Asymptotic analysis of finite difference methods

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If we consider a finite difference method simply as a set of equations containing a small parameter (the grid spacing), it is evident that the tools of asymptotic analysis can give us useful information about the method. The applicability of this approach for studying consistency, long time behavior and stability is demonstrated. As example, we use a simple lattice Boltzmann scheme for the 1D advection equation with constant advection velocity. Applications of the method to lattice Boltzmann schemes for the Navier-Stokes equation can be found in [1, 2, 3, 5, 6, 7]. It should be stressed that the results are not restricted to lattice Boltzmann methods but can readily be applied to any other finite difference scheme (see [4] for various examples and a short review of asymptotic methods in numerical analysis).

Lattice Boltzmann methods are based on discrete velocity particle models. Concretely, we consider fictitious particles that can move with unit speed in one space dimension either to the left or to the right. Hence the discrete velocities are given by $\mathcal{S} := \{-1, 1\}$. The particle distribution is described by a vector-valued function referred to as the *population function*. The first component represents the density of the particles traveling to the left, while the second component is associated with the other species:

$$\mathsf{F}(t,x) = \big[\mathsf{F}_k(t,x)\big]_{k\in\{1,2\}} = \begin{pmatrix} \mathsf{F}_1(t,x) \\ \mathsf{F}_2(t,x) \end{pmatrix}.$$

A compact notation is obtained with the help of the basis vectors

$$\mathsf{1} := \tbinom{1}{1} \in \mathbb{R}^2 \qquad \mathsf{s} := \tbinom{s_1}{s_2} = \tbinom{-1}{1} \in \mathbb{R}^2$$

and the componentwise product between vectors, e.g.

$$\mathsf{1f} = \mathsf{f}, \quad \mathsf{s}^2 = \begin{pmatrix} (-1)^2 \\ 1^2 \end{pmatrix} = \mathsf{1}, \quad \mathsf{s}^3 = \mathsf{s}\,.$$

Finally, $\langle \cdot, \cdot \rangle$ denotes the standard scalar product.

The algorithm we are going to investigate has the standard lattice Boltzmann form

$$\mathsf{F}_k(t+h, x+s_k h) = \mathsf{F}_k(t,x) + \big[J\mathsf{F}(t,x)\big]_k, \qquad k = 1,2 \tag{1}$$

where the discretization parameter $h = 1/N$, $N \in \mathbb{N}$ determines the space-time grid. More precisely, t can take the values $t_n = nh$ with $n \in \mathbb{N}_0$ and x ranges in $x_i = ih$ with $i \in \mathbb{Z}$. The collision operator J on the right hand side of (1) models the particle interaction. Here, we choose the simple BGK form

$$J\mathsf{F} = \omega(E - I)\mathsf{F}$$

with relaxation parameter ω and projection matrix

$$E := \tfrac{1}{2}\begin{pmatrix} 1-a & 1-a \\ 1+a & 1+a \end{pmatrix} \tag{2}$$

To keep things simple, we assume 1-periodic initial data which give rise to 1-periodic population functions so that no boundary data have to be prescribed.

In order to understand the behavior of the lattice Boltzmann solution $\mathsf{F}(t,x)$, we try to approximate it in the form of a regular h-expansion

$$\mathsf{F}(t,x) \approx \mathsf{f}^{[\alpha]}(t,x) := \mathsf{f}^{(0)}(t,x) + h\mathsf{f}^{(1)}(t,x) + \ldots + h^\alpha \mathsf{f}^{(\alpha)}(t,x) \tag{3}$$

with $t = t_n = nh$ and $x = x_j = jh$, $n \in \mathbb{N}, j \in \mathbb{Z}$. We refer to $\mathsf{f}^{[\alpha]}$ as *prediction function*. The *asymptotic order functions* $\mathsf{f}^{(\beta)}$ with $0 \le \beta \le \alpha$ are supposed to be h-independent, smooth in t and x and 1-periodic in x. The order functions are determined by inserting (3) into the update rule (1), performing a Taylor expansion and equating orders. Specifically, we find with $D = \partial_t + \mathsf{s}\partial_x$

$$\begin{aligned} (I-E)\mathsf{f}^{(0)} &= 0 \\ (I-E)\mathsf{f}^{(1)} &= -\tfrac{1}{\omega} D\mathsf{f}^{(0)} \\ (I-E)\mathsf{f}^{(2)} &= -\tfrac{1}{\omega} D\mathsf{f}^{(1)} - \tfrac{1}{2\omega} D^2 \mathsf{f}^{(0)} \end{aligned} \tag{4}$$

Since $I - E$ is not invertible (the range of $I - E$ is generated by the vector s which is orthogonal to $\mathsf{1}$), we encounter solvability conditions. In fact, the right hand sides of the equations (4) must be orthogonal to $\mathsf{1}$ which eventually can be cast into conditions on the so called *mass moments* $u^{(\beta)} = \langle \mathsf{f}^{(\beta)}, \mathsf{1} \rangle$ of the order functions. With the abbreviations

$$\mu := (\tfrac{1}{\omega} - \tfrac{1}{2})(1-a^2), \qquad \lambda := 2a(\tfrac{1}{\omega^2} - \tfrac{1}{\omega} + \tfrac{1}{6})(1-a^2).$$

