Valuation theory of exponential Hardy fields*

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Dedicated to Murray Marshall on the occasion of his 60th birthday

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1 Introduction

In this paper, we analyze the structure of the Hardy fields associated with o-minimal expansions of the reals with exponential function. In fact, we work in the following more general setting. We take T to be the theory of a polynomially bounded o-minimal expansion $\mathcal{P}$ of the ordered field of real numbers by a set $\mathcal{F}_T$ of real-valued functions. We assume that the language of T contains a symbol for every 0-definable function. Further, we assume that T defines the restricted exponential and logarithmic functions (cf. [D–M–M1]). Then also $T(\exp)$ is o-minimal (cf. [D–S2]). Here, $T(\exp)$ denotes the theory of the expansion $(\mathcal{P}, \exp)$ where exp is the un-restricted real exponential function. Finally, we take any model $\mathcal{R}$ of $T(\exp)$ which contains $(\mathbb{R}, +, \cdot, <, \mathcal{F}_T, \exp)$ as a substructure. Then we consider the Hardy field $H(\mathcal{R})$ (see Section 2.2 for the definition) as a field equipped with convex valuations. Theorem B of [D–S2] tells us that $T(\exp)$ admits quantifier elimination and a universal axiomatization in the language augmented by log. This implies that $H(\mathcal{R})$ is equal to the closure of its subfield $\mathcal{R}(x)$ under $\mathcal{F}_T$, exp and its inverse log; here, x denotes the germ of the identity function (cf. [D–M–M1], §5; the arguments also hold in the case where $\mathcal{R}$ is a non-archimedean model).

We shall analyze the valuation theoretical structure of this closure by explicitly showing how it can be built up from $\mathcal{R}(x)$ (cf. Section 3.3). Our construction method yields the following result (see Section 3.4 for definitions):

Theorem 1.1 *Every model $\mathcal{R}$ as chosen above is levelled.*

This implies that $T(\exp)$ *has levels with parameters*, in the sense of [M–M], and is exponentially bounded (cf. Theorem 3.11). We can determine the level of a function explicitly: it is the difference of two numbers which come up naturally in our construction method.

In Section 3.5 we use our main structure theorem (Theorem 3.11) to deduce:

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Theorem 1.2 *Suppose that for all $r \in \mathbb{R}$, $\mathcal{F}_T$ contains the power function*

$$\begin{array}{rccc} P_r : & (0,\infty) & \longrightarrow & \mathbb{R} \\ & x & \mapsto & x^r \, . \end{array}$$

Let $\mathcal{R}_T$ denote the reduct of $\mathcal{R}$ to the language of T. Then the Hardy field $H(\mathcal{R}_T)$ is maximal among the Hardy subfields of $H(\mathcal{R})$ associated with polynomially bounded reducts of $\mathcal{R}$.

L. v. d. Dries conjectured that

$$\mathbb{R}_{\mathrm{an,powers}} = (\mathbb{R}, +, \cdot, 0, 1, <, \mathcal{F}_{\mathrm{an}}, \{P_r \mid r \in \mathbb{R}\}) \, ,$$

the expansion of the ordered field of real numbers by the set $\mathcal{F}_{\mathrm{an}}$ of restricted analytic functions and the power functions P_r, is a *maximal* polynomially bounded reduct of

$$\mathbb{R}_{\mathrm{an,exp}} = (\mathbb{R}, +, \cdot, 0, 1, <, \mathcal{F}_{\mathrm{an}}, \exp\}) \, ,$$

At least on the level of Hardy fields, this is true: since the elementary theory of $\mathbb{R}_{\mathrm{an,powers}}$ is polynomially bounded and o-minimal and the power functions are definable in $\mathbb{R}_{\mathrm{an,exp}}$, the foregoing theorem shows (cf. Theorem 3.16 for a more general result):

$H(\mathbb{R}_{\mathrm{an,powers}})$ *is maximal among the Hardy subfields of* $H(\mathbb{R}_{\mathrm{an,exp}})$ *associated with polynomially bounded reducts of* $\mathbb{R}_{\mathrm{an,exp}}$.

In Sections 4 and 5 we answer a question raised by A. Macintyre. In the paper [D–M–M2], the authors give an explicit construction of a nonarchimedean model of the theory of the reals with restricted analytic functions and exponentiation, called the logarithmic exponential power series field. They use the results of [R–M] about truncation-closed embeddings in generalized power series fields to answer a problem raised by Hardy, and to show that certain functions, including the Gamma-function and the Riemann Zeta-function, cannot be defined using exponential function, logarithm and restricted analytic functions. Macintyre asked whether the results of [D–M–M2] can be deduced by a "more invariant" version of truncation. Indeed, we establish the results of [D–M–M2] without using embeddings in the logarithmic exponential power series field. We replace truncation results by an intrinsic property of the Hardy field of the expansion $\mathbb{R}_{\mathrm{an,exp}}$ of the reals by restricted analytic functions and the exponential function. This property is expressed by structure theorems for the residue fields of *arbitrary* convex valuations. It is invariant because it does not depend on an embedding in logarithmic exponential power series fields. Note that there is an abundance of convex valuation rings that are not $T(\exp)$-convex (cf. Theorem 3.11). For these, the methods of [D–L] are not applicable.

It is well known that the residue field of a real closed ordered field K is (up to order preserving isomorphism) a real closed ordered subfield of K. Now the question arises: if K has more structure, how much of it can be preserved on the embedded residue field? Take $K = H(\mathcal{R})$ and w a convex valuation which is trivial on $\mathcal{R}$. If $\mathcal{O}_w$ is not $T(\exp)$-convex, then the residue field will not be closed under exp. This problem can be approached as follows. Almost like building up $H(\mathcal{R}) = LE_{\mathcal{R},\mathcal{F}_T}(x)$ with our construction method, we can build up a subfield $LE^w_{\mathcal{R},\mathcal{F}_T}(x)$ of $\mathcal{O}_w$ by starting with any subfield of $\mathcal{O}_w$ which properly contains $\mathcal{R}$, and closing under the same functions as before, except for exp. For the function exp we apply the method only as long as it does not produce elements outside of $\mathcal{O}_w$. See Theorem 4.7.

Hardy conjectured that the compositional inverse of $(\log x)(\log\log x)$ is not asymptotic to an element of the Hardy field LE. This is defined to be the smallest subfield of $H(\mathbb{R}_{\text{an,exp}})$ which is real closed, exp- and log-closed and contains $\mathbb{R}(x)$. It coincides with the field of the germs of all compositions of semialgebraic functions, exp and log. For our intrinsic solution of the Hardy problem, we also need to know the residue fields of convex valuations w on the Hardy field LE. But this field is not definably closed in $H(\mathbb{R}_{\text{an,exp}})$. In fact, the compositional inverse of $(\log x)(\log\log x)$ is 0-definable over LE, but not an element of it. Hence, LE is not of the form $LE_{\mathcal{R},\mathcal{F}_T}(x)$. But from its definition we see that it is the closure of $\mathbb{R}(x)$ under a subset of $\mathcal{F}_T$ (for instance, the set of semi-algebraic functions), exp and log. Therefore, for $\mathcal{F} \subseteq \mathcal{F}_T$ we explicitly construct the smallest field which is real closed and closed under $\mathcal{F}$, exp and log; we denote it by $LE_{\mathcal{R},\mathcal{F}}(x)$. Similarly, we construct corresponding subfields $LE^w_{\mathcal{R},\mathcal{F}}(x)$ of $\mathcal{O}_w$. The only necessary condition on $\mathcal{F}$ for our construction is that it contains the restricted exp and log.

Under certain additional conditions on $\mathcal{F}$ (see Section 4), Theorem 4.7 then tells us that $LE^w_{\mathcal{R},\mathcal{F}}(x)$ coincides with the residue field $LE_{\mathcal{R},\mathcal{F}}(x)w$. The conditions are fulfilled by any set $\mathcal{F}$ of restricted analytic functions which is closed under partial derivations and contains the restricted exp and log. Furthermore, $H(\mathbb{R}_{\text{an,exp}})$ is equal to $LE_{\mathbb{R},\mathcal{F}_{\text{an}}}(x)$ (cf. Section 5 of [D–M–M1]). If $\mathcal{F}_{LE}$ is the smallest subset of $\mathcal{F}_{\text{an}}$ which contains the restricted exp and log and is closed under partial derivations, then LE is equal to $LE_{\mathbb{R},\mathcal{F}_{LE}}(x)$ (cf. Section 3 of [D–M–M2]). Thus, Theorem 4.7 gives us information about the residue fields of $H(\mathbb{R}_{\text{an,exp}})$ and of LE. An important point for our solution of the Hardy problem is that by our construction we obtain the residue field LEw inside of $H(\mathbb{R}_{\text{an,exp}})w$ (cf. Corollary 4.8). This is clear since if $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_T$, then $LE^w_{\mathcal{R},\mathcal{F}_1}(x) \subseteq LE^w_{\mathcal{R},\mathcal{F}_2}(x)$.

It would be interesting to verify that our condition (COMP) given in Section 4 is satisfied by sets of Gevrey functions (as it is the case for sets of restricted analytic functions), or by sets of convergent generalized power series for which the exponents of each variable form a sequence cofinal in $\mathbb{R}$ (indexed by the natural numbers), cf. [D–S]. Although the condition on the exponents is quite restrictive, it holds for the presently known applications of interest. In particular, the function $\zeta(-\log x) = \sum_{n=1}^{\infty} x^{\log n}$ on $[0, e^{-2}]$ (with ζ the Riemann zeta function) satisfies the condition. It is not known whether the results on residue fields can be established without the restriction on the exponents.

In Section 6, we introduce an intrinsic form of power series expansions for the elements of $LE_{\mathcal{R},\mathcal{F}}(x)$. For this, we use monomials (which are obtained from elements in the image of an arbitrary cross-section by multiplication with reals) together with coefficients from significant residue fields $LE_{\mathcal{R},\mathcal{F}}(x)w$. From such an expansion of a function $h \in H(\mathbb{R}_{\text{an,exp}})$, one can define the **principal part** of h, which turns out to carry information about the asymptotic behaviour of the function $\exp h(x)$ (Theorem 6.4). This puts the particular solution of the Hardy problem in a more general framework (Corollary 6.5).

2 Some preliminaries

If (K, w) is a valued field, then we write wa for the value of $a \in K$ and wK for its value group $\{wa \mid 0 \neq a \in K\}$. Further, we write aw for the residue of a, and Kw for the residue field. The valuation ring is denoted by $\mathcal{O}_w$. For generalities on valuation theory, see [R].

2.1 Convex valuations

A valuation w on an ordered field K is called **convex** if $\mathcal{O}_w$ is convex. The convex valuation rings of an ordered field are linearly ordered by inclusion. If $\mathcal{O}_w \subsetneqq \mathcal{O}_{w'}$ then w is said to be **finer** than w'. There is always a finest convex valuation, called the **natural valuation**. It is characterized by the fact that its residue field is archimedean. A valuation w on an ordered field is convex if and only if the natural valuation is finer or equal to w. **Throughout this paper,** v **will always denote the natural valuation,** unless stated otherwise.

If a, b are elements of an ordered group or an ordered field, then we write $a \ll b < 0$ if $a < b < 0$ and $\forall n \in \mathbb{N} : a < nb$. Similarly, $a \gg b > 0$ if $a > b > 0$ and $\forall n \in \mathbb{N} : a > nb$. We set $|a| := \max\{a, -a\}$. Then the natural valuation is characterized by:

$$va < vb \;\Leftrightarrow\; |a| \gg |b| \; . \tag{1}$$

Note that if $\mathbb{R} \subset K$ and $a \in K$ with $va = 0$, then there is some $r \in \mathbb{R}$ such that $v(a - r) > 0$. Further, $wr = 0$ for every $r \in \mathbb{R}$ and every convex valuation w.

Lemma 2.1 *Let v, w be arbitrary valuations on some field K. Suppose that v is finer than w. Then for all $a, b \in K$,*

$$va \leq vb \;\Rightarrow\; wa \leq wb \; . \tag{2}$$

In particular, $wa > 0 \Rightarrow va > 0$. Further, $H_w := \{vz \mid z \in K \wedge wz = 0\}$ is a convex subgroup of the value group vK of v. We have that $vz \in H_w \Leftrightarrow z \in \mathcal{O}_w^\times$. There is a canonical isomorphism $wK \simeq vK/H_w$. Conversely, every convex subgroup of vK is of the form H_w for some valuation w such that v finer or equal to w.

The valuation v of K induces a valuation v/w on Kw. There are canonical isomorphisms $v/w(Kw) \simeq H_w$ and $(Kw)v/w \simeq Kv$. If Kw is embedded in $\mathcal{O}_w$ such that the restriction of the residue map is the identity on Kw, then $v/w = v|_{Kw}$ (up to equivalence). Writing v instead of $v|_{Kw}$, we then have that $v(Kw) = H_w$ and $(Kw)v = Kv$.

We will call H_w the **convex subgroup associated with** w and w the **valuation associated with** H_w. Since the isomorphism is canonical, we will write $wK = vK/H_w$.

The order type of the chain of nontrivial convex subgroups of an ordered abelian group G is called the **rank** of G. If finite, then the rank is not bigger than the maximal number of rationally independent elements in G. In particular, G has finite rank if it is finitely generated or equivalently, if its divisible hull is a $\mathbb{Q}$-vector space of finite dimension.

From (1) and (2) it follows that for every convex valuation w,

$$|a| \leq |b| \;\Rightarrow\; wa \geq wb \; . \tag{3}$$

Lemma 2.2 *Let w be any valuation on $K(x_i \mid i \in I_1 \cup I_2)$ such that the values wx_i, $i \in I_1$, are rationally independent over wK, and the residues x_iw, $i \in I_2$, are algebraically independent over Kw. Then the elements x_i, $i \in I_1 \cup I_2$ are algebraically independent over K. Moreover,*

$$wK(x_i \mid i \in I_1 \cup I_2) \;=\; wK \oplus \bigoplus_{i\in I_1} \mathbb{Z}wx_i \quad \textit{and} \quad K(x_i \mid i \in I_1 \cup I_2)w \;=\; Kw(x_iw \mid i \in I_2)\;. \tag{4}$$

For the proof, see [B], chapter VI, §10.3, Theorem 1.

Corollary 2.3 *Suppose that $\mathbb{R}(x_i \mid i \in I)$ is an ordered field such that the values vx_i, $i \in I$ are rationally independent. Let w be a convex valuation on $\mathbb{R}(x_i \mid i \in I)$. Assume that there is a subset $I_w \subset I$ such that $wx_i = 0$ for all $i \in I_w$ and that the values wx_i, $i \in I \setminus I_w$ are rationally independent. Then*

$$w\mathbb{R}(x_i \mid i \in I) \;=\; \bigoplus_{i\in I\setminus I_w} \mathbb{Z}wx_i \quad \textit{and} \quad \mathbb{R}(x_i \mid i \in I)w \;=\; \mathbb{R}(x_i \mid i \in I_w)\;.$$

Proof: For $i \in I_w$, $wx_i = 0$ implies that $vx_i \in H_w$. By the foregoing lemma, $v\mathbb{R}(x_i \mid i \in I_w) = \bigoplus_{i\in I_w} \mathbb{Z}vx_i \subset H_w$. This proves that w is trivial on $\mathbb{R}(x_i \mid i \in I_w)$. So we can assume that the residue map is the identity on $\mathbb{R}(x_i \mid i \in I_w)$. Now apply the foregoing lemma with $K = \mathbb{R}(x_i \mid i \in I_w)$ (then $Kw = K$), $I_1 = I \setminus I_w$ and $I_2 = \emptyset$. □

A sequence of elements $a_\nu \in K$, $\nu < \lambda$ (λ some limit ordinal), is called a **pseudo Cauchy sequence** in (K, w) if $w(a_\rho - a_\sigma) < w(a_\sigma - a_\tau)$ for all ρ, σ, τ with $\rho < \sigma < \tau < \lambda$. It follows from the ultrametric triangle law that $w(a_\nu - a_\tau) = w(a_\nu - a_{\nu+1})$ whenever $\nu < \tau < \lambda$. The element a is called a (pseudo) limit of this pseudo Cauchy sequence if $w(a_\nu - a) = w(a_\nu - a_{\nu+1})$ for all $\nu < \lambda$. In general, there may be several distinct limits:

Lemma 2.4 *Let a be a limit of $(a_\nu)_{\nu<\lambda}$. Then b is also a limit of $(a_\nu)_{\nu<\lambda}$ if and only if $w(a - b) > w(a_\nu - a_{\nu+1})$ for all $\nu < \lambda$.*

An extension $(K, w) \subset (L, w)$ of valued fields is called **immediate** if the canonical embedding of wK in wL and the canonical embedding of Kw in Lw are surjective (we then write $wK = wL$ and $Kw = Lw$). The henselization of a valued field is an immediate extension.

Lemma 2.5 *Assume that $(K, w) \subset (L, w)$ is immediate and that $a \in L \setminus K$. Then there is a pseudo Cauchy sequence in (K, w) with limit a, but not having a limit in K.*

The next lemma follows from the Lemma of Ostrowski (cf. [R]) and the results of Kaplansky's important paper [KA]:

Lemma 2.6 *Let K be any real closed field and w a convex valuation on K. Assume that $(a_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence in (K, w), not having a limit in K. Assume further that in some extension of (K, w), there exists a limit a. Then the extension of w to $K(a)$ is uniquely determined and immediate.*

If $(K_1, w) \subset (K_2, w)$ is an immediate algebraic extension of ordered fields with convex valuation w, then their henselizations (in a fixed henselian extension field) are equal.

If the values $w(a_\nu - a_{\nu+1})$ are cofinal in wK, then $(a_\nu)_{\nu<\lambda}$ is called a **Cauchy sequence** in (K,w). Lemma 2.4 shows that if this sequence has a limit in K, then this limit is uniquely determined. Indeed, if $a,b \in K$ are limits, then $w(a-b) > wK$, that is, $w(a-b) = \infty$, or in other words, $a = b$. All elements in the completion of a valued field are limits of Cauchy sequences (and in particular, the completion is an immediate extension). Conversely:

Lemma 2.7 *Let the situation be as in Lemma 2.6, with $(a_\nu)_{\nu<\lambda}$ a Cauchy sequence. Then there is a unique embedding of $(K(a),w)$ over K in the completion of (K,w).*

Note that if wK is archimedean, then it follows from Newton's method together with this lemma that the henselization of (K,w) is embeddable in the completion of (K,w). If w and v are arbitrary valuations such that v is finer than w and $Kw \subset K$, then (K,v) is henselian if and only if (K,w) and (Kw,v) are henselian (cf. [R]). From these facts, one obtains:

Lemma 2.8 *Let K be an ordered field with convex valuation w. Suppose that $Kw \subset K$ and that (Kw,v) is henselian. Then the henselization of K with respect to v is equal to the henselization of K with respect to w. If in addition wK is archimedean, this henselization is embeddable in the completion of (K,w).*

If K is a formally real field, then K^{r} will denote its real closure. For the proof of the next lemma, see [P].

