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INHOMOGENEOUS BOUNDARY VALUE PROBLEMS IN SPACES OF HIGHER REGULARITY

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Dedicated to Yoshihiro Shibata on occasion of his 60th birthday

Abstract. Uniform a priori estimates for parameter-elliptic boundary value problems are well-known if the underlying basic space equals $L^p(\Omega)$. However, much less is known for the $W_p^s(\Omega)$-realization, $s > 0$, of a parameter-elliptic boundary value problem. We discuss a priori estimates and the generation of analytic semigroups for these realizations in various cases. The Banach scale method can be applied for homogeneous boundary conditions if the right-hand side satisfies certain compatibility conditions, while for the general case parameter-dependent norms are used. In particular, we obtain a resolvent estimate for the general situation where no analytic semigroup is generated.

1. Introduction

For the treatment of nonlinear parabolic equations, a priori estimates in L^p-Sobolev spaces are an important step. Based on the theory of parameter-ellipticity, resolvent estimates have been established for a large class of equations, implying sectoriality of the corresponding operator or even maximal regularity for the non-stationary problem. For a boundary value problem in a domain $\Omega \subset \mathbb{R}^n$, the basic space is usually $L^p(\Omega)$. This leads to a solution in $W_p^m(\Omega)$ where m denotes the order of the differential operator. For the boundary traces, one obtains non-integer Besov spaces.

The situation becomes more complicated and much less investigated if one is interested in spaces of higher regularity. Here we start with $W_p^s(\Omega)$, $s > 0$, as the basic space and expect the solution to be in $W_p^{m+s}(\Omega)$. Apart from its own interest, spaces of higher regularity naturally appear in mixed-order systems (Douglis-Nirenberg systems). Inhomogeneous boundary conditions and non-standard boundary spaces also appear in transmission problems and coupled systems. As an example, we mention the two-phase Stokes equations where the normal component of the velocity jumps across the interface. In the paper [22], Y. Shibata and S. Shimizu have shown maximal L^p-L^q-regularity for this system, introducing a special function space adapted to the inhomogeneous jump conditions. The proofs in this and many other papers in fluid mechanics (see, e.g., Shibata [21]) are based on partial Fourier transform and careful estimates of the solution operators. In the present text, we essentially follow the same approach, however, aiming at uniform a priori

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estimates where the basic space is $W_p^s(\Omega)$ instead of $L^p(\Omega)$. We will restrict ourselves to scalar parameter-elliptic equations which can be seen as a first step in the direction of the Stokes system and general mixed-order systems.

Let us consider the boundary value problem

$$\begin{aligned}(A-\lambda)u &= f \quad \text{in } \Omega,\\ Bu &= g \quad \text{on } \partial\Omega\end{aligned} \tag{1.1}$$

in a bounded sufficiently smooth domain $\Omega \subset \mathbb{R}^n$. Here A is a scalar differential operator of order $m \in 2\mathbb{N}$, and B is a column of $\frac{m}{2}$ boundary operators, $B = (B_1, \dots, B_{m/2})^T$, with $\operatorname{ord} B_j = m_j < m$. Classical parameter-elliptic theory states that, under suitable ellipticity and smoothness conditions, a uniform a priori estimate for the solution u holds. More precisely, we have

$$|\!|\!| u |\!|\!|_{m,p,\Omega} \le C\Big(\|f\|_{L^p(\Omega)} + \sum_{j=1}^{m/2} |\!|\!| g_j |\!|\!|_{m-m_j-\frac{1}{p},p,\partial\Omega}\Big). \tag{1.2}$$

Here for $s > 0$ the parameter-dependent norms $|\!|\!| \cdot |\!|\!|$ are defined by

$$|\!|\!| u |\!|\!|_{s,p,\Omega} := \|u\|_{W_p^s(\Omega)} + |\lambda|^{s/m}\|u\|_{L^p(\Omega)} \quad (s \ge 0)$$

(analogously for $|\!|\!| \cdot |\!|\!|_{s,p,\partial\Omega}$). For $s \ge 0$, $W_p^s(\Omega)$ stands for the standard Sobolev-Slobedeckii space. From the a priori estimate (1.2), we immediately obtain the resolvent estimate

$$\|\lambda(\lambda - A_B)^{-1}\|_{L(L^p(\Omega))} \le C \tag{1.3}$$

for the L^p-realization A_B of the boundary value problem (A, B). This unbounded operator in $L^p(\Omega)$ is defined by $D(A_B) := \{u \in W_p^m(\Omega) : Bu = 0\}$ and $A_B u := A(D)u$ ($u \in D(A_B)$). In particular, under suitable parabolicity assumptions, A_B is sectorial and generates an analytic semigroup. In fact, A_B is even $\mathcal{R}$-sectorial and therefore admits maximal L^p-regularity. A priori estimates of the form (1.2) are known since long; we refer to the classical works Agmon [1], Agranovich-Vishik [4], Geymonat-Grisvard [13], and Roitberg-Sheftel [19]. Concerning $\mathcal{R}$-sectoriality and maximal regularity, we mention Denk-Hieber-Prüss [10] and the references therein.

In spaces of higher regularity, however, the resolvent estimate (1.3) in general does not hold. In fact, it is easily seen (cf. Nesensohn [18]) that the Dirichlet-Laplacian Δ_D in $W_p^1(\Omega)$ with domain $D(\Delta_D) := \{u \in W_p^3(\Omega) : u|_{\partial\Omega} = 0\}$ does not generate an analytic semigroup; its resolvent decays like $|\lambda|^{-1/2-1/(2p)}$ as $|\lambda| \to \infty$. The paper Denk-Dreher [8] deals with resolvent estimates for mixed-order systems. Here conditions on the basic space $Y \subset W_p^s(\Omega)$ were formulated which are necessary and sufficient for a generation of an analytic semigroup. It was shown that additional conditions have to be included in the basic space; these conditions can be seen as compatibility relations. For scalar equations or systems with the same order in each component, the method of Banach scales developed by Amann in [5] can be applied and gives a rather complete answer to the question of generation of an analytic semigroup. We will comment on this in Section 2 below. Generation of analytic semigroups for parabolic equations was also studied by Guidetti in [17] (see also [16]). Here in particular mixed-order systems were studied which arise by the reduction of a higher-order system (in time) to a first-order system. A priori estimates in parameter-dependent norms have been studied, e.g., by Faierman and his coauthors in [2], [9], [12]. We also remark that a particular case of an a priori estimate in $W_p^s(\Omega)$ was used in the second author's thesis [20] to obtain a compactness property needed for a Schauder-type fixed-point argument in the context of a nonlinear elliptic-parabolic system.

