A Note on Schanuel's Conjectures for Exponential Logarithmic Power Series Fields

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Abstract

We consider a valued field of characteristic 0 with embedded residue field. We fix an additive complement to the valuation ring and its induced "constant term" map. We further assume that the valued field is endowed with an exponential map, and a derivation compatible with the exponential. We use a result of Ax to evaluate the transcendence degree of subfields generated by field elements which have constant term equal to 0 and are linearly independent. We apply our result to the examples of Logarithmic-Exponential power series fields, Exponential-Logarithmic power series fields, and Exponential Hardy fields.

In [1], J. Ax's establishes the following conjecture **(SD)** due to S. Schanuel. Let F be a field of characteristic 0 and D a derivation of F, i.e. $D(x+y) = D(x)+D(y)$ and $D(xy) = xD(y)+yD(x)$. We assume that the field of constants $C \supseteq \mathbb{Q}$. Below td denotes the transcendence degree.

(SD) Let $y_1, ..., y_n, z_1, ..., z_n \in F^\times$ be such that $Dy_k = \frac{Dz_k}{z_k}$ for $k = 1, ..., n$. If $\{Dy_k\ ;\ k = 1, \cdots, n\}$ is $\mathbb{Q}$-linearly independent, then

$$\mathrm{td}_C C(y_1, ..., y_n, z_1, ..., z_n) \geq n + 1$$

.

We rephrase **(SD)** as follows:

Theorem A: Let $y_1, ..., y_n, z_1, ..., z_n \in F^\times$ be such that $Dy_k = \frac{Dz_k}{z_k}$ for $k = 1, ..., n$. If $\mathrm{td}_C C(y_1, ..., y_n, z_1, ..., z_n) \leq n$, then $\sum_{i=1}^{n} m_i y_i \in C$ for some $m_1, ..., m_n \in \mathbb{Q}$ not all zero.

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Consider the field of Laurent series $\mathbb{C}((t))$, endowed with term by term derivation. The field of constants is $\mathbb{C}$. Let $\mathbb{C}[[t]]$ denote the ring of formal power series with complex coefficients in the variable t, i.e. $\mathbb{C}[[t]]$ is the ring of Laurent series with nonnegative exponents. Let $y_i(0)$ denote the constant term of the series y_i. The exponential map exp on $\mathbb{C}[[t]]$ is given by the Taylor series expansion and satisfies that $Dy_k = \frac{D\exp(y_k)}{\exp(y_k)}$.

A corollary to **(SD)** which appears in Ax's paper is:

Corollary B: Let $y_i \in \mathbb{C}[[t]]$ such that $y_i - y_i(0)$ are $\mathbb{Q}$-linearly independent, $i = 1, \cdots, n$. Then $\mathrm{td}_{\mathbb{C}}\mathbb{C}(y_1, ...y_n, \exp(y_1), \cdots, \exp(y_n)) \geq n + 1$.

Definition 1. A *differential valued exponential field* K is a field of characteristic 0, equipped with a derivation $D : K \to K$, a valuation $v : K^\times \to G$ with value group G, and an exponential map $\exp : K \to K^\times$ which satisfy the following:

- $\forall x \forall y(\exp(x + y) = \exp(x)\exp(y))$
- $\forall x(Dx = \frac{D\exp(x)}{\exp(x)})$
- The field of constants is isomorphic to the residue field of v.

We denote the valuation ring by $\mathcal{O}_v$, the maximal ideal by $\mathcal{M}_v$, and the residue field $\bar{K} \cong \mathcal{O}_v/\mathcal{M}_v$. We thus require that the field of constants $C \subseteq K$ is (isomorphic to) $\bar{K}$, i.e. every element y of $\mathcal{O}_v$ has a unique representation $y = c + \epsilon$ where $c \in C$ and $\epsilon \in \mathcal{M}_v$ (so $\mathcal{O}_v = C \oplus \mathcal{M}_v$). For $y \in \mathcal{O}_v$ we write $\bar{y}$ for the residue of y which is this $c \in C$. In particular $\bar{c} = c$ for $c \in C$.

In this note, we generalize Corollary B to $y_1, \cdots, y_n$ arbitrary elements (not necessarily in $\mathcal{O}_v$) of a differential valued exponential field K, see Corollary 3. In particular we apply this observation to exponential–logarithmic series (Example 9) and other examples, see below. Since K is not in general a field of series, we need to find an abstract substitute for the "constant term" $y(0)$ of a series: Since the additive group of K is a $\mathbb{Q}$-vector space, and $\mathcal{O}_v$ is a subspace, we choose and fix a vector space complement $\mathbf{A}$ such that $K = \mathbf{A} \oplus C \oplus \mathcal{M}_v$. For $y \in K^\times$ we define $\mathrm{co}_{\mathbf{A}}(y) := \overline{(y - a)}$ for the uniquely determined $a \in \mathbf{A}$ satisfying that $(y - a) \in \mathcal{O}_v$. Note that $\mathrm{co}_{\mathbf{A}}(y) = \bar{y}$ if $y \in \mathcal{O}_v$.