Lemma 2.9 *Let K be an ordered field with convex valuation w. Then K is real closed if and only if (K,w) is henselian, wK is divisible and Kw is real closed. Further, $wK^{\mathrm{r}} = \mathbb{Q} \otimes_{\mathbb{Z}} wK$ (the divisible hull of wK), and $K^{\mathrm{r}}w = (Kw)^{\mathrm{r}}$. If wK is divisible and Kw is real closed, then the real closure of K is equal to the henselization of K with respect to w (and embeddable in the completion of (K,w) if wK is archimedean).*

If x is a positive element in the real closed field K, then it has a unique positive k-th root, for every $k \in \mathbb{N}$. So if K contains the real closure of a field $\mathbb{R}(x_i \mid i \in I)$, with all x_i positive, then $x_i^q \in K$ for all $i \in I$ and all $q \in \mathbb{Q}$. This can be used to show that every real closed field K, with its natural (or any convex) valuation v, admits a **cross-section**, i.e., an embedding π of the group vK in the multiplicative group $K^\times$ such that $v\pi\alpha = \alpha$ for all $\alpha \in vK$. Indeed, take any maximal set $\mathcal{X} = \{x_i \mid i \in I\} \subset K$ such that the values vx_i are rationally independent. By the maximality of the set, together with Lemma 2.9, it follows that vK is the divisible hull of $v\mathbb{R}(x_i \mid i \in I) = \bigoplus_{i\in I} \mathbb{Z}vx_i$. For every $\alpha \in v\mathbb{R}(x_i \mid i \in I)$ there is a unique element x of the multiplicative group $\langle \mathcal{X} \rangle$ generated by the x_i, such that $vx = \alpha$. Consequently, there is a unique cross-section π of (K,v) whose image contains $\mathcal{X}$, and this image πvK is the divisible hull $\widetilde{\langle \mathcal{X} \rangle} = \{\prod_{i\in I_0} x_i^{q_i} \mid I_0 \subset I \text{ finite}, q_i \in \mathbb{Q}\}$ of $\langle \mathcal{X} \rangle$. If we have fixed a cross-section π, or a set $\mathcal{X}$ and take π to be the associated cross-section, then we call $\mathbb{R}^\times \cdot \pi vK$ the set of **monomials** of K. Hence the monomials are the elements of the form

$$d = r \prod_{i\in I_0} x_i^{q_i} \text{ with } 0 \neq r \in \mathbb{R},\ I_0 \subset I \text{ finite, and } q_i \in \mathbb{Q} \text{ for every } i \in I_0\,.$$

For the rest of this section, we will assume that $(M, \exp)$ **is a model of the elementary theory of** $(\mathbb{R}, +, \cdot, 0, 1, <, \exp)$ **such that** $\mathbb{R} \subset M$ **and the restriction of** $\exp$ **to** $\mathbb{R}$ **is the natural exponential** $\exp$ **on** $\mathbb{R}$**. Further, we take** w **to be any convex valuation on** M**.** Then the exponential exp of M is an order preserving isomorphism from the additive group of M onto its multiplicative group of positive elements. Its inverse is the logarithm log; it is order preserving and defined for all positive elements. Consequently, if $z \in M$ is positive infinite, that is, $z > \mathbb{R}$, then $\log z > \log(\{r \in \mathbb{R} \mid r > 0\}) = \mathbb{R}$. In other words,

$$vz < 0 \,\wedge\, z > 0 \;\Rightarrow\; v \log z < 0 \,\wedge\, \log z > 0 \;. \tag{5}$$

Further, exp satisfies the Taylor axiom scheme:

(TA) $\qquad |z| \leq 1 \;\Rightarrow\; |\exp z - \sum_{n=0}^{m} \frac{z^n}{n!}| < |z^m| \qquad (m \in \mathbb{N})\,.$

In order to derive a valuation theoretical property from this axiom, we need the following simple lemma:

Lemma 2.10 *Let* K *be an ordered field and* w *a convex valuation on* K*. Suppose that* $h \in K$ *satisfies*

$$\left| h \,-\, \sum_{k=0}^{m} s_k z_k \right| \;<\; |s'_m z_m| \quad \textit{for all } m \in \mathbb{N}, \tag{6}$$

where $s_k, s'_k \in \mathbb{R} \setminus \{0\}$*, and* $z_k \in K$ *are such that* $wz_{k+1} > wz_k$ *. Write*

$$S_m \;:=\; \sum_{k=0}^{m} s_k z_k \;.$$

Then $(S_m)_{m \in \mathbb{N}}$ *is a pseudo Cauchy sequence in* (K, w)*. Further,*

$$w(h - S_m) \;=\; wz_{m+1} \;=\; w(S_{m+1} - S_m) \;, \tag{7}$$

which shows that h *is a limit of this sequence.*

Proof: Recall that $ws = 0$ for $0 \neq s \in \mathbb{R}$, and that $w|a| = wa$ for every a in K. By (6) and (3), we have that

$$\begin{aligned} w(h - S_m - s_{m+1}z_{m+1} - s_{m+2}z_{m+2}) \;&=\; w(h - S_{m+2}) \;\geq\; ws'_{m+2}z_{m+2} \;=\; wz_{m+2} \\ &>\; wz_{m+1} \;=\; ws_{m+1}z_{m+1} \;. \end{aligned}$$

By the ultrametric triangle law,

$$w(s_{m+1}z_{m+1} + s_{m+2}z_{m+2}) \;=\; \min\{ws_{m+1}z_{m+1}, ws_{m+2}z_{m+2}\} \;=\; ws_{m+1}z_{m+1} \;.$$

Hence, again by the ultrametric triangle law,

$$\begin{aligned} w(h - S_m) \;&=\; \min\{w(h - S_m - s_{m+1}z_{m+1} - s_{m+2}z_{m+2}), w(s_{m+1}z_{m+1} + s_{m+2}z_{m+2})\} \\ &=\; ws_{m+1}z_{m+1} \;=\; w(S_{m+1} - S_m) \;. \end{aligned}$$

□

Lemma 2.11 *For every $z \in M$,*

$$wz > 0 \;\Rightarrow\; w\exp z = 0 \,\wedge\, w(\exp z - 1) = wz \tag{8}$$
$$vz = 0 \;\Rightarrow\; v\exp z = 0\,. \tag{9}$$

Proof: By Lemma 2.1, $wz > 0$ implies $vz > 0$, that is, z is infinitesimal. In particular, $|z| < 1$, and (TA) holds. Applying (7) of Lemma 2.10 with $m = 1$ and $z_m = z^m$, we find that $w(\exp z - 1 - z) = wz^2 = 2wz > wz$. By the ultrametric triangle law, this implies that $w\exp z = w(1+z) = w1 = 0$ and $w(\exp z - 1) = wz$. This proves (8).

Now assume that $vz = 0$. Then there is some $r \in \mathbb{R} \subset M$ such that $v(z - r) > 0$. We have that $\exp r \in \mathbb{R}$, hence $v\exp r = 0$. By (8) with $w = v$, $v\exp(z - r) = 0$. Thus, $v\exp z = v\exp r\exp(z - r) = v\exp r + v\exp(z - r) = 0$. This proves (9). $\square$

With M as before, exp also satisfies the following growth axiom scheme:

(GA) $\qquad z > m^2 \implies \exp z > z^m \qquad\qquad (m \in \mathbb{N})\,.$

From this, we derive:

Lemma 2.12 *For every $z \in M$,*

$$wz < 0 \,\wedge\, z > 0 \;\Rightarrow\; w\exp z \ll wz \ll w\log z < 0 \tag{10}$$
$$wz = 0 \,\wedge\, z > 0 \;\Rightarrow\; w\log z \geq 0 \tag{11}$$
$$vz \geq 0 \;\Leftrightarrow\; v\exp z = 0\,. \tag{12}$$

Proof: If $wz < 0$ and $z > 0$, then $z > \mathbb{R}$ and thus, $z > m^2$ for every $m \in \mathbb{N}$. So by (GA), $\exp z > z^m > 0$ for all m. Hence by (3), $w\exp z \leq mwz$ for all m, i.e., $w\exp z \ll wz < 0$. In view of (5), we can replace z by $\log z$ to get that $wz \ll w\log z < 0$. This proves (10).

Now assume that $wz = 0$ and $z > 0$. If $vz < 0$, then by (10), $vz < v\log z < 0$. If $vz > 0$, then $vz^{-1} < 0$ and by (10), $vz^{-1} < v\log z^{-1} = v(-\log z) = v\log z < 0$. In both cases, it follows from Lemma 2.1 that $0 = wz = wz^{-1} \leq w\log z \leq 0$, i.e., $w\log z = 0$. Now let $vz = 0$. If $v\log z < 0$, then by (10), $vz = v\exp\log z < 0$ if $\log z > 0$, and $vz = -vz^{-1} = -v\exp(-\log z) > 0$ if $\log z < 0$. Hence, $v\log z \geq 0$, and again by Lemma 2.1, $w\log z \geq 0$. This proves (11).

Implication "$\Rightarrow$" of (12) follows from (8) with $w = v$, together with (9). The converse implication follows from (11), where we take $w = v$ and replace z by $\exp z$. $\square$

For positive infinite elements $z \in M$ and $m \in \mathbb{Z}$, we set $\log_0 z = z$, $\log_{m+1} z = \log(\log_m z)$ if $m \geq 0$, and and $\log_{m-1} z = \exp(\log_m z)$ if $m \leq 0$; note that every $\log_m z$ is again positive infinite. Similarly, we define $\exp_m z$ for every $z \in M$.

Corollary 2.13 *Assume that $\mathcal{R}$ is an exp-closed subfield of M. If $x \in M$ such that $wx < w\mathcal{R}$ and $x > 0$, then for $m > 1$,*

$$wx \ll w\log x \ll \ldots \ll w\log_m x \ll \ldots < w\mathcal{R}\,. \tag{13}$$

Proof: The part "$wx \ll w\log x \ll \ldots \ll w\log_m x$" follows from (10) by induction on m. Now suppose that there is a positive integer m and some $\alpha \in w\mathcal{R}$ such that $\alpha \le w\log_m x$. Replacing α by $2\alpha \in w\mathcal{R}$ if necessary, we may assume that $\alpha < w\log_m x$. Take a positive element $a \in \mathcal{R}$ such that $wa = \alpha$. Then by virtue of (3), $0 < \log_m x < a$. It follows that $x < \exp_m a$, which implies that $wx \ge w\exp_m a \in w\mathcal{R}$. This proves that if $wx < w\mathcal{R}$ then $w\log_m x < w\mathcal{R}$ for all m. □

The valuation v is a homomorphism from the multiplicative group $M^{>0}$ of positive elements onto the value group vM. Its kernel is $\mathcal{U}^{>0} = \{z \in M \mid vz = 0 \wedge z > 0\}$, the subgroup of positive units. So v induces an isomorphism $M^{>0}/\mathcal{U}^{>0} \simeq vM$. (3) shows that it is order reversing. The exponential exp is an order preserving isomorphism from the additive group of M onto the multiplicative group $M^{>0}$. By (12), the preimage of $\mathcal{U}^{>0}$ under exp is precisely $\mathcal{O}_v$. Hence,

Lemma 2.14 *The map $z \mapsto v\exp(-z)$ induces an order preserving isomorphism $M/\mathcal{O}_v \simeq vM$ of ordered abelian groups. In particular, if $va < 0$, then the map $\mathbb{R} \ni r \mapsto v\exp(-ra) \in vM$ is order preserving.*

If the elements z_j, $j \in J$, are rationally independent over $\mathcal{O}_v$ in the additive group of M, then the values $v\exp z_j$, $j \in J$, are rationally independent in vM.

For further details on the valuation theory of exponential fields, see [KS] and [K–K1].

2.2 Hardy fields

Let us recall some basic facts about Hardy fields. Initially, they were only defined as fields consisting of germs at ∞ of real-valued functions. But we will work with a more general definition that has also been used by other authors lately. Assume that T is the theory of any o-minimal expansion of the ordered field of real numbers by real-valued functions, and that $\mathcal{R}$ is a model of T. The Hardy field of $\mathcal{R}$, denoted by $H(\mathcal{R})$, is the set of germs at ∞ of unary $\mathcal{R}$-definable functions $f : \mathcal{R} \to \mathcal{R}$. Then $H(\mathcal{R})$ is an ordered differential field which contains $\mathcal{R}$. Let $x \in H(\mathcal{R})$ be the germ of the identity function. Then $H(\mathcal{R})$ is the closure of $\mathcal{R}(x)$ under all 0-definable functions of $\mathcal{R}$.

By $v_{\mathcal{R}}$ we will denote the finest convex valuation on $H(\mathcal{R})$ which is trivial on $\mathcal{R}$. Then $v_{\mathcal{R}}a < 0$ if and only if $a > \mathcal{R}$. If f, g are non-zero unary $\mathcal{R}$-definable functions on $\mathcal{R}$, then we will denote their germs in $H(\mathcal{R})$ by the same letters. With this convention, the following holds:

$$v_{\mathcal{R}}f \;=\; v_{\mathcal{R}}g \iff \lim_{x\to\infty}\frac{f(x)}{g(x)} \text{ is a non-zero constant in } \mathcal{R}\,. \tag{14}$$

(Note that "$x \to \infty$" means letting x outgrows every element of $\mathcal{R}$.) The functions f and g are **asymptotic on** $\mathcal{R}$ if and only if this constant is 1, and we have:

$$v_{\mathcal{R}}(f-g) \;>\; v_{\mathcal{R}}g \iff f \text{ and } g \text{ are asymptotic on } \mathcal{R}\,, \tag{15}$$

or in other words,

$$v\left(\frac{f}{g}-1\right) \;>\; v\mathcal{R} \iff f \text{ and } g \text{ are asymptotic on } \mathcal{R}\,, \tag{16}$$

3 Closures of $\mathcal{R}(x)$ under $\mathcal{F}$, log and exp

General assumptions: Throughout this section, we will assume that T is the theory of a polynomially bounded o-minimal expansion $\mathcal{P}$ of the ordered field of real numbers by real-valued functions. Further, we assume that T defines the restricted exp and log. Then also $T(\exp)$ is o-minimal (cf. [D–S2]). Here, $T(\exp)$ denotes the theory of the expansion $(\mathcal{P}, \exp)$ where exp is the un-restricted real exponential function.

The archimedean field

$$\mathrm{Q} := \{r \in \mathbb{R} \mid \text{ the function } x \mapsto x^r : (0,\infty) \longrightarrow \mathbb{R} \text{ is 0-definable in } \mathcal{P}\}$$

is called the **field of exponents of** T.

We let $\mathcal{F}_T$ denote the set of function symbols in the language of T and assume that there is a function symbol in $\mathcal{F}_T$ for each 0-definable function of $\mathcal{P}$. This implies that T admits quantifier elimination and a universal axiomatization (cf. [D–L], §2). We let $\mathcal{F}$ denote any subset of $\mathcal{F}_T$.

Further, we assume that M is a model of T. (Later, we will assume that it is a model of $T(\exp)$, but we will not distinguish between this and its reduct to the language of T.) Suppose that the field K is a submodel (and hence elementary submodel) of M. Take $x_i \in M$, $i \in I$. By $K\langle x_i \mid i \in I\rangle$ we denote the 0-definable closure of $K \cup \{x_i \mid i \in I\}$ in M. By our assumption on the language of T, it is the closure of $K \cup \{x_i \mid i \in I\}$ under $\mathcal{F}_T$, that is, the smallest subfield of M containing $K \cup \{x_i \mid i \in I\}$ and closed under all functions which interpret the function symbols of $\mathcal{F}_T$ in M. Since T admits a universal axiomatization and $K\langle x_i \mid i \in I\rangle$ is a substructure of M, it is a model of T. Since T admits quantifier elimination, $K\langle x_i \mid i \in I\rangle$ is an elementary substructure of M.

For an arbitrary subfield $F \subset M$, the real closure F^{r} of F can be taken to lie in M since M is real closed. We denote by F^{h} the henselization of (F, v). It can be taken to lie in M since by Lemma 2.9, (M, v) is henselian.

We let $F^{\mathcal{F}}$ denote the smallest subfield of M which contains F and is $\mathcal{F}$-closed, that is, closed under all functions on M which are interpretations of function symbols in $\mathcal{F}$. Similarly, F^{Q} will denote the smallest subfield of M which contains F and is closed under the exponents from Q. Further, we let $F^{\mathrm{rQ}\mathcal{F}}$ denote the smallest real closed subfield of M which contains F and is $\mathcal{F}$-closed and closed under the exponents from Q; we will say that F is **rQ$\mathcal{F}$-closed** if $F = F^{\mathrm{rQ}\mathcal{F}}$. Analogously, we define $F^{\mathrm{h}\mathcal{F}}$ to be the smallest subfield of M which contains F and is $\mathcal{F}$-closed and henselian w.r.t. v. Note that $F^{\mathcal{F}} \subset F^{\mathrm{h}\mathcal{F}} \subset F^{\mathrm{rQ}\mathcal{F}}$.

If F is Q-closed, then for every convex valuation w, the value group wF is a Q-vector space with scalar multiplication defined by $qw(a) = w(a^q)$ for $q \in \mathrm{Q}$. If $\alpha \in wF$, then $\mathrm{Q}\alpha$ shall denote the Q-subvector space generated by α. As Q always contains $\mathbb{Q}$, we see that wF^{Q} is always divisible.

3.1 Value groups

The following property (Lemma 3.1) of polynomially bounded o-minimal expansions of the reals was proved in full generality in [D] (Lemma 5.4); see also Corollary 3.7 of [D–M–M1]. Note that in the case of a polynomially bounded expansion, every convex valuation w of a model is T-convex (cf. [D–L], §4).