In the present paper, we will discuss uniform a priori estimates for the boundary value problem (1.1) with inhomogeneous boundary conditions. To avoid technicalities, we will restrict ourselves to the model problems in the whole space $\mathbb{R}^n$ and the half-space $\mathbb{R}^n_+$. Here the operators are assumed to have constant coefficients and no lower-order terms. The generalization to bounded sufficiently smooth domains and to variable coefficients by localization and partition of unity is quite standard, and we will not dwell on this.

In Section 2, we will study the whole space case and the case of homogeneous boundary conditions. Whereas in the whole space the a priori estimates leading to the generation of an analytic semigroup follows quite directly from Michlin's theorem, the case of homogeneous boundary conditions can be treated by the Banach scale method. In Section 3, we will consider the case of inhomogeneous boundary conditions and derive the main a priori estimates of the present text.

2. The whole space case and the case of homogeneous boundary conditions

Let $A(D) = \sum_{|\alpha|=m} a_\alpha D^\alpha$ be a linear differential operator in $\mathbb{R}^n$, $n \geq 2$, of order $m \in 2\mathbb{N}$ with constant coefficients $a_\alpha \in \mathbb{C}$. Here and in the following, we set $D := -i\partial$ and use the standard multi-index notation $D^\alpha := (-i)^{|\alpha|}\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$. Let $\mathcal{L} \subset \mathbb{C}$ be a closed sector in the complex plane with vertex at the origin. Without loss of generality, we may assume that $\mathcal{L} = \overline{\Sigma}_\theta$ with $\Sigma_\theta := \{z \in \mathbb{C}\setminus\{0\} : |\arg z| < \theta\}$ for some $\theta \in (0, \pi]$. The operator $A(D)$ is called parameter-elliptic in $\overline{\Sigma}_\theta$ (see [4]) if $A(\xi) - \lambda \neq 0$ holds for all $(\xi, \lambda) \in (\mathbb{R}^n \times \overline{\Sigma}_\theta) \setminus \{0\}$. Here $A(\xi) := \sum_{|\alpha|=m} a_\alpha \xi^\alpha$ is the symbol of $A(D)$. If the latter condition is satisfied with $\theta \geq \frac{\pi}{2}$, the operator A is called parabolic.

We will consider the realization of the operator $A(D)$ in different scales of Sobolev spaces. For $s \in \mathbb{R}$ and $p \in (1, \infty)$, we denote by $H_p^s(\mathbb{R}^n)$ and $B_{pp}^s(\mathbb{R}^n)$ the standard Bessel potential and Besov space, respectively. The Sobolev-Slobodeckii space $W_p^s(\mathbb{R}^n)$, $s \geq 0$, coincides with $H_p^s(\mathbb{R}^n)$ for $s \in \mathbb{N}_0$ and with $B_{pp}^s(\mathbb{R}^n)$ for $s \in (0, \infty) \setminus \mathbb{N}$. We recall that a closed linear operator $A\colon X \supset D(A) \to X$ in a complex Banach space X is called sectorial if the domain and the range of A are dense in X and if there exists $\phi \in (0, \pi)$ such that $\rho(A) \supset \Sigma_\phi$ and the set $\{\lambda(\lambda - A)^{-1} : \lambda \in \Sigma_\phi\}$ is bounded in $L(X)$. In this case, the supremum over all angles satisfying this condition is called the spectral angle ϕ_A of A.

In the following, C stands for a generic constant which may vary from inequality to inequality but is independent of the variables appearing in the inequality (and in particular independent of λ).

In the whole space, it is easily seen that the realization of the operator $A(D)$ is sectorial:

Lemma 2.1. *Let $A(D)$ be parameter-elliptic in $\overline{\Sigma}_\theta$, and let $s \in \mathbb{R}$ and $p \in (1, \infty)$. Then for every $\lambda \in \overline{\Sigma}_\theta \setminus \{0\}$ and every $f \in H_p^s(\mathbb{R}^n)$, the equation $(A(D) - \lambda)u = f$ has a unique solution $u \in H_p^{m+s}(\mathbb{R}^n)$, and the a priori estimate*

$$\|u\|_{H_p^{m+s}(\mathbb{R}^n)} + |\lambda|\,\|u\|_{H_p^s(\mathbb{R}^n)} \leq C\|f\|_{H_p^s(\mathbb{R}^n)}$$

holds. In particular, if $\theta \geq \frac{\pi}{2}$*, the operator* $A^{(s)}$ *in* $H_p^s(\mathbb{R}^n)$*, defined by* $D(A^{(s)}) := H_p^{m+s}(\mathbb{R}^n)$*,* $A^{(s)}u := A(D)u$ $(u \in D(A^{(s)}))$*, is sectorial with spectral angle larger than* $\frac{\pi}{2}$ *and therefore generates an analytic semigroup.*

The analog results hold when $H_p^s(\mathbb{R}^n)$ *is replaced by the Besov space* $B_{pp}^s(\mathbb{R}^n)$*.*

Proof. This essentially follows from more general results on the existence of a bounded H^∞-calculus, see, e.g., Denk-Saal-Seiler [11]. However, the result can also easily be seen by an application of the Michlin's theorem. In fact, for each $\lambda \in \overline{\Sigma}_\theta \setminus \{0\}$, the unique solution u can be written as $u = \mathscr{F}^{-1}(A(\xi) - \lambda)^{-1}\mathscr{F} f$ where $\mathscr{F}$ stands for the Fourier transform in $\mathbb{R}^n$. Now it is immediately seen that

$$m(\xi) := (1 + |\xi|^2)^{m/2}(A(\xi) - \lambda)^{-1}$$

satisfies the conditions of Michlin's theorem (see [23], Section 2.2.4). By this and the definition of the Bessel potential spaces, the results in $H_p^s(\mathbb{R}^n)$ follow.

The analog results for Besov spaces are obtained by real interpolation. □

We will now consider boundary value problems where we will again restrict ourselves to the model problem in the half-space $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_n > 0\}$. As before, let $A(D) = \sum_{|\alpha|=m} a_\alpha D^\alpha$, $m \in 2\mathbb{N}$, be a differential operator with constant coefficients and let $B_j(D) = \sum_{|\beta|=m_j} b_{j\beta} D^\beta$, $j = 1, \dots, \frac{m}{2}$, be boundary operators with constant coefficients $b_{j\beta} \in \mathbb{C}$. We set $B := (B_1, \dots, B_{m/2})^T$ and consider the boundary value problem

$$\begin{aligned} (A(D) - \lambda)u &= f \quad \text{in } \mathbb{R}^n_+, \\ \gamma_0 B(D)u &= g \quad \text{on } \mathbb{R}^{n-1}. \end{aligned} \tag{2.1}$$

Here $\gamma_0 \colon u \mapsto u|_{\mathbb{R}^{n-1}}$ denotes the boundary trace operator. The boundary value problem $(A(D), B(D))$ is called parameter-elliptic in $\overline{\Sigma}_\theta$ if $A(D)$ is parameter-elliptic in $\overline{\Sigma}_\theta$ and if the following Shapiro-Lopatinskii condition holds:

For all $(\xi', \lambda) \in (\mathbb{R}^{n-1} \times \overline{\Sigma}_\theta) \setminus \{0\}$ and all $h = (h_1, \dots, h_{m/2})^T \in \mathbb{C}^{m/2}$, the ordinary differential equation

$$\begin{aligned} (A(\xi', D_n) - \lambda)v(x_n) &= 0 \quad (x_n > 0), \\ B(\xi', D_n)v(0) &= h, \\ v(x_n) &\to 0 \quad (x_n \to \infty) \end{aligned}$$

has a unique solution. If these conditions are satisfied with $\theta \geq \frac{\pi}{2}$, the boundary value problem $(A(D), B(D))$ is called parabolic. Throughout the following, we assume that the boundary value problem $(A(D), B(D))$ is parameter-elliptic in $\overline{\Sigma}_\theta$.

