha > 0$ satisfy $(\alpha + 1)\frac{k-c(c_1-c)}{c_1-c} \leq -c_1$ and

(iii) let $v_1 > 0$ fulfil $v_1 \left(\frac{-(c_1-c)}{k-c(c_1-c)}\right)^{\alpha} \leq k$.

In addition to that suppose

$$|F'(x)| \leq v_1|x|^{\alpha} \text{ for } x \in \left[\frac{c_1 - c}{k - c(c_1 - c)} - \delta, \frac{-(c_1 - c)}{k - c(c_1 - c)} + \delta\right],$$

for a $\delta > 0$. Then problem (4) has a unique solution $\phi \in C^1([0,\infty),\mathbb{R})$ that satisfies

$$|\dot{\phi}(t)| \leq e^{\frac{k-c(c_1-c)}{c_1-c}t} \text{ and } |\phi(t)| \leq \frac{-(c_1 - c)}{k - c(c_1 - c)} e^{\frac{k-c(c_1-c)}{c_1-c}t}.$$

Corollary 10. *Let $\varepsilon \in (0,1)$ and $f \in C^1\left(\left[-\frac{4}{3\varepsilon}, \frac{4}{3\varepsilon}\right], \mathbb{R}\right)$ twice differentiable in $x = 0$ and suppose $f(0) = f'(0) = 0$ and $f(1) > -1$. Then there exists a constant $\kappa_0 \in (0,1]$ such that the problem (4) with kernel-function $F := \kappa \cdot f$ has a unique solution $\phi \in C^1([0,\infty),\mathbb{R})$ for all $\kappa \in (0,\kappa_0]$, with*

$$|\phi(t)| \leq \frac{4}{3 + 3\kappa f(1)} e^{-\frac{3+3\kappa f(1)}{4}t} \text{ and } |\dot{\phi}(t)| \leq e^{-\frac{3+3\kappa f(1)}{4}t}.$$

Proof. We define for a $\kappa > 0$ to be determined later

$$c_\kappa := 1 + \kappa f(1), \ \alpha_\kappa := 1, \ k_\kappa := \frac{1}{8}c_\kappa^2, \ c_{1\kappa} := \frac{3}{2}c_\kappa \text{ and } v_{1\kappa} := \frac{3}{32}c_\kappa^3.$$

Let $\kappa_1 > 0$ such that for all $\kappa \in (0, \kappa_1]$

$$\frac{4}{3} \geq 1 + \kappa f(1) > \varepsilon.$$

Due to $\kappa \leq \kappa_1$ the constants defined above fulfil the conditions of Theorem 9. In consequence of the conditions on f it is easy to show that there exists $M > 0$ such that $|f'(x)| \leq M|x|$ for all $x \in \left[-\frac{4}{3\varepsilon}, \frac{4}{3\varepsilon}\right]$. Defining $\kappa_2 := \frac{3}{32}\varepsilon^3 \frac{1}{M}$ and $\kappa_0 := \min\{\kappa_1, \kappa_2\}$, we obtain for all $\kappa \in (0, \kappa_0]$ and $x \in \left[-\frac{4}{3\varepsilon}, \frac{4}{3\varepsilon}\right]$

$$\kappa |f'(x)| \leq v_{1\kappa} |x|^{\alpha_\kappa}.$$

Application of Theorem 9 to the kernel-function $\kappa \cdot f$ finishes the proof. □

As a consequence of Corollary 10, it is easy to prove the following

Corollary 11. *Let $\varepsilon \in (0,1)$ and $F \in C^1\left(\left[-\frac{4}{3\varepsilon}, \frac{4}{3\varepsilon}\right], \mathbb{R}\right)$ with $F(0) = F'(0) = 0$, $-1 < F(1) \leq \frac{1}{3}$ and*

$$|F'(x)| \leq \frac{3}{32}(1 + F(1))^3 |x|, \;\; x \in \left[-\frac{4}{3\varepsilon}, \frac{4}{3\varepsilon}\right],$$

then problem (4) with kernel-function F has a unique continuously differentiable solution, that decays exponentially.

Remark 12. *The results of this chapter can easily be extended to the more general case of inhomogeneous right-hand sides f and arbitrary initial conditions. Under the additional assumptions that the derivative of f decays exponentially and that the long-time limit of f is zero, one can construct a similar self-mapping as above. The smallness-parameters on the kernel-function F will additionally depend on the decay-parameters of f' and on $\phi(0)$.*

The condition $F'(0) = 0$ from Theorem 9 is too restrictive for some applications in physics. This restriction was necessary due to the fact that the convolution of an exponentially decaying solution with itself decays with a worse rate than the function. We will see that under the weaker expectation of polynomially decaying solutions, one can work without this restriction. We start formulating a special case of Theorem 2.2 from [18].

Lemma 13. *Let $d > 0$, $n > 1$ and $f(x) := \frac{1}{(d+x)^n}$ for $x \in [0, \infty)$. Then one has*

$$\left| \int_0^x f(x-y) f(y) dy \right| \leq \frac{2^{n+2}}{(n-1)d^{n-1}} \frac{1}{(d+x)^n}, \;\; x \in [0, \infty).$$

Theorem 14. *Assume $F \in C^1(\mathbb{R}, \mathbb{R})$ with $F(0) = 0$ and $F(1) > -1$. Furthermore, let $n > 1$, $K := n^n$, $k > K$ and $a > 0$ with $a \leq \frac{(k-K)(n-1)^2 n^{2n-2}}{32Kk^2 4^n}$. In addition to that suppose $|F'(x)| \leq a$ for $x \in \left[-\frac{k}{(n-1)n^{n-1}}, \frac{k}{(n-1)n^{n-1}}\right]$. Then there exists a unique solution $\phi \in C^1([0, \infty), \mathbb{R})$ of (4) with kernel-function F that satisfies*

$$|\dot{\phi}(t)| \leq \frac{k}{(n+t)^n} \;\; \textit{and} \;\; |\phi(t)| \leq \frac{k}{n-1} \frac{1}{(n+t)^{n-1}}.$$

Remark 15. *We easily see that* $a \leq \frac{1}{128}\frac{(n-1)^2}{4^n n^2} \overset{n\to\infty}{\longrightarrow} 0$, *i.e. better rates of decay for* ϕ *need stronger restrictions on* F.

