

**APPLICATION OF JACOBI'S REPRESENTATION THEOREM
TO LOCALLY MULTIPLICATIVELY CONVEX TOPOLOGICAL
 \mathbb{R} -ALGEBRAS**

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ABSTRACT. Let A be a commutative unital \mathbb{R} -algebra and let ρ be a seminorm on A which satisfies $\rho(ab) \leq \rho(a)\rho(b)$. We apply T. Jacobi's representation theorem [10] to determine the closure of a ΣA^{2d} -module S of A in the topology induced by ρ , for any integer $d \geq 1$. We show that this closure is exactly the set of all elements $a \in A$ such that $\alpha(a) \geq 0$ for every ρ -continuous \mathbb{R} -algebra homomorphism $\alpha : A \rightarrow \mathbb{R}$ with $\alpha(S) \subseteq [0, \infty)$, and that this result continues to hold when ρ is replaced by any locally multiplicatively convex topology τ on A . We obtain a representation of any linear functional $L : A \rightarrow \mathbb{R}$ which is continuous with respect to any such ρ or τ and non-negative on S as integration with respect to a unique Radon measure on the space of all real valued \mathbb{R} -algebra homomorphisms on A , and we characterize the support of the measure obtained in this way.

1. INTRODUCTION

It was known to Hilbert [9] that a nonnegative real multivariable polynomial $f = \sum_{\alpha} f_{\alpha} \underline{X}^{\alpha} \in \mathbb{R}[\underline{X}] := \mathbb{R}[X_1, \dots, X_n]$ is not necessarily a sum of squares of polynomials. However, every such polynomial can be approximated by elements of the cone $\Sigma \mathbb{R}[\underline{X}]^2 :=$ sums of squares of polynomials, with respect to the topology induced by the norm $\|\cdot\|_1$ (given by $\|\sum_{\alpha} f_{\alpha} \underline{X}^{\alpha}\|_1 := \sum_{\alpha} |f_{\alpha}|$). In fact, every polynomial $f \in \mathbb{R}[\underline{X}]$, nonnegative on $[-1, 1]^n$ is in the $\|\cdot\|_1$ -closure of $\Sigma \mathbb{R}[\underline{X}]^2$ [2, Theorem 9.1]. Moreover, it is known that for every $f \in \text{Pos}([-1, 1]^n) :=$ the cone of nonnegative polynomials on $[-1, 1]^n$, and $\epsilon > 0$, there exists $N > 0$ depending on $n, \epsilon, \deg f$ and the size of coefficients of f such that for every integer $r \geq N$, the polynomial $f_{\epsilon, r} := f + \epsilon(1 + \sum_{i=1}^n X_i^{2r}) \in \Sigma \mathbb{R}[\underline{X}]^2$. This gives an effective way of approximating f by sums of squares in $\|\cdot\|_1$ [11, Theorem 3.9]. The closure of $\Sigma \mathbb{R}[\underline{X}]^2$ with respect to the family of weighted $\|\cdot\|_p$ -norms has been studied in [4]. Note that an easy application of Stone-Weierstrass Theorem shows that the same result holds for the coarser norm $\|f\|_{\infty} := \sup_{x \in [-1, 1]^n} |f(x)|$; i.e., $\overline{\Sigma \mathbb{R}[\underline{X}]^2}^{\|\cdot\|_{\infty}} = \text{Pos}([-1, 1]^n)$, but in practice, finding $\|f\|_{\infty}$ is a computationally difficult optimization problem,

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whereas $\|f\|_1$ is easy to compute. Therefore to gain more computational flexibility it is interesting to study such closures with respect to various norms on $\mathbb{R}[\underline{X}]$.

The general set-up we consider is the following. Let C be a cone in $\mathbb{R}[\underline{X}]$, τ a locally convex topology on $\mathbb{R}[\underline{X}]$ and $K \subseteq \mathbb{R}^n$ be a closed set. Consider the condition:

$$(1) \quad \overline{C}^\tau \supseteq \text{Pos}(K),$$

(where as above, $\text{Pos}(K)$ denotes the set of polynomials nonnegative on K). An application of Hahn-Banach Separation Theorem together with Haviland's Theorem (see Theorem 2.2) shows that (1) holds if and only if for every τ -continuous linear functional L with $L(C) \subseteq [0, \infty)$, there exists a Borel measure μ on K such that

$$(2) \quad \forall f \in \mathbb{R}[\underline{X}] \quad L(f) = \int_K f \, d\mu.$$

In the present paper, we study closure results of type (1) and their corresponding representation results of type (2) for *any* locally multiplicatively convex (unital, commutative) topological \mathbb{R} -algebra.