We first discuss the case of homogeneous boundary conditions, i.e., we assume $g = 0$ in (2.1), so we discuss the $L^p(\mathbb{R}^n_+)$-realization of $(A(D), B(D))$ which is given by $D(A_B) := \{u \in W_p^m(\mathbb{R}^n_+) : \gamma_0 B(D)u = 0\}$ and $A_B u := A(D)u$ $(u \in D(A_B))$. For this, we apply the method of Banach scales (see [5], Chapter V). We recall the main definitions and results. Let X be a Banach space, $\{\cdot, \cdot\}$ an exact interpolation functor, and $A \colon X \supset D(A) \to X$ be the generator of a C_0-semigroup. Then for $k \in \mathbb{N}_0$, the space E_k is defined by $E_k := D(A^k)$, and the E_k-realization A_k of A_{k-1} is iteratively defined by

$$D(A_k) := \{u \in E_k \cap D(A_{k-1}) : A_{k-1}u \in E_k\}, \quad A_k u := A_{k-1}u \ (u \in D(A_k)).$$

Here we have set $E_0 := X$ and $A_0 := A$. For $s \in (0, \infty) \setminus \mathbb{N}$, we write $s = k + \theta$ with $k \in \mathbb{N}_0$ and $\theta \in (0, 1)$ and define the space $E_s := \{E_k, E_{k+1}\}_\theta$ and the operator A_s

as the E_s-realization of A_k, i.e.,

$$D(A_s) := \{u \in E_s \cap D(A_k) : A_k u \in E_s\}, \quad A_s u := A_k u \ (u \in D(A_k)).$$

Remark 2.2. In the above situation, $[(E_s, A_s) : s \geq 0]$ defines a scale of Banach spaces in the sense of [5], Definition V.1.1. Moreover, the operator A_s is again the generator of a C_0-semigroup in E_s for all $s \geq 0$. This follows from [5], Theorem V.2.1.3 and Corollary V.2.1.4, after introducing a shift, i.e. considering $\widetilde{A} := A - \omega$ with $\omega > 0$ sufficiently large such that $0 \in \rho(\widetilde{A})$. Moreover, we have $\rho(A_s) = \rho(A)$ for all $s \geq 0$, and the resolvent estimate carries over from A to A_s, see [5], inequality (2.1.16) in Theorem 2.1.3. In particular, if $A : E_0 \supset E_1 \to E_0$ is sectorial with angle ϕ then the same holds for $A_s : E_s \supset E_{s+1} \to E_s$.

To apply the above abstract definitions to the boundary value problem, we introduce the following spaces (see Amann [6], Section 4.9):

Definition 2.3. Assume that the L^p-realization A_B of $(A(D), B(D))$ generates a C_0-semigroup. For $p \in (1, \infty)$ and $s \in [0, \infty) \setminus \{km + m_j + \frac{1}{p} : k \in \mathbb{N}_0,\ j = 1, \dots, \frac{m}{2}\}$, we define the space $W^s_{p;(A,B)}(\mathbb{R}^n_+)$ as the set of all $u \in W^s_p(\mathbb{R}^n_+)$ which satisfy $\gamma_0 B_j A^k u = 0$ for all $k \in \mathbb{N}_0$ and $j = 1, \dots, \frac{m}{2}$ with $s - mk - m_j > \frac{1}{p}$.

The Banach scale method gives the following result.

Theorem 2.4. *Let the boundary value problem $(A(D), B(D))$ be parabolic, and let $p \in (1, \infty)$ and $s \in [0, \infty) \setminus \{km + m_j + \frac{1}{p} : k \in \mathbb{N}_0,\ j = 1, \dots, \frac{m}{2}\}$. Then for all $f \in W^s_{p;(A,B)}(\mathbb{R}^n_+)$ and all $\lambda \in \overline{\Sigma}_{\frac{\pi}{2}} \setminus \{0\}$ the problem $(A(D) - \lambda)u = f$, $\gamma_0 B(D)u = 0$, has a unique solution $u \in W^{m+s}_{p;(A,B)}(\mathbb{R}^n_+)$, and the a priori estimate*

$$\|u\|_{W^{m+s}_p(\mathbb{R}^n_+)} + |\lambda|\, \|u\|_{W^s_p(\mathbb{R}^n_+)} \leq C \|f\|_{W^s_p(\mathbb{R}^n_+)} \quad (u \in W^{m+s}_{p;(A,B)}(\mathbb{R}^n_+)) \tag{2.2}$$

holds. In particular, the $W^s_{p;(A,B)}(\mathbb{R}^n_+)$-realization $A^{(s)}_B$ given by

$$D(A^{(s)}_B) := W^{m+s}_{p;(A,B)}(\mathbb{R}^n_+), \quad A^{(s)}_B u := A(D)u \ (u \in D(A^{(s)}_B))$$

is sectorial with angle larger than $\frac{\pi}{2}$ and therefore generates an analytic semigroup in $W^s_{p;(A,B)}(\mathbb{R}^n_+)$.

Proof. We will apply the method of Banach scales as introduced above. We consider the L^p-realization A_B and remark that it is well known that A_B is sectorial with angle larger than $\frac{\pi}{2}$ (see, e.g., [2]).

(i) First we show that for each $k \in \mathbb{N}_0$, we have $W^{km}_{p;(A,B)}(\mathbb{R}^n_+) = D(A^k_B)$. By definition, the inclusion "$\subset$" is obvious. To show the converse inclusion, we have to prove that $E_k := D(A^k_B) := \{u \in D(A_B) : A^\ell u \in D(A_B)\ (\ell = 0, \dots, k-1)\}$ is contained in $W^{km}_{p;(A,B)}(\mathbb{R}^n_+)$.