Lemma 3.1 *Assume that $\mathcal{R}$ is a submodel of M. If $x \in M$ such that $wx \notin w\mathcal{R}$, then $w\mathcal{R}\langle x\rangle = w\mathcal{R} \oplus \mathbb{Q}wx$.*

Lemma 3.2 *Assume that $\mathcal{R}$ is a submodel of M. Take elements $x_i \in M$, $i \in I$, such that the values wx_i, $i \in I$, are $\mathbb{Q}$-linearly independent over $w\mathcal{R}$. Then*

$$w\mathcal{R}(x_i \mid i \in I)^{\mathrm{rQ}\mathcal{F}} = w\mathcal{R}(x_i \mid i \in I)^{\mathrm{Q}} = w\mathcal{R} \oplus \bigoplus_{i\in I} \mathbb{Q}wx_i \,. \tag{17}$$

Proof: Since every element of $\mathcal{R}(x_i \mid i \in I)^{\mathrm{rQ}\mathcal{F}}$ already lies in $\mathcal{R}(x_i \mid i \in I_0)^{\mathrm{rQ}\mathcal{F}}$ for a finite subset $I_0 \subseteq I$ and a similar assertion is true for the fields $\mathcal{R}(x_i \mid i \in I)^{\mathrm{Q}}$ and $\mathcal{R}\langle x_i \mid i \in I\rangle$, it suffices to prove our assertion for the case of I finite. We may write $I = \{1, \ldots, n\}$. By induction on n one shows that

$$v\mathcal{R}\langle x_1, \ldots, x_n\rangle \;=\; v\mathcal{R} \oplus \bigoplus_{i=1}^{n} \mathbb{Q}vx_i \,. \tag{18}$$

Since $\mathcal{R}\langle x_1, \ldots, x_n\rangle$ is $\mathrm{rQ}\mathcal{F}$-closed, we have that

$$\mathcal{R}(x_1, \ldots, x_n)^{\mathrm{Q}} \;\subseteq\; \mathcal{R}(x_1, \ldots, x_n)^{\mathrm{rQ}\mathcal{F}} \;\subseteq\; \mathcal{R}\langle x_1, \ldots, x_n\rangle \,.$$

As $w\mathcal{R}(x_1, \ldots, x_n)^{\mathrm{Q}}$ is a $\mathbb{Q}$-vector space and contains $wx_1, \ldots, wx_n$, we obtain that

$$\begin{aligned} w\mathcal{R} \oplus \bigoplus_{i=1}^{n} \mathbb{Q}wx_i \;\;&\subseteq\;\; w\mathcal{R}(x_1, \ldots, x_n)^{\mathrm{Q}} \;\subseteq\; w\mathcal{R}(x_1, \ldots, x_n)^{\mathrm{rQ}\mathcal{F}} \\ &\subseteq\;\; w\mathcal{R}\langle x_1, \ldots, x_n\rangle \;=\; w\mathcal{R} \oplus \bigoplus_{i=1}^{n} \mathbb{Q}wx_i \,, \end{aligned}$$

which shows that equality must hold everywhere. □

3.2 Linear independence of generating values

From now on, let M always be a model of $T(\exp)$, and $\mathcal{R}$ a submodel of M containing $(\mathbb{R}, +, \cdot, <, \mathcal{F}_T, \exp)$ as a substructure. We take $\mathcal{F}$ as before, but always assume in addition that $\mathcal{F}$ contains function symbols for the restricted exp and log. Hence, if a subfield F of M is $\mathcal{F}$-closed, then $\exp \varepsilon \in F$ and $\log(1+\varepsilon) \in F$ for every infinitesimal ε in F. Since $\mathbb{R} \subseteq \mathcal{R}$, we have that $\mathcal{R}v = \mathbb{R}$.

Note that in view of Theorem B of [D–S2], $\mathcal{R}$ is an elementary submodel of M, and $(\mathbb{R}, +, \cdot, <, \mathcal{F}_T, \exp)$ is an elementary submodel of both. However, we will not use this fact in our constructions.

For every subfield K of $\mathcal{O}_w$, its multiplicative group $K^\times$ is contained in the multiplicative group $\mathcal{O}_w^\times$ of all units of $\mathcal{O}_w$. We will say that K is **relatively exp-closed in** $\mathcal{O}_w^\times$ if $a \in K$ and $\exp(a) \in \mathcal{O}_w^\times$ implies that $\exp(a) \in K$. For example, $\mathbb{R}$ is relatively exp-closed in $\mathcal{O}_w^\times$ for every convex valuation w of M.

Lemma 3.3 *Let K be a log- and* r$\mathrm{Q}\mathcal{F}$*-closed subfield of M. Let w be a convex valuation of M. Assume that the residue field Kw is a subfield of $\mathcal{O}_w \cap K$, relatively exp-closed in $\mathcal{O}_w^\times$. Take any $a \in K$ such that $\exp a \notin K$. Then $w \exp a$ is* Q*-linearly independent over wK.*

Proof: Suppose that $w \exp a$ is not Q-linearly independent over wK. Since K is Q-closed, wK is a Q-vector space, and it follows that $w \exp a = wb \in wK$ for some positive $b \in K$. Then $w\frac{\exp a}{b} = 0$ and by Lemma 2.12, $w(a - \log b) = w \log(\frac{\exp a}{b}) \geq 0$. Since K is log-closed, $\log b \in K$. Hence, there is $c \in Kw$ such that $w(a - \log b - c) > 0$. By Lemma 2.11, this shows that $w\,\frac{\exp a}{b \exp c} = w \exp(a - \log b - c) = 0$. In particular, we find that $w \exp c = w\frac{\exp a}{b} = 0$, that is, $\exp c \in \mathcal{O}_w^\times$. By assumption on Kw, $\exp c \in Kw \subset K$.

By Lemma 2.1, $w(a - \log b - c) > 0$ yields that $v(a - \log b - c) > 0$. Therefore, $\exp(a - \log b - c) \in K^{\mathcal{F}} = K$, showing that $\exp a = \exp(a - \log b - c) \cdot b \cdot \exp c \in K$. We conclude: if $\exp a \notin K$, then $w \exp a$ is Q-linearly independent over wK. □

Lemma 3.4 *Assume that $K = \mathcal{R}(x_i \mid i \in I)^{\mathrm{rQ}\mathcal{F}} \subset M$ such that*
1) the values vx_i, $i \in I$, are Q*-linearly independent over $v\mathcal{R}$,*
2) $x_i > 0$ and $\log x_i \in K$ for all $i \in I$.
Then K is log-closed.

Proof: Take a positive $b \in K$. By virtue of Lemma 3.2, there is a finite subset $I_0 \subset I$ and $q_i \in$ Q such that $vb = vr' + \sum_{i \in I_0} q_i v x_i$ for some positive $r' \in \mathcal{R}$. So we can write $b = r' \prod_{i \in I_0} x_i^{q_i} \cdot r \cdot (1 + \varepsilon)$ with positive $r \in \mathbb{R}$ and some $\varepsilon \in K$ such that $v\varepsilon > 0$. We have that $\log(1 + \varepsilon) \in K$ since K is $\mathcal{F}$-closed. Moreover, $\log r' \in \mathcal{R} \subset K$ and $\log r \in \mathbb{R} \subset K$. Therefore,

$$\log b \;=\; \log r' + \sum_{i \in I_0} q_i \log x_i \;+\; \log r \;+\; \log(1 + \varepsilon) \;\in\; K \;.$$

□

Lemma 3.5 *Assume that K is of the form*

$$\left.\begin{array}{l} \mathcal{R}(x_i \mid i \in I)^{\mathrm{rQ}\mathcal{F}} \;\; \textit{log-closed, with } x_i > 0 \textit{ and} \\ vx_i,\ i \in I,\ \mathrm{Q}\textit{-linearly independent over } v\mathcal{R}. \end{array}\right\} \tag{19}$$

Take any $a \in K$ such that $\exp a \notin K$. Then $v \exp a$ is Q*-linearly independent over vK,*

$$vK(\exp a)^{\mathrm{rQ}\mathcal{F}} \;=\; vK \oplus \mathrm{Q}\, v \exp a \;. \tag{20}$$

Moreover, $K(\exp a)^{\mathrm{rQ}\mathcal{F}}$ is again log-closed, and therefore of the form (19). It contains $\exp b$ whenever $b \in K(\exp a)^{\mathrm{rQ}\mathcal{F}}$ and $v \exp b$ is Q*-linearly dependent over $vK(\exp a)^{\mathrm{rQ}\mathcal{F}}$.*

Proof: Applying Lemma 3.3 with $w = v$ and $Kw = \mathbb{R}$, we obtain that $v \exp a$ is Q-linearly independent over vK and that $\exp b \in K(\exp a)^{\mathrm{rQ}\mathcal{F}}$ whenever $b \in K(\exp a)^{\mathrm{rQ}\mathcal{F}}$ and $v \exp b$ is Q-linearly dependent over $vK(\exp a)^{\mathrm{rQ}\mathcal{F}}$. Equation (20) follows by an application of Lemma 3.2 to K and to $K(\exp a)^{\mathrm{rQ}\mathcal{F}}$. Finally, we infer from Lemma 3.4 that $K(\exp a)^{\mathrm{rQ}\mathcal{F}}$ is log-closed. □

Lemma 3.6 *Assume that $(K|\mathcal{R}, v)$ is an extension of valued fields and that w is a valuation on K, coarser than v and such that $Kw = \mathcal{R}$. Take $x_i \in K$ such that the values vx_i, $i \in I$, are Q-linearly independent over $v\mathcal{R}$. Then the values wx_i, $i \in I$, are Q-linearly independent.*

Proof: From $Kw = \mathcal{R}$ it follows that v is the composition of w with the restriction of v to $\mathcal{R}$. Thus, $v\mathcal{R}$ is a convex subgroup of vK and there is a canonical isomorphism $wK \simeq vK/v\mathcal{R}$. Hence $\sum_{i\in I} q_i wx_i = 0$ (where $q_i \in$ Q, almost all of them zero) implies $\sum_{i\in I} q_i vx_i \in v\mathcal{R}$. By assumption, this implies that $q_i = 0$ for all $i \in I$. □

3.3 A basic construction

First, we show how to construct log-closed fields K as in (19). **From now on, we always assume that $x \in M$ such that $x > \mathcal{R}$, that is, $vx < v\mathcal{R}$ and $x > 0$. By $v_{\mathcal{R}}$ we will denote the finest convex valuation on M which is trivial on $\mathcal{R}$.**

Lemma 3.7 *The field*

$$\mathcal{R}(\log_m x \mid m \geq 0)^{\mathrm{rQ}\mathcal{F}}$$

is log-closed. The convex hull of its value group in vM is equal to the smallest convex subgroup containing vx and $v\mathcal{R}$. If w is a convex valuation on M, trivial on $\mathcal{R}$ and such that $wx = 0$, then the field $\mathcal{R}(\log_m x \mid m \geq 0)^{\mathrm{rQ}\mathcal{F}}$ lies in $\mathcal{O}_w$.

Proof: From Corollary 2.13 we know that

$$vx \ll v \log x \ll \ldots \ll v \log_m x \ll \ldots < v\mathcal{R} \,. \tag{21}$$

In particular, the values $v \log_m x$ lie in distinct archimedean classes. As Q is archimedean, it follows that the values $v \log_m x$ are Q-linearly independent over $v\mathcal{R}$. So it follows from Lemma 3.4 that $\mathcal{R}(\log_m x \mid m \geq 0)^{\mathrm{rQ}\mathcal{F}}$ is log-closed.

From Lemma 3.2 we infer that $v\mathcal{R}(\log_m x \mid m \geq 0)^{\mathrm{rQ}\mathcal{F}} = v\mathcal{R} \oplus \bigoplus_{m\geq 0} \mathrm{Q}\, v \log_m x$. Now (21) yields that this group is contained in the smallest convex subgroup H of vM which contains vx and $v\mathcal{R}$. If w is as in our assumption, then H is contained in the convex subgroup H_w of vM associated with w. Thus, w is trivial on $\mathcal{R}(\log_m x \mid m \geq 0)^{\mathrm{rQ}\mathcal{F}}$, that is, this field lies in $\mathcal{O}_w$. □

Next, we build up $LE_{\mathcal{R},\mathcal{F}}(x)$. As a preparation for what we will need in the next section, we will keep our construction more general. We will construct a variety of fields (described in Lemma 3.8 below) of which $LE_{\mathcal{R},\mathcal{F}}(x)$ is just a special case. Let w be a convex valuation on M, trivial on $\mathcal{R}$, and H_w its associated convex subgroup of vM. Further, let $K_0^w \subset \mathcal{O}_w$ be any field of the form (19). For example, if $wx = 0$, then we can take $K_0^w = \mathcal{R}(\log_m x \mid m \geq 0)^{\mathrm{rQ}\mathcal{F}}$. We will see later that if $w \neq v_{\mathcal{R}}$ then there always exists such a field K_0^w which properly contains $\mathcal{R}$.

Now we construct K_1^w as follows. Assume that $a \in K_0^w$ such that $\exp a \notin K_0^w$, but $v \exp a \in H_w$. Then by Lemma 3.5, $K_0^w(\exp a)^{\mathrm{rQ}\mathcal{F}}$ is again of the form (19), with $vK_0^w(\exp a)^{\mathrm{rQ}\mathcal{F}} = vK_0^w \oplus \mathrm{Q}\, v \exp a \subset H_w$. The latter shows that it is again a subfield of $\mathcal{O}_w$. We repeat this procedure until we arrive at a field $K_1^w \subset \mathcal{O}_w$ of the form (19), which contains $\exp a$ for every $a \in K_0^w$ such that $\exp a \in \mathcal{O}_w^\times$. Then we construct K_2^w from K_1^w in the same way as we constructed K_1^w from K_0^w. We iterate to obtain fields $K_n^w \subset \mathcal{O}_w$, of the form (19). Their union

$$K_\infty^w := \bigcup_{n\in\mathbb{N}} K_n^w \subset \mathcal{O}_w$$

is $\mathrm{rQ}\mathcal{F}$-closed and of the form (19). By construction, we have:

Lemma 3.8 *K_∞^w is the uniquely determined smallest log- and $\mathrm{rQ}\mathcal{F}$-closed subfield of $\mathcal{O}_w$, relatively exp-closed in $\mathcal{O}_w^\times$ and containing K_0^w. It is of the form (19).*

We derive some further information from our construction.

Lemma 3.9 *Take $n \in \mathbb{N}$. If $a \in K_n^w$ with $va < 0$, $a > 0$, then*

$$v \log a \in vK_{n-1}^w\,, \qquad \textit{and} \qquad v \log_n a \in vK_0^w\,.$$

Proof: By the construction of K_n^w from K_{n-1}^w, there are elements $a_j \in K_{n-1}^w$, $j \in J$, such that $vK_n^w = vK_{n-1}^w \oplus \bigoplus_{j\in J} \mathrm{Q} v \exp a_j$. Hence, $a \in K_n^w$ can be written as

$$a = \prod_{j\in J_0} (\exp a_j)^{q_j} \cdot c \cdot r \cdot (1+\varepsilon)$$

with J_0 a finite subset of J, $q_j \in \mathrm{Q}$, $c \in K_{n-1}^w$, $r \in \mathbb{R}$ and $\varepsilon \in K_n^w$ with $v\varepsilon > 0$. Then $\log a = \sum_{j\in J_0} q_j a_j + \log c + \log r + \log(1+\varepsilon)$. Since $v \log a < 0$ by Lemma 2.12, but $v \log(1+\varepsilon) > 0$, we find that $v \log a = v(\sum_{j\in J_0} q_j a_j + \log c + \log r) \in vK_{n-1}^w$. By induction it follows that $v \log_n a \in vK_0^w$. □

If w is trivial on $\mathcal{R}$ and $wx = 0$ and we start our construction from $K_0^w = \mathcal{R}(\log_m x \mid m \geq 0)^{\mathrm{rQ}\mathcal{F}}$, then K_∞^w will be the uniquely determined smallest log- and $\mathrm{rQ}\mathcal{F}$-closed subfield of $\mathcal{O}_w$, relatively exp-closed in $\mathcal{O}_w^\times$ and containing $\mathcal{R}(x)$. We denote it by

$$LE_{\mathcal{R},\mathcal{F}}^w(x)\,.$$

Let u denote the trivial valuation on M. Then $\mathcal{O}_u = M$ and $H_u = vM$. In this case, $LE_{\mathcal{R},\mathcal{F}}^u(x)$ is exp-closed and contains x. Therefore,

$$LE_{\mathcal{R},\mathcal{F}}^u(x) = LE_{\mathcal{R},\mathcal{F}}(x)\,.$$

Lemma 3.10 *Suppose that $x > \mathcal{R}$. Then for every $y \in LE_{\mathcal{R},\mathcal{F}}(x)$, $y > \mathcal{R}$, the sequence* $\exp_m y$, *$m \geq 0$, is cofinal in $LE_{\mathcal{R},\mathcal{F}}(x)$, and the sequence* $\log_m y$, *$m \geq 0$, is coinitial in* $\{z \in LE_{\mathcal{R},\mathcal{F}}(x) \mid z > \mathcal{R}\}$.

Proof: It suffices to show the result for $y = x$. Indeed, if it holds in this case, then there is $\nu \in \mathbb{N}$ such that $\exp_\nu x > y > \log_\nu x$. It follows that $\exp_n y > \exp_{\nu+n} x$, showing that also the sequence $\exp_m y$, $m \geq 0$, is cofinal. It also follows that $\log_n x > \log_{\nu+n} y$, showing that also the sequence $\log_m y$, $m \geq 0$, is coinitial.

Take any $a \in LE_{\mathcal{R},\mathcal{F}}(x)$, $x > \mathcal{R}$. From Lemma 3.9 with $w = u$ and $K_0^w = \mathcal{R}(\log_m x \mid m \geq 0)^{\mathrm{rQ}\mathcal{F}}$ we infer that $v \log_n a \in v\mathcal{R}(\log_m x \mid m \geq 0)^{\mathrm{rQ}\mathcal{F}}$ for some $n \in \mathbb{N}$. By Lemma 3.7, every element $\alpha < 0$ in this value group is either archimedean equivalent to vx, or satisfies $vx \ll \alpha < 0$. Since $v \log_n a \ll v \log_{n+1} a < 0$ by Lemma 2.12, it follows that $vx \ll v \log_{n+1} a < 0$. Hence by (1), $x > \log_{n+1} a$ and therefore, $\exp_{n+1} x > a$.

Now let $a \in LE_{\mathcal{R},\mathcal{F}}(x)$, $a > \mathcal{R}$. As before, $v \log_n a \in v\mathcal{R}(\log_m x \mid m \geq 0)^{\mathrm{rQ}\mathcal{F}}$ for some $n \in \mathbb{N}$. As the sequence $v \log_m x$, $m \geq 0$, is cofinal in the negative part of this value group, there is some m_0 such that $v \log_n a < v \log_{m_0} x$. Hence by (1), $a \geq \log_n a > \log_{m_0} x$. □

Now we deduce our main theorem on the valuation theoretical structure of $LE_{\mathcal{R},\mathcal{F}}(x)$. If we take $\mathcal{F} = \mathcal{F}_T$ and $M = H(\mathcal{R})$, then the theorem describes the structure of the Hardy field $H(\mathcal{R})$.