Proof of Theorem 14. We define similar as in case of exponentially decaying solutions $X := \{f \in C^1([0,\infty),\mathbb{R}) | f, f' \text{ are bounded}\}$ with the norm $\|f\|_X := \max\{\|f\|_\infty, \|f'\|_\infty\}$ and

$$C := \left\{ f \in X \,\middle|\, f(0) = 1, \forall t \in [0,\infty) : \begin{array}{l} |f(t)| \leq \frac{k}{n-1}\frac{1}{(n+t)^{n-1}} \text{ and} \\ |f'(t)| \leq \frac{k}{(n+1)^n} \end{array} \right\}.$$

We consider the following mapping

$$T : C \to C,\ v \mapsto Tv := u_v,$$

where u_v is the unique solution of the linear problem

$$u_v(t) + \dot{u}_v(t) + \int_0^t m(t-s,s)ds = 0,\ \phi(0) = 1,$$

with $m(t,s) := F(v(t))\dot{v}(s)$ for $t, s \in [0,\infty)$. To show that T is well-defined, we consider the following equation using variation of constants formula

$$\dot{u}_v(t) = -e^{-t} - \int_0^t e^{-(t-s)} \int_0^s F'(v(s-r))\dot{v}(s-r)\dot{v}(r)drds - \int_0^t e^{-(t-s)} F(1)\dot{v}(s)ds.$$

Due to $e^{-t} \leq K\frac{1}{(n+t)^n}$ for $t \geq 0$, Lemma 13 and the conditions on F, it follows

$$|\dot{u}_v(t)| \leq \left(K + \frac{16Kak^2 4^n}{(n-1)^2 n^{2n-2}} + \frac{4K|F(1)|2^n k}{(n-1)n^{n-1}} \right) \frac{1}{(n+t)^n}.$$

Considering the conditions on the constants, we obtain $u_v \in C$ [2]. Using Schauder fixed-point theorem, one can easily prove the existence of a fixed-point $\phi \in C$ of T, which is a solution of (4). Uniqueness follows with the same argument as in case of exponentially decaying solutions by working with Banach fixed-point theorem on problem (5). □

5 More-parametric kernel-functions

In this chapter we aim to apply the techniques from Chapter 2 and 4 to more-parametric problems of the following kind

$$\Phi(t) + \dot{\Phi}(t) + \int_0^t F(\Phi(t-s), t-s, t)\dot{\Phi}(s)ds = 0,\ t \in [0,\infty), \qquad \Phi(0) = 1, \tag{13}$$

where $F : \mathbb{R} \times [0,\infty) \times [0,\infty) \to \mathbb{R}$. Physically relevant kernel-functions are of separate type, like $F(x,s,t) = f(x)g(s,t) + c$ with functions $f : \mathbb{R} \to \mathbb{R}$, $g : [0,\infty) \times [0,\infty) \to \mathbb{R}$ and constants $c \in \mathbb{R}$. We start formulating a result based on monotonicity-methods from Chapter 2.

[2] To obtain an estimate for $|u_v(t)|$, one can use similar techniques as used in the proof of Theorem 8.

Theorem 16. *Assume $f : \mathbb{R} \to \mathbb{R}$, $g : [0, \infty) \times [0, \infty) \to \mathbb{R}$, $(s, t) \mapsto g(s, t)$ and $c \in \mathbb{R}$ and suppose the following conditions:*

(i) $\exists \bar{g} := \lim_{t\to\infty} g(t, t)$,

(ii) $\exists x_0 < 1 : f(x_0)\bar{g} + c = \frac{x_0}{1-x_0}$,

(iii) f is differentiable and locally Lipschitz-continuous on $[x_0, 1]$,

(iv) g is partial differentiable with partial derivatives $g_1 := \frac{\partial g}{\partial s}$ and $g_2 := \frac{\partial g}{\partial t}$,

(v) g is locally bounded,

(vi) one of the two following conditions is fulfilled on $[x_0, 1] \times [0, \infty) \times [0, \infty)$:

$$\begin{aligned} &a)\ f' \geq 0, g \geq 0 \text{ and } f \geq 0, g_1 \leq 0, g_1 + g_2 \leq 0,\\ &b)\ f' \leq 0, g \leq 0 \text{ and } f \leq 0, g_1 \geq 0, g_1 + g_2 \geq 0. \end{aligned}$$

Then problem (13) with kernel-function $F := f \cdot g + c$ has a unique solution $\Phi \in C^1([0, \infty), \mathbb{R})$ that is monotonically decreasing with $x_0 \leq \Phi(t) \leq 1$ for all $t \in [0, \infty)$.

Proof. We define $\tilde{f} := \left\{ \begin{array}{ll} f(1), & x > 1 \\ f(x), & x_0 \leq x \leq 1 \\ f(x_0), & x < x_0 \end{array} \right\}$.
Let $(f_n)_{n\in\mathbb{N}} \subseteq C^0([0, \infty), \mathbb{R})$ be a sequence of differentiable locally Lipschitz-continuous functions that satisfies $\|f_n - \tilde{f}\|_\infty \overset{t\to\infty}{\longrightarrow} 0$, $f_n(x) \cdot \tilde{f}(x) \geq 0$ $(x \in \mathbb{R})$ and

$$f_n'(x) \left\{ \begin{array}{ll} \geq 0, & \text{if condition (vi) a) is satisfied} \\ \leq 0, & \text{else} \end{array} \right\}.$$