Due to the definition of $D(A_B)$, we have $\gamma_0 B_j A^\ell u = 0$ for all $\ell = 0, \dots, k-1$ and all $j = 1, \dots, \frac{m}{2}$. Therefore, we only have to show that $D(A^k_B) \subset W^{km}_p(\mathbb{R}^n_+)$. This is done iteratively. As $u \in D(A^2_B)$ and $\gamma_0 B_j u = 0$ for all $j = 1, \dots, \frac{m}{2}$, we have $u \in W^m_p(\mathbb{R}^n_+)$ and $Au \in W^m_p(\mathbb{R}^n_+)$. For $\lambda_0 \in \rho(A_B)$, the boundary value problem $(A - \lambda_0, B)$ is regular elliptic in the sense of Triebel [23], Def. 5.2.1/4. By elliptic regularity, we obtain $u \in W^{2m}_p(\mathbb{R}^n_+)$. Replacing u by Au and using $A^2 u \in W^m_p(\mathbb{R}^n_+)$, we can now prove $Au \in W^{2m}_p(\mathbb{R}^n_+)$. An iteration gives $u \in W^{km}_p(\mathbb{R}^n_+)$.

(ii) We consider $(A_B)_{s/m}$ and the scale generated by A_B. By real interpolation, we have for $k \in \mathbb{N}_0$ and $\theta \in (0,1)$ the identities

$$E_{k+\theta} = (E_k, E_{k+1})_{\theta,p} = \Big(W^{km}_{p;(A,B)}(\mathbb{R}^n_+), W^{(k+1)m}_{p;(A,B)}(\mathbb{R}^n_+) \Big)_{\theta,p} = W^{(k+\theta)m}_{p;(A,B)}(\mathbb{R}^n_+).$$

Here the last equality was shown in Amann [6], Corollary 4.9.2, in a more general setting. Due to Amann [5], Theorem 2.1.3, we see that $(A_B)_{s/m}$ generates an analytic semigroup and that $D((A_B)_{s/m}) = E_{\frac{s}{m}+1} = W^{m+s}_{p;(A,B)}(\mathbb{R}^n_+)$. Therefore, $(A_B)_{s/m}$ coincides with the $W^s_{p;(A,B)}$-realization $A_B^{(s)}$. In particular, $\rho(A_B^{(s)}) = \rho(A_B) \supset \overline{\Sigma}_{\frac{\pi}{2}} \setminus \{0\}$, and the resolvent estimate (2.2) holds. □

We remark that the exceptional cases $s = km + m_j + \frac{1}{p}$ arise due to the real interpolation results, see the discussion in Amann [6], Amann [7].

3. The case of inhomogeneous boundary conditions

Now we consider the boundary value problem (2.1) for $g \neq 0$, again restricting ourselves to the model problem in $\mathbb{R}^n_+$. For this, we will use an explicit representation of the solution. We start with the definition of the basic solutions. Throughout this section, we assume that $(A(D), B(D))$ is parameter-elliptic in a fixed sector $\overline{\Sigma}_\theta$.

Lemma 3.1. *For each $(\xi', \lambda) \in (\mathbb{R}^{n-1} \times \overline{\Sigma}_\theta) \setminus \{0\}$ and $j = 1, \ldots, \frac{m}{2}$, we define the basic solution $w_j = w_j(\xi', x_n, \lambda)$ as the unique stable solution of the ordinary differential equation*

$$\begin{aligned} (A(\xi', D_n) - \lambda) w_j(x_n) &= 0 \quad (x_n > 0), \\ B_k(\xi', D_n) w_j(0) &= \delta_{jk} \quad (k = 1, \ldots, \frac{m}{2}). \end{aligned}$$

Then w_j can be written in the form

$$w_j(\xi', x_n, \lambda) = \int_{\gamma(\xi',\lambda)} e^{i x_n \tau} (A(\xi', \tau) - \lambda)^{-1} N_j(\xi', \tau, \lambda) d\tau$$

where $\gamma(\xi', \lambda)$ is a smooth contour in the upper complex half-plane which encloses all roots of the polynomial $\tau \mapsto A(\xi', \tau) - \lambda$ with positive imaginary part. The functions N_j and w_j are smooth with respect to their arguments and continuous for all $(\xi', \lambda) \in (\mathbb{R}^{n-1} \times \overline{\Sigma}_\theta) \setminus \{0\}$, and we have the quasi-homogeneities

$$\begin{aligned} N_j(\rho\xi', \rho\tau, \rho^m \lambda) &= \rho^{m-m_j-1} N_j(\xi', \tau, \lambda), \\ w_j(\rho\xi', \tfrac{x_n}{\rho}, \rho^m \lambda) &= \rho^{-m_j} w_j(\xi', x_n, \lambda) \end{aligned}$$

for $\rho > 0$, $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$ and $\lambda \in \overline{\Sigma}_\theta$.

Proof. These assertions are stated in Denk-Faierman-Möller [9], Lemma 2.5. See also Volevich [24] for an explicit construction of N_j. □

To define the solution operators, we will use a parameter-dependent extension operator $E_\lambda \colon W_p^{k-\frac{1}{p}}(\mathbb{R}^{n-1}) \to W^k_p(\mathbb{R}^n_+)$ given by

$$(E_\lambda g)(x', x_n) := (\mathscr{F}')^{-1} \exp\big(-(|\xi'| + |\lambda|^{1/m}) x_n \big) (\mathscr{F}' g)(\xi').$$

Here $\mathscr{F}'$ stands for the partial Fourier transform with respect to the first $n-1$ variables. This operator was studied in Grubb-Kokholm [15] and in Agranovich-Denk-Faierman [2] in connection with the parameter-dependent norms above. It was shown that for all $k \in \mathbb{N}$, the trace operator

$$\gamma_0 \colon \big(W_p^k(\mathbb{R}^n_+), \|\!|\cdot|\!\|_{k,p,\mathbb{R}^n_+}\big) \to \big(W_p^{k-\frac{1}{p}}(\mathbb{R}^{n-1}), \|\!|\cdot|\!\|_{k-\frac{1}{p},p,\mathbb{R}^{n-1}}\big)$$

is continuous and E_λ is a continuous right-inverse to γ_0. Here and in the following, we call a linear operator continuous with respect to the parameter-dependent norms if for each $\lambda_0 > 0$ the norm of this operator can be estimated by a constant $C = C(\lambda_0)$ which does not depend on λ for each $\lambda \in \overline{\Sigma}_\theta$ with $|\lambda| \geq \lambda_0$.