In this setting we observe:

Lemma 2. *Let $y_1, \cdots, y_n \in K$ such that $\sum_{i=1}^n m_i y_i \in C$ for some $m_i \in \mathbb{Q}$, then $\sum_{i=1}^n m_i(y_i - co_{\mathbf{A}}(y_i)) = 0$.*

Proof. Write $y_i = a_i + c_i + \epsilon_i$ with $a_i \in \mathbf{A}$, $c_i = \mathrm{co}_{\mathbf{A}}(y_i) \in C$ and $\epsilon_i \in \mathcal{M}_v$. We compute:

$$\sum_{i=1}^n m_i y_i = \sum_{i=1}^n m_i a_i + \sum_{i=1}^n m_i \mathrm{co}_{\mathbf{A}}(y_i) + \sum_{i=1}^n m_i \epsilon_i \, .$$

Since $\sum_{i=1}^n m_i y_i \in C$, it follows by the uniqueness of the decomposition that $\sum_{i=1}^n m_i a_i = \sum_{i=1}^n m_i \epsilon_i = 0$ as required. $\square$

Corollary 3. *Let $y_1, ..., y_n \in K$ and suppose that*

$$y_1 - co_{\mathbf{A}}(y_1), \cdots, y_n - co_{\mathbf{A}}(y_n)$$

are $\mathbb{Q}$-linearly independent. Then

$$td_C C(y_1, ..., y_n, \exp(y_1), ..., \exp(y_n)) \geq n + 1$$

.

Proof. Follows immediately from Theorem A and Lemma 2 □

Example 4. Let H be a *Hardy field*, i.e. a set of germs at $+\infty$ of real functions which is a field and is closed under differentiation. It carries canonically a valuation v corresponding to the comparison relation between germs of real function in this context. Moreover we suppose that H carries an exponential [7, Definition p. 94], so H is a differential valued exponential field (see [10]). Assume that $f_1, \cdots, f_n \in H^\times$ such that $v(f_i) \geq 0$ i.e. $\lim_{x\to\infty} f_i \in \mathbb{R}$. If

$$f_1 - \lim_{x\to\infty} f_1, \cdots, f_n - \lim_{x\to\infty} f_n$$

are $\mathbb{Q}$-linearly independent, then

$$\mathrm{td}_C C(f_1, ..., f_n, \exp(f_1), ..., \exp(f_n)) \geq n + 1$$

As an explicit example, take H to be the field of logarithmic-exponential functions defined by G.H. Hardy himself in [4].

We now consider fields of generalized series. By Kaplansky's embedding theorem in [6], generalized series fields are universal domains for valued fields in the equal characteristic case.

Definition 5. Let k be a field of characteristic 0 and G a totally ordered Abelian group. A *generalized series* (with coefficients a_g in k and exponents g in G) is a formal sum $a = \sum_{g\in G} a_g t^g$ with well-ordered support Supp $a := \{g \in G \mid a(g) \neq 0\}$. These series form a field, say $K = k((G))$, under component-wise sum and convolution product [3]. We consider the canonical valuation v_{min} on K defined by $v_{min}(a) := \min(\text{Supp } a)$. The value group is G, the residue field is k, and the valuation ring is $k[[G^{\geq 0}]]$ (the ring of series with support in the positive cone of G). E.g. if $G = \mathbb{Z}$ then $\mathbb{C}((G))$ is the field of Laurent series and $\mathbb{C}[[G^{\geq 0}]] = \mathbb{C}[[t]]$ the ring of formal power series.

For $k((G))$ a canonical complement to the valuation ring is given by $\mathbf{A} := k[[G^{<0}]]$ (series with support in the negative cone of G). Then for $y \in K^\times$ we have $\mathrm{co}_{\mathbf{A}}(y) = y(0)$ the constant term of y. More generally, if F is a truncation closed subfield of K (i.e. $y \in F$ implies that every initial segment of $y \in F$ as well), then $\mathbf{A}_F := F \cap k[[G^{<0}]]$ is a canonical complement and $\mathrm{co}_{\mathbf{A_F}}(y) = y(0)$ for $y \in F^\times$.

Example 6. Take $K = \mathbb{R}((G))$ endowed with a series derivation D [8, Section 5]. For $\epsilon \in \mathbb{R}((G^{>0}))$, $\exp(\epsilon) := \sum \epsilon^i/i!$ is well-defined and satisfies $\exp(x + y) = \exp(x)\exp(y)$. The exponential is defined for a series $y(0) + \epsilon$ in the valuation ring $\mathbb{R}[[G^{\geq 0}]]$ via $\exp(y(0) + \epsilon) = \exp(y(0))\exp(\epsilon)$. Observe that $D(\epsilon) = \frac{D(\exp(\epsilon))}{\exp(\epsilon)}$, since D is a series derivation [5, Corollary 3.9]. Thus Corollary B holds for arbitrary value group G instead of just $G = \mathbb{Z}$.