Due to the boundedness of $\tilde{f}$ and f_n and to the conditions (iii)–(v), one can easily prove by using Banach's fixed-point theorem that problem (13) with kernel-function $F_n := f_n \cdot g + c$ has a unique solution $\Phi_n \in C^1([0, \infty), \mathbb{R})$ for all $n \in \mathbb{N}$ and that problem (13) with kernel-function $\tilde{F} := \tilde{f} \cdot g + c$ has a unique solution $\tilde{\Phi} \in C^1([0, \infty), \mathbb{R})$. Furthermore, this proves the uniqueness of any solution $\Phi \in C^1([0, \infty), \mathbb{R})$ of (13). Differentiating the equation from (13) with kernel-function F_n with respect to t, one obtains due to $\dot{\Phi}_n(0) < 0$ and condition (vi)

$$\ddot{\Phi}_n \leq -(1 + f_n(1)g(0, t) + c)\dot{\Phi}_n(t)$$

for $t \in [0, t_0)$, where $t_0 > 0$ is minimal such that $\dot{\Phi}_n(t) < 0$ for all $t \in [0, t_0)$.[3] Gronwall's inequality leads to $t_0 = \infty$, i.e. Φ_n is monotonically decreasing for all $n \in \mathbb{N}$. Using Gronwall's inequality once again one can easily show by considering the conditions (iii)–(v) that $\sup_{0\leq t\leq N} |\tilde{\Phi}(t) - \Phi_n(t)| \overset{n\to\infty}{\longrightarrow} 0$ for all $N > 0$, i.e. $\tilde{\Phi}$ is monotonically decreasing. With this one has for all $s_1, s_2, s_3 \in [0, \infty)$ with $s_2 \leq s_3$

$$\tilde{f}(\tilde{\Phi}(s_1))g(s_2, s_3) \overset{(vi)}{\geq} f(x_0)g(s_2, s_3) \overset{(i),(vi)}{\geq} f(x_0) \lim_{t\to\infty} g(t, t) \overset{(ii)}{=} \frac{x_0}{1 - x_0} - c.$$

[3] $t_0 = \infty$ is possible.

Using this, (13) and Gronwall's inequality, we obtain

$$\tilde{\Phi}(t) \geq e^{-\frac{1}{1-x_0}t} + \int\limits_0^t e^{-\frac{1}{1-x_0}(t-s)} \frac{x_0}{1-x_0} ds \overset{t\to\infty}{\longrightarrow} x_0,$$

i.e. one has $x_0 \leq \tilde{\Phi}(t) \leq 1$ for all $t \in [0,\infty)$. Due to $\tilde{f}(\tilde{\Phi}(s_1))g(s_2,s_3) = f(\tilde{\Phi}(s_1))g(s_2,s_3)$ for all $s_1, s_2, s_3 \in [0,\infty)$, $\tilde{\Phi}$ is a solution of (13) with kernel-function $F = f \cdot g + c$. □

If the limit $\bar{g}$ satisfies $\lim_{t\to\infty} g(t_n^1, t_n^2) = \bar{g}$ for all sequences $(t_n^i)_{n\in\mathbb{N}} \subseteq [0,\infty)$ with $t_n^i \overset{n\to\infty}{\longrightarrow} \infty$, $i = 1,2$, one has the convergency of ϕ to the maximal $\xi \in [x_0, 1]$ that fulfils

$$f(\xi)\bar{g} + c = \frac{\xi}{1-\xi}. \tag{14}$$

This can be proved analogously to Corollary 2, by using

$$\lim_{t\to\infty} f(\Phi(t-s))g(t-s,t) = f(\bar{\Phi})\bar{g},$$

where $\bar{\Phi}$ is the limit of Φ that exists due to Theorem 16.

Theorem 17. *Assume additionally to the conditions of Theorem 16*

(vii) $f(x)\bar{g} + c \begin{Bmatrix} < \frac{x}{1-x}, & x > 0 \\ = 0, & x = 0 \end{Bmatrix}, \ x \in [0,1],$

(viii) $f'(0)\bar{g} < 1$,

(ix) $g_2(s,t) \begin{Bmatrix} \leq 0, & f(x) \geq 0 \text{ for all } x \in [0,1], \\ \geq 0, & f(x) \leq 0 \text{ for all } x \in [0,1] \end{Bmatrix}, \ s,t \in [0,\infty).$

Then one has for alle $n \in \mathbb{N}$

$$\lim_{t\to\infty} t^n \Phi(t) = 0. \tag{15}$$

If additionally

(x) $\bar{g} = 0 \Rightarrow f(0)g(0,0) = 0$,

then one has the existence of a constant $s_0 > 0$ *such that*

$$\lim_{t\to\infty} e^{s_0 t}\Phi(t) = 0. \tag{16}$$

Proof. Due to Theorem 16 one has $\lim_{t\to\infty} \Phi(t) = 0$. We define $H(x) := f(x)\bar{g} + c$ $(x \in [0,1])$. Similar to the proof of Theorem 3, we obtain using the conditions (vii) and (viii) the existence of a $\varepsilon_0 > 0$ such that $H(x) \leq G_{\varepsilon_0}(x) < \frac{x}{1-x}$ for all $x \in [0,1)$ and this leads to

$$\exists \delta \in (0,1), t_0 \in [0,\infty) \ \forall t \geq t_0 : f(\Phi(t))g(t,t) + c \leq (1-\delta)\Phi(t),$$

which proves an analogue to estimate (6). Following the same steps as in the proof of Theorem 3 resp. of the proof of Theorem 5 from [13], one can prove

(15). Doing this, the following equation comes up

$$\int_0^t \left(f(\Phi(t-s))g(t-s,t)+c\right)\dot{\Phi}(s)ds$$

$$= \frac{d}{dt}\int_0^t \left(f(\Phi(s))g(s,t)+c\right)\Phi(t-s) - \left(f(\Phi(s))g(s,s)+c\right)ds$$

$$- \int_0^t f(\Phi(s))g_2(s,t)\Phi(t-s)ds. \quad (17)$$

Condition (ix) is needed to estimate the last integral-term of (17). To prove (16), we distinguish between two cases. In case of $\bar{g} \neq 0$ we obtain for $x \in [0,1]$, $s,t \in [0,\infty)$

$$f(x)g(s,t)+c \leq \kappa H(x) \leq \kappa G_{\varepsilon_0}(1)x,$$

where $\kappa := \max\left\{\frac{1}{\bar{g}}\sup_{s,t\in[0,\infty)} g(s,t), 1\right\}$. In case of $\bar{g}=0$, one has $c=0$ and

$$f(x)g(s,t) \leq f(x)g(0,0) \overset{(x)}{\leq} \sup_{x\in[0,1]} |f'(x)|g(0,0)x.$$