Let us now consider the boundary value problem

$$\begin{aligned} (A(D)-\lambda)u &= 0 \quad \text{in } \mathbb{R}^n_+, \\ \gamma_0 B(D) u &= g \quad \text{on } \mathbb{R}^{n-1} \end{aligned} \tag{3.1}$$

with $g \in \prod_{j=1}^{m/2} W_p^{m+k-m_j-\frac{1}{p}}(\mathbb{R}^{n-1})$. Following an idea from Volevich [24], we write $\mathscr{F}'u$ in the form

$$\begin{aligned} (\mathscr{F}'u)(\xi', x_n) &= \sum_{j=1}^{m/2} w_j(\xi', x_n, \lambda)(\mathscr{F}' g_j)(\xi') \\ &= -\sum_{j=1}^{m/2} \int_0^\infty \frac{\partial}{\partial y_n} \Big[w_j(\xi', x_n+y_n, \lambda)(\mathscr{F}' E_\lambda g_j)(\xi', y_n)\Big] dy_n \\ &= \sum_{j=1}^{m/2} \Big(T_j^{(1)}(\lambda) E_\lambda g_j + T_j^{(2)}(\lambda)(\partial_n E_\lambda g_j)\Big). \end{aligned}$$

Here the solution operators $T_j^{(1)}$, $T_j^{(2)}$ are given by

$$\begin{aligned} (T_j^{(1)}(\lambda)\varphi)(x) &:= -\int_0^\infty (\mathscr{F}')^{-1}(\partial_n w_j)(\xi', x_n+y_n, \lambda)(\mathscr{F}'\varphi)(\xi', y_n) dy_n, \\ (T_j^{(2)}(\lambda)\varphi)(x) &:= -\int_0^\infty (\mathscr{F}')^{-1} w_j(\xi', x_n+y_n, \lambda)(\mathscr{F}'\varphi)(\xi', y_n) dy_n. \end{aligned}$$

Lemma 3.2. *Let $k \in \mathbb{N}_0$ and $p \in (1,\infty)$. Then the operators*

$$\begin{aligned} T_j^{(1)}(\lambda) &\colon \big(W_p^{m+k-m_j}(\mathbb{R}^n_+), \|\!|\cdot|\!\|_{m+k-m_j,p,\mathbb{R}^n_+}\big) \to \big(W_p^{m+k}(\mathbb{R}^n_+), \|\!|\cdot|\!\|_{m+k,p,\mathbb{R}^n_+}\big), \\ T_j^{(2)}(\lambda) &\colon \big(W_p^{m+k-m_j-1}(\mathbb{R}^n_+), \|\!|\cdot|\!\|_{m+k-m_j-1,p,\mathbb{R}^n_+}\big) \to \big(W_p^{m+k}(\mathbb{R}^n_+), \|\!|\cdot|\!\|_{m+k,p,\mathbb{R}^n_+}\big), \end{aligned}$$

$j = 1, \ldots, \frac{m}{2}$, *are continuous with respect to the parameter-dependent norms.*

Proof. We only consider $T_j^{(1)}$, the proof for $T_j^{(2)}$ essentially being the same. Let $\lambda_0 > 0$. We make use of the equivalence of norms

$$\|\!|u|\!\|_{m+k,p,\mathbb{R}^n_+} \approx \sum_{\ell=0}^{m+k} \sum_{|\alpha|=\ell} \big\| \lambda^{\frac{m+k-\ell}{m}} (D')^{\alpha'} \partial_n^{\alpha_n} u \big\|_{L^p(\mathbb{R}^n_+)}.$$

Therefore, we have to estimate

$$\begin{aligned}
&\|T_j^{(1)}(\lambda)\varphi\|_{m+k,p,\mathbb{R}^n_+}^p \\
&\le C\sum_{\ell=0}^{m+k}\sum_{|\alpha|=\ell}\int_0^\infty \Big\|\int_0^\infty (\mathscr{F}')^{-1}\lambda^{\frac{m+k-\ell}{m}}(\xi')^{\alpha'}(\partial_n^{\alpha_n+1}w_j)(\xi',x_n+y_n,\lambda) \\
&\qquad\qquad (\mathscr{F}'\varphi)(\xi',y_n)dy_n\Big\|_{L^p(\mathbb{R}^{n-1})}^p dx_n \\
&\le C\sum_{\ell=0}^{m+k}\sum_{|\alpha|=\ell}\int_0^\infty \Big(\int_0^\infty \Big\|(\mathscr{F}')^{-1}M_{j,\ell,\alpha}(\xi',x_n+y_n,\lambda) \\
&\qquad\qquad (\mathscr{F}'\widetilde{\varphi})(\xi',y_n)\Big\|_{L^p(\mathbb{R}^{n-1})}dy_n\Big)^p dx_n.
\end{aligned}$$

Here we have defined

$$\widetilde{\varphi} := (\mathscr{F}')^{-1}(|\xi'|+|\lambda|^{1/m})^{m+k-m_j}\mathscr{F}'\varphi \in L^p(\mathbb{R}^n_+)$$

and

$$M_{j,\ell,\alpha}(\xi',x_n,\lambda) := (|\xi'|+|\lambda|^{1/m})^{-m-k+m_j}\lambda^{\frac{m+k-\ell}{m}}(\xi')^{\alpha'}(\partial_n^{\alpha_n+1}w_j)(\xi',x_n,\lambda).$$

We will apply Michlin's theorem to the functions $M_{j,\ell,\alpha}$. For this, we abbreviate $\rho := \rho(\xi',\lambda) := |\xi'|+|\lambda|^{1/m}$ and use the homogeneities stated in Lemma 3.1. In the integral representation for the basic solutions w_j in Lemma 3.1, we make the substitution $\tau \mapsto \widetilde{\tau} = \frac{\tau}{\rho}$ and use the fact that $\gamma(\frac{\xi'}{\rho},\frac{\lambda}{\rho^m})$ can be replaced by a contour $\widetilde{\gamma}$ which is independent of ξ' and λ. For $\beta' \in \mathbb{N}_0^{n-1}$, we obtain

$$\begin{aligned}
&\Big|(\xi')^{\beta'}D_{\xi'}^{\beta'}M_{j,\ell,\alpha}(\xi',x_n,\lambda)\Big| \\
&= \Big|(\xi')^{\beta'}\lambda^{\frac{m+k-\ell}{m}}\int_{\gamma(\xi',\lambda)}\tau^{\alpha_n+1}e^{i\tau x_n}D_{\xi'}^{\beta'}\Big(\rho^{-m-k+m_j}(\xi')^{\alpha'} \\
&\qquad\qquad (A(\xi',\tau)-\lambda)^{-1}N_j(\xi',\tau,\lambda)\Big)d\tau\Big| \\
&= \Big|\big(\tfrac{\xi'}{\rho}\big)^{\beta'}\big(\tfrac{\lambda}{\rho^m}\big)^{\frac{m+k-\ell}{m}}\int_{\gamma(\xi',\lambda)}\big(\tfrac{\tau}{\rho}\big)^{\alpha_n+1}e^{i\tau x_n}H_{j,\alpha',\beta'}\big(\tfrac{\xi'}{\rho},\tfrac{\tau}{\rho},\tfrac{\lambda}{\rho^m}\big)d\tau\Big| \\
&= \Big|\big(\tfrac{\xi'}{\rho}\big)^{\beta'}\big(\tfrac{\lambda}{\rho^m}\big)^{\frac{m+k-\ell}{m}}\int_{\widetilde{\gamma}}\rho\,\widetilde{\tau}^{\alpha_n+1}e^{i\rho\widetilde{\tau}x_n}H_{j,\alpha',\beta'}\big(\tfrac{\xi'}{\rho},\widetilde{\tau},\tfrac{\lambda}{\rho^m}\big)d\widetilde{\tau}\Big| \\
&\le C\rho\exp(-C\rho x_n) \le \frac{C}{x_n}.
\end{aligned}$$