In Section 2 we introduce some terminology and notation and recall Jacobi's Theorem and a generalized version of Haviland's Theorem, results which play a crucial role throughout the paper.

In Section 3 we consider the case of a submultiplicative seminorm ρ on an \mathbb{R} -algebra A . In Theorem 3.7 we prove that for any integer $d \geq 1$ and any $\sum A^{2d}$ -module S of A , \overline{S}^ρ consists of all elements of A with nonnegative image under every ρ -continuous \mathbb{R} -algebra homomorphism $\alpha : A \rightarrow \mathbb{R}$ such that $\alpha(S) \subseteq [0, \infty)$. This generalizes [5, Theorem 5.3] on the closure of $\sum A^{2d}$ with respect to a submultiplicative norm. The application of Theorem 3.7 to the representation of linear functionals by measures is explained in Corollary 3.8.

In Section 4 we explain how Theorem 3.7 and Corollary 3.8 apply in the case of a (unital, commutative) $*$ -algebra equipped with a submultiplicative $*$ -seminorm. Corollary 4.1 generalizes results on $*$ -semigroup algebras in [3, Theorem 4.2.5] and [6, Theorem 4.3 and Corollary 4.4].

In Section 5, specifically in Theorem 5.4, we explain how Theorem 3.7 extends to the class of locally multiplicatively convex topologies. Such topologies are induced by families of submultiplicative seminorms. Theorem 5.4 can be viewed as a strengthening (in the commutative case) of the result in [14, Lemma 6.1 and Proposition 6.2] about enveloping algebras of Lie algebras.

2. PRELIMINARIES

Throughout A denotes a unitary commutative \mathbb{R} -algebra. The set of all unitary \mathbb{R} -algebra homomorphisms from A to \mathbb{R} will be denoted by $\mathcal{X}(A)$. Note that $\mathcal{X}(A)$ as a subset of \mathbb{R}^A carries a natural topology, where \mathbb{R}^A is endowed with the product topology. This topology coincides with the weakest topology on $\mathcal{X}(A)$ which makes all the evaluation maps $\hat{a} : \mathcal{X}(A) \rightarrow \mathbb{R}$, defined by $\hat{a}(\alpha) = \alpha(a)$ continuous [13, section 5.7].

For an integer $d \geq 1$, ΣA^{2d} denotes the set of all finite sums of $2d$ powers of elements of A . A ΣA^{2d} -module of A is a subset S of A such that $1 \in S$, $S + S \subseteq S$ and $a^{2d} \cdot S \subseteq S$ for each $a \in A$. We say S is *archimedean* if for each $a \in A$ there exists an integer $n \geq 1$ such that $n + a \in S$. For any subset S of A , the non-negativity set of S , denoted by \mathcal{K}_S , is defined by

$$\mathcal{K}_S := \{\alpha \in \mathcal{X}(A) : \hat{a}(\alpha) \geq 0 \text{ for all } a \in S\}.$$

Also, for $K \subseteq \mathcal{X}(A)$, we define $\text{Pos}(K)$ by

$$\text{Pos}(K) := \{a \in A : \hat{a}(\alpha) \geq 0 \text{ for all } \alpha \in K\}.$$

Theorem 2.1 (Jacobi). *Suppose S is an archimedean ΣA^{2d} -module of A for some integer $d \geq 1$. Then for each $a \in A$,*

$$\hat{a} > 0 \text{ on } \mathcal{K}_S \Rightarrow a \in S.$$

Proof. See [10, Theorem 4]. □

Recall that a *Radon measure* on a Hausdorff topological space X is a measure on the σ -algebra of Borel sets of X that is locally finite and inner regular. *Locally finite* means that every point has a neighbourhood of finite measure. *Inner regular* means each Borel set can be approximated from within using a compact set. We will use the following version of Haviland's Theorem to get representations of linear functionals on A .

