Why Do Asset Prices Not Follow Random Walks?[1]

Erik Lüders[2] and Günter Franke[3]

This draft: March 2004

[2]Laval University and Stern School of Business, New York University, e-mail: elueders@stern.nyu.edu

[3]University of Konstanz, e-mail: guenter.franke@uni-konstanz.de.

Abstract

This paper analyzes the effect of non-constant elasticity of the pricing kernel on asset return characteristics in a rational expectations model. It is shown that declining elasticity of the pricing kernel can lead to predictability of asset returns and high and persistent volatility. Also, declining elasticity helps to motivate technical analysis and to explain stock market crashes. Moreover, based on a general characterization of the pricing kernel, we propose analytical asset price processes which can be tested empirically. The numerical analysis reveals strong deviations from the geometric Brownian motion which are caused by declining elasticity of the pricing kernel.

JEL classification: G12

Keywords: Pricing Kernel, Viable asset price processes, Serial correlation, Heteroskedasticity, Stock market crashes

Empirical research suggests that returns of broad based market indices as the S&P 500 exhibit significant serial correlation and that return volatility is random. Many studies indicate that short run returns are positively autocorrelated while long run returns are negatively autocorrelated. Asset returns appear to exhibit short-term momentum and long-term reversals.[1] While it is controversial whether the predictability in returns is economically significant - especially concerns related to data-snooping are often expressed - studies on return volatility provide clear evidence against constant volatility and therefore against the geometric Brownian motion. Volatility is also found to be highly persistent and negatively correlated with asset returns. Moreover, there is an extensive literature on excess volatility which was started by Shiller (1981) and LeRoy and Porter (1981). These articles claim that the volatility of asset prices is too high to be consistent with classical asset pricing models. Moreover, the occurrence of stock market crashes without any significant news and the widespread use of technical analysis are often claimed to be incompatible with rational, efficient markets.[2] To explain these findings many researchers argue in favor of investor irrationality and new behavioral postulates. Another strand of empirical research in option pricing suggests that the elasticity of the pricing kernel is not constant.[3] The elasticity of the pricing kernel can be interpreted as the relative risk aversion of the representative investor. Therefore it plays a vital role in asset return processes.

In this paper we show that a simple rational expectations model can explain these asset price characteristics if the elasticity of the pricing kernel is assumed to be non-constant. In spite of the vast literature on asset pricing little is known on return characteristics when the pricing kernel has

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[1] There are different definitions of momentum and reversals. In this paper we define positive [negative] serial correlation as momentum [reversal].

[2] For an overview on return predictability and return volatility as well as a discussion of the methodological problems, see Campbell et. al. (1997) and Cochrane (2001). Ghysels et. al. (1996) provide an extensive overview on the characteristics of return volatility. Shiller (2000) provides evidence that stock market crashes may occur without significant news. For a recent study on the effectiveness of technical analysis see Lo et. al. (2000).

[3] See Jackwerth (2000), Rosenberg and Engle (2002) but also Bliss and Panigirtzoglou (2003). Bliss and Panigirtzoglou (2003) restrict the pricing kernel to be consistent with either constant absolute risk aversion or constant relative risk aversion. However, they find that risk aversion declines with the forecast horizon and with the level of volatility. Analyzing the cross section of industry portfolios Dittmar (2002) also provides evidence against constant elasticity of the pricing kernel.

non-constant elasticity.[4] In order to analyze the relationship between the elasticity of the pricing kernel and the characteristics of asset returns, two alternative continuous-time models for the market portfolio are developed.

In the first model we consider the price of a performance index where the price at some terminal date T is assumed to be lognormally distributed, in the second model we consider the price of an infinite horizon continuous dividend stream. The first model is called the performance index model, the second the dividend model. In the performance index model the expectation of the terminal price is governed by a geometric Brownian motion without drift. In the dividend model the dividend is governed by a geometric Brownian motion with drift. Both models are standard in asset pricing. The performance index model with finite horizon is similar to Franke et. al. (1999) and Brennan et. al. (2003). The finite horizon assumption appears appropriate for the case of derivatives with a given maturity. When it comes to the analysis of primary securities such as the market portfolio, a model with an infinite horizon might be preferable. Therefore we also model the market portfolio in an infinite horizon framework with a continuous dividend stream, similar to the framework used in Brennan and Xia (2002). Our main results are similar in both frameworks. Also, for both, analytical discrete-time asset price processes are derived based on a new generalized characterization of the pricing kernel. They are especially valuable since they simplify numerical simulation and empirical estimation.

We find that expected asset returns under declining elasticity of the pricing kernel react stronger to changes in the asset price than under constant elasticity of the pricing kernel. The latter model is used as a benchmark. This kind of overreaction leads to higher volatility and serial correlation of returns. The numerical analysis shows that for certain parameter values asset returns exhibit short-term momentum and long-term reversals. Momentum effects are stronger in the performance index model than in the dividend model.

Moreover, we consider the link between the asset price and fundamental information. The latter is captured by a fundamental variable which is governed by a geometric Brownian motion. In the performance index model the expectation of the terminal asset price is the fundamental variable. In the

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[4]Since in this paper we consider only the characteristics of the market portfolio, we do not differentiate between the pricing kernel and the asset specific pricing kernel. For a discussion, see Camara (2003).

dividend model, it is the current dividend level. It is shown that the elasticity and the first derivative of the asset price with respect to the fundamental variable are not constant. For certain ranges of the fundamental variable the elasticity and the first derivative are relatively high while they are relatively low for other ranges. Thus, if fundamentals move within certain ranges, asset price reactions to changes in fundamentals are relatively weak. However, if fundamentals move into a range where the asset price is a relatively steep function, the asset price reacts strongly even to minor changes in fundamentals. This functional form of asset prices may explain stock market crashes which occur without any significant fundamental news. It also may explain why asset returns can be predicted to a certain extent by technical analysis. Especially, it provides a rationale for so-called support and resistance levels.

These results are new as shown by a brief discussion of the theoretical asset pricing literature. For finite horizon models it is known from Bick (1990) and Franke et. al. (1999) that if the price of the market portfolio is governed by a geometric Brownian motion, then the path-independent pricing kernel has constant elasticity and the expectations of the representative investor about the terminal asset value are also governed by a geometric Brownian motion. Hence, the geometric Brownian motion on which the Black and Scholes model is based can be derived as an equilibrium price process with a constant relative risk averse representative investor and an expectations process which is governed by another geometric Brownian motion.

Bick (1990) and He and Leland (1993) derive characteristics of asset price processes which are consistent with an equilibrium in a standard representative investor economy. They show that such an equilibrium rules out widely used stochastic processes such as the Ornstein Uhlenbeck process and constant elasticity of variance for the market portfolio. Despite the vast literature on time-series characteristics of asset returns, we still lack a sound economic understanding of their time-series characteristics. In particular, little is known on how asset returns depend on the shape of the pricing kernel. Moreover, simple characterizations of the pricing kernel are still prevalent, although empirical research suggests that the shape of the pricing kernel is more complicated. Among the few papers which analyze the impact of the shape of the pricing kernel on return characteristics is Stapleton and Subrahmanyam (1990). They assume that the cash flow process is governed by a geometric [arithmetic] Brownian motion. They show that if the pricing kernel is characterized by a power [exponential] function, the forward price is governed by a geometric [arithmetic] Brownian motion. Franke et. al. (1999)

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show that option prices are higher for declining than for constant elasticity of the pricing kernel and that asset returns are serially correlated in case of declining elasticity of the pricing kernel. Neither Franke et. al. (1999) nor Stapleton and Subrahmanyam (1990) give a characterization of the volatility function or the autocorrelation function. Also, they do not provide any quantification of the effects of the pricing kernel on the asset price process. Recent papers have analyzed the implications of heterogeneous preferences on the pricing kernel. Benninga and Mayshar (2000) show in a two date economy that if all investors are constant relative risk averse but with different levels of relative risk aversion then the pricing kernel exhibits declining elasticity. Chan and Kogan (2002) analyze a continuous time economy with a continuum of agents who have 'catching up with the Joneses' preferences and differ in the level of constant relative risk aversion. Although they do not provide an analytical solution for asset prices, they show that this kind of heterogeneity can generate mean reversion in asset returns. Their analysis, however, does not provide any rationale for short-term momentum or stock market crashes.

Related to this paper is the research on the effect of learning on return characteristics. Brennan and Xia (2002) assume that the representative investor cannot observe the growth rate of dividends but estimates it from realized data. Their model can explain high volatility of stock prices. Johnson (2002) builds on their results to show that stochastic expected growth rates of the dividend process lead to momentum. Brennan et. al. (2003) and Brennan and Xia (2003) also work within a similar framework. They emphasize the importance of a time-varying investment opportunity set to explain the predictability of asset returns.[5]

The remainder of the paper is organized as follows. In Section 1 the two models are introduced and the general relationship between the pricing kernel and the information process respectively the dividend process, and the asset price process is derived. In Section 2 the asset price process is analyzed for the performance index model in the finite horizon case. The process is analyzed in continuous- and discrete-time. An analytical version is derived based on a general new characterization of the pricing kernel. Numerical simulations are also provided. In Section 3 the same analysis is presented for the dividend model in the infinite horizon case. Section 4 concludes.

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[5]See also Timmermann (1993), David (1997), Veronesi (2000) and Pastor and Veronesi (2003) for the effect of learning on asset pricing.

1 The Models

1.1 The Performance Index Model

Consider a pure exchange economy with rational investors and a given time horizon T. First, we assume that the price of the single asset in the economy (the market portfolio) is given by a performance index which pays no dividends. Also, we assume that the information process $I_{t \in [0,T]}$ (investors' expectations about the terminal value of the market portfolio, F_T) is governed by a geometric Brownian motion without drift, i.e. $I_t \equiv E(F_T | \mathcal{F}_t)$, $0 \leq t \leq T$, is governed by

$$
\begin{aligned}
dI_t &= \sigma_I I_t dW_t, \ 0 \leq t \leq T, \\
I_0 &> 0,
\end{aligned}
\tag{1}
$$

where σ_I is the constant instantaneous volatility of the information process, W_t is a one-dimensional standard Brownian motion and $\mathcal{F}_t$ is the time t information set. Equation (1) describes a standard information process as used for example in Brennan et. al. (2003) or Franke et. al. (1999). To emphasize the effect of the pricing kernel on asset return characteristics, the instantaneous volatility of the information process is assumed to be constant. This assumption implies that the market portfolio F_T is lognormally distributed at the terminal date T with $F_T = I_T$ and

$$
Var\left(\ln F_T | \mathcal{F}_t\right) = \sigma_I^2 \left(T - t\right) , \quad 0 \leqslant t \leqslant T.
$$

Interpreting the information process in terms of the implied information flow, equation (1) implies a constant information flow since the resolution of uncertainty as measured by the decrease in variance of the terminal value is constant over time.