Using this, one can use the techniques in section 7 from [13] to prove (16). □

Remark 18. *Theorem 17 only considers the case of a maximal fixed-point $\xi = 0$ of (14), which leads to the limit of the solution $\bar{\Phi} = \xi = 0$. In case of $\bar{\Phi} \neq 0$, we define $\tilde{\Phi}(t) := \frac{\Phi((1-\bar{\Phi})t)-\bar{\Phi}}{1-\bar{\Phi}}$, $\tilde{f}(x) := f((1-\bar{\Phi})x+\bar{\Phi})(1-\bar{\Phi})$, $\tilde{g}(s,t) := g((1-\bar{\Phi})s,(1-\bar{\Phi})t)$ and $\tilde{c} := -f(\bar{\Phi})\bar{g}(1-\bar{\Phi})$. Using (14), one has*

$$\tilde{\Phi}(t)+\dot{\tilde{\Phi}}(t)+\int_0^t \left(\tilde{f}\left(\tilde{\Phi}(t-s)\right)\tilde{g}(t-s,s)+\tilde{c}\right)\dot{\tilde{\Phi}}(s)ds = 0,\ \tilde{\Phi}(0)=1. \quad (18)$$

Applying Theorem 17 to (18), one obtains asymptotic results for this case.

We will now formulate a result for the problem (13) using smallness-conditions based on Chapter 4. We start considering the related linear problem

$$\Phi(t)+\dot{\Phi}(t)+\int_0^t m(t-s,t)\dot{\Phi}(s)ds = 0,\ \Phi(0)=1, \quad (19)$$

where $m \in C^1([0,\infty)\times[0,\infty),\mathbb{R})$ is a fixed kernel. Problem (19) is equivalent to the following problem of an integral-equation

$$\Phi(t) = 1+\int_0^t m(s,s)-\Phi(s)-m(s,t)\Phi(t-s)ds+\int_0^t\int_0^s m_2(r,s)\Phi(s-r)drds, \quad (20)$$

where $m_2(s,t) := \frac{d}{dt}m(s,t)$ $(m_1(s,t) := \frac{d}{ds}m(s,t))$. Banach's fixed-point theorem leads to a unique solution $\Phi \in C^1([0,\infty),\mathbb{R})$ of (20) resp. (19).

Lemma 19. *Assume the following conditions:*

(i) $m(0,t) \geq -1+\varepsilon$ *for a* $\varepsilon > 0$ *and for all* $t \in [0,\infty)$.

(ii) $|m(0,t)| \leq c$ *for a* $c > 0$ *and for all* $t \in [0,\infty)$.

(iii) $|m_1(s,t) + m_2(s,t)| \leq ke^{-c_1 s}$ *for all* $s,t \in [0,\infty)$, *where* $c_1 > 1+c$ *and* $k > 0$ *such that* $\frac{k}{c_1-c-1} < \varepsilon$.

(iv) $\lim\limits_{s,t\to\infty} m(s,t) = 0$.

Then the solution Φ *of (19) satisfies for all* $t \in [0,\infty)$

$$|\Phi(t)| \leq \frac{1}{\kappa}e^{-\kappa t} \text{ and } \left|\dot{\Phi}(t)\right| \leq e^{-\kappa t},$$

with $\kappa := \varepsilon - \frac{k}{c_1-c-1} > 0$.

Proof. Differentiation of (19) with respect to t and variation of constants formula lead to

$$e^{c(t)}\dot{\Phi}(t) = -1 + \int\limits_0^t \int\limits_r^t e^{c(s)}(m_1+m_2)(s-r,s)\dot{\Phi}(s)dsdr,$$

with $c(t) := \int\limits_0^t 1 + m(0,s)ds$. One has with (i) and (ii)

$$|c(t) - c(s)| \leq (1+c)|t-s| \text{ and } c(t) \geq \varepsilon t.$$

Using (iii), we obtain

$$e^{c(t)}\left|\dot{\Phi}(t)\right| \leq 1 + \frac{k}{c_1-c-1}\int\limits_0^t e^{c(r)}\left|\dot{\Phi}(r)\right| dr.$$

Gronwall's inequality and condition (iv) finish the proof. □

Using Lemma 19 we will extend the result to the nonlinear problem (13). Assume $F \in C^1(\mathbb{R}\times[0,\infty)\times[0,\infty),\mathbb{R})$ with derivatives $F_1(x,s,t) := \frac{\partial}{\partial x}F(x,s,t)$, $F_2(x,s,t) := \frac{\partial}{\partial s}F(x,s,t)$ and $F_3(x,s,t) := \frac{\partial}{\partial t}F(x,s,t)$ and suppose $F(1,0,t) \geq -1+\varepsilon$ for a $\varepsilon > 0$ and for all $t \in [0,\infty)$.

(i) Let $v_3, \gamma > 0$.

(ii) Let $k > 0$ such that $\kappa := \varepsilon - \frac{k}{c_1-c-1} > 0$ and $\kappa \leq 1$ with $c := v_3$ and $c_1 > c+1$.

(iii) Let $a_3 \geq 0$ such that $a_3 < \gamma\kappa$.

(iv) Let $v_1, v_2, \alpha, \beta, a_1, a_2 \geq 0$ such that $v_1\frac{1}{\kappa^\alpha} + v_2\frac{1}{\kappa^\beta} \leq k$, $(\alpha+1)\kappa - a_1 \geq c_1$ and $\beta\kappa - a_2 \geq c_1$.

Furthermore, let F satisfy the following smallness-conditions:

(v) $|F(x,s,t)| \leq v_3|x|^\gamma e^{a_3 s}$,

(vi) $|F_1(x,s,t)| \leq v_1|x|^\alpha e^{a_1 s}$,

(vii) $|F_2(x,s,t) + F_3(x,s,t)| \leq v_2|x|^\beta e^{a_2 s}$.