Theorem 2.2. *Suppose A is an \mathbb{R} -algebra, X is a Hausdorff space, and $\hat{\cdot} : A \rightarrow C(X)$ is an \mathbb{R} -algebra homomorphism such that for some $p \in A$, $\hat{p} \geq 0$ on X , the set $X_i = \hat{p}^{-1}([0, i])$ is compact for each $i = 1, 2, \dots$. Then for every linear functional $L : A \rightarrow \mathbb{R}$ satisfying*

$$L(\{a \in A : \hat{a} \geq 0 \text{ on } X\}) \subseteq [0, \infty),$$

there exists a Radon measure μ on X such that $\forall a \in A \quad L(a) = \int_X \hat{a} \, d\mu$.

Here, $C(X)$ denotes the ring of all continuous real valued functions on X . A proof of Theorem 2.2 can be found in [12, Theorem 3.1] or [13, Theorem 3.2.2] (also see [7, 8] for the original version). Note that the hypothesis of Theorem 2.2 implies in particular that X is locally compact (so μ is actually a Borel measure).

3. SEMINORMED \mathbb{R} -ALGEBRAS

Definition 3.1. A *seminorm* ρ on A is a map $\rho : A \rightarrow [0, \infty)$ such that

- (1) for $x \in A$ and $r \in \mathbb{R}$, $\rho(rx) = |r|\rho(x)$, and
- (2) for all $x, y \in A$, $\rho(x + y) \leq \rho(x) + \rho(y)$.

Moreover, ρ is called a *submultiplicative* seminorm if in addition:

- (3) for all $x, y \in A$, $\rho(xy) \leq \rho(x)\rho(y)$.

The algebra A together with a submultiplicative seminorm ρ on A is called a *seminormed algebra* and is denoted by the symbolism (A, ρ) . We denote the set of all ρ -continuous \mathbb{R} -algebra homomorphisms from A to \mathbb{R} by $\mathfrak{sp}(\rho)$, which we refer to as the *Gelfand spectrum* of (A, ρ) . The topology on $\mathfrak{sp}(\rho)$ is the topology induced as a subspace of $\mathcal{X}(A)$.

Lemma 3.2. *For any submultiplicative seminorm ρ on A ,*

$$\mathfrak{sp}(\rho) = \{\alpha \in \mathcal{X}(A) : |\alpha(x)| \leq \rho(x) \text{ for all } x \in A\}.$$

Proof. Suppose $\alpha \in \mathcal{X}(A)$ and there exists $x \in A$ such that $|\alpha(x)| > \rho(x)$. Set $y = \frac{x}{\delta}$ where $\delta \in \mathbb{R}$ is such that $|\alpha(x)| > \delta > \rho(x)$. Then $\rho(y) < 1$ and $|\alpha(y)| > 1$ so, as $n \rightarrow \infty$, $\rho(y^n) \rightarrow 0$ and $|\alpha(y^n)| \rightarrow \infty$. This proves (\subseteq) . The other inclusion is clear. \square

Corollary 3.3. *For any submultiplicative seminorm ρ on A , $\mathfrak{sp}(\rho)$ is compact.*

Proof. The map $\alpha \mapsto (\hat{a}(\alpha))_{a \in A}$ identifies $\mathfrak{sp}(\rho)$ with a closed subset of the compact space $\prod_{a \in A} [-\rho(a), \rho(a)]$. \square

Remark 3.4. For a seminormed algebra (A, ρ) , the set $I_\rho := \{a \in A : \rho(a) = 0\}$ is a closed ideal of A and the map

$$\bar{\rho} : \bar{A} = A/I_\rho \longrightarrow [0, \infty)$$

defined by $\bar{\rho}(\bar{a}) = \rho(a)$ is a well-defined norm on \bar{A} . Thus $(\bar{A}, \bar{\rho})$ is a normed \mathbb{R} -algebra and hence $(\bar{A}, \bar{\rho})$ admits a completion $(\tilde{A}, \tilde{\rho})$ which is a Banach \mathbb{R} -algebra.