Given this information structure we will now derive the forward price of the asset. To focus on risk preferences we consider forward prices and, thus, excess returns instead of spot prices and total returns. It is well known that in an arbitrage-free market an equivalent martingale measure exists. Moreover, in a complete market the equivalent martingale measure $\widetilde{P}$ is unique. The transformation from the subjective probability measure P to $\widetilde{P}$ is given by the pricing kernel $\Phi_{t,T} = \frac{\Phi_{0,T}}{\Phi_{0,t}}$ where $\Phi_{0,t} = E(\Phi_{0,T} | \mathcal{F}_t)$, $0 \leqslant t \leqslant T$. Thus, the forward price F_t is given by

$$
\begin{aligned}
F_t &= E^{\widetilde{P}}(F_T | \mathcal{F}_t) = E(F_T \Phi_{t,T} | \mathcal{F}_t) \\
&= E(I_T \Phi_{t,T} | \mathcal{F}_t) , \quad 0 \leqslant t \leqslant T.
\end{aligned}
\tag{2}
$$

In general, the pricing kernel is characterized by the Girsanov-functional and, thus, it is not necessarily a deterministic function of I_t or F_t. However, in a representative investor economy with a state-independent von Neumann-Morgenstern utility function over the terminal asset price $F_T = I_T$, the pricing kernel is characterized by a deterministic function of time t and either I_t or F_t. This follows from the equilibrium condition

$$\Phi_{0,T} \equiv \Phi(F_T) = a\frac{\partial}{\partial F_T}U(F_T), \tag{3}$$

with a being some positive scalar and U the state-independent utility function of the representative investor. The pricing kernels considered in this article are assumed to be path-independent and therefore consistent with a representative investor economy. The elasticity of the pricing kernel $\left(\eta_t^{\Phi,F} \equiv -\frac{\partial \ln \Phi_{0,t}}{\partial \ln F_t}\right)$ can then be interpreted as the relative risk aversion of the representative investor.[6] Moreover, unless stated differently, we always consider monotonically decreasing pricing kernels which imply a risk averse representative investor.

Given the information process of equation (1) with $I_T = F_T$, the forward price F_t can then be characterized by the following backward stochastic differential equation

$$dF_t = \underbrace{\left\{\frac{\partial}{\partial t}v(t, I_t) + \frac{1}{2}\frac{\partial^2}{\partial I_t^2}v(t, I_t)(\sigma_I I_t)^2\right\}}_{=\mu_F(t)F_t}dt + \underbrace{\frac{\partial}{\partial I_t}v(t, I_t)\sigma_I I_t}_{=\Sigma_F(t)F_t}dW_t,$$

$$0 \leqslant t \leqslant T, \tag{4}$$

$$v(T, I_T) = I_T,$$

with $v(t, I_t) = F_t$ and the instantaneous drift (expected excess return) $\mu_F(t)$ and the instantaneous volatility $\Sigma_F(t)$ being deterministic functions of time t and F_t. An important characteristic of asset prices is the elasticity of the price with respect to the information process, $\eta_t^{F,I} \equiv \frac{\partial \ln F_t}{\partial \ln I_t}$. As can be seen from equation (4) $\eta_t^{F,I}$ determines, for example, the instantaneous volatility of the price process, $\Sigma_F(t) = \eta_t^{F,I}\sigma_I$. Applying Ito's Lemma and making use of the fact that $\Phi_{0,t}$ is a martingale, we get the following stochastic differential equation for the pricing kernel as a function of the forward price and time t

$$d\Phi_{0,t} = -\eta_t^{\Phi,F}\Sigma_F(t)\Phi_{0,t}dW_t, \quad 0 \leqslant t \leqslant T,$$

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[6]For a more detailed discussion see for example Decamps and Lazrak (2000).

$$\Phi_{0,0} = 1.$$

By definition, $F_t\Phi_{0,t}$ is also a martingale. Hence we get the following characterization of the expected excess return $\mu_F(t)$

$$\mu_F(t) = \eta_t^{\Phi,F}(\Sigma_F(t))^2 = \eta_t^{\Phi,F}\left(\eta_t^{F,I}\sigma_I\right)^2 , \quad 0 \leqslant t \leqslant T.$$

Alternatively the pricing kernel can also be characterized as a function of the information process. This yields the following equivalent stochastic differential equation for the pricing kernel

$$d\Phi_{0,t} = -\eta_t^{\Phi,I}\sigma_I\Phi_{0,t}dW_t , \quad 0 \leqslant t \leqslant T,$$
$$\Phi_{0,0} = 1,$$

where $\eta_t^{\Phi,I}$ is the elasticity of the pricing kernel with respect to I_t, i.e. $\eta_t^{\Phi,I} \equiv -\frac{\partial\Phi_{0,t}}{\partial I_t}\frac{I_t}{\Phi_{0,t}}$. In this notation the expected excess return is characterized by

$$\mu_F(t) = \eta_t^{\Phi,I}\eta_t^{F,I}\sigma_I^2 , \quad 0 \leqslant t \leqslant T.$$

Both characterizations are equivalent, but sometimes it is more convenient to work with $\eta_t^{\Phi,I}$ instead of $\eta_t^{\Phi,F}$. Hence we have a complete characterization of the forward price process in terms of the underlying variables, i.e. the information process and the pricing kernel. However, note that the elasticity of the pricing kernel $\eta_t^{\Phi,F}$ may be time-dependent. If we specify the utility function over terminal wealth, we also specify the elasticity of the pricing kernel at the terminal date T but not for $t < T$. The following proposition states that the elasticity of F_t with respect to I_t is equal to 1 for constant elasticity of the pricing kernel $\eta_T^{\Phi,F}$ and it is higher [lower] than 1 for declining [increasing] elasticity of the pricing kernel, i.e. $\frac{\partial\eta_T^{\Phi,F}}{\partial F_T} < [>]\,0$.

Proposition 1 *Assume that the information process is governed by a geometric Brownian motion with constant instantaneous volatility and no drift. Then*
$$\eta_t^{F,I} > [=]\,[<]1, \forall I_t \Leftrightarrow \frac{\partial\eta_T^{\Phi,F}}{\partial F_T} < [=]\,[>]\,0 \;, \forall F_T.$$

Proof The forward price is given by

$$F_t = E\left(I_T\Phi_{t,T}\middle|\mathcal{F}_t\right) , \quad 0 \leqslant t \leqslant T, \tag{5}$$

with

$$\Phi_{t,T} = \frac{\Phi_{0,T}}{E\left(\left.\Phi_{0,T}\right|\mathcal{F}_t\right)}\ ,\quad 0 \leqslant t \leqslant T,$$

and $\Phi_{0,T} \equiv \Phi\left(I_T\right)$ being a deterministic function of I_T, see equation (3). Hence, since $\frac{\partial \ln I_T}{\partial \ln I_t} = 1$,

$$
\begin{aligned}
\frac{\partial \ln \Phi_{t,T}}{\partial \ln I_t} &= \frac{\partial \ln \Phi_{0,T}}{\partial \ln I_T}\frac{\partial \ln I_T}{\partial \ln I_t} - \frac{E\left(\left.\frac{\partial \ln \Phi_{0,T}}{\partial \ln I_T}\frac{\partial \ln I_T}{\partial \ln I_t}\Phi_{0,T}\right|\mathcal{F}_t\right)}{E\left(\left.\Phi_{0,T}\right|\mathcal{F}_t\right)} \\
&= -\eta_T^{\Phi,F} + \frac{E\left(\left.\eta_T^{\Phi,F}\Phi_{0,T}\right|\mathcal{F}_t\right)}{E\left(\left.\Phi_{0,T}\right|\mathcal{F}_t\right)} \\
&= -\eta_T^{\Phi,F} + E\left(\left.\eta_T^{\Phi,F}\Phi_{t,T}\right|\mathcal{F}_t\right)\ ,\quad 0 \leqslant t \leqslant T.
\end{aligned}
$$

Differentiating $\ln F_t = \ln E\left(\left.I_T\Phi_{t,T}\right|\mathcal{F}_t\right)$ with respect to $\ln I_t$ yields

$$
\begin{aligned}
\frac{\partial \ln F_t}{\partial \ln I_t} &= \frac{E\left(\left.\frac{\partial \ln I_T}{\partial \ln I_t}\left(I_T\Phi_{t,T} + I_T\Phi_{t,T}\frac{\partial \ln \Phi_{t,T}}{\partial \ln I_T}\right)\right|\mathcal{F}_t\right)}{F_t} \\
&= 1 + \frac{E\left(\left.I_T\Phi_{t,T}\left(-\eta_T^{\Phi,F} + E\left(\left.\eta_T^{\Phi,F}\Phi_{t,T}\right|\mathcal{F}_t\right)\right)\right|\mathcal{F}_t\right)}{F_t} \\
&= 1 - E^{\tilde{P}}\left(\left.\frac{I_T}{F_t}\left(\eta_T^{\Phi,F} - E^{\tilde{P}}\left(\left.\eta_T^{\Phi,F}\right|\mathcal{F}_t\right)\right)\right|\mathcal{F}_t\right) \\
&= 1 - E^{\tilde{P}}\left(\left.\eta_T^{\Phi,F}\left(\frac{I_T}{F_t} - 1\right)\right|\mathcal{F}_t\right) \\
&= 1 - cov^{\tilde{P}}\left(\left.\eta_T^{\Phi,F}, \frac{I_T}{F_t}\right|\mathcal{F}_t\right),\quad 0 \leqslant t \leqslant T,
\end{aligned}
$$

since $E^{\tilde{P}}\left(\left.\frac{I_T}{F_t}\right|\mathcal{F}_t\right) = 1$.