(viii) $\forall N, M > 0\, \exists L > 0\, \forall x, y \in [-M, M]\, \forall s, t \in [0, N] : |F_3(x,s,t) - F_3(y,s,t)| \leq L|x-y|$.

We define $X := \{f \in C^1([0,\infty),\mathbb{R}) | f, f' \text{ are bounded}\}$, with the norm $\|f\|_X := \max\{\|f\|_\infty, \|f'\|_\infty\}$ and

$$C := \left\{ f \in X \,\middle|\, f(0) = 1, \forall t \in [0,\infty) : |f(t)| \leq \frac{1}{\kappa} e^{-\kappa t}, |f'(t)| \leq e^{-\kappa t} \right\}.$$

We consider the following self-mapping

$$T : C \to C,\ v \mapsto Tv := u_v,$$

where u_v is the solution of the linear problem (19) with kernel-function $m(s,t) := F(u(s), s, t)$. Due to the conditions (i)–(vii), T is well-defined. Since $C \subseteq X$ is bounded, closed and convex, Schauder's fixed-point theorem leads to a fixed-point $\Phi \in C$ of T, i.e. to an exponentially decaying solution of (13). Uniqueness follows from condition (viii) by applying Banach's fixed-point theorem to (20). With this we have proved the following

Theorem 20. *Assume $F \in C^1(\mathbb{R} \times [0,\infty) \times [0,\infty), \mathbb{R})$ and suppose $F(1,0,t) \geq -1 + \varepsilon$ for a $\varepsilon > 0$ ($t \in [0,\infty)$) and the conditions (i)–(viii). Then there exists a unique solution $\Phi \in C^1([0,\infty),\mathbb{R})$ such that*

$$|\Phi(t)| \leq \frac{1}{\kappa} e^{-\kappa t} \text{ and } \left|\dot{\Phi}(t)\right| \leq e^{-\kappa t}$$

for all $t \in [0,\infty)$.

6 Comments on systems with real- and complex-valued equations

In this chapter we consider the following problem for a system of a real- with a complex-valued equation[4]

$$\begin{array}{lrcl} (i) & \dot{\phi_1}(t) + \omega_1 \phi_1(t) + \omega_1 \int\limits_0^t \frac{f_1(\overline{\phi_1(t-s)}, \phi_2(t-s), t-s)}{1 - ip_1} \dot{\phi_1}(s) ds & = & 0, \\ & \phi_1(0) & = & \phi_1^0, \\ (ii) & \dot{\phi_2}(t) + \omega_2 \phi_2(t) + \omega_2 \int\limits_0^t \frac{f_2(\phi_2(t-s), \Re\phi_1(t-s), t-s)}{1 + p_2} \dot{\phi_2}(s) ds & = & 0, \\ & \phi_2(0) & = & \phi_2^0, \end{array} \tag{21}$$

where $\phi_1^0 \in \mathbb{C}$, $\phi_2^0 \in \mathbb{R}$, $\omega_1 \in \mathbb{C}$, $\omega_2, p_1, p_2 \in \mathbb{R}$, $f_1 : \mathbb{C} \times \mathbb{R} \times [0,\infty) \to \mathbb{C}$ and $f_2 : \mathbb{R} \times \mathbb{R} \times [0,\infty) \to \mathbb{R}$. The functions f_1 and f_2 are of linear type $f_1(x_1, x_2, s) = \alpha_1 x_1 \phi(s) + \alpha_2 x_2 \phi(s)$ and $f_2(x_1, x_2, s) = \beta_1 x_1 \phi(s) + \beta_2 x_2 \phi(s)$, with $\alpha_{1,2}, \beta_{1,2} \in (0,\infty)$ and $\phi : [0,\infty) \to \mathbb{R}$ is the solution of an ordinary

[4]See [10], $\Re(z)$ denotes the real-part of a complex number $z \in \mathbb{C}$.

integro-differential equation with kernel-function $F(x) = v_1 x + v_2 x^2$ $(v_{1,2} > 0)$ that satisfies[5]

$$|\phi(t)| \leq \frac{k}{n-1}\frac{1}{(d+t)^{n-1}} \quad \text{und} \quad |\dot{\phi}(t)| \leq \frac{k}{(d+t)^n}$$

where $k, d > 0$ and $n > 1$.
We will sketch techniques which will lead to well-posedness and asymptotic behaviour resuls for (21). As compared to Chapter 4, we will need to work with the related linear problems

$$\begin{array}{llll} (i) & \dot{\phi_1}(t) + \omega_1\phi_1(t) + \omega_1 \int\limits_0^t m_1(t-s,s)ds &=& 0, \quad \phi_1(0) = \phi_1^0 \in \mathbb{C}, \\ (ii) & \dot{\phi_2}(t) + \omega_2\phi_2(t) + \omega_2 \int\limits_0^t m_2(t-s,s)ds &=& 0, \quad \phi_2(0) = \phi_2^0 \in \mathbb{R}, \end{array} \tag{22}$$

with $m_1 : [0,\infty)\times[0,\infty) \to \mathbb{C}$ and $m_2 : [0,\infty)\times[0,\infty) \to \mathbb{R}$ both differentiable.

Lemma 21. *(i) One has for all $t \in [0,\infty)$*

$$e^{-\omega_1 t} \leq \left(\frac{n}{\omega_1}\right)^n \frac{1}{\left(\frac{n}{\omega_1}+t\right)^n} \quad \textit{and} \quad e^{-\omega_2 t} \leq \left(\frac{n}{\omega_2}\right)^n \frac{1}{\left(\frac{n}{\omega_2}+t\right)^n}.$$

(ii) There are $M_1, M_2 > 0$ such that for all $t \in [0,\infty)$

$$\frac{1}{\left(\frac{n}{\omega_1}+t\right)^n} \leq \frac{M_1}{(d+t)^n} \quad \textit{and} \quad \frac{1}{\left(\frac{n}{\omega_2}+t\right)^n} \leq \frac{M_2}{(d+t)^n}.$$

Let

$$\begin{aligned} X_1 &:= \left\{f \in C^1([0,\infty),\mathbb{C}) | f, f' \text{ are bounded}\right\} \quad \text{and} \\ X_2 &:= \left\{f \in C^1([0,\infty),\mathbb{R}) | f, f' \text{ are bounded}\right\}, \end{aligned}$$

with norms

$$\|f\|_{X_1} = \max\left\{\|f\|_\infty, \|f'\|_\infty\right\} \quad \text{and} \quad \|f\|_{X_2} = \max\left\{\|f\|_\infty, \|f'\|_\infty\right\}.$$

To construct convenient self-mappings, we define the following constants:

C1. Let $k_1 > 0$ such that $k_1 > \left(\frac{n}{\omega_1}\right)^n M_1|\phi_1^0|$ and $k_1 \geq (n-1)d^{n-1}|\phi_1^0|$.