Lemma 3.5. *For any unital Banach \mathbb{R} -algebra (B, φ) , any $a \in A$ and $r \in \mathbb{R}$ such that $r > \varphi(a)$, and any integer $k \geq 1$, there exists $p \in B$ such that $p^k = r + a$.*

Proof. This is well-known. The standard power series expansion

$$(r + x)^{1/k} = r^{1/k} \left(1 + \frac{x}{r}\right)^{1/k} = r^{1/k} \sum_{i=0}^{\infty} \frac{\frac{1}{k}(\frac{1}{k}-1) \cdots (\frac{1}{k}-i)}{i!} \left(\frac{x}{r}\right)^i$$

converges absolutely for $|x| < r$. This implies that

$$p := r^{1/k} \sum_{i=0}^{\infty} \frac{\frac{1}{k}(\frac{1}{k}-1) \cdots (\frac{1}{k}-i)}{i!} \left(\frac{a}{r}\right)^i$$

is a well-defined element of B and $p^k = r + a$. \square

Corollary 3.6. *For any unital Banach \mathbb{R} -algebra (B, φ) and any linear functional $L : B \rightarrow \mathbb{R}$, if $L(b^{2d}) \geq 0$ for all $b \in B$ for some $d \geq 1$ then L is φ -continuous. In particular, each $\alpha \in \mathcal{X}(B)$ is φ -continuous.*

Proof. By Lemma 3.5, $\frac{1}{n} + \varphi(a) \pm a = \frac{1}{n} + \varphi(\pm a) + (\pm a) \in B^{2d}$ for all $a \in B$ and all $n \geq 1$. Applying L this yields $|L(a)| \leq (\frac{1}{n} + \varphi(a))L(1)$ for all $a \in B$ and all $n \geq 1$ so $|L(a)| \leq \varphi(a)L(1)$ for all $a \in B$. \square

We come now to the main result of the section.

Theorem 3.7. *Let ρ be a submultiplicative seminorm on A and let S be a $\sum A^{2d}$ -module of A . Then $\overline{S}^\rho = \text{Pos}(\mathcal{K}_S \cap \mathfrak{sp}(\rho))$. In particular, $\overline{\sum A^{2d}}^\rho = \text{Pos}(\mathfrak{sp}(\rho))$.*

Proof. Since each $\alpha \in \mathcal{K}_S \cap \mathfrak{sp}(\rho)$ is continuous and

$$\text{Pos}(\mathcal{K}_S \cap \mathfrak{sp}(\rho)) = \bigcap_{\alpha \in \mathcal{K}_S \cap \mathfrak{sp}(\rho)} \alpha^{-1}([0, \infty)),$$

$\text{Pos}(\mathcal{K}_S \cap \mathfrak{sp}(\rho))$ is ρ -closed. Since $S \subseteq \text{Pos}(\mathcal{K}_S \cap \mathfrak{sp}(\rho))$ this implies $\overline{S}^\rho \subseteq \text{Pos}(\mathcal{K}_S \cap \mathfrak{sp}(\rho))$. For the reverse inclusion we have to show that if $b \in \text{Pos}(\mathcal{K}_S \cap \mathfrak{sp}(\rho))$ then $b \in \overline{S}^\rho$. Let \tilde{S} denote the closure of the image of S in $(\tilde{A}, \tilde{\rho})$. Then \tilde{S} is a $\Sigma \tilde{A}^{2d}$ -module of \tilde{A} . By Lemma 3.5, $\frac{1}{n} + \tilde{\rho}(a) + a \in \tilde{A}^{2d} \subseteq \tilde{S}$ for all $a \in \tilde{A}$ and all $n \geq 1$, so $\tilde{\rho}(a) + a \in \tilde{S}$ for all $a \in \tilde{A}$. This implies that \tilde{S} is archimedean. By Corollary 3.6 every element of $\mathcal{K}_{\tilde{S}}$ restricts to an element of $\mathcal{K}_S \cap \mathfrak{sp}(\rho)$ ¹ so, by our hypothesis on b , $\alpha(\tilde{b}) = \alpha|_A(b) \geq 0$ for all $\alpha \in \mathcal{K}_{\tilde{S}}$, where \tilde{b} denotes the image of b in \tilde{A} . Then $\alpha(\tilde{b} + \frac{1}{n}) > 0$ for all $\alpha \in \mathcal{K}_{\tilde{S}}$ so, by Jacobi's Theorem 2.1, $\tilde{b} + \frac{1}{n} \in \tilde{S}$ for all $n \geq 1$. Then $\tilde{b} \in \tilde{S}$, so $b \in \overline{S}^\rho$. \square