they are

$$\begin{aligned} \partial_t u^{(0)} + a\partial_x u^{(0)} &= 0 \\ \partial_t u^{(1)} + a\partial_x u^{(1)} &= \mu \partial_x^2 u^{(0)} \\ \partial_t u^{(2)} + a\partial_x u^{(2)} &= \mu \partial_x^2 u^{(1)} + \lambda \partial_x^3 u^{(0)} \end{aligned} \tag{5}$$

and the precise form of the leading order coefficients is:

$$\begin{aligned} \mathsf{f}^{(0)} &= \tfrac{1}{2}(\mathsf{1} + a\mathsf{s})u^{(0)} \\ \mathsf{f}^{(1)} &= \tfrac{1}{2}(\mathsf{1} + a\mathsf{s})u^{(1)} - \tfrac{1}{2\omega}(1-a^2)\mathsf{s}\partial_x u^{(0)} \\ \mathsf{f}^{(2)} &= \tfrac{1}{2}(1 + a\mathsf{s})u^{(2)} - \tfrac{1}{2\omega}(1-a^2)\mathsf{s}\partial_x u^{(1)} - \tfrac{1}{2\omega}(\tfrac{1}{\omega} - \tfrac{1}{2})(1-a^2)a\mathsf{s}\partial_x^2 u^{(0)}. \end{aligned} \tag{6}$$

If the algorithm is initialized compatibly to (6), i.e.

$$\mathsf{F}(0,x) = \tfrac{1}{2}(1 + a\mathsf{s})v_0(x) - h\tfrac{1}{2\omega}(1-a^2)\mathsf{s}\partial_x v_0(x) - h^2 \tfrac{1}{2\omega}(\tfrac{1}{\omega} - \tfrac{1}{2})(1-a^2)a\mathsf{s}\partial_x^2 v_0(x)$$

with a 1-periodic function v_0, we deduce initial conditions for the mass moments $u^{(0)}(0,x) = v_0(x)$ and $u^{(1)}(0,x) = u^{(2)}(0,x) = 0$ which completely determine the order functions in view of (5) and (6).

Assuming that $\mathsf{f}^{[2]}$ correctly captures the h-behavior of F up to the expanded order, i.e. $\mathsf{F}(t,x) - \mathsf{f}^{[2]}(t,x) = \mathrm{O}(h^3)$ we find for the mass moment at every grid point

$$U(t,x) = u^{(0)}(t,x) + hu^{(1)}(t,x) + h^2 u^{(2)}(t,x) + \mathrm{O}(h^3). \tag{7}$$

In particular, U coincides with the solution $u^{(0)}$ of the advection equation (see (5)) up to an error which is at least proportional to h. In this sense, our lattice Boltzmann algorithm is *consistent* to the advection equation. The order of consistency can also be deduced from (7). If $\omega \neq 2$ and $a^2 \neq 1$ and hence $\mu \neq 0$, the equation for $u^{(1)}$ generally involves a non-zero source term. Thus $u^{(1)}$ will be different from zero and the coincidence of U and $u^{(0)}$ is of first order

$$U(t,x) - u^{(0)}(t,x) = hu^{(1)}(t,x) + \mathrm{O}(h^2).$$

We say that the algorithm is *first order consistent* to the advection equation in that case. If, however, $\omega = 2$ or $a^2 = 1$, the source term in the $u^{(1)}$-equation vanishes and since $u^{(1)}(0,x) = 0$, the solution $u^{(1)}(t,x)$ turns out to be zero everywhere. In this case,

$$U(t,x) - u^{(0)}(t,x) = h^2 u^{(2)}(t,x) + \mathrm{O}(h^3)$$

where $u^{(2)}$ is non-zero for non-trivial $u^{(0)}$ and $a^2 \notin \{0,1\}$. Hence, the lattice Boltzmann method is second order accurate in the case $\omega = 2$.

Summarizing these considerations, we can say that a regular expansion of the algorithm essentially amounts to a *consistency* analysis. In contrast to this, information about the *stability* of the method can be obtained by investigating the long-time behavior with the help of a multi-scale expansion.

Starting with an ansatz of the form $\mathsf{F}(t,x) \approx \mathsf{f}^{[\alpha]}(t,ht,x)$, where

$$\mathsf{f}^{[\alpha]}(t_1,t_2,x) := \mathsf{f}^{(0)}(t_1,t_2,x) \; + \; h\mathsf{f}^{(1)}(t_1,t_2,x) \; + \; \ldots \; + \; h^\alpha \mathsf{f}^{(\alpha)}(t_1,t_2,x), \tag{8}$$

we find the following equation for the leading order mass moment

$$\partial_{t_1} u^{(0)}(t_1,t_2,x) + a\partial_x u^{(0)}(t_1,t_2,x) = 0$$
$$\partial_{t_2} u^{(0)}(t_1,t_2,x) - \mu\partial_x^2 u^{(0)}(t_1,t_2,x) = 0$$

with the initial value $u^{(0)}(0,0,x) = v_0(x)$. We see that the numerical solution is governed by the advection equation for short times but that the diffusion equation dictates the behavior on the long time scale. In particular, an unwanted behavior of the scheme can be expected for $\mu = (\frac{1}{\omega} - \frac{1}{2})(1-a^2) < 0$ since the analysis leads to the ill-posed backward heat equation in that case. This reflects the findings of a detailed stability analysis which reveals that the scheme runs stable only for $0 \leq \omega \leq 2$ and $a^2 \leq 1$. At the same time, it should be stressed that $\mu > 0$ is possible also in unstable situations (e.g. $a = 2$ and $\omega = 4$) which shows that the long-time asymptotics cannot capture all the instabilities.

References

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