C2. Let $\varepsilon_1 := k_1 - \left(\frac{n}{\omega_1}\right)^n M_1|\phi_1^0|$.

C3. Let $\alpha_1, \alpha_2 > 0$ such that

$$\begin{aligned} &\left(\frac{n}{\omega_1}\right)^n \frac{16\omega_1 M_1 k_1 4^n}{|1-ip_1|(n-1)^2 d^{2n-2}} \left(2\alpha_1 \frac{k_1 k}{d^{n-1}(n-1)} + 2\alpha_2 \frac{k_2 k}{d^{n-1}(n-1)}\right) \leq \frac{\varepsilon_1}{2}, \\ &\left(\frac{n}{\omega_1}\right)^n \omega_1 M_1 \left|\frac{\alpha_1\overline{\phi_1^0}\phi(0) + \alpha_2\phi_2^0\phi(0)}{1-ip_1}\right| \frac{2^{n+2}}{(n-1)d^{n-1}} k_1 \leq \frac{\varepsilon_1}{2}. \end{aligned}$$

[5] See Theorem 14.

C4. Let $k_2 > 0$ such that $k_2 > \left(\frac{n}{\omega_2}\right)^n M_2|\phi_2^0|$ and $k_2 \geq (n-1)d^{n-1}|\phi_2^0|$.

C5. Let $\varepsilon_2 := k_2 - \left(\frac{n}{\omega_2}\right)^n M_2|\phi_2^0|$.

C6. Let $\beta_1, \beta_2 > 0$ such that

$$\left(\frac{n}{\omega_2}\right)^n \frac{16\omega_2 M_2 k_2 4^n}{|1+p_2|(n-1)^2 d^{2n-2}} \left(2\beta_1 \frac{k_2 k}{d^{n-1}(n-1)} + 2\beta_2 \frac{k_1 k}{d^{n-1}(n-1)}\right) \leq \frac{\varepsilon_2}{2},$$
$$\left(\frac{n}{\omega_2}\right)^n \omega_2 M_2 \left|\frac{\beta_1 \phi_2^0 \phi(0) + \beta_2 \Re \phi_1^0 \phi(0)}{1+p_2}\right| \frac{2^{n+2}}{(n-1)d^{n-1}} k_2 \leq \frac{\varepsilon_2}{2}.$$

We look for solutions $(\Phi_1, \Phi_2) \in \mathscr{C}_1 \times \mathscr{C}_2$, where

$$\mathscr{C}_1 := \left\{ f \in X_1 : f(0) = \phi_1^0, \begin{array}{l} |f'(t)| \leq \frac{k_1}{(d+t)^n} \\ |f(t)| \leq \frac{k_1}{n-1}\frac{1}{(d+t)^{n-1}} \end{array} \right\} \quad \text{and}$$
$$\mathscr{C}_2 := \left\{ f \in X_2 : f(0) = \phi_2^0, \begin{array}{l} |f'(t)| \leq \frac{k_2}{(d+t)^n} \\ |f(t)| \leq \frac{k_2}{n-1}\frac{1}{(d+t)^{n-1}} \end{array} \right\}.$$

Due to conditions **C1** and **C4**, $\mathscr{C}_1$ and $\mathscr{C}_2$ are not empty.
Let $u \in \mathscr{C}_2$ be arbitrary but fixed. We consider $T_1 : \mathscr{C}_1 \to \mathscr{C}_1,\ w \mapsto T_1 w$ solution of (22(i)) with kernel-function

$$m_1(t,s) := \frac{f_1(\overline{w(t)}, u(t), t)\dot{w}(s)}{1 - ip_1}.$$

Due to Lemma 13, Lemma 21 and conditions **C1**, **C2** and **C3**, T_1 is well-definied. Applying Schauder fixed-point theorem, we obtain a fixed-point $F_1(u) \in \mathscr{C}_1$ for T_1. With this, we define $T_2 u$ as the solution of (22(ii)) with kernel-function

$$m_2(t,s) = \frac{f_2(u(t), \Re(F_1(u)(t)), t)}{1+p_2}\dot{u}(s).$$

By using conditions **C4**, **C5** and **C6**, we obtain $T_2 u \in \mathscr{C}_2$. This defines a self-mapping $T_2 : \mathscr{C}_2 \to \mathscr{C}_2$, that has a fixed-point $\Phi_2 \in \mathscr{C}_2$ as one can prove similarly as for T_1. By construction, the pair $(\Phi_1, \Phi_2) \in \mathscr{C}_1 \times \mathscr{C}_2$ with $\Phi_1 := F_1(\Phi_2)$ is a solution of (21). This proves the following[6]

Theorem 22. *Let $\omega_1 \in \mathbb{C}$, $\omega_2, p_1, p_2 \in \mathbb{R}$, $\phi_1^0 \in \mathbb{C}$ and $\phi_2^0 \in \mathbb{R}$. Let $\phi \in C^1([0,\infty), \mathbb{R})$ with*

$$|\phi(t)| \leq \frac{k}{n-1}\frac{1}{(d+t)^{n-1}} \quad \textit{and} \quad |\dot{\phi}(t)| \leq \frac{k}{(d+t)^n}$$

where $k, d > 0$ and $n > 1$. Furthermore, let $M_1, M_2 > 0$ such that for all $t \in [0,\infty)$

$$\frac{1}{\left(\frac{n}{\omega_1} + t\right)^n} \leq \frac{M_1}{(d+t)^n} \quad \textit{and} \quad \frac{1}{\left(\frac{n}{\omega_2} + t\right)^n} \leq \frac{M_2}{(d+t)^n}.$$

[6] We skip proof of uniqueness here. For more details we refer the reader to [16, Lemma 6.1].