Theorem 3.11 *$LE_{\mathcal{R},\mathcal{F}}(x)$ is of the form*

$$\mathcal{R}(x_i \mid i \in I)^{\mathrm{rQ}\mathcal{F}} \quad \textit{with } x_i > 0 \textit{ and } v_{\mathcal{R}} x_i,\ i \in I,\ \mathrm{Q}\textit{-linearly independent.} \tag{22}$$

Moreover,

$$LE_{\mathcal{R},\mathcal{F}}(x) v_{\mathcal{R}} \;=\; \mathcal{R}\,. \tag{23}$$

The elements x_i can be chosen so as to include x and $\log_m x$ for all $m \in \mathbb{N}$.

If $\mathcal{R} = \mathbb{R}$, then $LE_{\mathcal{R},\mathcal{F}}(x)$ has exponential rank 1, in the sense of [K–K2]. In general, $\operatorname{exprk} LE_{\mathcal{R},\mathcal{F}}(x) = \operatorname{exprk} \mathcal{R} + 1$.

Proof: By our construction, we get that $LE_{\mathcal{R},\mathcal{F}}(x)$ is of the form (19). Since $\mathcal{F} \subseteq \mathcal{F}_T$, we have that $LE_{\mathcal{R},\mathcal{F}}(x) \subseteq LE_{\mathcal{R},\mathcal{F}_T}(x)$. By definition of the valuation $v_{\mathcal{R}}$, its valuation ring is the convex hull of $\mathcal{R}$ in M. As $\mathcal{R}$ is an elementary submodel of $LE_{\mathcal{R},\mathcal{F}_T}(x)$, we can deduce from [D–L], p. 75, (1), that this valuation ring is $T(\exp)$-convex in $LE_{\mathcal{R},\mathcal{F}_T}(x)$. Since $LE_{\mathcal{R},\mathcal{F}_T}(x)$ is the $T(\exp)$-definable closure of $\mathcal{R}(x)$ in itself, we can apply Corollary 5.4 of [D–L] to obtain that $LE_{\mathcal{R},\mathcal{F}_T}(x) v_{\mathcal{R}} = \mathcal{R}$. Since $\mathcal{R} \subset LE_{\mathcal{R},\mathcal{F}}(x) \subseteq LE_{\mathcal{R},\mathcal{F}_T}(x)$, this proves (23). By Lemma 3.6, this also implies that $v_{\mathcal{R}} x_i$, $i \in I$, are Q-linearly independent.

The exponential rank is the order type of the set of proper $T(\exp)$-convex valuation rings, ordered by inclusion. Lemma 3.10 shows that $LE_{\mathcal{R},\mathcal{F}}(x)$ has exactly one more than $\mathcal{R}$, namely $\mathcal{R}$ itself. This proves our assertions about the exponential rank. □

3.4 Levels

An infinitely increasing unary function f on $\mathcal{R}$ **has level** s if $s \in \mathbb{Z}$ and there is $N \in \mathbb{N}$ such that $\log_{N+s} \circ f$ is asymptotic to $\log_N$ on $\mathcal{R}$. Note that if the latter holds, then it also holds for every integer $N' > N$ in the place of N. If a denotes the germ of f in $H(\mathcal{R})$, then by (16) the condition is equivalent to

$$v\left(\frac{\log_{N+s} a}{\log_N x} - 1\right) \;>\; v\mathcal{R}\;.$$

Here, N can be chosen such that $N+s \geq 0$. Suppose that $s < s' \in \mathbb{Z}$. Since $a > \mathcal{R}$ we have that $va < v\mathcal{R}$; hence by Corollary 2.13, $v\log_{N+s} a \neq v\log_{N+s'} a$ which shows that the above inequality cannot hold for s' in the place of s. Thus, the level s is uniquely determined (see also [M–M]).

We say that $\mathcal{R}$ is **levelled** if every $\mathcal{R}$-definable ultimately strictly increasing and unbounded unary function on $\mathcal{R}$ has a level. In this section, we will prove that every definable function on $\mathcal{R}$ has a level, and we will determine this level explicitly.

Take any $a \in LE_{\mathcal{R},\mathcal{F}}(x)$ such that $a > \mathcal{R}$. According to our construction, we write $LE_{\mathcal{R},\mathcal{F}}(x) = K_\infty$ with $K_0 = \mathcal{R}(\log_m x \mid m \geq 0)^{\mathrm{rQ}\mathcal{F}}$. By Lemma 3.9 there is some $n \in \mathbb{N}$ such that $v\log_n a \in vK_0$. Similarly as in the proof of Lemma 3.4, we write $\log_n a = r' \prod_{i\geq 0} (\log_i x)^{q_i} \cdot r \cdot (1+\varepsilon)$ with $q_i \in \mathrm{Q}$, only finitely many of them nonzero, $r' \in \mathcal{R}$, $r \in \mathbb{R}$ and $\varepsilon \in K$ such that $v\varepsilon > 0$. It follows that

$$\log_{n+1} a \;=\; \log r' + \sum_{i\geq 0} q_i \log_{i+1} x \;+\; \log r \;+\; \log(1+\varepsilon)\;.$$

As $a > \mathcal{R}$ by assumption, there must be at least one nonzero q_i. Let i_0 be the smallest of all $i \geq 0$ for which $q_i \neq 0$. We have that $v\log r = 0$, $v\log(1+\varepsilon) > 0$ and $v\log_{i_0+1} x < v\log_{i+1} x$ for $i > i_0$. Also, $v\log_{i_0+1} x < vr'$. Thus, we can write $\log_{n+1} a = q_{i_0} \log_{i_0+1} x \cdot (1+\varepsilon')$ with $v\varepsilon' > 0$. Then

$$\log_{n+2} a \;=\; \log q_{i_0} \;+\; \log_{i_0+2} x \;+\; \log(1+\varepsilon')\;.$$

Again, $v\log_{i_0+2} x < 0 = v\log q_{i_0} < v\varepsilon' = v\log(1+\varepsilon')$. Hence,

$$v\left(\log_{n+2} a \;-\; \log_{i_0+2} x\right) \;=\; v\left(\log q_{i_0} + \log(1+\varepsilon')\right) \;=\; v\log q_{i_0} \;=\; 0\;.$$

Thus,

$$v\left(\frac{\log_{n+2} a}{\log_{i_0+2} x} \;-\; 1\right) \;=\; -v\log_{i_0+2} x \;>\; v\mathcal{R}\;. \tag{24}$$

We have now proved a result which in fact constitutes an abstract notion of levels, without referring to Hardy fields:

Proposition 3.12 *Take any element $a \in LE_{\mathcal{R},\mathcal{F}}(x)$ such that $a > \mathcal{R}$. Then a "has level over $\mathcal{R}$" in the following sense: there is some $s \in \mathbb{Z}$ and $N \in \mathbb{N}$ such that*

$$v_{\mathcal{R}}(\log_{N+s} a \;-\; \log_N x) \;>\; v_{\mathcal{R}} \log_N x\;.$$

Now take any $\mathcal{R}$-definable, ultimately strictly increasing and unbounded function f on $\mathcal{R}$. Let a be the germ of f at infinity. Then $a > \mathcal{R}$. Hence, a is an element of the Hardy field $H(\mathcal{R}) = LE_{\mathcal{R},\mathcal{F}_T}(x)$ of $\mathcal{R}$ (where $x > \mathcal{R}$). Then (24) shows that $\log_{n+2} f(x)$ and $\log_{i_0+2} x$ are asymptotic as functions on $\mathcal{R}$. That is,

the function f has level $n - i_0$.

This proves Theorem 1.1.

3.5 A maximality property of the T-definable closure in the $T(\exp)$-definable closure

Lemma 3.13 *Assume that T has field of exponents $\mathbb{R}$ and that $\mathbb{R} \subset \mathcal{R} \subset M$ are models of $T(\exp)$. Let $x \in M$, $x > \mathcal{R}$. Then $\mathcal{R}(x)^{\mathcal{F}_T}$ (the T-definable closure of $\mathcal{R} \cup \{x\}$ in M) has the following maximality property:*
1) $v_{\mathcal{R}}\mathcal{R}(x)^{\mathcal{F}_T} \simeq \mathbb{R}$,
2) $\mathcal{R}(x)^{\mathcal{F}_T}$ is maximal among all subfields of $LE_{\mathcal{R},\mathcal{F}_T}(x)$ whose value group w.r.t. $v_{\mathcal{R}}$ is archimedean.

Proof: Assertion 1) follows from Lemma 3.2. In order to prove assertion 2), we show the following: Take any $a \in LE_{\mathcal{R},\mathcal{F}_T}(x) \setminus \mathcal{R}(x)^{\mathcal{F}_T}$. Then $v_{\mathcal{R}}\mathcal{R}(x)^{\mathcal{F}_T}(a)$ is not archimedean.

By Theorem 3.11 we can write $LE_{\mathcal{R},\mathcal{F}_T}(x) = \mathcal{R}(x_i \mid i \in I)^{\mathcal{F}_T}$ with $x_i > 0$ and $v_{\mathcal{R}}x_i$, $i \in I$, $\mathbb{R}$-linearly independent, and x among the x_i. As $a \in \mathcal{R}(x_i \mid i \in I)^{\mathcal{F}_T}$, there are $x_{i_1}, \ldots, x_{i_n}$ ($n \geq 1$) such that $a \in \mathcal{R}(x, x_{i_1}, \ldots, x_{i_n})^{\mathcal{F}_T}$, and we choose n minimal with this property. By the Exchange Lemma for o-minimal theories ([P–S]) applied to T, we then obtain that

$$x_{i_1} \in \mathcal{R}(x, a, x_{i_2}, \ldots, x_{i_n})^{\mathcal{F}_T} \ . \tag{25}$$

Suppose that $v_{\mathcal{R}}\mathcal{R}(x,a)^{\mathcal{F}_T} = v_{\mathcal{R}}\mathcal{R}(x)^{\mathcal{F}_T}$. Then by Lemma 3.2,

$$\begin{aligned} v_{\mathcal{R}}\mathcal{R}(x, a, x_{i_2}, \ldots, x_{i_n})^{\mathcal{F}_T} &= v_{\mathcal{R}}\mathcal{R}(x,a)^{\mathcal{F}_T}(x_{i_2}, \ldots, x_{i_n})^{\mathcal{F}_T} \;=\; v_{\mathcal{R}}\mathcal{R}(x,a)^{\mathcal{F}_T} \oplus \bigoplus_{j=2}^{n} \mathbb{R}v_{\mathcal{R}}x_{i_j} \\ &= v_{\mathcal{R}}\mathcal{R}(x)^{\mathcal{F}_T} \oplus \bigoplus_{j=2}^{n} \mathbb{R}v_{\mathcal{R}}x_{i_j} \;=\; \mathbb{R}v_{\mathcal{R}}x \oplus \bigoplus_{j=2}^{n} \mathbb{R}v_{\mathcal{R}}x_{i_j} \ . \end{aligned}$$

But this does not contain $v_{\mathcal{R}}x_{i_1}$. This contradiction to (25) shows that

$$v_{\mathcal{R}}\mathcal{R}(x,a)^{\mathcal{F}_T} \;\neq\; v_{\mathcal{R}}\mathcal{R}(x)^{\mathcal{F}_T} \ .$$

By the Valuation Property ([D–S2], Proposition 9.2) it follows that

$$v_{\mathcal{R}}\mathcal{R}(x)^{\mathcal{F}_T} \subsetneq v_{\mathcal{R}}\mathcal{R}(x)^{\mathcal{F}_T}(a) \ .$$

Since $v_{\mathcal{R}}\mathcal{R}(x)^{\mathcal{F}_T} \simeq \mathbb{R}$ it follows that $v_{\mathcal{R}}\mathcal{R}(x)^{\mathcal{F}_T}(a)$ is not archimedean. □

Lemma 3.14 *Let $H \subset H(\mathcal{R})$ be a subfield containing $\mathcal{R}(x)$ and closed under compositions and compositional inverses for $v_{\mathcal{R}}$-positive infinite germs (i.e., germs $a \in H$ such that $a > \mathcal{R}$). If H is polynomially bounded (i.e., every germ in H is bounded by a power x^n for some $n \in \mathbb{N}$), then $v_{\mathcal{R}}(H)$ is archimedean.*

Proof: Assume for a contradiction that there is $g \in H(\mathcal{R})$ such that $g > \mathcal{R}$ and $v_{\mathcal{R}} g \ll v_{\mathcal{R}} x$ or $v_{\mathcal{R}} x \ll v_{\mathcal{R}} g$. The former implies that $g > x^n$ for all $n \in \mathbb{N}$, a contradiction to the fact that H is polynomially bounded. So assume that $v_{\mathcal{R}} x \ll v_{\mathcal{R}} g$. But this implies that for all $n \in \mathbb{N}$,

$$x^n < g^{-1} ,$$

where g^{-1} denotes the compositional inverse of g. This again contradicts the assumption that H is polynomially bounded. Indeed, let $n \in \mathbb{N}$. Since $g^n < x$, there exists $r \in \mathcal{R}$ (and we may assume $r > 1$) such that for $a \in \mathcal{R}$ with $a > r$ we have $g(a)^n < a$. On the other hand, g is invertible, ultimately. So for b large enough, $g^{-1}(b) = a$ exists with $a > r$. Thus, $g(g^{-1}(b))^n < g^{-1}(b)$. □

Corollary 3.15 *The field $\mathcal{R}(x)^{\mathcal{F}_T}$ (i.e., the Hardy field associated with the reduct of $\mathcal{R}$ to the language of T) is maximal among the polynomially bounded subfields of $H(\mathcal{R})$ which are closed under compositions and compositional inverses for $v_{\mathcal{R}}$-positive infinite germs.*

Proof: Let H be a polynomially bounded subfield of $H(\mathcal{R})$ closed under compositions and compositional inverses for $v_{\mathcal{R}}$-positive infinite germs, and containing $\mathcal{R}(x)^{\mathcal{F}_T}$. Then by Lemma 3.14, $v_{\mathcal{R}} H$ is archimedean. Hence by Lemma 3.13, H cannot be a proper extension of $\mathcal{R}(x)^{\mathcal{F}_T}$. □

Let us note that there exist polynomially bounded subfields of $H(\mathcal{R})$ which properly contain $\mathcal{R}(x)^{\mathcal{F}_T}$. For instance, $\mathcal{R}(x, \log x)^{\mathcal{F}_T}$ and $\mathcal{R}(\log_m x \mid m \geq 0)^{\mathcal{F}_T}$ are such fields. But they are not closed under compositions and compositional inverses for $v_{\mathcal{R}}$-positive infinite germs.

3.6 A maximality property of the Hardy field $H(\mathcal{R}_{\text{an,powers}})$

Now we consider the special case where $\mathcal{F}_T$ is the set of 0-definable functions in $\mathbb{R}_{\text{an,powers}}$. We let $\mathcal{R}_{\text{an,powers}}$ denote the reduct of $\mathcal{R}$ to the language of $\mathbb{R}_{\text{an,powers}}$, and $\mathcal{R}_{\text{an,exp}}$ the reduct of $\mathcal{R}$ to the language of $\mathbb{R}_{\text{an,exp}}$. Since

$$x^r = \exp(r \log x)$$

for all $r \in \mathbb{R}$, the power functions are $\mathbb{R}$-definable (actually, already 0-definable) in $\mathcal{R}_{\text{an,exp}}$. Therefore,

$$H(\mathcal{R}_{\text{an,exp}}) = H(\mathcal{R}) .$$

On the other hand, $H(\mathcal{R}_{\text{an,powers}})$ is a proper subfield of $H(\mathcal{R})$. It has the following maximality property:

Theorem 3.16 *Let $H \subseteq H(\mathcal{R})$ be a polynomially bounded field containing $H(\mathcal{R}_{\text{an,powers}})$ and closed under compositions and compositional inverses for $v_{\mathcal{R}}$-positive infinite germs. Then $H = H(\mathcal{R}_{\text{an,powers}})$.*

In particular, $H(\mathcal{R}_{\text{an,powers}})$ is maximal among the Hardy subfields of $H(\mathcal{R})$ associated with polynomially bounded reducts of $\mathcal{R}$.

Proof: We take T to be the elementary theory of $\mathcal{R}_{\text{an,powers}}$. We know that $H(\mathcal{R}_{\text{an,powers}}) = \mathcal{R}(x)^{\mathcal{F}_T}$ with $x \in H(\mathcal{R})$, $x > \mathcal{R}$ the germ of the identity function. Now our first assertion follows from Corollary 3.15.

If H is the Hardy field of a polynomially bounded reducts of $\mathcal{R}$, then H is closed under compositions and compositional inverses for $v_{\mathcal{R}}$-positive infinite germs. Hence our second assertion follows from the first. □

4 Residue fields of $\mathcal{F}$-closures

In this section we wish to determine the residue fields of $LE_{\mathcal{R},\mathcal{F}}(x)$ with respect to any convex valuation which is trivial on $\mathcal{R}$; such a valuation is not necessarily $T(\exp)$-convex. In addition to our earlier assumptions (see Section 3.2), we consider the following conditions:

(PADE) $\mathcal{F}$ is closed under partial derivations;

(COMP) if w is a convex valuation on a model N of $T(\exp)$ and F is a subfield of N such that $Fw \subset F$ is $\mathcal{F}$-closed and wF is archimedean, then either $F^{\mathcal{F}}$ is embeddable in the completion of (F, w), or there is some $y \in F^{\mathcal{F}}$, $y \neq 0$, such that $wy > wF$.

Note that if $F^{\mathcal{F}}$ is embeddable in the completion of (F, w), then $wF^{\mathcal{F}} = wF$ and $F^{\mathcal{F}}w = Fw$. If on the other hand, $0 \neq y \in F^{\mathcal{F}}$ such that $wy > wF$, then $wF^{\mathcal{F}}$ is not archimedean.

We denote by T_{an} the theory of the expansion $\mathbb{R}_{\text{an}} = (\mathbb{R}, +, \cdot, 0, 1, <, \mathcal{F}_{\text{an}}\}$.