Here we have set

$$H_{j,\alpha',\beta'}(\xi',\tau,\lambda) := D_{\xi'}^{\beta'}\Big(\rho^{-m-k+m_j}(\xi')^{\alpha'}(A(\xi',\tau)-\lambda)^{-1}N_j(\xi',\tau,\lambda)\Big)$$

and used the fact that $H_{j,\alpha',\beta'}$ is quasi-homogeneous in its arguments of degree $|\alpha'|-m-k-1-|\beta'|$. The two inequalities follow by a compactness argument and by the elementary inequality $te^{-t}\le 1$ for $t\ge 0$.

Now an application of Michlin's theorem in $\mathbb{R}^{n-1}$ gives

$$\begin{aligned}
&|||T_j^{(1)}(\lambda)\varphi|||_{m+k,p,\mathbb{R}^n_+} \\
&\le C\sum_{\ell=0}^{m+k}\sum_{|\alpha|=\ell}\Big(\int_0^\infty\Big(\int_0^\infty \frac{\|\widetilde{\varphi}(\cdot,y_n)\|_{L^p(\mathbb{R}^{n-1})}}{x_n+y_n}dy_n\Big)^p dx_n\Big)^{1/p} \\
&\le C\Big(\int_0^\infty \|\widetilde{\varphi}(\cdot,y_n)\|^p_{L^p(\mathbb{R}^{n-1})}dy_n\Big)^{1/p} \\
&= C\|\widetilde{\varphi}\|_{L^p(\mathbb{R}^n_+)}.
\end{aligned}$$

Here we have used the continuity of the Hilbert transform in $L^p(\mathbb{R}_+)$ for the second inequality. Finally, we have

$$\begin{aligned}
\|\widetilde{\varphi}\|_{L^p(\mathbb{R}^n_+)} &= \Big\|(\mathscr{F}')^{-1}(|\xi'|+|\lambda|^{1/m})^{m+k-m_j}\mathscr{F}'\varphi\Big\|_{L^p(\mathbb{R}^n_+)} \\
&\le C\Big(\|\varphi\|_{m+k-m_j,p,\mathbb{R}^n_+} + |\lambda|^{\frac{m+k-m_j}{m}}\|\varphi\|_{L^p(\mathbb{R}^n_+)}\Big) \\
&\le C|||\varphi|||_{m+k-m_j,p,\mathbb{R}^n_+},
\end{aligned}$$

which shows the continuity of $T_j^{(1)}(\lambda)$. □

The last lemma is the main step in the proof of uniform a priori estimates with respect to parameter-dependent norms. We obtain the following result, cf. Agranovich [3], Theorem 3.2.1 for the case $p = 2$.

Theorem 3.3. *Let $s \in [0,\infty)$ and $p \in (1,\infty)$ with $m + s - m_j - \frac{1}{p} \notin \mathbb{N}_0$ for all $j = 1,\dots,\frac{m}{2}$. Then for every $\lambda \in \overline{\Sigma}_\theta \setminus \{0\}$, all $f \in W_p^s(\mathbb{R}^n_+)$ and all $g \in \prod_{j=1}^{m/2} W_p^{m+s-m_j-\frac{1}{p}}(\mathbb{R}^n)$ there exists a unique solution $u \in W_p^{m+s}(\mathbb{R}^n_+)$ of (2.1). Moreover, the operator*

$$(A(D)-\lambda,\gamma_0 B(D))\colon W_p^{m+s}(\mathbb{R}^n_+) \to W_p^s(\mathbb{R}^n_+)\times\prod_{j=1}^{m/2} W_p^{m+s-m_j-\frac{1}{p}}(\mathbb{R}^{n-1}) \tag{3.2}$$

is an isomorphism of Banach spaces with respect to the parameter-dependent norms. In particular, for every $\lambda_0 > 0$ there exists a constant $C = C(\lambda_0)$ such that

$$|||u|||_{m+s,p,\mathbb{R}^n_+} \le C\Big(|||f|||_{s,p,\mathbb{R}^n_+} + \sum_{j=1}^{m/2}|||g_j|||_{m+s-m_j-\frac{1}{p},p,\mathbb{R}^{n-1}}\Big) \tag{3.3}$$

holds for all $\lambda \in \overline{\Sigma}_\theta$ with $|\lambda| \ge \lambda_0$.

Proof. (i) First we assume $s \in \mathbb{N}_0$. The case $s = 0$ is well-known, see, e.g., Agranovich-Denk-Faierman [2], Theorem 2.1. In particular, we already know unique solvability of (2.1) with the solution u being at least in $W_p^m(\mathbb{R}^n_+)$. Moreover, the continuity of the operator in (3.2) with respect to the parameter-dependent norms is an immediate consequence of the continuity of the derivatives and of the trace operator. Therefore, we only have to show the a priori estimate (3.3) which also gives the smoothness of the solution.

Let $r_+ \in L(W_p^s(\mathbb{R}^n), W_p^s(\mathbb{R}^n_+))$, $r_+f := f|_{\mathbb{R}^n_+}$, denote the operator of restriction onto $\mathbb{R}^n_+$. Using the fact that there exists a coretraction e_+ of r_+ (see Amann [6], Section 4.4) with $e_+ \in L(W_p^\ell(\mathbb{R}^n_+), W_p^\ell(\mathbb{R}^n))$ for all $\ell \in [0,s]$, we see that both r_+ and e_+ are continuous with respect to the parameter-dependent norms, too.