(i) *Let* $k_1 > 0$ *such that* $k_1 > \left(\frac{n}{\omega_1}\right)^n M_1|\phi_1^0|$ *and* $k_1 \geq (n-1)d^{n-1}|\phi_1^0|$.

(ii) *Let* $\varepsilon_1 := k_1 - \left(\frac{n}{\omega_1}\right)^n M_1|\phi_1^0|$.

(iii) *Let* $\alpha_1, \alpha_2 > 0$ *such that*

$$\left(\frac{n}{\omega_1}\right)^n \frac{16\omega_1 M_1 k_1 4^n}{|1-ip_1|(n-1)^2 d^{2n-2}} \left(2\alpha_1 \frac{k_1 k}{d^{n-1}(n-1)} + 2\alpha_2 \frac{k_2 k}{d^{n-1}(n-1)}\right) \leq \frac{\varepsilon_1}{2},$$
$$\left(\frac{n}{\omega_1}\right)^n \omega_1 M_1 \left|\frac{\alpha_1 \overline{\phi_1^0}\phi(0) + \alpha_2 \phi_2^0 \phi(0)}{1-ip_1}\right| \frac{2^{n+2}}{(n-1)d^{n-1}} k_1 \leq \frac{\varepsilon_1}{2}.$$

(iv) *Let* $k_2 > 0$ *such that* $k_2 > \left(\frac{n}{\omega_2}\right)^n M_2|\phi_2^0|$ *and* $k_2 \geq (n-1)d^{n-1}|\phi_2^0|$.

(v) *Let* $\varepsilon_2 := k_2 - \left(\frac{n}{\omega_2}\right)^n M_2|\phi_2^0|$.

(vi) *Let* $\beta_1, \beta_2 > 0$ *such that*

$$\left(\frac{n}{\omega_2}\right)^n \frac{16\omega_2 M_2 k_2 4^n}{|1+p_2|(n-1)^2 d^{2n-2}} \left(2\beta_1 \frac{k_2 k}{d^{n-1}(n-1)} + 2\beta_2 \frac{k_1 k}{d^{n-1}(n-1)}\right) \leq \frac{\varepsilon_2}{2},$$
$$\left(\frac{n}{\omega_2}\right)^n \omega_2 M_2 \left|\frac{\beta_1 \phi_2^0 \phi(0) + \beta_2 \Re\phi_1^0 \phi(0)}{1+p_2}\right| \frac{2^{n+2}}{(n-1)d^{n-1}} k_2 \leq \frac{\varepsilon_2}{2}.$$

Then there exists a unique solution $(\phi_1, \phi_2) \in C^1([0,\infty), \mathbb{C}) \times C^1([0,\infty), \mathbb{R})$ *of (21) that satisfies*

$$|\phi_1(t)| \leq \frac{k_1}{n-1}\frac{1}{(d+t)^{n-1}}, \quad |\dot{\phi}_1(t)| \leq \frac{k_1}{(d+t)^n} \quad \text{and}$$
$$|\phi_2(t)| \leq \frac{k_2}{n-1}\frac{1}{(d+t)^{n-1}}, \quad |\dot{\phi}_2(t)| \leq \frac{k_2}{(d+t)^n}.$$

7 Examples and applications

Example 23 (results of Chapter 2)**.**

(i) *We consider problem (4) with kernel-function* $F(x) = \frac{1}{2}\sin(x)$. *Applying Theorem 3, we obtain a unique solution* $\Phi \in C^1([0,\infty), \mathbb{R})$ *that decays exponentially. In case of* $F(x) = \sin(x)$ *condition* $F'(0) < 1$ *is not fulfilled. Using Theorem 4, we obtain* $\Phi(t) \leq t^{-\frac{1}{2}}$.

(ii) *The rate of convergency for problem (4) with kernel-function* $F(x) = x + x^2$ *is not answered, yet. With Corollary 2, we obtain* $\Phi(t) \to 0$ *if* $t \to \infty$.

Example 24 (results of Chapter 3)**.**

(i) *Considering* $F(x) = -x^2 + 2x - 2 - \tau$ *for* $\tau > 0$, *Theorem 7 proves the existence of a unique solution* $\Phi \in C^1([0,T), \mathbb{R})$, *with* $T \in (0,\infty)$ *such that* $\Phi(t) \to -\infty$ *if* $t \to T$, *i.e. there is no global solution for problem (4).*

(ii) *Condition (9) is not fulfilled for* $F(x) = x - 2$, *i.e. this condition can be interpreted, that* F *has to decay stronger than any linear function for* $x \to -\infty$.

Example 25 (results of Chapter 4).

(i) *We consider for* $\varepsilon \in (0,1)$ *the kernel-function*

$$F_\varepsilon(x) = \left(\frac{3}{64}\frac{9\varepsilon^2}{32 - 9\varepsilon^2}\right)(x^2 - x^4).$$

Applying Corollary 11 to F_ε *and considering* $\varepsilon \to 1$, *one obtains the existence of a unique solution* $\Phi \in C^1([0,\infty),\mathbb{R})$ *for problem (4) with kernel-function* $F(x) = \frac{27}{1472}(x^2 - x^4)$, *that fulfils*

$$|\Phi(t)| \leq \frac{4}{3}e^{-\frac{3}{4}t} \quad \textit{and} \quad |\dot{\Phi}(t)| \leq e^{-\frac{3}{4}t}.$$

(ii) *Let* $w_0 > 0$ *be the unique real root of the polynomial* $P(x) = 3 - 73x + 9x^2 - 3x^3$. *In case of* $F(x) = \pm w_0 x^2$ *we obtain a unique solution* $\Phi \in C^1([0,\infty),\mathbb{R})$ *that satisfies*

$$|\Phi(t)| \leq \frac{4}{3 - 3w_0}e^{-\frac{3-3w_0}{4}t} \quad \textit{and} \quad |\dot{\Phi}(t)| \leq e^{-\frac{3-3w_0}{4}t}.$$

(iii) *Let* $n = 2$, $K = 4$, $k = 8$ *and* $a = \frac{1}{8192}$. *From Theorem 14, every function* $F : [-4,4] \to \mathbb{R}$ *that satisfies* $|F'(x)| \leq a$ *leads to a unique solution* $\Phi \in C^1([0,\infty),\mathbb{R})$ *with*

$$|\Phi(t)| \leq \frac{8}{2+t} \quad \textit{and} \quad |\dot{\Phi}(t)| \leq \frac{8}{(2+t)^2},$$

e.g. $F(x) = \pm\frac{1}{8192}x$ *or* $F(x) = \pm\frac{1}{73728}(x + x^2)$.