Corollary 3.8. *Let ρ be a submultiplicative seminorm on A , S a ΣA^{2d} -module of A . If $L : A \rightarrow \mathbb{R}$ is a ρ -continuous linear functional such that $L(s) \geq 0$ for all $s \in S$ then there exists a unique Radon measure μ on $\mathcal{X}(A)$ such that*

$$\forall a \in A \quad L(a) = \int \hat{a} \, d\mu.$$

Moreover, $\text{supp}(\mu) \subseteq \mathcal{K}_S \cap \mathfrak{sp}(\rho)$.

Proof. By our hypothesis and Theorem 3.7 L is non-negative on $\text{Pos}(\mathcal{K}_S \cap \mathfrak{sp}(\rho))$. Applying Theorem 2.2, with $X := \mathcal{K}_S \cap \mathfrak{sp}(\rho)$ and $\hat{\cdot} : A \rightarrow C(X)$ the map defined by $a \mapsto \hat{a}|_X$, yields a Radon measure μ' on X such that $L(a) = \int_X \hat{a} \, d\mu'$ for all $a \in A$. Observe that X is compact, by Corollary 3.3, so we can take $p = 1$. The Radon measure μ on $\mathcal{X}(A)$ that we are looking for is just the extension of μ' to $\mathcal{X}(A)$, i.e., $\mu(E) := \mu'(E \cap X)$ for all Borel sets E in $\mathcal{X}(A)$. Uniqueness of μ is a consequence of the following easy result. \square

Lemma 3.9. *Suppose μ is a Radon measure on $\mathcal{X}(A)$ having compact support. Then μ is determinate, i.e., if ν is any Radon measure on $\mathcal{X}(A)$ satisfying $\int \hat{a} \, d\nu = \int \hat{a} \, d\mu$ for all $a \in A$ then $\nu = \mu$.*

Proof. Set $Y = \text{supp}(\mu)$. Suppose first that $\text{supp}(\nu) \not\subseteq Y$. Then there exists a compact set $Z \subseteq \mathcal{X}(A) \setminus Y$ with $\nu(Z) > 0$. Choose $\epsilon > 0$ so that $\epsilon < \frac{\nu(Z)}{\mu(Y) + \nu(Z)}$. Since Y, Z are compact and disjoint, the Stone-Weierstrass Theorem implies there exists $a \in A$ such that $|\hat{a}(\alpha)| \leq \epsilon$ for all $\alpha \in Y$ and $|\hat{a}(\alpha) - 1| \leq \epsilon$ for all $\alpha \in Z$. Replacing a by a^2 if necessary, we can suppose $\hat{a} \geq 0$ on $\mathcal{X}(A)$. Then $\int \hat{a} \, d\mu \leq \epsilon \mu(Y)$, but $\int \hat{a} \, d\nu \geq \int_Z \hat{a} \, d\nu \geq (1 - \epsilon)\nu(Z)$, which is a contradiction. It follows that $\text{supp}(\nu) \subseteq Y$, so μ, ν both have support in the same compact set Y . Then, using the Stone-Weierstrass Theorem again, $\int \varphi \, d\mu = \int \varphi \, d\nu$ for all $\varphi \in C(Y)$ so $\mu = \nu$, by the Riesz Representation Theorem. \square

Remark 3.10. (i) The converse of Corollary 3.8 holds trivially: If $L(a) = \int \hat{a} \, d\mu$ for all $a \in A$ for some Radon measure μ with $\text{supp}(\mu) \subseteq \mathcal{K}_S \cap \mathfrak{sp}(\rho)$ then $L(s) \geq 0$ for all $s \in S$ and $|L(a)| \leq \rho(a)L(1)$ for all $a \in A$, so L is ρ -continuous.

(ii) Theorem 3.7 and Corollary 3.8 should be viewed as ‘two sides of the same coin’. We have shown how Corollary 3.8 can be deduced from Theorem 3.7 using Theorem 2.2. Conversely, one can deduce Theorem 3.7 from Corollary 3.8 by an easy application of the Hahn-Banach Separation Theorem.