Lemma 4.1 *If $\mathcal{F} \subseteq \mathcal{F}_{\text{an}}$ satisfies condition (PADE), then it satisfies condition (COMP) in each model of T_{an}.*

Proof: Assume the hypothesis as given in condition (COMP). By Zorn's Lemma, we find a maximal subfield F_0 of $F^{\mathcal{F}}$ containing F and embeddable in the completion of (F, w). Suppose that $F^{\mathcal{F}}$ is not embeddable in the completion of (F, w). Then $F_0 \neq F^{\mathcal{F}}$, that is, F_0 is not $\mathcal{F}$-closed. So let $f(X_1, \ldots, X_k) \in \mathcal{F}$ and $a = (a_1, \ldots, a_k) \in F_0^k$ with $va_i > 0$ such that $f(a) \in F^{\mathcal{F}} \setminus F_0$. We write $a_i = c_i + \varepsilon_i$ with $c_i \in F_0 w = Fw$ and $w\varepsilon_i > 0$; let $c = (c_1, \ldots, c_k)$. By the Taylor expansion, the following assertions hold (they are elementary sentences in the language of T_{an} and thus hold in the T_{an}-model N): for all $m \in \mathbb{N}$,

$$\left| f(a_1, \ldots, a_k) - \sum_{\nu=(0,\ldots,0)}^{(m,\ldots,m)} \frac{\partial^\nu f}{\partial X^\nu}(c_1, \ldots, c_k) \frac{\varepsilon^\nu}{\nu!} \right| \leq |\varepsilon_1 \cdot \ldots \cdot \varepsilon_k|^m$$

(for $\nu = (\nu_1, \ldots, \nu_k) \in \mathbb{N}^k$, $\frac{\partial^\nu f}{\partial X^\nu}$ stands for $\frac{\partial^{\nu_1} \ldots \partial^{\nu_k} f}{\partial X_1^{\nu_1} \ldots \partial X_k^{\nu_k}}$, and $\nu!$ stands for $\nu_1! \cdot \ldots \cdot \nu_k!$). By (3) it follows that for all $m \in \mathbb{N}$,

$$w\left(f(a_1, \ldots, a_k) - \sum_{\nu=(0,\ldots,0)}^{(m,\ldots,m)} \frac{\partial^\nu f}{\partial X^\nu}(c_1, \ldots, c_k) \frac{\varepsilon^\nu}{\nu!} \right) \geq m(w\varepsilon_1 + \ldots + w\varepsilon_k) .$$

Since wF_0 is archimedean and $w\varepsilon_i > 0$, the sequence $m(w\varepsilon_1 + \ldots + w\varepsilon_k)$, $m \in \mathbb{N}$, is cofinal in wF_0. This shows that the partial sums form a Cauchy sequence in (F_0, w), with limit $f(a)$. Note that since $\mathcal{F}$ is closed under partial derivatives and Fw is $\mathcal{F}$-closed, the coefficients $\frac{\partial^\nu f}{\partial X^\nu}(c_1, \ldots, c_k)$ lie in $Fw \subset F_0$. So the partial sums are indeed elements of F_0.

Suppose that the sequence has no limit in F_0. Then we can apply Lemma 2.7 to obtain that $F_0(f(a))$ is embeddable in the completion of (F_0, w) and hence also in the completion of (F, w). But this contradicts the maximality of F_0. Hence, there is some $b \in F_0$ which is also a limit of this sequence (observe that it is not necessarily a Cauchy sequence in (N, w)). Then by Lemma 2.4, $w(f(a) - b) > wF_0$. With $y := f(a) - b \neq 0$, we have found the desired element y which satisfies $wy > wF$. □

At this point, it may be helpful to give an example which shows that an element y as in the assertion of the above lemma can indeed exist. Take L to be any T_{an}-model with non-archimedean value group. Choose $y, t \in L$ such that $vy \gg vt > 0$. Then $vy > \mathbb{Q}vt = v\mathbb{R}(t)^{\mathrm{r}}$. It is well known that in general, $(\mathbb{R}(t)^{\mathrm{r}})^{\mathcal{F}} \neq \mathbb{R}(t)^{\mathrm{r}}$. Take any $a \in (\mathbb{R}(t)^{\mathrm{r}})^{\mathcal{F}} \setminus \mathbb{R}(t)^{\mathrm{r}}$. As $\mathbb{R}(t)^{\mathrm{r}} \subset (\mathbb{R}(t)^{\mathrm{r}})^{\mathcal{F}} \subset \mathbb{R}(t)^{\mathrm{r}\mathcal{F}}$, Lemma 3.2 yields that $v(\mathbb{R}(t)^{\mathrm{r}})^{\mathcal{F}} = v\mathbb{R}(t)^{\mathrm{r}}$. Further, $(\mathbb{R}(t)^{\mathrm{r}})^{\mathcal{F}}v = \mathbb{R} = \mathbb{R}(t)^{\mathrm{r}}v$. Hence also $v\mathbb{R}(t, a)^{\mathrm{r}} = v\mathbb{R}(t)^{\mathrm{r}}$ and $\mathbb{R}(t, a)^{\mathrm{r}}v = \mathbb{R}(t)^{\mathrm{r}}v$. That is, the extension $(\mathbb{R}(t, a)^{\mathrm{r}} | \mathbb{R}(t)^{\mathrm{r}}, v)$ is immediate. Hence by Lemma 2.5, a is a limit of a pseudo Cauchy sequence without limit in $(\mathbb{R}(t)^{\mathrm{r}}, v)$. Set $z := a + y$. Then $v(z - a) = vy > v\mathbb{R}(t)^{\mathrm{r}}$, and Lemma 2.4 shows that z is also a limit of this pseudo Cauchy sequence. Hence by Lemma 2.6, the extension $(\mathbb{R}(t)^{\mathrm{r}}(z) | \mathbb{R}(t)^{\mathrm{r}}, v)$ is immediate. It follows from Lemma 2.9 that also $(\mathbb{R}(t, z)^{\mathrm{r}} | \mathbb{R}(t)^{\mathrm{r}}, v)$ is immediate. On the other hand, $a \in (\mathbb{R}(t, z)^{\mathrm{r}})^{\mathcal{F}}$ and consequently, $y \in (\mathbb{R}(t, z)^{\mathrm{r}})^{\mathcal{F}}$ with $vy > v\mathbb{R}(t)^{\mathrm{r}} = v\mathbb{R}(t, z)^{\mathrm{r}}$. So $(F, w) = (\mathbb{R}(t, z)^{\mathrm{r}}, v)$ is our desired example.

Throughout this section, we will assume that $\mathcal{F}$ satisfies conditions (PADE) and (COMP).

Lemma 4.2 *Let $x_i \in M$ be such that the values vx_i, $i \in I$ are* Q*-linearly independent over $v\mathcal{R}$. Further, let w be any convex valuation which is trivial on $\mathcal{R}$. Assume that there is a subset $I_w \subset I$ such that $wx_i = 0$ for all $i \in I_w$ and that the values wx_i, $i \in I \setminus I_w$ are* Q*-linearly independent. Then*

$$w\mathcal{R}(x_i \mid i \in I)^{\mathrm{rQ}\mathcal{F}} = \bigoplus_{i \in I \setminus I_w} \mathrm{Q}wx_i \quad \textit{and} \quad w\mathcal{R}(x_i \mid i \in I)^{\mathrm{h}\mathcal{F}} = \bigoplus_{i \in I \setminus I_w} \mathbb{Z}wx_i .$$

Further,

$$\begin{aligned} \mathcal{R}(x_i \mid i \in I)^{\mathrm{rQ}\mathcal{F}}w &= \mathcal{R}(x_i \mid i \in I_w)^{\mathrm{rQ}\mathcal{F}} = (\mathcal{R}(x_i \mid i \in I)w)^{\mathrm{rQ}\mathcal{F}} \\ \mathcal{R}(x_i \mid i \in I)^{\mathrm{h}\mathcal{F}}w &= \mathcal{R}(x_i \mid i \in I_w)^{\mathrm{h}\mathcal{F}} = (\mathcal{R}(x_i \mid i \in I)w)^{\mathrm{h}\mathcal{F}} . \end{aligned}$$

Proof: We set $L := \mathcal{R}(x_i \mid i \in I)$ and $K := \mathcal{R}(x_i \mid i \in I_w)$. By Corollary 2.3, $vL = v\mathcal{R} \oplus \bigoplus_{i\in I} \mathbb{Z}vx_i$, $vK = v\mathcal{R} \oplus \bigoplus_{i\in I_w} \mathbb{Z}vx_i$, $wL = \bigoplus_{i\in I\setminus I_w} \mathbb{Z}wx_i$ and $Lw = K$. From Lemma 3.2 we infer that $vL^{\mathrm{rQ}\mathcal{F}} = v\mathcal{R} \oplus \bigoplus_{i\in I} \mathrm{Q}vx_i = \mathrm{Q}\otimes_{\mathbb{Z}} vL$ and that $vK^{\mathrm{rQ}\mathcal{F}} = v\mathcal{R} \oplus \bigoplus_{i\in I_w} \mathrm{Q}vx_i = \mathrm{Q}\otimes_{\mathbb{Z}} vK$ (recall that vM and $v\mathcal{R}$ are Q-vector spaces). The former implies that $wL^{\mathrm{rQ}\mathcal{F}} = \mathrm{Q}\otimes_{\mathbb{Z}} wL$, which in view of the Q-linearly independence of the wx_i implies our assertion on the value groups for the r$\mathrm{Q}\mathcal{F}$-closure.

We prove the assertions of our lemma for the h$\mathcal{F}$-closure. The proof for the residue field of the r$\mathrm{Q}\mathcal{F}$-closure is analogous. If our assertions are not true, then there is some $b \in L^{\mathrm{h}\mathcal{F}}$ such that $wb \notin \bigoplus_{i\in I\setminus I_w} \mathbb{Z}wx_i$ or $bw \notin K^{\mathrm{h}\mathcal{F}}$. But b is already contained in some subfield $\mathcal{R}(x_1,\ldots,x_n)^{\mathrm{h}\mathcal{F}} \subset L^{\mathrm{h}\mathcal{F}}$, where $x_1,\ldots,x_n$ are suitably chosen from the x_i's. So we see that it suffices to prove our lemma in the case of a finite set $I = \{1,\ldots,n\}$.

As usual, we let H_w denote the convex subgroup of vM associated with w. Since vK is contained in H_w and since Q is archimedean, we find that also $vK^{\mathrm{rQ}\mathcal{F}} = \mathrm{Q}\otimes_{\mathbb{Z}} vK \subset H_w$. That is, w is trivial on $K^{\mathrm{rQ}\mathcal{F}}$ and thus also on $K^{\mathrm{h}\mathcal{F}}$. Therefore, $K^{\mathrm{h}\mathcal{F}} \subset L^{\mathrm{h}\mathcal{F}}w$. We will show that equality holds.

First assume that wL is archimedean. Then $wL^{\mathrm{rQ}\mathcal{F}} = \mathrm{Q}\otimes_{\mathbb{Z}} wL$ is archimedean, and so is $wL^{\mathrm{h}\mathcal{F}} \subset wL^{\mathrm{rQ}\mathcal{F}}$. Set $F_0 := K^{\mathrm{h}\mathcal{F}}(x_i \mid i \in I\setminus I_w)$. Then $L^{\mathrm{h}\mathcal{F}} = K^{\mathrm{h}\mathcal{F}}(x_i \mid i \in I\setminus I_w)^{\mathrm{h}\mathcal{F}} = F_0^{\mathrm{h}\mathcal{F}}$, and by Lemma 2.2, $F_0w = K^{\mathrm{h}\mathcal{F}}$ and $wF_0 = \bigoplus_{i\in I\setminus I_w} \mathbb{Z}wx_i = wL$. By Zorn's Lemma, we find a maximal subfield F of $F_0^{\mathrm{h}\mathcal{F}}$ containing F_0 and embeddable in the completion of (F_0,w). Since $wF = wF_0$ is archimedean and $Fw = F_0w$ is $\mathcal{F}$-closed, we can apply condition (COMP) to see that F is $\mathcal{F}$-closed. From Lemma 2.8 we infer that F must be equal to its henselization, i.e., it is henselian. Therefore, $F = F_0^{\mathrm{h}\mathcal{F}} = L^{\mathrm{h}\mathcal{F}}$, showing that $w\mathcal{R}(x_i \mid i\in I)^{\mathrm{h}\mathcal{F}} = wL^{\mathrm{h}\mathcal{F}} = wF = \bigoplus_{i\in I\setminus I_w}\mathbb{Z}wx_i$ and $\mathcal{R}(x_i \mid i \in I)^{\mathrm{h}\mathcal{F}}w = L^{\mathrm{h}\mathcal{F}}w = Fw = K^{\mathrm{h}\mathcal{F}}$. (For the r$\mathrm{Q}\mathcal{F}$-closure, one takes F to be a maximal subfield of $F_0^{\mathrm{rQ}\mathcal{F}}$ containing F_0^{rQ} and embeddable in the completion of (F_0^{rQ},w), and uses Lemma 2.9 in the place of Lemma 2.8.)

Now let wL be non-archimedean. Since it is finitely generated, it has finite rank. So we can proceed by induction on the rank. Let H be the largest proper convex subgroup of wL. Since H is finitely generated, we can choose a system $\alpha_1,\ldots,\alpha_\ell \in L$ of rationally independent generators of H. We take w' to be a convex valuation on M whose restriction to L is the valuation associated with H. Since wL is finitely generated, we can choose a system $\alpha_{\ell+1},\ldots,\alpha_m \in L$ of rationally independent generators of wL/H. It follows that α_i, $\ell < i \le m$, are rationally independent over H, and that α_i, $1 \le i \le m$, form a rationally independent system of generators of wL. W.l.o.g., we may assume that there is some $m' \le n$ such that $I\setminus I_w = \{1,\ldots,m'\}$. Then also wx_i, $1 \le i \le m'$, form a rationally independent set of generators of wL, and we find that $m = m'$. Therefore, there is an invertible matrix (μ_{ij}), $\mu_{ij} \in \mathbb{Z}$, such that $\alpha_i = \sum_{j=1}^m \mu_{ij}wx_j$. We set

$$y_i \;:=\; \prod_{j=1}^m x_j^{\mu_{ij}} \;\in K(x_1,\ldots,x_m) \qquad (1 \le i \le m)$$

so that $wy_i = \alpha_i$. If (ν_{ij}) denotes the inverse of (μ_{ij}), then $x_i = \sum_{j=1}^m \nu_{ij}y_j \in K(x_1,\ldots,x_n)$. This implies that

$$L \;=\; K(x_1,\ldots,x_n) \;=\; K(y_1,\ldots,y_m)\;.$$

The rank of $wK(y_1,\ldots,y_\ell) = H$ is smaller than that of wL. Hence by induction hypothesis,

$$wK(y_1,\ldots,y_\ell)^{\mathrm{h}\mathcal{F}} \;=\; \bigoplus_{1\le i\le \ell} \mathbb{Z}wy_i \quad\text{and}\quad K(y_1,\ldots,y_\ell)^{\mathrm{h}\mathcal{F}}w \;=\; K^{\mathrm{h}\mathcal{F}}\;. \tag{26}$$

On the other hand, the value group $w'K(y_1,\ldots,y_m) = wL/H$ is archimedean since H was chosen to be the largest proper convex subgroup of wL. By our choice of the elements y_i, $w'y_i = 0$ for $1\le i\le \ell$, and the values $w'y_{\ell+1},\ldots,w'y_m$ are rationally independent. Thus, we can replace w by w' and apply the assertion of our lemma, which is already proved in the archimedean case, to deduce that

$$w'K(y_1,\ldots,y_m)^{\mathrm{h}\mathcal{F}} \;=\; \bigoplus_{\ell< i\le m} \mathbb{Z}w'y_i \;\text{ and }\; K(y_1,\ldots,y_m)^{\mathrm{h}\mathcal{F}}w' \;=\; K(y_1,\ldots,y_\ell)^{\mathrm{h}\mathcal{F}}\;. \tag{27}$$

Replacing w' by an equivalent valuation if necessary, we can write

$$w'K(y_1,\ldots,y_m)^{\mathrm{h}\mathcal{F}} \;=\; wK(y_1,\ldots,y_m)^{\mathrm{h}\mathcal{F}}/w(K(y_1,\ldots,y_m)^{\mathrm{h}\mathcal{F}}w')\;.$$

Note that $w'y_i$ is the coset of wy_i in this quotient group. Therefore we obtain, using also (26) and (27),

$$\begin{aligned}
wK(y_1,\ldots,y_m)^{\mathrm{h}\mathcal{F}} &= \bigoplus_{\ell< i\le m} \mathbb{Z}wy_i \,\oplus\, w(K(y_1,\ldots,y_m)^{\mathrm{h}\mathcal{F}}w')\\
&= \bigoplus_{\ell< i\le m} \mathbb{Z}wy_i \,\oplus\, wK(y_1,\ldots,y_\ell)^{\mathrm{h}\mathcal{F}} \;=\; \bigoplus_{\ell< i\le m} \mathbb{Z}wy_i \,\oplus\, \bigoplus_{1\le i\le \ell} \mathbb{Z}wy_i\\
&= \bigoplus_{1\le i\le m} \mathbb{Z}wy_i \;=\; H \;=\; \bigoplus_{i\in I\setminus I_w} \mathbb{Z}wx_i\;.
\end{aligned}$$

It follows that

$$w\mathcal{R}(x_1,\ldots,x_n)^{\mathrm{h}\mathcal{F}} \;=\; wK(x_1,\ldots,x_m)^{\mathrm{h}\mathcal{F}} \;=\; wK(y_1,\ldots,y_m)^{\mathrm{h}\mathcal{F}} \;=\; \bigoplus_{i\in I\setminus I_w} \mathbb{Z}wx_i\;.$$

Again by (26) and (27),

$$\begin{aligned}
\mathcal{R}(x_1,\ldots,x_n)^{\mathrm{h}\mathcal{F}}w &= K(x_1,\ldots,x_m)^{\mathrm{h}\mathcal{F}}w \;=\; K(y_1,\ldots,y_m)^{\mathrm{h}\mathcal{F}}w\\
&= (K(y_1,\ldots,y_m)^{\mathrm{h}\mathcal{F}}w')w \;=\; K(y_1,\ldots,y_\ell)^{\mathrm{h}\mathcal{F}}w\\
&= K^{\mathrm{h}\mathcal{F}} \;=\; \mathcal{R}(x_i\mid i\in I_w)^{\mathrm{h}\mathcal{F}}\;.
\end{aligned}$$

□

Let us note that the result of this lemma remains true if the henselization with respect to v is replaced by the henselization with respect to any convex valuation. — The lemma shows in particular that if the values vx_i, $i\in I$, are Q-linearly independent, then

$$v\mathbb{R}(x_i\mid i\in I)^{\mathrm{rQ}\mathcal{F}} \;=\; \bigoplus_{i\in I} \mathrm{Q}vx_i \quad\text{and}\quad v\mathbb{R}(x_i\mid i\in I)^{\mathrm{h}\mathcal{F}} \;=\; \bigoplus_{i\in I} \mathbb{Z}vx_i\;. \tag{28}$$

Corollary 4.3 *Take $x_i \in M$ such that the values vx_i, $i \in I$ are* Q*-linearly independent over $v\mathcal{R}$. Further, let w be any convex valuation. Then there exist some index set J_w and algebraically independent elements $y_j \in \mathcal{R}(x_i \mid i \in I)$, $j \in J_w$, such that*

$$\mathcal{R}(x_i \mid i \in I)^{\mathrm{rQ}\mathcal{F}} w \;=\; \mathcal{R}(y_j \mid j \in J_w)^{\mathrm{rQ}\mathcal{F}} \;.$$

Proof: By Zorn's Lemma, choose a maximal subset $I'_w \subset I$ such that the values wx_i, $i \in I'_w$, are Q-linearly independent. We set $J_w := I \setminus I'_w$. Then for every $j \in J_w$, there are $i_1, \ldots, i_\ell \in I'_w$ and $q, q_1, \ldots, q_\ell \in \mathrm{Q}$, $q \neq 0$, such that $wy_j = 0$ for $y_j := x_j^q \cdot x_{i_1}^{q_1} \cdot \ldots \cdot x_{i_\ell}^{q_\ell}$. Then $x_j^q \in \mathcal{R}(x_i, y_j \mid i \in I'_w, j \in J_w)^{\mathrm{rQ}} =: L$ and thus, $x_j \in L$. Therefore, $L^{\mathrm{rQ}\mathcal{F}} = \mathcal{R}(x_i \mid i \in I)^{\mathrm{rQ}\mathcal{F}}$. From Lemma 4.2 it follows that w is trivial on $\mathcal{R}(y_j \mid j \in J_w)^{\mathrm{rQ}\mathcal{F}}$. By Lemma 4.2, $L^{\mathrm{rQ}\mathcal{F}} w = \mathcal{R}(y_j \mid j \in J_w)^{\mathrm{rQ}\mathcal{F}}$. □

For use in Sections 5 and 6, we add the following lemma:

Lemma 4.4 *Let $x_i \in M$ such that $x_i > 0$ and the values vx_i, $i \in I$ are* Q*-linearly independent over $v\mathcal{R}$. Then*

$$\mathcal{R}(x_i \mid i \in I)^{\mathrm{r}\mathcal{F}} \;=\; \bigcup_{I_0 \subset I \text{ finite}} \; \bigcup_{k \in \mathbb{N}} \; \mathcal{R}(x_i^{1/k} \mid i \in I_0)^{\mathrm{h}\mathcal{F}} \tag{29}$$

with

$$v\mathcal{R}(x_i^{1/k} \mid i \in I_0)^{\mathrm{h}\mathcal{F}} \;=\; v\mathcal{R} \oplus \bigoplus_{i \in I_0} \mathbb{Z}\frac{vx_i}{k} \;,$$

finitely generated over $v\mathcal{R}$ (as a $\mathbb{Z}$-module). Therefore, $v_{\mathcal{R}}\mathcal{R}(x_i^{1/k} \mid i \in I_0)^{\mathrm{h}\mathcal{F}}$ is finitely generated.