We write $u = r_+u_1 + u_2$, where u_1 is the unique solution of $(A(D)-\lambda)u_1 = e_+f$ in $\mathbb{R}^n$. By the explicit representation of u_1 (see the proof of Lemma 2.1), we see that

$$\|r_+u_1\|_{m+s,p,\mathbb{R}^n_+} \le \|u_1\|_{m+s,p,\mathbb{R}^n} \le C\|e_+f\|_{s,p,\mathbb{R}^n} \le C\|f\|_{s,p,\mathbb{R}^n_+}.$$

For u_2 we obtain the boundary value problem

$$\begin{aligned} (A(D)-\lambda)u_2 &= 0 \quad \text{in } \mathbb{R}^n_+,\\ \gamma_0 B(D)u_2 &= \widetilde{g} \quad \text{on } \mathbb{R}^{n-1} \end{aligned}$$

with $\widetilde{g} := g - \gamma_0 B(D) r_+ u_1$. By the continuity of r_+, $B(D)$ and γ_0 (with respect to the parameter-dependent norms), we see that

$$\|\widetilde{g}_j\|_{m+s-m_j-\frac{1}{p},p,\mathbb{R}^{n-1}} \le C\Big(\|g_j\|_{m+s-m_j-\frac{1}{p},p,\mathbb{R}^{n-1}} + \|f\|_{s,p,\mathbb{R}^n_+}\Big). \tag{3.4}$$

Due to Lemma 3.2, we have

$$u_2 = \sum_{j=1}^{m/2} \Big(T_j^{(1)}(\lambda)E_\lambda \widetilde{g}_j + T_j^{(2)}(\lambda)\partial_n E_\lambda \widetilde{g}_j\Big).$$

Now the continuity of E_λ, ∂_n, and $T_j^{(1)}, T_j^{(2)}$ yields

$$\|u_2\|_{m+s,p,\mathbb{R}^n_+} \le C\sum_{j=1}^{m/2} \|\widetilde{g}_j\|_{m+s-m_j-\frac{1}{p},p,\mathbb{R}^{n-1}}$$

which in connection with (3.4) gives the a priori estimate (3.3) and the proof for $s \in \mathbb{N}_0$.

(ii) For $s \in (0,\infty)\setminus\mathbb{N}$ with $m+s-m_j-\frac{1}{p} \notin \mathbb{N}_0$, the result follows by real interpolation. Here we use the fact that for $k \in \mathbb{N}_0$ and $\theta \in (0,1)$ we have

$$\big(W_p^k(\mathbb{R}^n_+), W_p^{k+1}(\mathbb{R}^n_+)\big)_{\theta,p} = W_p^{k+\theta}(\mathbb{R}^n_+)$$

uniformly in λ with respect to the parameter-dependent norms, see Grubb-Kokholm [15], Theorem 1.1. □

A combination of Theorem 2.4 and Theorem 3.3 immediately yields the following result.

Corollary 3.4. *Let $(A(D), B(D))$ be parabolic, and let $s \in [0,\infty)\setminus\{km+m_j-\frac{1}{p} : k \in \mathbb{N}, j = 1,\dots,\frac{m}{2}\}$. Then for all $\lambda \in \overline{\Sigma}_{\frac{\pi}{2}}\setminus\{0\}$, $|\lambda| \ge \lambda_0 > 0$, and all $f \in W^s_{p;(A,B)}(\mathbb{R}^n_+)$ and $g \in \prod_{j=1}^{m/2} W_p^{m+s-m_j-\frac{1}{p}}(\mathbb{R}^{n-1})$ there exists a unique solution $u \in W_p^{m+s}(\mathbb{R}^n_+)$ of (2.1), and*

$$\|u\|_{W_p^{m+s}(\mathbb{R}^n_+)} + |\lambda|\,\|u\|_{W_p^s(\mathbb{R}^n_+)} \le C\Big(\|f\|_{W_p^s(\mathbb{R}^n_+)} + \sum_{j=1}^{m/2}\|g_j\|_{m+s-m_j-\frac{1}{p},p,\mathbb{R}^{n-1}}\Big). \tag{3.5}$$

In particular, this holds for all $f \in W_p^s(\mathbb{R}^n_+)$ if $s < m_j + \frac{1}{p}$ for all $j = 1,\dots,\frac{m}{2}$.

Proof. We write $u = u_1 + u_2$, where u_1 solves $(A(D) - \lambda)u_1 = f$, $\gamma_0 B(D)u_1 = 0$ and u_2 solves $(A(D) - \lambda)u_2 = 0$, $\gamma_0 B(D)u_2 = g$, and apply Theorem 2.4 and Theorem 3.3, respectively. For the application of Theorem 3.3, we note that by the interpolation inequality, the left-hand side of (3.5) is not larger than a constant times $|\!|\!| u |\!|\!|_{m+s,p,\mathbb{R}^n_+}$. The last statement follows directly from the fact that for these s, the spaces $W^s_p(\mathbb{R}^n_+)$ and $W^s_{p;(A,B)}(\mathbb{R}^n_+)$ coincide. □

Remark 3.5. On the right-hand side of (3.5) large powers of λ may appear, although we only have $|\lambda|$ on the left-hand side. The following elementary example in $\mathbb{R}^1_+$ shows that even in the one-dimensional case this cannot be avoided: Consider the boundary value problem $\lambda u(x_n) - u''(x_n) = 0\,(x_n > 0)$, $u(0) = g \in \mathbb{C}$. Then for $\lambda > 0$ the stable solution is $u(x_n) = \exp(-\sqrt{\lambda}x_n)g$, and a direct calculation shows that for $s \in \mathbb{N}_0$ we have

$$\|u\|_{W^{2+s}_p(\mathbb{R}_+)} \geq C|\lambda|^{\frac{2+s-1/p}{2}}|g|.$$

The power on the right-hand side is consistent with the exponent $(m+s-m_j-\frac{1}{p})/m$ appearing on the right-hand side of (3.5).

In some sense the parameter-dependent norms in Theorem 3.3 are natural for parameter-elliptic problems. However, they do not yield resolvent estimates as the parameter λ appears on both sides. Moreover, on the left-hand side we have $|\lambda|^{(m+s)/m}$ instead of $|\lambda|$. We will now derive a resolvent estimate for the $W^s_p(\mathbb{R}^n_+)$-realization. As it was shown in [8], for a decay like $|\lambda|^{-1}$ additional conditions on f are necessary. The following result gives a general resolvent estimate.

Theorem 3.6. *a) In the situation of Theorem 3.3, assume that*

$$\sigma := \max_{j=1,\dots,m/2}(s - m_j - \tfrac{1}{p}) > 0. \tag{3.6}$$

Then for all $\lambda_0 > 0$ there exists a constant $C = C(\lambda_0) > 0$ such that

(3.7)

$$\|u\|_{m+s,p,\mathbb{R}^n_+} + |\lambda|\,\|u\|_{s,p,\mathbb{R}^n_+} \leq C\Big(|\lambda|^{\sigma/m}\|f\|_{s,p,\mathbb{R}^n_+} + \sum_{j=1}^{m/2} |\!|\!| g_j |\!|\!|_{m+s-m_j-\frac{1}{p},p,\mathbb{R}^{n-1}}\Big).$$

b) Let $(A(D), B(D))$ be parabolic, and define σ as in (3.6)*. For $s \in [0,\infty) \setminus \{mk + m_j - \frac{1}{p} : k \in \mathbb{N}_0,\ j = 1,\dots,m/2\}$ define the unbounded operator $\widetilde{A}^{(s)}_B$ in $W^s_p(\mathbb{R}^n_+)$ by*

$$D(\widetilde{A}^{(s)}_B) := \{u \in W^{m+s}_p(\mathbb{R}^n_+) : \gamma_0 B(D)u = 0\},$$
$$\widetilde{A}^{(s)}_B u := A(D)u\ (u \in D(\widetilde{A}^{(s)}_B)).$$