¹In fact one can show that the restriction map $\mathcal{K}_{\tilde{S}} \rightarrow \mathcal{K}_S \cap \mathfrak{sp}(\rho)$ is a homeomorphism.

4. *-SEMINORMED *-ALGEBRAS

In this section we consider a $*$ -algebra R equipped with a submultiplicative $*$ -seminorm φ , i.e., R is a (unital, commutative) \mathbb{C} -algebra equipped with an involution $*$: $R \rightarrow R$ satisfying

$$(\lambda a)^* = \bar{\lambda}a^*, (a+b)^* = a^* + b^*, (ab)^* = a^*b^* \text{ and } a^{**} = a$$

for all $\lambda \in \mathbb{C}$ and all $a, b \in R$, and $\varphi : R \rightarrow [0, \infty)$ satisfies

$$\varphi(\lambda a) = |\lambda|\varphi(a), \varphi(a+b) \leq \varphi(a) + \varphi(b), \varphi(ab) \leq \varphi(a)\varphi(b) \text{ and } \varphi(a^*) = \varphi(a)$$

for all $\lambda \in \mathbb{C}$ and all $a, b \in R$.

We denote by $\mathcal{X}(R)$ set of all $*$ -algebra homomorphisms $\alpha : R \rightarrow \mathbb{C}$ equipped with its natural topology as a subspace of the product space \mathbb{C}^R and by $\mathfrak{sp}(\varphi)$ the subspace of $\mathcal{X}(R)$ consisting of all φ -continuous $*$ -algebra homomorphisms $\alpha : R \rightarrow \mathbb{C}$. The *symmetric part* of R is

$$H(R) := \{a \in R : a^* = a\}.$$

Since $R = H(R) \oplus iH(R)$, one sees that $\mathcal{X}(R)$ and $\mathfrak{sp}(\varphi)$ are naturally identified via restriction with $\mathcal{X}(H(R))$ and $\mathfrak{sp}(\varphi|_{H(R)})$, respectively, and φ continuous $*$ -linear functionals $L : R \rightarrow \mathbb{C}$ are naturally identified via restriction with $\varphi|_{H(R)}$ -continuous \mathbb{R} -linear functionals $L : H(R) \rightarrow \mathbb{R}$.

Applying Theorem 3.7 and Corollary 3.8 to the symmetric part of (R, φ) yields the following result.

Corollary 4.1. *Let R be a $*$ -algebra equipped with a submultiplicative $*$ -seminorm φ , S a $\sum H(R)^{2d}$ -module of $H(R)$. Then $\overline{S}^\varphi = \text{Pos}(\mathcal{K}_S \cap \mathfrak{sp}(\varphi))$. If $L : R \rightarrow \mathbb{C}$ is any φ -continuous $*$ -linear functional such that $L(s) \geq 0$ for all $s \in S$ then there exists a unique Radon measure on $\mathcal{X}(R)$ such that $L(a) = \int \hat{a}d\mu$ for all $a \in R$. Moreover, $\text{supp}(\mu) \subseteq \mathcal{K}_S \cap \mathfrak{sp}(\varphi)$.*

Corollary 4.1 applies, in particular, to any $*$ -semigroup algebra $\mathbb{C}[W]$ equipped with a $*$ -seminorm $\|\cdot\|_\phi$ arising from an absolute value ϕ on the $*$ -semigroup W , i.e., $\|\sum \lambda_w w\|_\phi := \sum_w |\lambda_w|\phi(w)$. In this way Corollary 4.1 extends [3, Theorem 4.2.5] and [6, Theorem 4.3 and Corollary 4.4].

5. LOCALLY MULTIPLICATIVELY CONVEX TOPOLOGIES

Let A be an \mathbb{R} -algebra. A subset U of A is called a *multiplicative set* (an *m-set* for short) if $U \cdot U \subseteq U$. A locally convex vector space topology on A is said to be *locally multiplicatively convex* (*lmc* for short) if there exists a system of neighbourhoods for 0 consisting of *m*-sets. It is immediate from the definition that multiplication is continuous in any lmc-topology. We recall the following result.