Proof: The assertion for the value group follows from Lemma 4.2. Let U denote the union on the right hand side of (29). Every field in the union is henselian, so U is henselian. The value group vU is divisible and the residue field $Uv = \mathbb{R}$ is real closed. Hence by Lemma 2.9, U is real closed. By construction, U is also $\mathcal{F}$-closed. Since all $x_i^{1/k}$ are in the real closed field $\mathcal{R}(x_i \mid i \in I)^{\mathrm{r}\mathcal{F}}$, we find that U is contained in $\mathcal{R}(x_i \mid i \in I)^{\mathrm{r}\mathcal{F}}$. Since this field is the smallest real closed and $\mathcal{F}$-closed field containing $\mathcal{R}(x_i \mid i \in I)$, it follows that $U = \mathcal{R}(x_i \mid i \in I)^{\mathrm{r}\mathcal{F}}$. □

Now we are able to improve the results from Section 3.3 for the present more special setting.

Lemma 4.5 *Assume that w is trivial on $\mathcal{R}$. Set $L = LE^w_{\mathcal{R},\mathcal{F}}(x)$ if $w \neq v_{\mathcal{R}}$, and $L = \mathcal{R}$ if $w = v_{\mathcal{R}}$. Suppose that K is of the form*

$$L(x_i \mid i \in J)^{\mathrm{rQ}\mathcal{F}} \text{ log-closed, with } x_i > 0 \text{ and } wx_i,\ i \in J,\ \mathrm{Q}\text{-linearly independent.} \tag{30}$$

Take any $a \in K$ such that $\exp a \notin K$. Then $w \exp a$ is Q*-linearly independent over wK, and*

$$wK(\exp a)^{\mathrm{rQ}\mathcal{F}} \;=\; wK \oplus \mathrm{Q}\, v \exp a \;. \tag{31}$$

Moreover, $K(\exp a)^{\mathrm{rQ}\mathcal{F}}$ is again log-closed, and therefore of the form (30). It contains $\exp b$ whenever $b \in K(\exp a)^{\mathrm{rQ}\mathcal{F}}$ and $w \exp b$ is Q*-linearly dependent over $wK(\exp a)^{\mathrm{rQ}\mathcal{F}}$.*

Proof: By our choice of L, it is a subfield of $\mathcal{O}_w \cap K$, relatively exp-closed in $\mathcal{O}_w^\times$. Also, it is of the form $\mathcal{R}(x_i \mid i \in I)^{\mathrm{rQ}\mathcal{F}}$ with $x_i > 0$ and vx_i, $i \in I$, Q-linearly independent over $v\mathcal{R}$. Since the values wx_i, $i \in J$, are Q-linearly independent over $\{0\} = wL$, the values vx_i, $i \in I_w \cup J$, are Q-linearly independent over $v\mathcal{R}$. With $I = I_w \cup J$ we infer from Lemma 4.2 that $Kw = L$. Thus, we can apply Lemma 3.3. We obtain that $w \exp a$ is Q-linearly independent over wK and that $\exp b \in K(\exp a)^{\mathrm{rQ}\mathcal{F}}$ whenever $b \in K(\exp a)^{\mathrm{rQ}\mathcal{F}}$ and $w \exp b$ is Q-linearly dependent over $wK(\exp a)^{\mathrm{rQ}\mathcal{F}}$. Equation (31) follows by an application of Lemma 4.2 to $K(\exp a)^{\mathrm{rQ}\mathcal{F}}$. Since the values $w \exp a$, wx_i, $i \in J$, are Q-linearly independent, the values $v \exp a$, vx_i, $i \in I$, are Q-linearly independent over $v\mathcal{R}$. Thus, we can infer from Lemma 3.4 that $K(\exp a)^{\mathrm{rQ}\mathcal{F}}$ is log-closed. □

After these preparations, we can determine the residue field of $LE_{\mathcal{R},\mathcal{F}}(x)$ with respect to a given convex valuation w of M which is trivial on $\mathcal{R}$. As we know already that $LE_{\mathcal{R},\mathcal{F}}(x)v_{\mathcal{R}} = \mathcal{R}$, we can assume that $w \neq v_{\mathcal{R}}$. As usual, we assume that $x \in M$ is such that $x > \mathcal{R}$, that is, $v_{\mathcal{R}}x < 0$. We also assume that $wx = 0$. By our construction, $LE^w_{\mathcal{R},\mathcal{F}}(x) \subset LE_{\mathcal{R},\mathcal{F}}(x)$. So we can rerun our construction of $LE^u_{\mathcal{R},\mathcal{F}}(x) = LE_{\mathcal{R},\mathcal{F}}(x)$ starting with $K_0^u = LE^w_{\mathcal{R},\mathcal{F}}(x)$. We note that K_0^u is of the form $\mathcal{R}(x_i \mid i \in I_w)^{\mathrm{rQ}\mathcal{F}}$, where the x_i, $i \in I_w$, are obtained from the above construction (and thus, their values $v_{\mathcal{R}}x_i \in H_w$ are Q-linearly independent). Since $K_0^u \subset \mathcal{O}_w$, we have that $K_0^u w = K_0^u$ and that $wx_i = 0$ for $i \in I_w$. Suppose that while building up $LE^u_{\mathcal{R},\mathcal{F}}(x)$ from this field by the above construction, we have reached a field K of the form $\mathcal{R}(x_i \mid i \in I)^{\mathrm{rQ}\mathcal{F}}$ with $Kw = K_0^u$ and such that the values wx_i, $i \in I \setminus I_w$, are Q-linearly independent. If $a \in K$, but $\exp a \notin K$, then Lemma 4.5 shows that $w \exp a$ is Q-linearly independent over wK. Therefore, the values $w \exp a, wx_i$, i, are Q-linearly independent, and Lemma 4.2 shows that

$$K(\exp a)^{\mathrm{rQ}\mathcal{F}}w \;=\; \mathcal{R}(\exp a, x_i \mid i \in I)^{\mathrm{rQ}\mathcal{F}}w \;=\; \mathcal{R}(x_i \mid i \in I_w)^{\mathrm{rQ}\mathcal{F}} \;=\; K_0^u \,.$$

Hence, $K(\exp a)^{\mathrm{rQ}\mathcal{F}}$ is again of the same form as K. By induction, it follows that

$$LE_{\mathcal{R},\mathcal{F}}(x)w \;=\; LE^w_{\mathcal{R},\mathcal{F}}(x) \quad \text{if } wx = 0.$$

In our above considerations, the only assumption on x was that it is a positive infinite element. So we can well replace it by $\log_{m_0} x$, for arbitrary $m_0 \in \mathbb{N}$. Note that $LE_{\mathcal{R},\mathcal{F}}(x) = LE_{\mathcal{R},\mathcal{F}}(\log_{m_0} x)$. If $w \log_{m_0} x = 0$, then we find that

$$LE_{\mathcal{R},\mathcal{F}}(x)w \;=\; LE^w_{\mathcal{R},\mathcal{F}}(\log_{m_0} x)\,. \tag{32}$$

Theorem 4.6 *Let w be an arbitrary convex valuation of M, trivial on $\mathcal{R}$ but different from $v_{\mathcal{R}}$. Then there is an integer $m_0 \geq 0$ such that $w \log_{m_0} x = 0$. With every such m_0, equation (32) holds. If $wx = 0$, then we can choose $m_0 = 0$.*

Proof: Starting our above construction from $K_0^u = \mathcal{R}(\log_m x \mid m \geq 0)^{\mathrm{rQ}\mathcal{F}}$, we can write $LE_{\mathcal{R},\mathcal{F}}(x) = \bigcup_{n\in\mathbb{N}} K_n^u$. Take any negative element α in the convex subgroup H_w of $vLE_{\mathcal{R},\mathcal{F}}(x)$ associated with w. Then there is some $n \in \mathbb{N}$ and a positive $a \in K_n^u$ such that

$\alpha = va$. By Lemma 3.9, $v \log_n a \in K_0^u$. Corollary 2.13 tells us that $va < v\log_n a < v\mathcal{R}$. On the other hand, the values $v\log_m x$ are not bounded away from the subgroup $v\mathcal{R}$ in the value group $v\mathcal{R}(\log_m x \mid m \geq 0)^{\mathrm{rQ}\mathcal{F}}$. So there is some m_0 such that $v\log_n a < v\log_{m_0} x < 0$. Thus, $\alpha < v\log_{m_0} x < 0$, which yields that $v\log_{m_0} x \in H_w$. That is, $w\log_{m_0} x = 0$, and equation (32) holds. □

From this theorem together with the uniqueness of $LE^w_{\mathcal{R},\mathcal{F}}(x)$ (which also works with $\log_{m_0} x$ in the place of x), we obtain:

Theorem 4.7 *Let w be an arbitrary convex valuation of $LE_{\mathcal{R},\mathcal{F}}(x)$, and denote its valuation ring by $\mathcal{O}_w$. Then there exists a real closed subfield $K \subset \mathcal{O}_w$ which is log-closed and $\mathcal{F}$-closed, relatively exp-closed in $\mathcal{O}_w^\times$ and satisfies $LE_{\mathcal{R},\mathcal{F}}(x)w = K$. If w is not the natural valuation, then there is some integer $m_0 \geq 0$ such that K can be chosen to be the uniquely determined smallest subfield of $\mathcal{O}_w$ which is real closed, log- and $\mathcal{F}$-closed, relatively exp-closed in $\mathcal{O}_w^\times$ and contains $\mathbb{R}(\log_{m_0} x)$. If $wx = 0$, then we can choose $m_0 = 0$, so that K contains x.*

Further, if $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_T$, then the smallest log- and $\mathrm{rQ}\mathcal{F}_2$-closed subfield of $\mathcal{O}_w$, relatively exp-closed in $\mathcal{O}_w^\times$ and containing $\mathcal{R}(x)$, is contained in the smallest log- and $\mathrm{rQ}\mathcal{F}_1$-closed subfield of $\mathcal{O}_w$, relatively exp-closed in $\mathcal{O}_w^\times$ and containing $\mathbb{R}(x)$. Hence:

Corollary 4.8 *Suppose that $\mathcal{F}_1 \subset \mathcal{F}_2$ are subsets of $\mathcal{F}_T$, both satisfying conditions (PADE) and (COMP). Then for every convex valuation w such that $wx = 0$,*

$$LE^w_{\mathcal{R},\mathcal{F}_1}(x) \subset LE^w_{\mathcal{R},\mathcal{F}_2}(x) .$$

5 Further applications

In this section we show how our approach can be used to deduce the applications which van den Dries, Macintyre and Marker give in their paper [D–M–M2].

We take $M = H(\mathbb{R}_{\mathrm{an,exp}})$ and x to be the germ of the identity function. Recall that this choice yields that $H(\mathbb{R}_{\mathrm{an,exp}}) = LE_{\mathcal{F}_{\mathrm{an}}}(x)$ and $LE = LE_{\mathcal{F}_{LE}}(x)$.

We deduce Corollary 2.10 of [D–M–M2]:

Corollary 5.1 *If $f : \mathbb{R} \to \mathbb{R}$ is definable in $\mathbb{R}_{\mathrm{an,exp}}$, then there are $c \in \mathbb{R}$ and $n \in \mathbb{N}$ such that $f(z) < \exp_n(z)$ for all $z > c$.*

Proof: Let $f \in H(\mathbb{R}_{\mathrm{an,exp}})$ denote the germ of the function $f(z)$. By Lemma 3.10, there is some $n \in \mathbb{N}$ such that $f < \exp_n x$ (as elements in the ordered field $H(\mathbb{R}_{\mathrm{an,exp}})$). Since this says that the germ of $\exp_n z$ is bigger than that of $f(z)$, it follows that $f(z) < \exp_n(z)$ for all large enough $z \in \mathbb{R}$. □

From now on, we will not any more distinguish the variable x from the germ x of the identity function. Note that if f is definable in $\mathbb{R}_{\mathrm{an,exp}}$ and $g \in H(\mathbb{R}_{\mathrm{an,exp}})$ is the germ of the function $g(x)$, then the element $f(g) \in H(\mathbb{R}_{\mathrm{an,exp}})$ is defined to be the germ of the function $f(g(x))$; in this way, f is made into a function on $H(\mathbb{R}_{\mathrm{an,exp}})$. In particular, the element $f(x) \in H(\mathbb{R}_{\mathrm{an,exp}})$ is the germ of the function $f(x)$.

5.1 The Hardy problem

Take two functions $f, g : \mathbb{R} \to \mathbb{R}$, definable in $\mathbb{R}_{\mathrm{an,exp}}$. Assume that $\exp f(x)$ is asymptotic to $g(x)$, that is, $\lim_{x\to\infty} \frac{\exp f(x)}{g(x)} = 1$. This is equivalent to $\lim_{x\to\infty} f(x) - h(x) = 0$, where $h : (r, \infty) \to \mathbb{R}$ for suitable $r \in \mathbb{R}$ is the function $\log g(x)$, which again is definable in $\mathbb{R}_{\mathrm{an,exp}}$. This means that the function $f(x) - h(x)$ is ultimately smaller than every nonzero constant function. Equivalently, its germ $f - h$ in $H(\mathbb{R}_{\mathrm{an,exp}})$ is infinitesimal, or in other words, $v(f - h) > 0$.

As in [D–M–M2], let the function $i(x)$ denote the compositional inverse of the function $x \log x$. Identifying $i(x)$ with its germ, we have that $i(x) \in H(\mathbb{R}_{\mathrm{an,exp}})$. But by an argument about Liouville extensions of the Hardy field $\mathbb{R}(x)$, Corollary 4.6 of [D–M–M2] shows that $i(x) \notin LE$. Assume that $\exp i(x)$ were asymptotic to a function $g(x)$ which is a composition of semialgebraic functions, exp and log. Through identification with its germ, the latter means that $g(x) \in LE$. Then also $h(x) := \log g(x) \in LE$, and $v(i(x) - h(x)) > 0$. Further, one shows as in [D–M–M2] that there is a convergent power series $f(X, Y)$ such that

$$i(x) \;=\; \frac{x}{\log x}\left(1 + f\left(\frac{\log\log x}{\log x}, \frac{1}{\log x}\right)\right) \;.$$

Now let w be the convex valuation corresponding to the largest convex subgroup not containing vx. This contains $v \log x$. Therefore, $w \log x = 0$ and $w\frac{\log x}{x} = -wx > 0$. With

$$\tilde{h} \;:=\; \frac{\log x}{x}\, h(x) - 1 \;\in\; LE \;,$$

and

$$\tilde{f} := f\left(\frac{\log\log x}{\log x}, \frac{1}{\log x}\right) \in \mathbb{R}(\log x, \log\log x)^{r\mathcal{F}_{\mathrm{an}}} \;\subset\; LE^{w}_{\mathcal{F}_{\mathrm{an}}}(\log x) \;, \tag{33}$$

we find that

$$v(\tilde{f} - \tilde{h}) > v\,\frac{\log x}{x} \;. \tag{34}$$

By Lemma 2.1 it follows that $w(\tilde{f} - \tilde{h}) \geq w\frac{\log x}{x} > 0$. Note that $w\tilde{f} = 0$ according to (33), so it follows that also $w\tilde{h} = 0$. Hence, by Theorem 4.7 we can choose an element $\tilde{h}_w \in LE^{w}_{\mathcal{F}_{LE}}(\log x)$ such that $w(\tilde{h} - \tilde{h}_w) > 0$. It follows that $w(\tilde{f} - \tilde{h}_w) > 0$. By Corollary 4.8, $LE^{w}_{\mathcal{F}_{LE}}(\log x) \subset LE^{w}_{\mathcal{F}_{\mathrm{an}}}(\log x)$. Therefore, $\tilde{f} - \tilde{h}_w \in LE^{w}_{\mathcal{F}_{\mathrm{an}}}(\log x)$. Since $wg \in \{0, \infty\}$ for every $g \in LE^{w}_{\mathcal{F}_{\mathrm{an}}}(\log x)$ (as this is a field of representatives for the w-residue field), we find that $w(\tilde{f} - \tilde{h}_w) = \infty$. In other words,

$$\tilde{f} = \tilde{h}_w \in LE^{w}_{\mathcal{F}_{LE}}(\log x) \subset LE \;.$$

But then $i(x) = \frac{x}{\log x}(1 + \tilde{f}) \in LE$, a contradiction. This proves that $\exp i(x)$ is not asymptotic to any function with germ in LE.