Then $\rho(\widetilde{A}^{(s)}_B) \supset \overline{\Sigma}_{\frac{\pi}{2}} \setminus \{0\}$, and for all $\lambda_0 > 0$ there exists $C = C(\lambda_0) > 0$ such that

$$\big\|(\widetilde{A}^{(s)}_B - \lambda)^{-1}\big\|_{L(W^s_p(\mathbb{R}^n_+))} \leq C|\lambda|^{-1+\max\{0,\sigma/m\}} \quad (\lambda \in \overline{\Sigma}_{\frac{\pi}{2}},\ |\lambda| \geq \lambda_0).$$

Proof. a) We write the solution u in the form $u = u_1 + u_2 + u_3$. Here u_1 solves $(A(D) - \lambda)u_1 = 0$, $\gamma_0 B(D)u_1 = g$, while $u_2 := r_+\widetilde{u}_2$ with $\widetilde{u}_2$ being the solution of $(A(D) - \lambda)\widetilde{u}_2 = e_+ f$ in $\mathbb{R}^n$. Due to Corollary 3.3, we obtain

$$\|u_1\|_{m+s,p,\mathbb{R}^n_+} + |\lambda|\,\|u_1\|_{s,p,\mathbb{R}^n_+} \leq C\sum_{j=1}^{m/2} |\!|\!| g_j |\!|\!|_{m+s-m_j-\frac{1}{p},p,\mathbb{R}^{n-1}}.$$

Moreover, by Lemma 2.1, we have

$$\|u_2\|_{m+s,p,\mathbb{R}^n_+} + |\lambda|\,\|u_2\|_{s,p,\mathbb{R}^n_+} \le C\|f\|_{s,p,\mathbb{R}^n_+}. \tag{3.8}$$

The function u_3 is the solution of the boundary value problem

$$\begin{aligned} (A(D)-\lambda)u_3 &= 0 \quad \text{in } \mathbb{R}^n_+,\\ \gamma_0 B(D)u_3 &= -\gamma_0 B(D)u_2 \quad \text{on } \mathbb{R}^{n-1}. \end{aligned}$$

We apply Corollary 3.3 again and get

$$\begin{aligned} \|u_3\|_{m+s,p,\mathbb{R}^n_+} + |\lambda|\,\|u_3\|_{s,p,\mathbb{R}^n_+} &\le C\sum_{j=1}^{m/2} |\!|\!|\gamma_0 B(D)u_2|\!|\!|_{m+s-m_j-\frac{1}{p},p,\mathbb{R}^{n-1}}\\ &= C\sum_{j=1}^{m/2}\Big(\|\gamma_0 B(D)u_2\|_{m+s-m_j-\frac{1}{p}} + |\lambda|^{\frac{m+s-m_j-1/p}{m}}\|\gamma_0 B(D)u_2\|_{L^p(\mathbb{R}^{n-1})}\Big). \end{aligned}$$

We estimate the norms on the right-hand side for each j. First, we have

$$\begin{aligned} \|\gamma_0 B(D)u_2\|_{m+s-m_j-\frac{1}{p},p,\mathbb{R}^{n-1}} &\le C\|B(D)u_2\|_{m+s-m_j,p,\mathbb{R}^n_+}\\ &\le C\|u_2\|_{m+s,p,\mathbb{R}^n_+} \end{aligned}$$

for all $j=1,\dots,\frac{m}{2}$.

If $s > m_j + \frac{1}{p}$, we can estimate

$$\begin{aligned} |\lambda|^{\frac{m+s-m_j-1/p}{m}}\|\gamma_0 B(D)u_2\|_{L^p(\mathbb{R}^{n-1})} &\le |\lambda|^{\frac{m+s-m_j-1/p}{m}}\|\gamma_0 B(D)u_2\|_{s-m_j-\frac{1}{p},p,\mathbb{R}^{n-1}}\\ &\le C|\lambda|^{\frac{m+s-m_j-1/p}{m}}\|B(D)u_2\|_{s-m_j,p,\mathbb{R}^n_+}\\ &\le C|\lambda|^{\frac{m+s-m_j-1/p}{m}}\|u_2\|_{s,p,\mathbb{R}^n_+}\\ &\le C|\lambda|^{\sigma/m}\,|\lambda|\,\|u_2\|_{s,p,\mathbb{R}^n_+}\\ &\le C|\lambda|^{\sigma/m}\,\|f\|_{s,p,\mathbb{R}^n_+}. \end{aligned}$$

Here we applied (3.8) in the last step.

If $s < m_j + \frac{1}{p}$, we similarly write

$$\begin{aligned} |\lambda|^{\frac{m+s-m_j-1/p}{m}}\|\gamma_0 B(D)u_2\|_{L^p(\mathbb{R}^{n-1})} &\le |\lambda|^{\frac{m+s-m_j-1/p}{m}}\|\gamma_0 B(D)u_2\|_{\sigma,p,\mathbb{R}^{n-1}}\\ &\le C|\lambda|^{\sigma/m}\,|\lambda|^{\frac{m+s-m_j-\sigma-1/p}{m}}\,\|u_2\|_{\sigma+m_j+\frac{1}{p},p,\mathbb{R}^n_+}\\ &\le C|\lambda|^{\sigma/m}\Big(\|u_2\|_{m+s,p,\mathbb{R}^n_+} + |\lambda|\,\|u\|_{s,p,\mathbb{R}^n_+}\Big)\\ &\le C|\lambda|^{\sigma/m}\|f\|_{s,p,\mathbb{R}^n_+}, \end{aligned}$$

where we used the interpolation inequality (see Grisvard [14], Theorem 1.4.3.3) for the third inequality. This finishes the proof of the a priori estimate (3.7) and of part a).

b) In the case $\sigma < 0$ the a priori estimate (and the fact that the operator $\widetilde{A}_B^{(s)}$ is sectorial) follows from the last statement in Corollary 3.4. For $\sigma > 0$, we apply part a) with $g = 0$. Note that the case $\sigma = 0$ is excluded. □

Remark 3.7. For the Dirichlet-Laplacian Δ_D, we have $m = 2$, $m_1 = 0$, and therefore $\sigma = s - \frac{1}{p}$. For the resolvent in $W_p^s(\mathbb{R}_+^n)$, we obtain from Theorem 3.6

$$\|(\lambda - \Delta_D)^{-1}\|_{L(W_p^s(\mathbb{R}_+^n))} \le C|\lambda|^{-1+\frac{s}{2}-\frac{1}{2p}}.$$

For $s = 1$, we have a decay like $|\lambda|^{-1/2-1/(2p)}$. It was shown in [18] that this is the exact decay rate.

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