Thus, if $\eta_T^{\Phi,F}$ is constant, then $\eta_t^{F,I} \equiv \frac{\partial \ln F_t}{\partial \ln I_t} = 1$. For declining elasticity, $\frac{\partial \eta_T^{\Phi,F}}{\partial F_T} < 0$, $\eta_t^{F,I} \equiv \frac{\partial \ln F_t}{\partial \ln I_t} > 1$ and for increasing elasticity , $\frac{\partial \eta_T^{\Phi,F}}{\partial F_T} > 0$, $\eta_t^{F,I} \equiv \frac{\partial \ln F_t}{\partial \ln I_t} < 1$. $\blacksquare$

Since constant elasticity of the pricing kernel yields $\eta_t^{F,I} = 1$, it follows also from Proposition 1 that the forward price is governed by a geometric Brownian motion (see also Bick, 1990 and Franke et. al., 1999). What

else do we learn from Proposition 1? $\eta_t^{F,I} > [<]$ 1 implies that a 1 percent change in I_t leads to a higher [less] than 1 percent change in F_t. Hence, the proposition establishes that the forward price overreacts [underreacts] compared to the case of constant elasticity of the pricing kernel if the elasticity of the pricing kernel is declining [increasing]. To get the intuition for the overreaction [underreaction], think about the elasticity of the pricing kernel in terms of relative risk aversion of the representative investor. A representative investor with decreasing [increasing] relative risk aversion requires a lower [higher] excess return for the same risk, the wealthier he is. Compared to an investor with constant relative risk aversion, his required relative risk premium $\left(\frac{I_t - F_t}{F_t} \right)$ decreases [increases], the wealthier he is. Hence, the price he is willing to pay for the asset increases with increasing expected terminal wealth more [less] than under constant relative risk aversion. Thus, with non-constant relative risk aversion a change in the expected terminal value I_t also induces a change of the required risk premium. This change of the risk premium reinforces [diminishes] the purely 'information based' change of the asset price. Thus, since the required risk premium decreases [increases] with the level of I_t for declining [increasing] elasticity of the pricing kernel, the forward price overreacts [underreacts].

1.2 The Dividend Model

Before we turn to the detailed analysis of the excess return characteristics let us consider the pricing of an asset in a similar pure exchange economy with two modifications: 1) an infinite instead of a finite time horizon and 2) a continuous dividend stream instead of an exogenously given distribution of the asset price at the terminal date T.

To keep the information structure as simple as in the finite horizon case, we assume that the cash flow process (dividend stream) $D_{t \in [0,\infty)}$ is governed by a geometric Brownian motion

$$
\begin{aligned}
dD_t &= \mu_D D_t dt + \sigma_D D_t dW_t , \quad 0 \leqslant t < \infty, \\
D_0 &> 0,
\end{aligned}
\tag{6}
$$

where the instantaneous drift μ_D and the instantaneous volatility σ_D are assumed constant. Since there is no finite time horizon, we now consider the spot price of the asset. In order to have a finite price, we introduce a risk-free rate r_f which is assumed to be constant over time and sufficiently high. The

spot price of the asset, S_t, is then given by

$$S_t = E \left(\left. \int_t^\infty \exp\left(-r_f \left(s - t\right)\right) D_s \Phi_{t,s} ds \right| \mathcal{F}_t \right) , \quad 0 \leqslant t < \infty. \qquad (7)$$

Due to the simple information structure generated by a one-dimensional geometric Brownian motion, there is a deterministic relation between the dividend payment D_t and the spot price S_t. While for the finite horizon case we assume the pricing kernel $\Phi_{0,T}$ to be a deterministic function of $F_T = I_T$, in the infinite horizon case we assume $\forall t \in [0, \infty)$ $\Phi_{0,t}$ to depend on date t-wealth and, thus, to be a deterministic function of S_t and t. This assumption again rules out path-dependence of the pricing kernel. It is consistent with an economy in which the representative investor's marginal utility of any date depends only on his wealth at that date and the date itself. This is in line with intertemporal models in which pricing is driven by an indirect utility of wealth function. Since S_t is a deterministic function of D_t, the pricing kernel $\Phi_{0,t}$ can also be written as a function of D_t and t, $\Phi_{0,t} = \Phi\left(D_t, t\right)$ Hence, the pricing kernel can be characterized as a deterministic function of the exogenous process $D_{t \in [0,\infty)}$ and time t,

$$\begin{aligned} d\Phi_{0,t} &= -\eta_t^{\Phi,D} \sigma_D \Phi_{0,t} dW_t , \quad 0 \leqslant t \leqslant T, \\ \Phi_{0,0} &= 1. \end{aligned}$$

Since the dividend stream is exogenously given, $\Phi_{0,t}$ should not be interpreted as the representative investor's marginal utility from consuming D_t. $\Phi_{0,t}$ should be understood as a stochastic discount factor that depends on the present value of future dividends, S_t, which in turn depends on the dividend D_t. For the spot price we get the following stochastic differential equation

$$dS_t = \underbrace{\left\{ \frac{\partial}{\partial t} v\left(t, D_t\right) + \frac{\partial}{\partial D_t} v\left(t, D_t\right) \mu_D D_t + \frac{1}{2} \frac{\partial^2}{\partial D_t^2} v\left(t, D_t\right) \left(\sigma_D D_t\right)^2 \right\}}_{=\mu_S(t)S_t = \eta_t^{\Phi,D} \eta_t^{S,D} \sigma_D^2 S_t - D_t + r_f S_t} dt$$

$$+ \underbrace{\frac{\partial}{\partial D_t} v\left(t, D_t\right) \sigma_D D_t}_{=\Sigma_S(t)S_t = \eta_t^{S,D} \sigma_D S_t} dW_t, \qquad (8)$$

$$0 \leqslant t \leqslant T, \ v\left(t, D_t\right) = S_t, \ S_0 > 0.$$

It should be noted that $\mu_S\left(t\right)$ denotes the expected excess return on S_t plus the risk-free rate r_f. To get an understanding of the characteristics of the

spot price in the infinite horizon case consider the elasticity of the spot price S_t with respect to the dividend D_t, i.e. $\eta_t^{S,D} \equiv \frac{\partial \ln S_t}{\partial \ln D_t}$. If the elasticity is equal to 1, then the spot price is also governed by a geometric Brownian motion. If the elasticity is higher [lower] than 1, then the spot price overreacts [underreacts] compared to a geometric Brownian motion. The following proposition establishes the relationship between the overreaction [underreaction] and the elasticity of the pricing kernel with respect to the dividend D_t, i.e. $\eta_t^{\Phi,D} \equiv -\frac{\partial \ln \Phi_t}{\partial \ln D_t}$, similar to the relationship derived in Proposition 1.

Proposition 2 *Assume that the cash flow process $D_{t \in [0,\infty)}$ is governed by a geometric Brownian motion with constant instantaneous volatility and constant instantaneous drift. Then*

$$\eta_t^{S,D} > [=][<]1, \text{ for all } D_t \text{ and } t \in [0,\infty)$$

$$\Leftrightarrow$$

$$\frac{\partial \eta_t^{\Phi,D}}{\partial D_t} < [=][>]0 , \text{ for all } D_t \text{ and } t \in [0,\infty).$$

Proof The proof is similar to that of Proposition 1. The spot price is given by equation (7). Differentiating $\ln S_t$ with respect to $\ln D_t$ yields after some manipulation

$$\eta_t^{S,D} = 1 + \frac{E^{\widetilde{P}}\left(\int_t^\infty \exp\left(-r_f\left(s-t\right)\right) D_s \left(-\eta_s^{\Phi,D} + E^{\widetilde{P}}\left(\eta_s^{\Phi,D}\big|\mathcal{F}_t\right)\right) ds \bigg| \mathcal{F}_t\right)}{S_t},$$

$$0 \leqslant t < \infty.$$

Let $S_{t,s} \equiv E^{\widetilde{P}}\left(\exp\left(-r_f\left(s-t\right)\right) D_s | \mathcal{F}_t\right)$. Then

$$\eta_t^{S,D} = 1 + \int_t^\infty \frac{S_{t,s}}{S_t} E^{\widetilde{P}}\left(\frac{D_s}{S_{t,s}}\left(-\eta_s^{\Phi,D} + E^{\widetilde{P}}\left(\eta_s^{\Phi,D}\big|\mathcal{F}_t\right)\right)\bigg|\mathcal{F}_t\right) ds$$

$$= 1 + \int_t^\infty \frac{S_{t,s}}{S_t} E^{\widetilde{P}}\left(\left(\left(\frac{D_s}{S_{t,s}} - \exp\left(r_f\left(s-t\right)\right)\right)\left(-\eta_s^{\Phi,D}\right)\bigg|\mathcal{F}_t\right) ds\right.$$

$$= 1 - \int_t^\infty \frac{S_{t,s}}{S_t} cov^{\widetilde{P}}\left(\frac{D_s}{S_{t,s}}, \eta_s^{\Phi,D}\bigg|\mathcal{F}_t\right) ds,$$

$$0 \leqslant t < \infty.$$

Thus, if $\eta_t^{\Phi,D}$ is constant, then $\eta_t^{S,D} \equiv \frac{\partial \ln S_t}{\partial \ln D_t} = 1$. For declining elasticity, $\frac{\partial \eta_t^{\Phi,D}}{\partial D_t} < 0, \forall t \in [0,\infty)$, $\eta_t^{S,D} > 1$ and for increasing elasticity , $\frac{\partial \eta_t^{\Phi,D}}{\partial D_t} > 0$, $\forall t \in [0,\infty)$, $\eta_t^{S,D} < 1$.

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Proposition 2 is the analog to Proposition 1. If the pricing kernel has constant elasticity with respect to the dividend, then the spot price follows a geometric Brownian motion. If the elasticity of the pricing kernel is declining [increasing], then the price overreacts [underreacts]. To draw conclusions about the behavior of excess returns we need to derive the behavior of the total return index (performance index) V_t. Since the total return index includes the reinvested dividend payments, its return minus the risk-free rate is the excess return that we are interested in,

$$\frac{dV_t}{V_t} - r_f dt = \frac{dS_t}{S_t} + \frac{D_t}{S_t} dt - r_f dt.$$

Note that $V_t = \alpha_t S_t$ with α_t being independent of D_t. Therefore $\frac{\partial \ln V_t}{\partial \ln D_t} = \frac{\partial \ln S_t}{\partial \ln D_t}$. This implies that Proposition 2 holds equally for $\eta_t^{S,D}$ and $\eta_t^{V,D}$. Hence, for declining [increasing] elasticity of the pricing kernel the total return index also overreacts [underreacts].

Thus, so far the qualitative results on the relationship between the elasticity of the pricing kernel and the characteristics of the asset price process indicate that declining [increasing] elasticity of the pricing kernel leads to overreaction [underreaction] of asset prices compared to the geometric Brownian motion. This result is independent of whether we chose an infinite or a finite horizon setting. In order to further scrutinize the relationship between the shape of the pricing kernel and the time-series characteristics, we analyze in the next section the performance index model more deeply before turning to the dividend model in Section 3.

2   Performance Index Model

2.1   Predictability of Excess Returns

The following proposition establishes for the performance index model that predictability of asset excess returns and non-constant elasticity of the pricing kernel $\Phi(F_T)$ are closely related.