5.2 Undefinable functions

We choose a representation $H(\mathbb{R}_{\mathrm{an,exp}}) = \mathbb{R}(x_i \mid i \in I)^{\mathrm{r}\mathcal{F}_{\mathrm{an}}}$ with vx_i, $i \in I$, rationally independent, which exists by Theorem 3.11. For the applications, we will assume in addition that x is among the x_i.

Lemma 5.2 *Take any positive infinitesimal element t in $H(\mathbb{R}_{\mathrm{an,exp}})$. Suppose that the element $h \in H(\mathbb{R}_{\mathrm{an,exp}})$ satisfies*

$$\left| h - \sum_{n=0}^{m} r_n t^n \right| < r'_m t^m \quad \textit{for all } m \in \mathbb{N},$$

where $r_n, r'_n \in \mathbb{R}$, $r'_n > 0$. Then $\sum_{n=0}^{\infty} r_n X^n$ converges in $\mathbb{R}$ near 0.

Proof: The assertion is trivial if there is some n_0 such that $r_n = 0$ for all $n > n_0$ (which in particular will hold if $r'_n = 0$ for some n). So let us assume that this is not the case. If n_k ($k \geq 0$) denotes the $k+1$-th among the indeces n for which $r_n \neq 0$, then we set $s_k = r_{n_k} \neq 0$, $s'_k = r'_{n_k} \neq 0$ and $z_k = t^{n_k}$. Then by Lemma 2.10, h is a limit of the pseudo Cauchy sequence formed by the partial sums $S_m = \sum_{k=0}^{m} r_{n_k} t^{n_k}$. Note that $vh = vS_0 = vt^{n_0} = n_0 vt$ with $n_0 \in \mathbb{N}$.

Let H be the convex subgroup of $vH(\mathbb{R}_{\mathrm{an,exp}})$ generated by vt, and w the convex valuation associated with H. Then $wt = 0$ and $wh = 0$. By Theorem 4.7, we can choose an element $h_w \in LE^w_{\mathcal{F}_{\mathrm{an}}}(t^{-1})$ such that $w(h - h_w) > 0$. That is, $v(h - h_w) > vt^{n_{m+1}} = v(S_{m+1} - S_m)$ for all m. So Lemma 2.4 shows that h_w is a limit of $(S_m)_{m \geq 0}$, too. Since $v\mathbb{R}(t)^{\mathrm{r}\mathcal{F}_{\mathrm{an}}} = \mathbb{Q}vt$ (cf. Lemma 3.2), this is a Cauchy sequence in $(\mathbb{R}(t)^{\mathrm{r}\mathcal{F}_{\mathrm{an}}}, v)$. Since $\mathbb{Q}vt$ is cofinal in $vLE^w_{\mathcal{F}_{\mathrm{an}}}(t^{-1})$ by our choice of H, it is also a Cauchy sequence in $(LE^w_{\mathcal{F}_{\mathrm{an}}}(t^{-1}), v)$. Hence, h_w is the only limit that the sequence admits in this field. If $h_w \in \mathbb{R}(t)^{\mathrm{r}\mathcal{F}_{\mathrm{an}}}$, then trivially, $v\mathbb{R}(t)^{\mathrm{r}\mathcal{F}_{\mathrm{an}}}(h_w) = \mathbb{Q}vt$. Otherwise, this follows by Lemma 2.6. Thus by Corollary 3.7 of [D–M–M1], $v\mathbb{R}(h_w, t)^{\mathrm{r}\mathcal{F}_{\mathrm{an}}} = \mathbb{Q}vt$.

In view of (28), we can write $vt = \sum_{i \in I} q_i vx_i$ with $q_i \in \mathbb{Q}$, only finitely many of them nonzero. Take $i_0 \in I$ with $q_{i_0} \neq 0$. Then by the rational independence of the values vx_i,

$$vt \notin \sum_{i \in I \setminus \{i_0\}} \mathbb{Q}vx_i = v\mathbb{R}(x_i \mid i \in I \setminus \{i_0\})^{\mathrm{r}\mathcal{F}_{\mathrm{an}}} .$$

So $t \notin \mathbb{R}(x_i \mid i \in I \setminus \{i_0\})^{\mathrm{r}\mathcal{F}_{\mathrm{an}}}$. An application of the Exchange Lemma for o-minimal theories ([P–S]) to this model of T_{an} shows that $x_{i_0} \in \mathbb{R}(t, x_i \mid i \in I \setminus \{i_0\})^{\mathrm{r}\mathcal{F}_{\mathrm{an}}}$. Hence, $H(\mathbb{R}_{\mathrm{an,exp}}) = \mathbb{R}(t, x_i \mid i \in I \setminus \{i_0\})^{\mathrm{r}\mathcal{F}_{\mathrm{an}}}$. Moreover, the values vt, vx_i, $i \in I \setminus \{i_0\}$, are rationally independent. Now choose $\{x_1, \ldots, x_\ell\} \subset \{x_i \mid i \in I \setminus \{i_0\}\}$ with ℓ minimal such that $h_w \in \mathbb{R}(t, x_1, \ldots, x_\ell)^{\mathrm{r}\mathcal{F}_{\mathrm{an}}}$. Suppose that $\ell > 0$. Because of the minimality of ℓ, it follows from the Exchange Lemma that $x_\ell \in \mathbb{R}(h_w, t, x_1, \ldots, x_{\ell-1})^{\mathrm{r}\mathcal{F}_{\mathrm{an}}} = \mathbb{R}(h_w, t)^{\mathrm{r}\mathcal{F}_{\mathrm{an}}}(x_1, \ldots, x_{\ell-1})^{\mathrm{r}\mathcal{F}_{\mathrm{an}}}$. By Lemma 3.2 and what we have shown for $\mathbb{R}(h_w, t)^{\mathrm{r}\mathcal{F}_{\mathrm{an}}}$, we know that $v\mathbb{R}(h_w, t)^{\mathrm{r}\mathcal{F}_{\mathrm{an}}}(x_1, \ldots, x_{\ell-1})^{\mathrm{r}\mathcal{F}_{\mathrm{an}}} = \mathbb{Q}vt \oplus \mathbb{Q}vx_1 \oplus \ldots \oplus \mathbb{Q}vx_{\ell-1}$. But this group does not contain vx_ℓ. This contradiction shows that $\ell = 0$, i.e., $h_w \in \mathbb{R}(t)^{\mathrm{r}\mathcal{F}_{\mathrm{an}}}$.

Now let $\mathbb{R}\langle t \rangle$ denote the set of convergent Puiseux series in t, that is, the subset of the completion of $\mathbb{R}(t)^r$ consisting of all series $\sum_{n=n_0}^{\infty} r_n t^{n/k}$, where $n_0 \in \mathbb{Z}$, $k \in \mathbb{N}$, $r_n \in \mathbb{R}$, and

$\sum_{n=0}^{\infty} r_n X^n$ converges near 0. Then $\mathbb{R}\langle t\rangle$ is a real closed field such that it $f(X_1,\ldots,X_m)$ is a power series over $\mathbb{R}$ converging near 0 and $\varepsilon_1,\ldots,\varepsilon_m$ are infinitesimals in $\mathbb{R}\langle t\rangle$, then $f(\varepsilon_1,\ldots,\varepsilon_m)\in\mathbb{R}\langle t\rangle$. This shows that $\mathbb{R}\langle t\rangle$ is r$\mathcal{F}_{\rm an}$-closed. By its definition it is clear that the r$\mathcal{F}_{\rm an}$-closure of $\mathbb{R}(t)$ in $\mathbb{R}\langle t\rangle$ must be equal to $\mathbb{R}\langle t\rangle$. By induction along the lines of the proof of Lemma 4.1, one shows that there is a unique isomorphism $\mathbb{R}(t)^{{\rm r}\mathcal{F}_{\rm an}}\simeq\mathbb{R}\langle t\rangle$ (of valued fields) which is the identity on $\mathbb{R}(t)$. Since h_w is the limit of the Cauchy sequence $(S_m)_{m\geq 0}$, this isomorphism sends h_w to the unique limit $\sum_{k=0}^{\infty} r_{n_k}t^{n_k}$, which consequently must lie in $\mathbb{R}\langle t\rangle$. By definition of $\mathbb{R}\langle t\rangle$, $\sum_{k=0}^{\infty} r_{n_k}t^{n_k}=\sum_{n=0}^{\infty} r_nX^n$ must be convergent near 0. □

If a definable function $f:(r,+\infty)\to\mathbb{R}$ has an asymptotic expansion $f(x)\sim\sum r_nf_n(x)$ in the sense of [D–M–M2], then for some $C\in\mathbb{R}$, $C>0$,

$$\left|f(x)\;-\;\sum_{n=0}^{m} r_nf_n(x)\right|\;<\;Cf_m(x)$$

holds in $H(\mathbb{R}_{\rm an,exp})$ for all $m\in\mathbb{N}$. Hence if $f(x)\sim\sum r_nx^{-n}$, then with $t:=x^{-1}$, it follows from the foregoing lemma that $\sum_{n=0}^{m} r_nX^n$ is a convergent series. Using the asymptotic expansions as given in [D–M–M2], it follows that the Gamma-function and the functions

$$\int_0^x e^{-t^2}dt\,,\;\;\int_x^\infty \frac{e^{-t}}{t}\,dt\,,\;\;\int_0^\infty \frac{e^{-t}}{t+x}\,dt\,,\;\;\int_0^\infty \frac{e^{-t}}{1+xt}\,dt\,,\;\;\int_0^x e^{t^2}dt\,,\;\;\int_0^x e^{e^t}dt$$

on $(0,+\infty)$ are not definable in $\mathbb{R}_{\rm an,exp}$.

Lemma 5.3 *Suppose that the element $h\in H(\mathbb{R}_{\rm an,exp})$ satisfies*

$$\left|h\;-\;\sum_{n=0}^{m} d_n\right|\;<\;r'_m d_m\quad\textit{for all } m\in\mathbb{N}, \tag{35}$$

where $0<r'_n\in\mathbb{R}$, and the d_n are positive monomials such that the values vd_n are strictly increasing. Then these values are contained in a finitely generated subgroup of $vH(\mathbb{R}_{\rm an,exp})$.

Proof: From Lemma 4.4 we infer that $h\in\mathbb{R}(x_i^{1/k}\mid i\in I_0)^{{\rm h}\mathcal{F}_{\rm an}}=:K$ for some $k\in\mathbb{N}$ and some finite subset $I_0\subset I$, and that the value group of this field is the finitely generated subgroup $vK=\bigoplus_{i\in I_0}\mathbb{Z}\frac{vx_i}{k}$ of $vH(\mathbb{R}_{\rm an,exp})$. From the rational independence of the values vx_i it follows for every monomial d that $vd\in vK$ if and only if $d\in K$.

Suppose that $vd_n\notin vK$ for some $n\in\mathbb{N}$, and take n to be the smallest integer with this property. Then $d_j\in K$ for $1\leq j<n$. Consequently, $h-S_{n-1}\in K$. But by Lemma 2.10, $v(h-S_{n-1})=vd_n\notin vK$, a contradiction. □

For the application to the Riemann zeta function, we run our construction of $LE_{\mathcal{F}_{\rm an}}(x)$ with a slight refinement. We choose a $\mathbb{Q}$-basis $\mathcal{B}$ of $\mathbb{R}$ containing the $\mathbb{Q}$-linearly independent elements $\log p$, where p runs through all primes. Starting our construction from

$K_0^u = \mathbb{R}(\log_m x \mid m \geq 0)^{\mathrm{r}\mathcal{F}_{\mathrm{an}}}$, we may first adjoin all elements $\exp(rx)$ as new x_i's. Indeed, as the elements rx, $\log_m x$, $r \in \mathcal{B}$, $m \geq 1$, are rationally independent over the valuation ring, Lemma 2.14 shows that the values $v \exp(rx)$, $v \log_m x$, $r \in \mathcal{B}$, $m \geq 0$, are rationally independent. Hence, the values $v \exp(rx)$, $r \in \mathcal{B}$, are rationally independent over vK_0^u. Therefore, for all $s \in \mathcal{B}$, $\exp(sx) \notin K_0^u(\exp(rx) \mid r \in \mathcal{B} \setminus \{s\})$. So we can assume the elements $\exp(x \log p)$ to be among the x_i.

The restriction ζ of the zeta function to $(1, +\infty)$ has the asymptotic expansion $\zeta(x) \sim \sum \exp(-x \log n)$. Writing $n = \prod_{p\,\mathrm{prime}} p^{\nu_p}$ with integers $\nu_p \geq 0$, we obtain that

$$\begin{aligned} d_n := \exp(-x \log n) &= \exp(-x \log \prod_{p\,\mathrm{prime}} p^{\nu_p}) \\ &= \exp(-x \sum_{p\,\mathrm{prime}} \nu_p \log p) = \prod_{p\,\mathrm{prime}} (\exp(x \log p))^{-\nu_p}, \end{aligned}$$

is a monomial. As the sequence $\log n$, $n \in \mathbb{N}$, is strictly increasing, Lemma 2.14 shows that also vd_n, $n \in \mathbb{N}$, is strictly increasing.

If ζ were definable in $\mathbb{R}_{\mathrm{an,exp}}$, it would follow from the foregoing lemma that the values $v \exp(-x \log n)$, and in particular the values $v \exp(-x \log p)$, p prime, lie in a finitely generated group. But this is not the case since the latter values are rationally independent. This proves that the restriction of the zeta function to $(1, +\infty)$ is not definable in $\mathbb{R}_{\mathrm{an,exp}}$.

6 Principal parts of elements in $LE_{\mathcal{R},\mathcal{F}}(x)$

Throughout this section, we keep our general assumptions on $\mathcal{R}$ and $\mathcal{F}$ and also assume that $\mathcal{F}$ satisfies (PADE) and (COMP). We work with a representation

$$LE_{\mathcal{R},\mathcal{F}}(x) = \mathcal{R}(x_i \mid i \in I)^{\mathrm{r}\mathcal{R},\mathcal{F}} \quad \text{with } v_{\mathcal{R}} x_i,\ i \in I, \text{ rationally independent,} \tag{36}$$

which exists by Theorem 3.11. Note that it follows from our construction that for any two convex valuations w, w', trivial on $\mathcal{R}$,

$$LE_{\mathcal{R},\mathcal{F}}(x)w = LE^{w}_{\mathcal{R},\mathcal{F}}(x) \subset LE^{w'}_{\mathcal{R},\mathcal{F}}(x) = LE_{\mathcal{R},\mathcal{F}}(x)w' \quad \text{if } w \text{ is finer than } w'. \tag{37}$$

We note in advance that Lemmas 6.1 and 6.2 as well as Theorem 6.4 do not require any further assumptions on the x_i (not even that x is among them). Similarly, we only need (37), without any further information on the structure induced on the residue fields. So the results will hold for every field of the form (36), since for every real closed field, one can construct embeddings of the residue fields such that (37) holds. This can be done by transfinite induction on an arbitrary enumeration of the elements of the field. However, it should not be overlooked that Corollaries 6.5 and 6.6 and the application to the Hardy problem need additional assumptions on the x_i and the residue fields.

Let (K, w) be a valued field and $a \in K$. We write $a =_w \sum_{n=0}^{\infty} a_n$ if all $a_n \in K$ and
– either there is some m_0 such that $a = \sum_{n=0}^{m_0} a_n$, and $a_n = 0$ for all $n > m_0$,
– or the values wa_n form a strictly monotone cofinal sequence in wK and a is the (unique) limit of the Cauchy sequence $(\sum_{n=0}^{m} a_n)_{m \geq 0}$ in (K, w).

Lemma 6.1 *Let (K,w) be a valued field and assume that $Kw \subset K$. Further, let wK be archimedean. Suppose that $K = Kw\,(z_j \mid j \in J)$, where the values $wz_j\,,\ j \in J$, are rationally independent. Let*

$$\mathcal{D} := \{\prod_{j\in J_0} z_j^{n_j} \mid \ J_0 \subset J \text{ finite and } n_j \in \mathbb{Z} \text{ for every } j \in J_0\}.$$

Take any $a \in K$. Then there are uniquely determined elements $c_n \in Kw$ and $d_n \in \mathcal{D}$ such that

$$a \ =_w \ \sum_{n=0}^{\infty} c_n d_n \;.$$

The same holds for every element a in the henselization or the completion of (K,w).

Proof: Let R denote the subring of K consisting of all finite sums $c_1d_1+\ldots+c_md_m$ with $c_i \in Kw$ and $d_i \in \mathcal{D}$. We show that R is w-dense in K, that is, for every $a \in K$ and every $\alpha \in wK$ there is $a' \in R$ such that $w(a-a') > \alpha$. From the rational independence of the values wz_j it follows that every two distinct elements $d,d' \in \mathcal{D}$ have distinct values. On the other hand, every $a \in K$ can be written as a quotient of two polynomials in finitely many of the z_j, and therefore also as a quotient $\frac{b_1d_1+\ldots+b_md_m}{b'_1d'_1+\ldots+b'_\ell d'_\ell}$ where $d_1,\ldots,d_m \in \mathcal{D}$ are distinct and $d'_1,\ldots,d'_\ell \in \mathcal{D}$ are distinct, and $b_i,b'_i \in Kw$. We may assume that $b'_1d'_1$ is the summand of least value in the denominator. We write

$$b'_1d'_1+\ldots+b'_\ell d'_\ell \ = \ b'_1d'_1(1+d') \quad \text{with} \ \ d' \ := \ \frac{b'_2d'_2}{b'_1d'_1}+\ldots+\frac{b'_\ell d'_\ell}{b'_1d'_1} \;.$$

Note that $\frac{d'_2}{d'_1},\ldots,\frac{d'_\ell}{d'_1}$ are elements of $\mathcal{D}$ of positive value. Hence, also $wd' > 0$. By the geometric expansion,

$$w\left(\frac{1}{1+d'} - \sum_{i=0}^{k}(-d')^i\right) \ = \ (k+1)wd'$$

for every integer $k \geq 1$. Take $\alpha \in wK$. Since wK is archimedean, we can choose k as big as to obtain that $(k+1)wd' > \alpha - w(b_1d_1+\ldots+b_md_m)(b'_1d'_1)^{-1}$. For

$$a' := (b_1d_1+\ldots+b_md_m)(b'_1d'_1)^{-1}\sum_{i=0}^{k}(-d')^i \in R\,,$$

this yields that $w(a-a') > \alpha$.