Theorem 5.1. *A locally convex vector space topology τ on A is lmc if and only if τ is generated by a family of submultiplicative seminorms on A .*

Proof. See [1, 4.3-2]. □

A family \mathcal{F} of submultiplicative seminorms of A is said to be *saturated* if, for any $\rho_1, \rho_2 \in \mathcal{F}$, the seminorm ρ of A defined by

$$\rho(x) := \max\{\rho_1(x), \rho_2(x)\} \text{ for all } x \in A$$

belongs to \mathcal{F} . For an lmc topology τ on A one can always assume that the family \mathcal{F} of submultiplicative seminorms generating τ is saturated. In this situation the topology τ is the inductive limit topology, i.e., the balls $B_r^\rho(0) := \{a \in A : \rho(a) < r\}$, $\rho \in \mathcal{F}$, $r > 0$ form a system of τ -neighbourhoods of zero. This is clear.

We record the following more-or-less obvious result:

Lemma 5.2. *Suppose τ is an lmc topology on A generated by a saturated family \mathcal{F} of submultiplicative seminorms of A and $L : A \rightarrow \mathbb{R}$ is a τ -continuous linear functional. Then there exists $\rho \in \mathcal{F}$ such that L is ρ -continuous.*

Proof. The set $\{a \in A : |L(a)| < 1\}$ is an open neighbourhood of 0 in A so there exists $\rho \in \mathcal{F}$ and $r > 0$ such that $B_r^\rho(0) \subseteq \{a \in A : |L(a)| < 1\}$. Then $B_{r\epsilon}^\rho(0) = \epsilon B_r^\rho(0)$ so

$$L(B_{r\epsilon}^\rho(0)) = L(\epsilon B_r^\rho(0)) = \epsilon L(B_r^\rho(0)) \subseteq \epsilon(-1, 1) = (-\epsilon, \epsilon)$$

for all $\epsilon > 0$, i.e., L is ρ -continuous. \square

We denote the Gelfand spectrum of (A, τ) , i.e., the set of all τ -continuous $\alpha \in \mathcal{X}(A)$, by $\mathfrak{sp}(\tau)$ for short.

Corollary 5.3. *Suppose τ is an lmc topology on A generated by a saturated family \mathcal{F} of submultiplicative seminorms of A . Then $\mathfrak{sp}(\tau) = \bigcup_{\rho \in \mathcal{F}} \mathfrak{sp}(\rho)$.*

Our main result in the previous section extends to general lmc topologies, as follows:

Theorem 5.4. *Let τ be an lmc topology on A and let S be any ΣA^{2d} -module of A . Then $\overline{S}^\tau = \text{Pos}(\mathcal{K}_S \cap \mathfrak{sp}(\tau))$. In particular, $\overline{\Sigma A^{2d}}^\tau = \text{Pos}(\mathfrak{sp}(\tau))$.*

Proof. Let \mathcal{F} be a saturated family of submultiplicative seminorms generating τ . Then $\overline{S}^\tau = \bigcap_{\rho \in \mathcal{F}} \overline{S}^\rho = \bigcap_{\rho \in \mathcal{F}} \text{Pos}(\mathcal{K}_S \cap \mathfrak{sp}(\rho)) = \text{Pos}(\mathcal{K}_S \cap \mathfrak{sp}(\tau))$. \square

In view of Lemma 5.2, Corollary 3.8 also extends to general lmc topologies in an obvious way. The unique Radon measure corresponding to a τ -continuous linear functional $L : A \rightarrow \mathbb{R}$ such that $L(s) \geq 0$ for all $s \in S$ has support contained in the compact set $\mathcal{K}_S \cap \mathfrak{sp}(\rho)$ for some $\rho \in \mathcal{F}$.

The finest lmc topology on A is the lmc topology generated by the family of all submultiplicative seminorms of A . Theorem 5.4 can be thought of as a strengthening (in the commutative case) of the result of [14, Lemma 6.1 and Proposition 6.2] about enveloping algebras for \mathbb{R} -algebras. Note also the following:

Corollary 5.5. *Let η be the finest lmc topology on A . Then, for any ΣA^{2d} -module S of A , $\overline{S}^\eta = \text{Pos}(\mathcal{K}_S)$. In particular, $\overline{\Sigma A^{2d}}^\eta = \text{Pos}(\mathcal{X}(A))$*

Proof. Apply Theorem 5.4 with $\tau = \eta$, using the fact that $\mathfrak{sp}(\eta) = \mathcal{X}(A)$. \square

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