Proposition 3 *Assume that the information process is governed by a geometric Brownian motion with constant instantaneous volatility and no drift.*

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Then

$$\frac{\partial \eta_T^{\Phi,F}}{\partial F_T} < [=] [>] 0 \; , \forall F_T \Leftrightarrow \frac{\partial}{\partial F_t} E\left(\ln\frac{F_T}{F_t}\middle|\mathcal{F}_t\right) < [=] [>] 0 \; , \quad \forall F_t \; , \; t < T.$$

and the relation between the instantaneous volatility of the forward price, $\Sigma_F(t)$, and the instantaneous volatility of the information process σ_I is driven by the elasticity of the pricing kernel,

$$\frac{\partial \eta_T^{\Phi,F}}{\partial F_T} < [=] [>] 0 \; , \forall F_T \Leftrightarrow \Sigma_F(t) > [=] [<] \sigma_I \; , \; t < T.$$

Proof.

$$E\left(\ln\frac{F_T}{F_t}\middle|\mathcal{F}_t\right) = \ln I_t - \frac{1}{2}\sigma_I^2(T-t) - \ln v(t, I_t) \; ,$$

with $F_t = v(t, I_t)$. Hence

$$\frac{\partial E\left(\ln\frac{F_T}{F_t}\middle|\mathcal{F}_t\right)}{\partial \ln I_t} = 1 - \eta_t^{F,I}.$$

Thus, the first assertion of the Proposition follows from Proposition 1. It follows also from Proposition 1 that the instantaneous volatility $\Sigma_F(t)$ of the forward price process $F_{t\in[0,T]}$, i.e.

$$\Sigma_F(t) = \underbrace{\frac{\frac{\partial}{\partial I_t} v(t, I_t)}{v(t, I_t)}I_t}_{=\eta_t^{F,I}} \sigma_I \; , \quad 0 \leqslant t \leqslant T \; , \tag{9}$$

is higher [lower] than σ_I under declining [increasing] elasticity of the pricing kernel. ∎

What do we learn from Proposition 3 and which economic mechanism drives the results? First, expected excess returns depend negatively [positively] on the level of the forward price if the elasticity of the pricing kernel is declining [increasing]. Since a high [low] forward price F_t implies that past excess returns have been relatively high [low], we define excess returns as mean reverting [mean averting] if $\frac{\partial}{\partial F_t} E\left(\ln\frac{F_T}{F_t}\middle|\mathcal{F}_t\right) < [>] 0$. With this definition Proposition 3 states the condition for mean reversion [mean aversion]. Mean reversion [aversion] is due to the changing risk premium. The higher

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I_t, the lower [higher] will be the risk premium under declining [increasing] elasticity. Therefore the expected excess return decreases [increases] with the level of I_t. Note that only long-term excess returns are considered since we measure the impact on the excess return for the longest possible horizon, i.e. $(T - t)$.

This mean reversion [mean aversion] can also be related to the overreaction [underreaction] effect. Note, first, that the distribution of the terminal asset price is independent of the pricing kernel and equal to the distribution of I_T. However, under declining [increasing] elasticity of the pricing kernel the forward price overreacts [underreacts]. Hence, since $I_T = F_T$, this overreaction [underreaction] has to be compensated and, thus, excess returns exhibit mean reversion [mean aversion]. Second, the higher [lower] instantaneous volatility is related to the overreaction [underreaction] effect. The instantaneous volatility of the forward price $\Sigma_F(t)$ measures the instantaneous reaction of the forward price to innovations of the Brownian motion W. This Brownian motion drives both processes $I_{t \in [0,T]}$ and $F_{t \in [0,T]}$. Since F_t overreacts [underreacts] relatively to I_t, $\Sigma_F(t)$ must be higher [lower] than σ_I.

The following proposition sheds some light on the serial correlation of excess returns. The proposition states that excess returns exhibit negative [positive] serial correlation in the long run if the elasticity of the pricing kernel is declining [increasing].

Proposition 4 *Suppose that the information process is governed by a geometric Brownian motion with constant instantaneous volatility and no drift. Then final period excess returns $(r_{\tau,T} = \ln \frac{F_T}{F_\tau})$ are conditionally negatively [positively] correlated with preceding excess returns $(r_{t,\tau} = \ln F_\tau - \ln F_t$ with $0 \le t < \tau < T)$, i.e. $Corr(r_{\tau,T}, r_{t,\tau} | \mathcal{F}_t) < [>]0$, if the elasticity of the pricing kernel $\eta_T^{\Phi,F}$ is declining [increasing].*

Proof. Since

$$Cov(r_{\tau,T}, r_{t,\tau} | \mathcal{F}_t) = Cov\left(\ln \frac{E(F_T | \mathcal{F}_\tau)}{F_\tau}, \ln F_\tau | \mathcal{F}_t\right) ,$$

the covariance is positive [zero] [negative] if $\ln \frac{E(F_T | \mathcal{F}_\tau)}{F_\tau}$ is increasing [independent] [declining] in $\ln F_\tau$ or F_τ. By Proposition 3, this is true if the elasticity $\eta_T^{\Phi,F}$ is increasing [zero] [declining]. ∎

Analyzing more generally the serial correlation of other excess returns, i.e. $Corr\left(r_{\tau_1,\tau_2}, r_{\tau_2,\tau_3}\right)$, $0 \le \tau_1 < \tau_2 < \tau_3 < T$, does not lead to such clear results. For the correlation of final excess returns with preceding excess returns, the compensation of the overreaction [underreaction] of previous excess returns always dominates and leads to negative [positive] serial correlation. But for the other excess returns this need not be true because excess return-distributions are non-stationary. The following proposition shows that the excess return characteristics depend also on time t. Stationarity requires that the forward price process is governed by a time-homogeneous stochastic differential equation, i.e. $\mu_F(t)$ and $\Sigma_F(t)$ of the stochastic differential equation for the price F_t may depend on F_t, but they must not depend on time t.

Proposition 5 *Assume that the information process is governed by a geometric Brownian motion with constant instantaneous volatility and no drift. Then the forward price process is governed by a time-homogeneous stochastic differential equation if and only if the elasticity of the pricing kernel is constant.*

Proof.

a) *Sufficiency:* If the information process is governed by a geometric Brownian motion with constant instantaneous volatility and the pricing kernel has constant elasticity, then the forward price is governed by a geometric Brownian motion.

b) *Necessity:* Note that because $\frac{\partial v(T,I_T)}{\partial I_T}\frac{I_T}{v(T,I_T)} = 1$ and $\Sigma_F(t) = \frac{\partial v(t,I_t)}{\partial I_t}\frac{I_t}{v(t,I_t)}\sigma_I$, $0 \le t \le T$, the instantaneous volatility $\Sigma_F(t)$ is constant in T with $\Sigma_F(T) = \sigma_I$. By Proposition 3, for $t < T$, $\Sigma_F(t) \equiv \Sigma(t,F_t)$ deviates from σ_I if $\eta_T^{\Phi,F}$ is not constant. Since $\Sigma(T,F_T) = \sigma_I$, the instantaneous volatility of the forward price depends on time t if the elasticity of the pricing kernel is not constant. ∎

Proposition 5 states that given our information process only constant elasticity of the pricing kernel yields a time-homogeneous stochastic process for the forward price. Otherwise, asset returns are non-stationary.

It should be emphasized that this kind of non-stationarity is different from the non-stationarity which is usually discussed. While usually "non-stationary processes" is used almost synonymously to "integrated processes", in our case the process is not stationary even after taking first differences. The intuition behind this result is as follows. With declining elasticity of the

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pricing kernel, for example, the asset price instantaneously overreacts. This overreaction is then compensated by the mean reversion. However, both effects depend on the distance to the terminal date T since at the terminal date the forward price is equal to the lognormally distributed I_T. The important point of Proposition 5 is that most estimation techniques rely on the assumption of time-homogeneity.[7] Hence, empirical studies might suffer from the non-stationarity of asset returns. However, it might be questioned whether this non-stationarity is important in reality. Two points might weaken the time-dependence of transition densities. First, in contrast to the asset considered here, assets usually pay dividends regularly so that the time to maturity effect becomes less severe. Moreover, the terminal date T is not known in reality. A random terminal date T would also lead to a less pronounced time to maturity effect.

2.2   Volatility

We now analyze the characteristics of the instantaneous volatility $\Sigma_F(t) = \eta_t^{F,I} \sigma_I$ in more detail. Proposition 3 states that the functional form of the instantaneous volatility depends on the elasticity of the pricing kernel $\eta_T^{\Phi,F}$. For declining [increasing] elasticity the instantaneous volatility is higher [lower] than for constant elasticity. For constant elasticity of the pricing kernel the volatility of the forward price process and the volatility of the information process are equal.

We now turn to the relation between the instantaneous volatility and the level of the forward price. The following corollary provides a new explanation for the empirically well documented asymmetric volatility phenomenon.

Corollary 1 *Assume that the information process is governed by a geometric Brownian motion with constant instantaneous volatility and no drift. Then, the instantaneous volatility $\Sigma_F(t)$ converges from above [below] to σ_I for $F_t \to \infty$, i.e.*

$$\Sigma_F(t) \searrow [\nearrow] \sigma_I \text{ for } F_t \to \infty \,,$$

if the elasticity of the pricing kernel $\eta_T^{\Phi,F}$ is declining [increasing] and converges to a lower [upper] bound.

---

[7] For an overview on the estimation of diffusion models see Gourieroux and Jasiak (2001). A recent development on the estimation of diffusion processes is found in Elerian, Chib and Shephard (2001).

Proof. See appendix.

Corollary 1 states for declining [increasing] elasticity $\eta_T^{\Phi,F}$ converging to a lower [upper] bound that the instantaneous volatility of the forward price converges from above [below] to σ_I. This convergence need not be monotonic. Convergence in volatility for $I_t \to \infty$ (which implies $F_t \to \infty$) comes from the fact that we assume finite risk aversion over $\mathbb{R}^+$. Thus, for $I_t \to \infty$ the representative investor's relative risk aversion converges to a lower [upper] bound and, thus, the relative risk aversion becomes constant. Convergence in volatility tends to induce negative [positive] correlation between asset returns and volatility since prices tend to be high, if past returns were high.