Every valued field (K,w) is w-dense in its completion (by definition). Since wK is archimedean, then the henselization of (K,w) lies in the completion and thus, (K,w) is also w-dense in its henselization. Since density is transitive, we find that R is also w-dense in the henselization and in the completion of (K,w).

Every element of the ring R can be written as a sum $c_1d_1+\ldots+c_md_m$ with distinct $d_i \in \mathcal{D}$, and such that $wd_1 < wd_2 < \ldots < wd_m$. Its value is equal to wd_1. Therefore, such a sum can only be equal to 0 if it is trivial. Consequently, the representation of every element as a sum of this form is uniquely determined.

Now we choose $\alpha \in wK$, $\alpha > 0$. Then the sequence $k\alpha$, $k > 0$, is cofinal in the archimedean group wK. For every k, we choose $a_k \in R$ such that $w(a-a_k) > k\alpha$. For

$k' > k > 0$, $w(a_{k'} - a_k) \geq \min\{w(a - a_{k'}), w(a - a_k)\} > k\alpha$. Thus, the summands of value $\leq k\alpha$ in the representations of a_k and $a_{k'}$ have to be the same. So we take $c_n d_n$ to be the uniquely determined n-th summand appearing in the representation of all a_m, for m large enough. Since distinct elements of $\mathcal{D}$ have distinct values, d_n and thus also c_n is uniquely determined from the element $c_n d_n$. □

Lemma 6.2 *Take $h \in LE_{\mathcal{R},\mathcal{F}}(x)$. Then there are convex valuations w, w', trivial on $\mathcal{R}$, such that:*
a) the value group of $(LE_{\mathcal{R},\mathcal{F}}(x)w', w)$ is archimedean,
b) $h \in LE_{\mathcal{R},\mathcal{F}}(x)w' \setminus LE_{\mathcal{R},\mathcal{F}}(x)w$,
c) there are monomials $d_n \in LE_{\mathcal{R},\mathcal{F}}(x)w'$ and elements $c_n \in LE_{\mathcal{R},\mathcal{F}}(x)w$ such that in $(LE_{\mathcal{R},\mathcal{F}}(x)w', w)$,

$$h =_w \sum_{n=0}^{\infty} c_n d_n \,, \tag{38}$$

d) the summands $c_n d_n$ are uniquely determined,
e) the values $v_{\mathcal{R}} c_n d_n$ lie in a finitely generated subgroup of $v_{\mathcal{R}} LE_{\mathcal{R},\mathcal{F}}(x)$.

Proof: From Lemma 4.4 we infer that $h \in \mathcal{R}(x_i^{1/k} \mid i \in I_0)^{\mathrm{h}\mathcal{R},\mathcal{F}} =: K$ for some $k \in \mathbb{N}$ and some finite subset $I_0 \subset I$. Since $v_{\mathcal{R}} K$ is finitely generated, it has finite rank. That is, there are only finitely many distinct convex valuations on K, trivial on $\mathcal{R}$. Therefore, there are convex valuations w_0', w_0 on $LE_{\mathcal{R},\mathcal{F}}(x)$, trivial on $\mathcal{R}$, such that the value group $w_0(Kw_0')$ is archimedean and $h \in LE_{\mathcal{R},\mathcal{F}}(x)w_0' \setminus LE_{\mathcal{R},\mathcal{F}}(x)w_0$.

Every element in $v_{\mathcal{R}} K$ is the value of a monomial built up from the elements x_i, $i \in I_0$. Hence, we can choose monomials $z_1, \ldots, z_\ell \in K$ such that:
- $v_{\mathcal{R}} z_1, \ldots, v_{\mathcal{R}} z_{\ell_1}$ form a set of rationally independent generators of $v_{\mathcal{R}}(Kw_0)$,
- $w_0 z_{\ell_1+1} = v_{\mathcal{R}} z_{\ell_1+1} + v_{\mathcal{R}}(Kw_0), \ldots, w_0 z_{\ell_2} = v_{\mathcal{R}} z_{\ell_2} + v_{\mathcal{R}}(Kw_0)$ form a set of rationally independent generators of $w_0(Kw_0')$, and
- $w_0' z_{\ell_2+1} = v_{\mathcal{R}} z_{\ell_2+1} + v_{\mathcal{R}}(Kw_0'), \ldots, w_0' z_\ell = v_{\mathcal{R}} z_\ell + v_{\mathcal{R}}(Kw_0')$ form a set of rationally independent generators of $w_0' K$.

Similarly as in the proof of Lemma 4.2, one finds that $\ell = |I_0|$ and that $\mathcal{R}(z_1, \ldots, z_\ell)^{\mathrm{h}\mathcal{F}} = \mathcal{R}(x_i^{1/k} \mid i \in I_0)^{\mathrm{h}\mathcal{F}}$. From Lemma 4.2 it follows that $Kw_0 = \mathcal{R}(z_1, \ldots, z_{\ell_1})^{\mathrm{h}\mathcal{F}}$ and $Kw_0' = \mathcal{R}(z_1, \ldots, z_{\ell_2})^{\mathrm{h}\mathcal{F}}$. Now we have that $Kw_0 \subset Kw_0' \subset K$. Since $Kw_0 = K \cap LE_{\mathcal{R},\mathcal{F}}(x)w_0$ and $Kw_0' = K \cap LE_{\mathcal{R},\mathcal{F}}(x)w_0'$, we obtain that $h \in Kw_0' \setminus Kw_0$. Since $w_0(Kw_0') = \bigoplus_{\ell_1 < j \leq \ell_2} \mathbb{Z} w_0 z_j$ is archimedean, we can apply Lemma 6.1, where we set $J = \{\ell_1+1, \ldots, \ell_2\}$, to obtain the unique representation (38). Here, the d_n are monomials built up from $z_{\ell_1+1}, \ldots, z_{\ell_2}$. Thus, they are also monomials built up from x_i, $i \in I_0$. Note that the d_n depend on our choice of the elements z_j, $j = \ell_1 + 1, \ldots, \ell_2$. These in turn are uniquely determined only up to multiplication with monomials with trivial w-value. Thus, the d_n are in general not uniquely determined. However, the uniqueness of the summands $c_n d_n$ can be shown as in the proof of the foregoing lemma. The values $v_{\mathcal{R}} c_n d_n$ lie in the value group $v_{\mathcal{R}} K$, which is finitely generated, according to Lemma 4.4.

It remains to find appropriate valuations w, w' on $LE_{\mathcal{R},\mathcal{F}}(x)$. Since $h \notin Kw_0$, there is at least one summand $c_n d_n$ such that $w_0 c_n d_n \neq 0$. We take w to be the valuation associated with the smallest convex subgroup H of $v_{\mathcal{R}} LE_{\mathcal{R},\mathcal{F}}(x)$ containing $v_{\mathcal{R}} c_n d_n$. Then w is the finest convex valuation on $LE_{\mathcal{R},\mathcal{F}}(x)$ which coincides with w_0 on K. Similarly, the valuation w' associated with the largest convex subgroup H' of $v_{\mathcal{R}} LE_{\mathcal{R},\mathcal{F}}(x)$ not containing $v_{\mathcal{R}} c_n d_n$ is the coarsest convex valuation on $LE_{\mathcal{R},\mathcal{F}}(x)$ coinciding with w_0' on K. Finally, $w(LE_{\mathcal{R},\mathcal{F}}(x)w') = H/H'$ is archimedean. □

For each monomial $d \in LE_{\mathcal{R},\mathcal{F}}(x)$ we define w_d to be the convex valuation associated with the largest convex subgroup not containing $v_{\mathcal{R}} d$. Then w_d is the coarsest convex valuation such that $w_d d \neq 0$. The residue field $LE_{\mathcal{R},\mathcal{F}}(x)w_d$ can be thought of as the largest residue field "below d"; it is the largest residue field in which the residue of either d or d^{-1} is 0. Note that if $w_d d < w_d d' < 0$, then the largest convex subgroup not containing $v_{\mathcal{R}} d$ must be equal to that not containing $v_{\mathcal{R}} d'$ and therefore, $w_d = w_{d'}$. The following theorem is the intrinsic version of "truncation at 0".

Theorem 6.3 *Take $h \in LE_{\mathcal{R},\mathcal{F}}(x)$ such that $v_{\mathcal{R}} h < 0$. Then there exist $m \in \mathbb{N}$, monomials $d_n \in LE_{\mathcal{R},\mathcal{F}}(x)$, elements $c_n \in LE_{\mathcal{R},\mathcal{F}}(x)w_{d_n}$, $1 \leq n \leq m$, some $r_h \in \mathcal{R}$, and $h^+ \in LE_{\mathcal{R},\mathcal{F}}(x)$ of value $v_{\mathcal{R}} h^+ > 0$, such that*

$$h \;=\; c_0 d_0 + \ldots + c_m d_m + r_h + h^+ \quad \text{with} \;\; v_{\mathcal{R}} c_0 d_0 < \ldots < v_{\mathcal{R}} c_m d_m < 0\,, \qquad (39)$$

and such that $w_{d_n} d_n < w_{d_n} d_{n+1}$ for all $n < m$. The summands $c_n d_n$ and the elements r_h and h^+ are uniquely determined.

Proof: In the representation (38) of h given by Lemma 6.2, there are only finitely many summands $c_0 d_0, \ldots, c_{m_1} d_{m_1}$ of negative $v_{\mathcal{R}}$-value. Note that the valuation w' of that lemma coincides with w_{d_0}. Therefore, $c_0, \ldots, c_{m_1} \in LE_{\mathcal{R},\mathcal{F}}(x)w_{d_0}$. Assume that $w_{d_0} c_{m_1} d_{m_1} < 0$. Then also $w_{d_0} c_n d_n < 0$ and thus, $w_{d_0} = w_{d_n}$ for $0 \leq n \leq m_1$. It follows that $c_n \in LE_{\mathcal{R},\mathcal{F}}(x)w_{d_0} = LE_{\mathcal{R},\mathcal{F}}(x)w_{d_n}$ for $0 \leq n \leq m_1$ and $w_{d_n} d_n < w_{d_n} d_{n+1}$ for $0 \leq n < m_1$. We have that $v_{\mathcal{R}}(h - c_1 d_1 - \ldots - c_{m_1} d_{m_1}) \geq 0$. Consequently, there is a unique $r_h \in \mathcal{R}$ such that $h^+ := h - c_1 d_1 - \ldots - c_{m_1} d_{m_1} - r_h$ has value $v_{\mathcal{R}} h^+ > 0$, and we are done.

Now assume that $w_{d_0} c_{m_1} d_{m_1} = 0$ (note that $w_{d_0} c_n d_n < 0$ for $n < m_1$ by condition c) of Lemma 6.2). It follows that $c_{m_1} d_{m_1} \in LE_{\mathcal{R},\mathcal{F}}(x)w_{d_0}$, and we apply the lemma again to this element in the place of h. We repeat this procedure, thereby descending through the convex valuations of $LE_{\mathcal{R},\mathcal{F}}(x)$. But we are actually working with elements inside of the r$\mathcal{F}$-closure of the field K which we used in the proof of Lemma 6.2. Since the value group of K has finite rank, there are only finitely many distinct convex valuations on K. Therefore, after a finite repetition of our procedure, we reach a convex valuation which coincides with $v_{\mathcal{R}}$ on K (if the procedure doesn't stop before). If $c_\ell d_\ell, \ldots, c_m d_m$ are the summands obtained from Lemma 6.2 at this step, then by their choice we have that $w_{d_\ell} c_n d_n = v_{\mathcal{R}} c_n d_n < 0$ for $\ell \leq n \leq m$, and our procedure will stop here. Now necessarily $v_{\mathcal{R}}(h - c_1 d_1 - \ldots - c_m d_m) \geq 0$ since otherwise, we would have to obtain a further summand from our procedure. — The uniqueness of the summands $c_n d_n$ follows from the uniqueness assertion of Lemma 6.2. □

Given the representation (39) of an element h according to this theorem, the uniquely determined finite sum

$$\mathrm{pp}(h) \;:=\; c_1 d_1 + \ldots + c_m d_m$$

will be called the **principal part of** h; we set $\mathrm{pp}(h) := 0$ if $v_{\mathcal{R}} h \geq 0$. The principal part is uniquely determined, once the set of monomials in $LE_{\mathcal{R},\mathcal{F}}(x)$ is fixed. Note that $v_{\mathcal{R}}(h - \mathrm{pp}(h) - r_h) > 0$ with $r_h \in \mathcal{R}$.

Theorem 6.4 *Let $f, g : \mathcal{R} \to \mathcal{R}$ be ultimately positive $\mathcal{R}$-definable functions. Then f is asymptotic to rg on $\mathcal{R}$ for some positive $r \in \mathcal{R}$ if and only if the germs $\log f$ and $\log g$ in $H(\mathcal{R})$ have the same principal part.*

Proof: We know already that f is asymptotic to rg on $\mathcal{R}$ if and only if $v_{\mathcal{R}}(\log f - \log rg) > 0$. This in turn is equivalent to $v_{\mathcal{R}}(\log f - \log g) \geq 0$, since if the latter holds, then there is some $r_0 \in \mathcal{R}$ such that $v_{\mathcal{R}}(\log f - \log g - r_0) > 0$, and we set $r = \exp r_0$. By the uniqueness of the principal part, $v_{\mathcal{R}}(\log f - \log g) \geq 0$ if and only if $\mathrm{pp}(\log f) = \mathrm{pp}(\log g)$. □

To apply this theorem in the spirit of the Hardy problem, we take $\mathcal{F}$ to be any set of restricted analytic functions, closed under partial derivations. Then by running our construction of Section 3 simultaneously for $\mathcal{F}$ and $\mathcal{F}_{\mathrm{an}}$, we find index sets $I_{\mathcal{F}} \subset I$ and elements x_i such that $LE_{\mathbb{R},\mathcal{F}}(x) = \mathbb{R}(x_i \mid i \in I_{\mathcal{F}})^{\mathrm{r}\mathcal{F}}$ and $LE_{\mathbb{R},\mathcal{F}_{\mathrm{an}}}(x) = \mathbb{R}(x_i \mid i \in I)^{\mathrm{r}\mathcal{F}_{\mathrm{an}}}$. So the monomials of $LE_{\mathbb{R},\mathcal{F}}(x)$ will also be monomials of $LE_{\mathbb{R},\mathcal{F}_{\mathrm{an}}}(x)$. Moreover, we can take

$$LE_{\mathbb{R},\mathcal{F}}(x)w \;=\; LE^{w}_{\mathbb{R},\mathcal{F}}(\log_{m_0} x) \;\subset\; LE^{w}_{\mathbb{R},\mathcal{F}_{\mathrm{an}}}(\log_{m_0} x) \;=\; LE_{\mathbb{R},\mathcal{F}_{\mathrm{an}}}(x)w$$

for each convex valuation w and suitable m_0, according to Theorem 4.6 and Corollary 4.8. Using principal parts determined by this choice of the x_i and the residue fields, we get:

Corollary 6.5 *Assume that $h : \mathbb{R} \to \mathbb{R}$ is definable in $\mathbb{R}_{\mathrm{an,exp}}$. Then $\exp h$ is asymptotic to a composition of semialgebraic functions, exp, log and restricted analytic functions in $\mathcal{F}$, if and only if $\mathrm{pp}(h) \in LE_{\mathbb{R},\mathcal{F}}(x)$.*

As an example, let us reconsider the Hardy problem. Here we assume in addition that the x_i include x (cf. Theorem 3.11). We choose w as in Section 5.1. The representation of $i(x)$ is just $i(x) = cx$, where $c = \frac{1}{\log x}(1 + \tilde{f}) \in H(\mathbb{R}_{\mathrm{an,exp}})w$. Thus, $\mathrm{pp}(i(x)) = i(x) \notin LE$. Hence by our corollary, $\exp i(x)$ is not asymptotic to any element of LE.

Let us give a further application of Theorem 6.3. Denote by $\mathcal{L}_{\mathcal{F}}$ the language of ordered rings, enriched by symbols for the functions from $\mathcal{F}$. Recall that every generalized power series field $\mathbb{R}((G))$ has a canonical cross-section, sending $\alpha \in G$ to the element $1_\alpha \in \mathbb{R}((G))$ which has a 1 at α and zeros everywhere else. (1_α is the characteristic function of the singleton $\{\alpha\}$.)

Corollary 6.6 *Take any $\mathcal{L}_{\mathcal{F}}$-embedding of $LE_{\mathbb{R},\mathcal{F}}(x)$ in some generalized power series field $\mathbb{R}((G))$, and denote by L its image in $\mathbb{R}((G))$. Assume that the restriction of the canonical cross-section of $\mathbb{R}((G))$ to vL is a cross-section π of (L, v), and that $L = \mathbb{R}(\pi v L)^{\mathrm{r}\mathcal{F}}$. Then the nonzero elements of the support of each element in L are bounded away from 0.*

Proof: For every convex valuation w with associated convex subgroup $H_w \subset G$, we have that $\mathbb{R}((G))w = \mathbb{R}((H_w))$. These residue fields satisfy (37).

Let $I \subset vL$ be a maximal set of rationally independent values. Set $x_i := 1_\alpha$ for $i = \alpha \in I$. Then $\mathbb{R}(x_i \mid i \in I)^{\rm r} = \mathbb{R}(\pi vL)^{\rm r}$ and hence, $\mathbb{R}(x_i \mid i \in I)^{{\rm r}\mathcal{F}} = \mathbb{R}(\pi vL)^{{\rm r}\mathcal{F}} = L$ by hypothesis. The monomials obtained from the x_i are precisely the elements of the form $r \cdot 1_\alpha$ with $r \in \mathbb{R}$ and $\alpha \in vL$. Note that if $\alpha < H_w$, then for every $c \in \mathbb{R}((H_w))$, the support of $c\,r1_\alpha$ is bounded away from 0 by every element β which satisfies $\alpha + H_w < \beta < 0$. For example, $\beta = \alpha/2$ is a good choice.

Take $h \in L$ and consider the representation (39) with respect to the monomials x_i and the residue fields $\mathbb{R}((H_w))$. Now support$(h) \setminus \{0\}$ is the union of the support of $c_1d_1 + \ldots + c_md_m$ and the support of h^+. The latter is bounded away from 0 by vh^+. The support of $c_1d_1 + \ldots + c_md_m$ is the union of the supports of $c_1d_1, \ldots, c_md_m$. This union is bounded away from 0 by $\frac{1}{2}vd_m$. □

Note that the embeddings of $H(\mathbb{R}_{\rm an,exp})$ and of LE in the logarithmic power series field $\mathbb{R}((t))^{LE}$ given in [D–M–M2] satisfy the conditions of the corollary. (Recall that $\mathbb{R}((t))^{LE}$ can be viewed as a subfield of a suitable power series field.)

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