Example 7. To make the preceding example more explicit, take G to be a *Hahn group* over some totally ordered set Φ, i.e. the lexicographical product of Φ copies of $\mathbb{R}$ restricted to elements with well-ordered support. (Recall that, by Hahn's embedding theorem in [3], the Hahn groups are universal domains for ordered abelian groups). We denote by $\mathbb{1}_\phi$ the element of G corresponding to 1 for ϕ and to 0 for the others elements of Φ. We consider the following cases. Suppose that Φ carries an order preserving map σ into itself which is a right-shift, i.e. $\sigma(\phi) > \phi$. We define D as follows:

- set $D(t^{\mathbb{1}_\phi}) := t^{\mathbb{1}_\phi - \mathbb{1}_{\sigma(\phi)}}$;
- for any $g = \sum_\phi g_\phi \mathbb{1}_\phi$, set $D(t^g) := \sum_\phi g_\phi t^{g - \mathbb{1}_{\sigma(\phi)}}$;
- for any $a = \sum_g a_g t^g$, set $D(a) := \sum_g a_g D(t^g)$.

Suppose that Φ is isomorphic to a subset of $\mathbb{R}$. We define D as follows:

- take f to be an embedding of Φ into $\mathbb{R}_{>1}$;
- consider an increasing sequence $\phi_n < \phi_{n+1}$ cofinal in Φ, with $\phi_0 = \inf(\Phi)$, which is either infinite if Φ has no greatest element, or terminating at $\phi_{n_0} = \phi_M$ for some $n_0 \in \mathbb{N}$ if $\phi_M = \max \Phi$;
- consider the corresponding partition of Φ made of sub-intervals of the form $S_n := [\phi_n, \phi_{n+1})$ (with possibly $S_{n_0} := \{\phi_{n_0}\}$);
- for any $\phi \in \Phi$, there is n such that $\phi \in S_n$, then set $D(t^{\mathbb{1}_\phi}) := t^{\mathbb{1}_\phi - f(\phi)\mathbb{1}_{\phi_{n+1}}}$, with possibly $D(t^{\mathbb{1}_{\phi_{n_0}}}) := t^{\mathbb{1}_{\phi_{n_0}}}$;
- for any $g = \sum_\phi g_\phi \mathbb{1}_\phi$, set $D(t^g) := \sum_\phi g_\phi t^{g - f(\phi)\mathbb{1}_{\phi_{n+1}}}$;
- for any $a = \sum_g a_g t^g$, set $D(a) := \sum_g a_g D(t^g)$.

By [9, Proposition 5.2] each of these two constructions of D define a series derivation on the corresponding field $\mathbb{R}((G))$ making it into a differential valued field with exponential for the series in the valuation ring.

Example 8. The field of Logarithmic-Exponential (LE) series is a differential valued exponential field [2]. Moreover, as it is the increasing union of power series $\mathbb{R}((G_n))$ it is a truncation closed subfield of $\mathbb{R}((G_\omega))$ where $G_\omega := \cup G_n$. So Corollary 3 applies to LE-series $y_1, \cdots, y_n$ such that $y_1 - y_1(0), ..., y_n - y_n(0)$ are $\mathbb{Q}$-linearly independent. This generalizes Ax's result Corollary B to Laurent series that are not necessarily in the valuation ring.

Example 9. The fields of Exponential-Logarithmic series $\mathrm{EL}(\sigma)$ are differential valued exponential fields [9, Section 5.3 (2) and Theorem 6.2]. They are truncation closed, so again Corollary 3 applies to $y_1, \cdots, y_n \in \mathrm{EL}(\sigma)$ such that $y_1 - y_1(0), ..., y_n - y_n(0)$ are $\mathbb{Q}$-linearly independent. More explicitely, if we consider G as the Hahn group over some totally ordered set Φ endowed with a right-shift automorphism σ, the construction given in [9, Section 5.3 (2)] is as follows:

- for any $\phi \in \Phi$, set $\log(t^{\mathbb{1}_\phi}) := t^{-\mathbb{1}_{\sigma(\phi)}}$ and $D(t^{\mathbb{1}_\phi}) := t^{\mathbb{1}_\phi - \sum_n \mathbb{1}_{\sigma^n(\phi)}}$;
- for any $g = \sum_\phi g_\phi \mathbb{1}_\phi$, set $\log(t^g) := \sum_\phi g_\phi t^{-\mathbb{1}_{\sigma(\phi)}}$ and $D(t^g) := \sum_\phi g_\phi t^{g - \sum_n \mathbb{1}_{\sigma^n(\phi)}}$;
- for any $a := \sum_g a_g t^g = a_{g_0} t^{g_0}(1+\epsilon)$ with $a_{g_0} > 0$, set $\log(a) = \log(a_0) + \log(t^{g_0}) + \sum_{n \geq 1} \epsilon^n / n$ and $D(a) := \sum_g a_g D(t^g)$.

D is a series derivation making $\mathrm{EL}(\sigma)$ into a differential valued exponential field.

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