Although we do not establish a monotonic relationship between $\Sigma_F(t)$ and F_t, Corollary 1 shows an effect which might contribute to the observed asymmetric volatility of asset returns. This explanation for low volatility in bull markets differs from the two explanations proposed in the literature, i.e. the leverage effect and the volatility feedback effect. The leverage effect relates the lower volatility in bull markets to the decreased leverage of companies due to the higher value of equity while the volatility feedback effect states that volatility and asset prices are negatively correlated since higher uncertainty (volatility) leads to higher risk premia and thus to lower asset prices. Recent empirical results show that the negative correlation between volatility and returns is more pronounced for market returns than for individual stocks. This suggests that the leverage effect may be less important than the preference based arguments, i.e. volatility feedback and declining elasticity.[8]

It is shown in Proposition 3 that the elasticity of the forward price with respect to I_t is greater [smaller] than 1 for declining [increasing] elasticity of the pricing kernel. Thus, the variance of $\ln F_t$ and the instantaneous variance $(\Sigma_F(t))^2$ are higher [lower] than the variance and the instantaneous variance of $\ln I_t$ for $0 \leqslant t < T$.[9] Which conclusions can be drawn for the properties of the return volatility over finite time intervals? The following proposition shows that for the case of declining elasticity of the pricing kernel, the conditional variance of returns over finite periods, $Var\left(\ln\frac{F_\tau}{F_t}\,\middle|\,\mathcal{F}_t\right)$, and the unconditional variance of returns over finite periods, $Var\left(\ln\frac{F_\tau}{F_t}\,\middle|\,\mathcal{F}_0\right)$,

---

[8] See for example Anderson, Bollerslev, Diebold and Ebens (2001) and Hentschel (1995).

[9] See also Franke et. al. (1999) who have shown that the variance of the forward price is higher under the declining elasticity pricing kernel.

are higher than under constant elasticity of the pricing kernel. Moreover, it should be noted that the results are not sensitive to whether we consider $Var\left(\ln\frac{F_\tau}{F_t}\Big|\mathcal{F}_0\right)$ or $Var\left(\ln\frac{F_\tau}{F_t}\Big|\mathcal{F}_{t-\theta}\right)$ with $t > \theta > 0$. Important is only whether $\ln F_t$ is measurable with respect to the filtration on which the variance is conditioned. This means, it is only important whether $\ln F_t$ is known.

Proposition 6 *Suppose that the information process is governed by a geometric Brownian motion with constant instantaneous volatility and no drift. Then*
a) for the conditional and unconditional variance

$$Var\left(\ln\frac{F_\tau}{F_t}\Big|\mathcal{F}_{t-\theta}\right) > Var\left(\ln\frac{I_\tau}{I_t}\Big|\mathcal{F}_{t-\theta}\right), \quad 0 \le \theta \le t < \tau < T,$$

if the elasticity of the pricing kernel is declining,
b) for the conditional variance

$$Var\left(\ln\frac{F_\tau}{F_t}\Big|\mathcal{F}_t\right) < Var\left(\ln\frac{I_\tau}{I_t}\Big|\mathcal{F}_t\right), \quad 0 \le t < \tau < T,$$

if the elasticity of the pricing kernel is increasing.

Proof.
Proof of a) Since with declining elasticity of the pricing kernel $Var\left(\ln F_\tau|\mathcal{F}_t\right) > Var\left(\ln I_\tau|\mathcal{F}_t\right)$ for $0 \le t < \tau < T$, it follows immediately that also

$$Var\left(\ln\frac{F_\tau}{F_t}\Big|\mathcal{F}_t\right) > Var\left(\ln\frac{I_\tau}{I_t}\Big|\mathcal{F}_t\right), \quad 0 \le t < \tau < T.$$

Hence, the conditional variance of returns is higher under declining elasticity of the pricing kernel.
Consider now the unconditional variance:

$$Var\left(\ln\frac{F_\tau}{F_t}\right) = Var\left(E\left(\ln F_\tau|\mathcal{F}_t\right) - \ln F_t\right) + E\left(Var\left(\ln F_\tau|\mathcal{F}_t\right)\right), \quad (10)$$
$$0 \le t \le \tau \le T,$$

with

$$E\left(\ln F_\tau|\mathcal{F}_t\right) - \ln F_t = E\left(\int_t^\tau \left(\mu_F(s) - \frac{1}{2}\Sigma_F(s)^2\right) ds\Big|\mathcal{F}_t\right), \quad (11)$$
$$0 \le t \le \tau \le T.$$

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We need to show that $Var\left(\ln\frac{F_\tau}{F_t}\right)$ is greater than

$$Var\left(\ln\frac{I_\tau}{I_t}\right) = Var\left(\ln I_\tau|\mathcal{F}_t\right),$$

$$0 \leqslant t \leqslant \tau \leqslant T.$$

Equation (11) shows that, except for constant elasticity of the pricing kernel with non-random term $\mu_F - \frac{1}{2}\Sigma_F^2$,

$$Var\left(E\left(\ln F_\tau|\mathcal{F}_t\right) - \ln F_t\right) > 0.$$

Thus, since $Var\left(\ln F_\tau|\mathcal{F}_t\right) > Var\left(\ln I_\tau|\mathcal{F}_t\right)$, we have

$$Var\left(\ln\frac{F_\tau}{F_t}\right) > Var\left(\ln\frac{I_\tau}{I_t}\right).$$

Proof of b) Since with increasing elasticity of the pricing kernel $Var\left(\ln F_\tau\mid\mathcal{F}_t\right) < Var\left(\ln I_\tau|\mathcal{F}_t\right)$ for $0 \leq t < \tau < T$, it follows immediately that also

$$Var\left(\ln\frac{F_\tau}{F_t}\bigg|\mathcal{F}_t\right) < Var\left(\ln\frac{I_\tau}{I_t}\bigg|\mathcal{F}_t\right), \quad 0 \leq t < \tau < T.$$

Hence, the conditional variance of returns is smaller under increasing elasticity of the pricing kernel. ∎

While the conditional variance $Var\left(\ln\frac{F_\tau}{F_t}\bigg|\mathcal{F}_t\right)$ is lower for increasing elasticity of the pricing kernel, this is not necessarily true for $Var\left(\ln\frac{F_\tau}{F_t}\bigg|\mathcal{F}_{t-\theta}\right)$ with $\theta > 0$. To see this, consider

$$Var\left(\ln\frac{F_\tau}{F_t}\bigg|\mathcal{F}_{t-\theta}\right) = Var\left(E\left(\ln F_\tau|\mathcal{F}_t\right) - \ln F_t|\mathcal{F}_{t-\theta}\right)$$

$$+ E\left(Var\left(\ln F_\tau|\mathcal{F}_t\right)|\mathcal{F}_{t-\theta}\right), \qquad (12)$$

$$0 < \theta \leqslant t \leqslant \tau \leqslant T,$$

and $E\left(Var\left(\ln F_\tau|\mathcal{F}_t\right)|\mathcal{F}_{t-\theta}\right)$ being lower under increasing elasticity than under constant elasticity of the pricing kernel. However, while in the case of constant elasticity of the pricing kernel

$$Var\left(E\left(\ln F_\tau|\mathcal{F}_t\right) - \ln F_t|\mathcal{F}_{t-\theta}\right) = 0,$$

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this is positive for non-constant elasticity. Hence, in contrast to the case of declining elasticity, the effect of increasing elasticity on the unconditional variance is ambiguous, since the first term on the right hand side of equation (12) is higher than under constant elasticity, but the second term is lower.

The intuition for the higher variance of returns when the pricing kernel has declining elasticity is the same as for the instantaneous volatility. The change in the risk premium increases the reaction to a change in expectations compared to the case of constant elasticity of the pricing kernel.

2.3 An Analytical Price Process

The purpose of this section is to further analyze the impact of the shape of the pricing kernel on the characteristics of excess returns, especially on the serial correlation and the volatility. To get a better understanding of this relationship, we propose a very general characterization of the pricing kernel which allows to derive analytical solutions for the forward price.

2.3.1 A New Class of Pricing Kernels

To get analytical solutions of the forward price we propose a generalized polynomial as a characterization of the pricing kernel, i.e.

$$\Phi_{t,T}^{general} = \frac{\sum_{i=1}^{N} \alpha_i I_T^{\delta_i}}{E\left(\sum_{i=1}^{N} \alpha_i I_T^{\delta_i} \,\middle|\, \mathcal{F}_t\right)} , \quad 0 \le t \le T , \tag{13}$$

with $\alpha_i, \delta_i \in \mathbb{R}$ and $I_T = F_T$. This specification is rather general so that many different characteristics of the pricing kernel can be matched. For example, the classical pricing kernels derived from a representative investor with exponential utility and from a representative investor with power utility are covered by (13), i.e.
exponential utility:

$$\Phi_{t,T}^{exponential} = \frac{\sum_{i=0}^{\infty} \frac{1}{i!} (-aI_T)^i}{E\left(\sum_{i=0}^{\infty} \frac{1}{i!} (-aI_T)^i \,\middle|\, \mathcal{F}_t\right)} , \quad 0 \le t \le T ,$$

with $a \in \mathbb{R}^+$, which has increasing elasticity,
power utility:

$$\Phi_{t,T}^{power} = \frac{\alpha I_T^{\delta}}{E\left(\alpha I_T^{\delta} \,\middle|\, \mathcal{F}_t\right)} , \quad 0 \le t \le T ,$$

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with $\alpha > 0, \delta < 0$, which has constant elasticity.

Comparing the generalized polynomial to a Taylor-series approximation of a function $f(x)$ around x_0, i.e.

$$\sum_{i=0}^{N} \frac{f^{(i)}(x_0)}{i!} (x - x_0)^i = \sum_{i=0}^{N} \frac{f^{(i)}(x_0)}{i!} \left(\sum_{k=0}^{i} \binom{i}{k} x^{i-k} (-x_0)^k \right) \tag{14}$$

where $f^{(i)}$ is the ith derivative of f shows that our characterization approximates any pricing kernel at least as well as a Taylor-series expansion, since the right hand side of (14) is a special case of our weighted sum of power functions.

Our proposed class of pricing kernels has the convenient property that the pricing kernel is characterized by a series of non-central moments of the random variable. Given our information process, the pricing kernel and the asset price are easily computed since the terminal value of the information process is lognormally distributed. The forward price admits the following characterization

$$
\begin{aligned}
F_t &= E \left(\frac{\sum_{i=1}^{N} \alpha_i I_T^{\delta_i+1}}{E \left(\sum_{i=1}^{N} \alpha_i I_T^{\delta_i} \middle| \mathcal{F}_t \right)} \middle| \mathcal{F}_t \right) \\
&= I_t \frac{\sum_{i=1}^{N} \alpha_i I_t^{\delta_i} A_i(t) \exp \left(\sigma^2 (T-t) \delta_i \right)}{\sum_{i=1}^{N} \alpha_i I_t^{\delta_i} A_i(t)} , \\
0 &\leq t \leq T ,
\end{aligned}
\tag{15}
$$

with $A_i(t) = \exp \left(\frac{1}{2} (\delta_i - 1) \sigma^2 (T-t) \delta_i \right)$. Thus, (15) provides a general analytical characterization of asset prices with lognormally distributed terminal value. Of course, appropriate parameters have to be chosen to assure monotonically declining positive pricing kernels.