Remark 26. *Some of the functions from Examples 23 and 25 are not absolutely monotone on* $[0,1]$. *By this we see, that the results in this work extend the class of kernel-functions introduced in [13].*

Example 27 (results of Chapter 5).

(i) *Let* $f(x) := x + x^2 + \tau$ $(\tau > 0)$, $g(s,t) := \frac{1}{1+s^2}$, $c := 0$. *From Theorem 17 the solution* Φ *of problem (13) with kernel-function* $F := f \cdot g + c$ *satisfies*

$$\forall n \in \mathbb{N} : \lim_{t\to\infty} t^n \Phi(t) = 0.$$

Condition (x) is not fulfilled.

(ii) *In case of* $f(x) = x + x^2$, $g(s,t) = \frac{1}{1+s^2} + \tau$ $(\tau \in [0,1))$, $c = 0$ *one has*

$$\exists s_0 > 0 : \lim_{t\to\infty} e^{s_0 t}\Phi(t) = 0.$$

(iii) *Let* $f(x) := x + x^2$, $g(s,t) := 1 + \frac{1}{1+s^2}$, $c := 0$, *then Theorem 17 is not applicable. We have from Theorem 16*

$$\lim_{t\to\infty} \Phi(t) = 0.$$

(iv) We consider the following physically relevant problem introduced in [9]:

$$\Phi(t) + \dot{\Phi}(t) + \int_0^t \frac{f(\Phi(t-s))}{1+\gamma^2(t-s)^2}\dot{\Phi}(s)ds = 0, \quad \Phi(0) = 1, \tag{23}$$

where $\gamma \in \mathbb{R}$ and $f : [0,1] \to \mathbb{R}$. If f is differentiable and locally Lipschitz-continous such that $f(x) \geq 0$ and $f'(x) \geq 0$ for all $x \in [0,1]$, then application of Theorem 17 to (23) proves the existence of a unique solution $\Phi \in C^1([0,\infty),\mathbb{R})$ that is monotonically decreasing and satisfies

$$\forall n \in \mathbb{N} : \lim_{t\to\infty} t^n \Phi(t) = 0.$$

If additionally $f(0) = 0$, one has

$$\exists s_0 > 0 : \lim_{t\to\infty} e^{s_0 t}\Phi(t) = 0.$$

(v) We consider the following problem from [5]:

$$\Phi(t) + \dot{\Phi}(t) + \int_0^t \frac{f(\Phi(t-s))}{1+\gamma^2 \sin^2(\omega(t-s))}\dot{\Phi}(s)ds = 0, \quad \Phi(0) = 1, \tag{24}$$

with $\gamma, \omega \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$. If one defines $F(x,s,t) := \frac{f(x)}{1+\gamma^2\sin^2(\omega s)}$, one can apply Theorem 20 to problem (24). In case of $\omega = \gamma = 1$ the kernel-functions $f(x) = \pm\frac{75}{1324}x^2$ and $f(x) = \pm\frac{16875}{3018752}(x^2 + x^4)$ lead to unique solutions $\Phi \in C^1([0,\infty),\mathbb{R})$ that satisfies

$$|\Phi(t)| \leq \frac{16}{15}e^{-\frac{15}{16}t} \quad \text{and} \quad |\dot{\Phi}(t)| \leq e^{-\frac{15}{16}t}.$$

(vi) If $\tilde{f} \in C^1(\mathbb{R},\mathbb{R})$ satisfies $\tilde{f}(0) = \tilde{f}'(0) = 0$ twice differentiable in $x = 0$, one can prove the existence of a $\tau_0 \in (0,1]$ such that problem (24) with kernel-function $f := \tau_0\tilde{f}$ has a unique solution that decays exponentially (see [16, Corollary 5.21]).

Example 28 (results of Chapter 6)**.**

Let $\Phi \in C^1([0,\infty),\mathbb{R})$ with $\Phi(0) = 1$ and

$$|\Phi(t)| \leq \frac{8}{2+t} \quad \text{and} \quad |\dot{\Phi}(t)| \leq \frac{8}{(2+t)^2}$$

(compare Example 25(iii)). Furthermore, let $\Phi_1^0 = \Phi_2^0 = 1$, $\omega_1 = \omega_2 = 2$ and $p_1 = p_2 = 1$. If one sets $n = 2$, $k = 8$, $d = 2$, $M_1 = M_2 = 4$, $k_1 = k_2 = 5$ and $\varepsilon_1 = \varepsilon_2 = 1$, then application of Theorem 22 to problem (21) proves under the restrictions $\alpha_1 + \alpha_2 \leq \frac{\sqrt{2}}{204800}$ and $\beta_1 + \beta_2 \leq \frac{1}{102400}$ the existence of a unique solution $(\Phi_1, \Phi_2) \in C^1([0,\infty),\mathbb{R}) \times C^1([0,\infty),\mathbb{R})$ that satisfies

$$|\phi_1(t)| \leq \frac{5}{2+t}, \ |\dot{\phi}_1(t)| \leq \frac{5}{(2+t)^2}, \ |\phi_2(t)| \leq \frac{5}{2+t} \text{ and } |\dot{\phi}_2(t)| \leq \frac{5}{(2+t)^2}.$$

Remark 29.

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