2.3.2 Numerical Simulations

To illustrate the effect of declining elasticity of the pricing kernel, let us consider the following specification which generates analytical asset price processes for constant and declining elasticity of the pricing kernel

$$\Phi_{t,T} = \frac{\frac{1}{I_T} + \beta I_T^\delta}{E \left(\frac{1}{I_T} + \beta I_T^\delta \middle| \mathcal{F}_t \right)} , \quad 0 \leq t \leq T . \tag{16}$$

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This pricing kernel implies a representative investor with a utility function over terminal wealth which is a linear combination of a log-utility and a power-utility function. However, in order to have a well-defined pricing kernel, respectively utility function of the representative investor, we make the additional assumptions:

no-arbitrage: $\beta \geq 0$. Hence, the pricing kernel is positive for $I_T > 0$.

risk aversion: $\delta \leq 0$. This implies a negative slope of the pricing kernel (i.e. marginal utility of the representative investor would be declining).

While the power and the log specification yield constant elasticity of the pricing kernel, our extended log-power-utility generates constant ($\beta = 0$ or $\delta = -1$) or declining ($\beta > 0$ and $\delta \neq -1$) elasticity of the pricing kernel.

The elasticity $\eta_T^{\Phi,F} = 1 - \frac{(1+\delta)\beta}{\left(I_T^{-\delta-1}+\beta\right)}$ approaches $-\delta$ for $I_T \to 0$ and declines monotonically in I_T. It converges to 1 for $I_T \to \infty$. The slope $\frac{\partial \eta_T^{\Phi,F}}{\partial I_T} \to 0$ for $I_T \to 0$ and $I_T \to \infty$.[10] As

$$\frac{\partial \ln\left(-\frac{\partial \eta_T^{\Phi,F}}{\partial I_T}\right)}{\partial \ln I_T} = -(2+\delta) + \frac{2(1+\delta)I_T^{-\delta-1}}{I_T^{-\delta-1}+\beta} \ , \tag{17}$$

the elasticity is a declining, concave/convex function of I_T for $\delta < -2$.

This choice of the pricing kernel permits to obtain more general shapes of the elasticity function than assuming a HARA-utility function. In the HARA-case, the elasticity is either a convex or a concave function of I_T. Here the elasticity first is concave, then convex, moreover, it is declining and bounded from above and below. The available empirical evidence suggests that the elasticity is not a simple, declining convex, but a more complicated function. Our pricing kernel (16) is a first step towards a more complicated function. This first parsimonious generalization already generates interesting insights even though further generalizations might be desirable.

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[10]Note that an economy with investors having different levels of constant relative risk aversion would generate a pricing kernel with an upper and a lower bound for the elasticity of the pricing kernel, e.g. Benninga and Mayshar (2000).

With pricing kernel (16) we get the following analytical solution for the forward price:

$$
\begin{aligned}
F_t &= E\left(I_T \Phi_{t,T} \mid \mathcal{F}_t\right) \\
&= I_t \exp\left(-\sigma^2 (T-t)\right) \frac{1 + \beta I_t^{\delta+1} \exp\left(\frac{\delta^2+\delta}{2}\sigma^2 (T-t)\right)}{1 + \beta I_t^{\delta+1} \exp\left(\frac{\delta^2-\delta-2}{2}\sigma^2 (T-t)\right)} \ , \\
0 &\leq t \leq T \ .
\end{aligned}
\tag{18}
$$

Equation (18) nests the geometric Brownian motion. For $\beta = 0$ or $\delta = -1$ we get the same asset price as under log-utility. Moreover, analyzing equation (18) shows that in this case the forward price is governed by a geometric Brownian motion. Applying Ito's Lemma unveils even for the relatively simple pricing kernel of equation (16) the complex structure of the relationship between asset returns and volatility as well as the complex intertemporal dependencies.

To get a better sense for these dependencies, numerical simulations are useful. In each simulation run, we generate 240 observations of the information process. This corresponds to 20 years of monthly data. Every simulation is repeated 1000 times. We assume $\sigma = 0.037$ and a starting point $I_0 = 1$. Thus, the standard deviation of annual returns of the information process of approximately 12.8 percent is comparable to the 12.9 percent standard deviation of the real annual dividend growth rate for the U.S. over the period 1871-1996 (Brennan and Xia, 2002). The choice of the upper bound of the elasticity of the pricing kernel, $-\delta$, is more difficult. Empirical results on the elasticity of the pricing kernel are mixed. Recent studies focus on the elasticity of the pricing kernel implied by option prices. The empirical results documented in Ait-Sahalia and Lo (2000), Jackwerth (2000) and Rosenberg and Engle (2002) suggest U-shaped elasticities of the pricing kernel. Also the documented levels of the elasticity are astonishingly high. Ait-Sahalia and Lo (2000), for example, document levels up to 60 for S&P 500 index values about 15 percent below the current future price. Empirical elasticities of the pricing kernel estimated in Jackwerth (2000) are displayed in Figure 1. Assuming constant elasticity of the pricing kernel, Bliss and Panigirtzoglou (2003) estimate elasticities of the pricing kernel between 1.97 and 9.52. However, empirically little is known on the level of the elasticity of the pricing kernel for index values more than 15 percent above or below the current future price. Extrapolating the empirical estimates would lead

to unreasonable values for high and low index values. Moreover, the empirical estimates are subject to various methodological concerns (see Bliss and Panigirtzoglou, 2003 and Hentschel, 2003). For the numerical simulation we stick to more conservative pricing kernel specifications. Table 1 shows the parameter combinations for four specifications.

Specification:	1	2	3	4
δ	-5	-7.5	-15	-40
β	0.5	1	0.5	0.1

Table 1. Pricing kernel specifications of the numerical simulations (performance index model)

The table shows the parameter combinations for the 4 presented numerical simulations. δ and β are the parameters of the pricing kernel. Setting $\beta = 0$ or $\delta = -1$ leads to a geometric Brownian motion.

The elasticity of the pricing kernel specification 3 is shown in Figure 1.

- insert Figure 1 here -

Figure 1 reveals that although the pricing kernels of Table 1 exhibit declining elasticity, the elasticity converges for higher values of wealth to 1, the elasticity of a pricing kernel implied by a logarithmic utility function. Also the elasticities of specifications 1 to 3 are conservative compared to the empirical elasticities displayed in Figure 1.

The following table summarizes the characteristics of the simulated excess returns $(\ln F_t - \ln F_{t-1})$. GBM gives the results for constant elasticity of the pricing kernel so that the asset price follows a geometric Brownian motion.

Specification:	1	2	3	4	GBM
mean annualized volatility of monthly returns	0.163	0.151	0.152	0.154	0.128
mean annualized volatility of 4-year returns	0.163	0.155	0.163	0.138	0.128
mean autocorrelation (lag 1) of monthly returns	0.005	0.007	0.012	0.036	0
mean autocorrelation (lag 1) of 4-year returns	-0.056	-0.005	-0.085	-0.036	0
autocorrelation (lag 1) in monthly return volatility	0.951	0.948	0.956	0.934	0
autocorrelation (lag 4) in monthly return volatility	0.952	0.942	0.953	0.886	0

Table 2. Characteristics of excess returns and excess return volatility (performance index model)

The table shows the mean annualized volatility of monthly and 4-year-returns, the lag 1-serial correlation of these returns as well as lag 1- and lag 4-serial correlations in return volatility. For comparison we also show the theoretical values for a geometric Brownian motion (constant elasticity of the pricing kernel).

Table 2 illustrates that declining elasticity of the pricing kernel leads to higher volatility of excess returns and to long-term reversals, i.e. negative serial correlation in long-term returns. Table 2 also indicates that declining elasticity of the pricing kernel causes short-term momentum, i.e. positive serial correlation in short-term returns. While, for example, the volatility is close to historical values of about 18 percent (Brennan and Xia, 2002), the short-term positive serial correlation appears to be relatively low compared to the serial correlation estimates of major stock indices found in the literature (see for example Campbell et. al., 1997).

The standard deviation and the autocorrelation of lag 1 (AC(1)) for the monthly returns ($\ln F_t - \ln F_{t-1}$) for every $t \in \{1, 2, \ldots 240\}$ are displayed in Figure 2. The figure shows that there are significant variations in volatility and serial correlation of monthly excess returns over time. For example, for Specification 4 the autocorrelation at certain times clearly exceeds the average of 3.6 percent. Thus, also the short-term correlation of the simulated time series is consistent with empirical estimates of serial correlation as high as 18 percent (see Campbell et. al., 1997). However, as suggested by the

theoretical analysis, positive and negative autocorrelations exist.

- insert Figure 2 here -

Figure 2 also shows that asset returns overreact for declining elasticity compared to the geometric Brownian motion which leads to the higher volatility. It is also obvious from Figure 2 that volatility is not only higher under declining elasticity of the pricing kernel but also persistent.

Since the terminal distribution is exogenously given, the overreaction has to be compensated over time. The following figure shows this long-term reversal for declining elasticity of the pricing kernel.

- insert Figure 3 here -

We see from Figure 3 that the negative serial correlation tends to be more pronounced, the shorter the remaining time to the terminal date T. Also, the long-term reversal tends to be stronger when asset returns overreact more heavily on the short run.

To sum up, the numerical simulations suggest that declining elasticity of the pricing kernel causes excess volatility, high persistence in volatility, some momentum on the short run and strong long-term reversal effects. All these findings are consistent with recent empirical research.

2.4   Technical Analysis and Market Crashes

The technical reason for the overreaction and the predictability of excess returns is that for non-constant elasticity of the pricing kernel the forward price is not a linear function of investors' expectations I_t. This may also explain that in some ranges of I_t, the forward price F_t reacts little to changes in I_t while in other ranges it reacts very strongly. If $\frac{\partial F_t}{\partial I_t}$ is quite high, then small changes in fundamental news I_t trigger a large change in F_t. Thus, a small decline in corporate profits might lead to a crash. Conversely, a small increase in profits might lead to a stock market boom. The important insight is that these phenomena can happen in a rational expectations model. Hence there is no need to invoke irrational behavior to explain these phenomena.

In order to obtain some theoretical insights before looking at the numerical results, we analyze the valuation equation (18) for the special case of a small risk as defined by Pratt (1964). Pratt has shown that, for a small risk $\widehat{\sigma}^2$, the risk premium equals the agent's absolute risk aversion times $\frac{\widehat{\sigma}^2}{2}$. Therefore, in our case we have

$$F_t = I_t \left(1 - \eta_t^{\Phi,F} \frac{\widehat{\sigma}^2 (T-t)}{2} \right) \ , \tag{19}$$

with $\eta_t^{\Phi,F}$ being the representative investor's relative risk aversion. Hence $\frac{F_t}{I_t}$ depends on $\eta_t^{\Phi,F}$ only, given $\widehat{\sigma}^2 (T-t)$. For a small risk, $\eta_t^{\Phi,F} \rightarrow \eta_t^{\Phi,I} \rightarrow \eta_T^{\Phi,F}$. Therefore differentiating F_t twice with respect to I_t we obtain

$$\frac{\partial F_t}{\partial I_t} = 1 - \frac{\widehat{\sigma}^2 (T-t)}{2} \left(\eta_t^{\Phi,I} + I_t \frac{\partial \eta_t^{\Phi,I}}{\partial I_t} \right) \ , \tag{20}$$

$$\frac{\partial^2 F_t}{\partial I_t^2} = -\frac{\widehat{\sigma}^2 (T-t)}{2} \left(2 + \frac{\partial \ln \left(-\frac{\partial \eta_t^{\Phi,I}}{\partial I_t} \right)}{\partial \ln I_t} \right) \frac{\partial \eta_t^{\Phi,I}}{\partial I_t} \ . \tag{21}$$

For the pricing kernels of our specifications, $\frac{\partial \eta_t^{\Phi,I}}{\partial I_t} \rightarrow 0$ for $I_t \rightarrow 0$ and $I_t \rightarrow \infty$. Hence

$$\frac{\partial F_t}{\partial I_t} \rightarrow \begin{cases} 1 + \delta \frac{\widehat{\sigma}^2 (T-t)}{2} & \text{for } I_t \rightarrow 0 \\ 1 - \frac{\widehat{\sigma}^2 (T-t)}{2} & \text{for } I_t \rightarrow \infty \end{cases} \tag{22}$$

For $\delta < -2$, equation (17) shows that $2 + \frac{\partial \ln \left(-\frac{\partial \eta_t^{\Phi,I}}{\partial I_t} \right)}{\partial \ln I_t}$ is positive for low values of I_t, declines monotonically and converges to $2 + \delta < 0$ for $I_t \rightarrow \infty$. Hence F_t is a convex function of I_t for $I_t \leq I_t^0$ and concave for $I_t \geq I_t^0$ with I_t^0 denoting the inflection point. $\frac{\partial F_t}{\partial I_t}$ obtains its maximum at the inflection point. From equation (17), $\frac{\partial \ln \left(-\frac{\partial \eta_t^{\Phi,I}}{\partial I_t} \right)}{\partial \ln I_t} = -2$ implies $I_t^{-\delta-1} = \frac{\beta \delta}{\delta+2}$. Inserting this in equation (20) yields

$$\left. \frac{\partial F_t}{\partial I_t} \right|_{I_t=I_t^0} = 1 + \frac{\widehat{\sigma}^2 (T-t)}{2} \left(\left(\frac{\delta+2}{2} \right)^2 - 1 \right) \tag{23}$$

Equation (22) shows that for low and high values of I_t, the slope $\frac{\partial F_t}{\partial I_t}$ is smaller than 1. At the inflection point, the slope is higher than 1 (equation (23)).

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This effect increases with $\left(\frac{\delta+2}{2}\right)^2$. For $\delta = -10$, for example, the slope equals $1 + 15\frac{\widehat{\sigma}^2(T-t)}{2}$. Therefore, a high $|\delta|$ implies a strong sensitivity of the asset price with respect to fundamental information as is typical of a crash.

The results also hold for large risks as illustrated by the numerical results. Figure 4 displays the functional form of the simulated asset prices.

- insert Figure 4 here -

The asset prices displayed in Figure 4 have the described shape. Since $|\delta|$ is relatively small for specifications 1 and 2, the slope $\frac{\partial E_t}{\partial I_t}$ does not vary much. This is different for specifications 3 and 4 with higher $|\delta|$. Here the slope is quite high for certain ranges of the fundamental variable. Consequentially, there are critical asset price levels. If the asset price crosses such levels from below [above], then a small improvement [deterioration] in the fundamental variable triggers a strong asset price increase [decline]. This might explain why support and resistance levels play an important role in technical analysis.[11] Such relatively steep areas of the asset price as a function of the fundamental variable are typical of a stock market crash without any significant change in fundamentals. These effects depend on the time span $(T - t)$ and, thus, are time-dependent. This underscores the non-stationarity of asset returns.

The existence of crash scenarios is illustrated more dramatically in Figure 5. This figure shows for specification 3 the instantaneous serial covariance, the instantaneous Sharpe ratio, the instantaneous volatility and the instantaneous elasticity of the asset price with respect to the information.

- insert Figure 5 here -

This elasticity which is closely related to the slope $\frac{\partial E_t}{\partial I_t}$, is very high for certain information ranges. In these ranges asset prices react very strongly to changes in expectations, like in a crash. In these ranges, also the volatility of asset returns is very high. The serial return-covariance is, first, close to

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[11] Chart analysts think of resistance and support levels as asset price levels at which the asset price tends to bounce back. If the price crosses theses price levels then it moves strongly up or down. This feature is reproduced by our model.

zero, then strongly positive in a crash. After the crash it is, first, strongly negative and then returns to about zero. Note also that the Sharpe ratio at early dates, i.e. pre-crash, seems unexplainable high. The risk premium, however, is high during this low volatility period as compensation for the risk of a future stock market crash.

3   Dividend Model

3.1   Predictability of Excess Returns

We now turn to the dividend model in an infinite horizon setting. As shown by Proposition 2, also for the infinite horizon setting declining [increasing] elasticity of the pricing kernel leads to overreaction [underreaction] of the spot price. Does the overreaction [underreaction] also induce serial correlation? The instantaneous Sharpe ratio is given by

$$\frac{\mu_S\left(t\right) + \frac{D_t}{S_t} - r_f}{\Sigma_S\left(t\right)} = \frac{\mu_V\left(t\right) - r_f}{\Sigma_S\left(t\right)} = \eta_t^{\Phi,D}\sigma_D\ , \quad 0 \le t \le T.$$

It depends negatively [positively] on D_t for declining [increasing] elasticity of the pricing kernel, $\eta_t^{\Phi,D}$. If the volatility $\Sigma_S\left(t\right)$ was non-random, then the variations in the Sharpe ratio would lead to negative [positive] serial correlation in excess returns. However, changes in volatility might outweigh this effect. To analyze the serial correlation of excess returns, we consider the instantaneous cross variation between the expected excess return $\mu_V\left(\tau\right) - r_f$ and the cumulated excess return $CER_{t,\tau} \equiv \int_t^\tau \frac{dV_s}{V_s} - \int_t^\tau r_f ds$.[12] The covariance $Cov\left(CER_{t,\tau}, \mu_V\left(\tau\right) - r_f\middle|\mathcal{F}_t\right)$ is the integral over the time span $[t, \tau]$ of the instantaneous cross variation between the processes $d\left(\mu_V\left(\tau\right) - r_f\right)$ and $dCER_{t,\tau}$. The instantaneous cross variation is also called the instantaneous covariance. The following proposition states that for declining [increasing] elasticity of the pricing kernel $\eta_t^{\Phi,D}$, excess returns are negatively [positively] autocorrelated if the instantaneous volatility of excess returns $\Sigma_S\left(t\right)$ depends negatively [positively] on D_t and therefore negatively [positively] on S_t.

Proposition 7 *Suppose that the dividend process $D_{t\in[0,\infty)}$ is governed by a geometric Brownian motion with constant instantaneous volatility and constant instantaneous drift. Then, the cumulated excess return $CER_{t,\tau}$ and*

---

[12]For a similar analysis see Johnson (2002).

the expected excess return $\mu_V(\tau) - r_f$ *are negatively [positively] correlated for declining [increasing] elasticity of the pricing kernel if the volatility of excess returns depends negatively [positively] on* D_t.

Proof. See appendix. ∎

The conditions established in Proposition 7 are sufficient but not necessary. However, as in the performance index model (finite horizon setting), declining [increasing] elasticity of the pricing kernel does not unequivocally generate negative [positive] serial correlation. The reason for this result is that the sign of the serial correlation depends on whether the change in the Sharpe ratio is dominated by the change in the volatility. Hence, whether excess returns are negatively [positively] serially correlated depends also on the relation between the volatility and the dividend D_t. In the following section we characterize this relationship. In Section 3.3 we again provide an analytical discrete-time approximation of the price process. The ensuing simulation will allow us to scrutinize the effect of the shape of the pricing kernel on serial correlation.

3.2 Volatility

Due to the overreaction [underreaction] for declining [increasing] elasticity of the pricing kernel $\eta_t^{\Phi,D}$, the instantaneous volatility $\Sigma_S(t)$ is higher [lower] than for constant elasticity of the pricing kernel. The implications for the variance of asset returns over finite periods are stated in Proposition 8, similar to Proposition 6 for the finite horizon setting.

Proposition 8 *Suppose that the dividend process* $D_{t\in[0,\infty)}$ *is governed by a geometric Brownian motion with constant instantaneous volatility and constant instantaneous drift. Then*
a) for the conditional and unconditional variance

$$Var\left(\ln V_\tau - \ln V_t | \mathcal{F}_{t-\theta}\right) > Var\left(\ln D_\tau - \ln D_t | \mathcal{F}_{t-\theta}\right), \quad 0 \le \theta \le t < \tau < T,$$

if the elasticity of the pricing kernel $\eta_t^{\Phi,D}$ *is declining,*
b) for the conditional variance

$$Var\left(\ln V_\tau - \ln V_t | \mathcal{F}_t\right) < Var\left(\ln D_\tau - \ln D_t | \mathcal{F}_t\right), \quad 0 \le t < \tau < T,$$

if the elasticity of the pricing kernel $\eta_t^{\Phi,D}$ *is increasing.*

Proof. The proof follows from the proof of Proposition 6. ∎

Hence, as in the finite horizon setting, the conditional and the unconditional variance of log asset returns are higher for declining than for constant elasticity of the pricing kernel. Also the analog to Corollary 1 holds. If the elasticity of the pricing kernel declines to a lower bound, then the instantaneous volatility of asset returns declines to the instantaneous volatility of the dividend process for high dividend levels which generate high asset price levels.

Corollary 2 *Assume that the dividend process $D_{t \in [0,\infty)}$ is governed by a geometric Brownian motion with constant instantaneous volatility and constant instantaneous drift. Then, the instantaneous volatility $\Sigma_F(t) = \Sigma_V(t)$ converges from above [below] to σ_D for $F_t \to \infty$, i.e.*

$$\Sigma_V(t) = \Sigma_F(t) \searrow [\nearrow] \sigma_D \text{ for } F_t \to \infty ,$$

if the elasticity of the pricing kernel $\eta_t^{\Phi,D}$ is declining [increasing] and converges to a lower [upper] bound.

Proof. The proof is the same as the proof of Corollary 1 with D_t instead of I_t. ∎

To conclude, many results for the return characteristics seem to be robust against whether we chose a finite or an infinite horizon setting. Non-constant elasticity of the pricing kernel generates serial correlation in excess returns and random volatility. Asset prices overreact [underreact] if the elasticity of the pricing kernel is declining [increasing]. Overall the qualitative asset price characteristics are similar in the finite and in the infinite horizon setting.

To get a deeper understanding of the infinite horizon setting we present numerical simulations in the following section. We again derive an analytical approximation of the asset price based on our general pricing kernel characterization.

3.3 An Analytical Price Process

The discrete time version for the infinite horizon setting can be derived in a similar way as for the finite horizon setting. However, we approximate the sum over an infinite number of periods of dividends by a sum of the

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dividends until a given horizon and a suitable approximation for the horizon wealth generated by subsequent dividends. The following procedure is chosen. We assume that the investor has a finite horizon h up to which dividends are explicitly discounted, the remainder is modeled by a term which approximates the expected asset price at time $t+h$. The horizon h is constant over time. The asset price at the horizon is assumed to be a deterministic function of the dividend paid at the horizon. We approximate the horizon wealth, i.e. the asset price at time $t + h$, by the dividend D_{t+h} multiplied by the price-dividend ratio dp_{t+h}, i.e. $S_{t+h} = dp_{t+h}D_{t+h}$. In a traditional framework with constant elasticity of the pricing kernel dp would be constant. As we have seen, this is not true for non-constant elasticity. Since declining elasticity of the pricing kernel leads to a lower risk premium for high levels of the dividend, the price-dividend ratio dp increases with D. To get analytical results, a generalized polynomial approximation as for the pricing kernel appears suitable, i.e. $dp_{t+h} = \overline{dp} \sum_{j=1}^{J} D_{t+h}^{\vartheta_j}$, where $\overline{dp}$ is some benchmark price-dividend ratio. To keep the calculation as simple as possible we further restrict the price-dividend ratio to $dp_{t+h} = \overline{dp}D_{t+h}^{\vartheta}$, $\vartheta \geq 0$, so that the horizon wealth is $\overline{dp}D_{t+h}^{\vartheta+1}$. Though this might be a rough approximation, the approximation error will be small if h is high. Analyzing the annual S&P 500 real price and price-dividend data for the time period 1871-2002[13], one finds that the price-dividend ratio is reasonably approximated by $\frac{S_t}{AD_t} = 7.27AD_t^{0.53}$ with an R^2 of almost 40 percent where $AD_t = 12D_t$ is the annual and D_t the monthly dividend . This implies for the numerical simulation based on monthly data $dp_{t+h} = 325.6D_{t+h}^{0.53}$.

The pricing kernel $\Phi_{t,s}$ is a function of S_s and, hence, of D_s. We specify this function in our simulation by assuming that $\Phi_{t,s} = \Phi(I_{s,t+h})$ with $I_{s,t+h} = E\left(\overline{dp}D_{t+h}^{\vartheta+1}\middle| \mathcal{F}_s\right)$ being the expected horizon wealth conditional on the information set $\mathcal{F}_s$. Then for a geometric Brownian motion of the dividend, $I_{s,t+h}$ is related to D_s by

$$I_{s,t+h} = E\left(\overline{dp}D_{t+h}^{\vartheta+1}\middle| \mathcal{F}_s\right) = \overline{dp}D_s^{\vartheta+1} \exp\left((\vartheta + 1)\left(\mu_D + \frac{\vartheta\sigma_D^2}{2}\right)(t + h - s)\right) .$$

Hence, the general pricing kernel $\Phi_{t,s}$ is given by

$$\Phi_{t,s} = \frac{\Phi(I_{s,t+h})}{E(\Phi(I_{s,t+h})| \mathcal{F}_t)} = \frac{\sum_{i=1}^{N} \alpha_i (I_{s,t+h})^{\delta_i}}{E\left(\sum_{i=1}^{N} \alpha_i (I_{s,t+h})^{\delta_i} \middle| \mathcal{F}_t\right)}, \tag{24}$$

---

[13]Source: Shiller (http://www.econ.yale.edu/~shiller/data.htm)

with $\alpha_i, \delta_i \in \mathbb{R}$.

The asset price is given by

$$
\begin{aligned}
S_t &= E\left(\sum_{s=t}^{t+h-1} D_s \exp\left(r_f\left(t-s\right)\right) \frac{\Phi\left(I_{s,t+h}\right)}{E\left(\left.\Phi\left(I_{s,t+h}\right)\right| \mathcal{F}_t\right)} \middle| \mathcal{F}_t \right) \\
&\quad + E\left(\overline{dp} D_{t+h}^{\vartheta} D_{t+h} \exp\left(-r_f h\right) \frac{\Phi\left(I_{t+h,t+h}\right)}{E\left(\left.\Phi\left(I_{t+h,t+h}\right)\right| \mathcal{F}_t\right)} \middle| \mathcal{F}_t \right) .
\end{aligned}
\tag{25}
$$

Inserting the pricing kernel definition into equation (25) yields the general analytical pricing formula. Equations (24) and (25) show that $S_t = S\left(D_t\right)$ since the dividend process is a geometric Brownian motion. Therefore S_t does not depend on t so that the asset returns are time-homogeneous.

For the numerical simulation, we again assume the pricing kernel to be

$$
\Phi_{t,s} = \frac{\frac{1}{I_{s,t+h}} + \beta I_{s,t+h}^{\delta}}{E\left(\left.\frac{1}{I_{s,t+h}} + \beta I_{s,t+h}^{\delta}\right| \mathcal{F}_t\right)} .
$$

This yields the following formula for the asset price.

$$
\begin{aligned}
S_t &= \sum_{s=t}^{t+h-1} \exp\left(r_f\left(t-s\right)\right) \\
&\quad \times \frac{\frac{D_t^{-\vartheta}}{dp} H_1 + \beta \overline{dp}^{\delta} D_t^{(\vartheta+1)\delta+1} A\left(s,\left(\vartheta+1\right)\delta+1\right) B\left(s,\delta\right)}{\frac{D_t^{-(\vartheta+1)}}{dp} H_2 + \beta \overline{dp}^{\delta} D_t^{(\vartheta+1)\delta} A\left(s,\left(\vartheta+1\right)\delta\right) B\left(s,\delta\right)} \\
&\quad + \exp\left(-r_f h\right) \\
&\quad \times \frac{1 + \beta \overline{dp}^{\delta+1} D_t^{(\delta+1)(\vartheta+1)} A\left(t+h,\left(\vartheta+1\right)\left(\delta+1\right)\right)}{\frac{D_t^{-(\vartheta+1)}}{dp} A\left(t+h,-\left(\vartheta+1\right)\right) + \beta \overline{dp}^{\delta} D_t^{(\vartheta+1)\delta} A\left(t+h,\left(\vartheta+1\right)\delta\right)} ,
\end{aligned}
\tag{26}
$$

with

$$
\begin{aligned}
H_1 &= A\left(s,-\vartheta\right) B\left(s,-1\right), \\
H_2 &= A\left(s,-\left(\vartheta+1\right)\right) B\left(s,-1\right), \\
A\left(s,\left(\vartheta+1\right)\delta\right) &= \exp\left(\left(\left(\vartheta+1\right)\delta\right)\left(s-t\right)\left(\mu_D + \left(\left(\vartheta+1\right)\delta-1\right)\frac{\sigma_D^2}{2}\right)\right), \\
B\left(s,\delta,\vartheta\right) &= \exp\left(\delta\left(\vartheta+1\right)\left(t+h-s\right)\left(\mu_D + \frac{\vartheta\sigma_D^2}{2}\right)\right)
\end{aligned}
$$

For constant elasticity of the pricing kernel ($\beta = 0$ or $\delta = -1$) we also have $\vartheta = 0$. Then equation (26) simplifies to

$$S_t = D_t \sum_{s=t}^{t+h-1} \exp\left(r_f \left(t - s\right)\right) \exp\left(\left(s - t\right) \left(\mu_D - \sigma_D^2\right)\right) \tag{27}$$

$$+ D_t \exp\left(-r_f h\right) \overline{dp} \exp\left(h \left(\mu_D - \sigma_D^2\right)\right).$$

Hence, in this case the asset price S_t is a linear function of the dividend D_t. Since the dividend payments are governed by a geometric Brownian motion, the same is true of the asset price.

Again, we simulate the asset price for 240 months, i.e. 20 years. The constant investment horizon of the investor is also set to $h = 240$ months. Choosing such a large h should render the approximation error very small. Also, for every parameter combination we run 1000 simulations. The real riskless interest rate is set to 2.5 percent p.a. which is consistent with the historical average (see also Brennan and Xia, 2002). We also use pricing kernel specifications 1 to 3 as in the finite horizon setting. Specification 4 is replaced by specification 5 with $\delta = -20$ instead of -40 because of extreme fluctuations of the present value of horizon wealth generated by $\delta = -40$. In the infinite horizon setting we have some more parameters to choose. Since $dp_{t+h} = 325.6 D_{t+h}^{0.53}$ seems to be a reasonable approximation for the monthly price-dividend ratio, we use it to compute the horizon wealth. Consistent with the historical mean and volatility of real dividend growth we choose $\sigma_D = 0.037$ and $\mu_D = 0.002$. D_0 is assumed to be 1. The following table shows the parameter combinations of the pricing kernel for our simulations.

Specification	1	2	3	5
δ	-5	-7.5	-15	-20
β	0.5	1	0.5	0.1

Table 3. Pricing kernel specifications of the numerical simulations (dividend model)
The table shows the parameter combinations for the 4 presented numerical simulations. δ and β are the parameters of the pricing kernel. Setting $\beta = 0$ or $\delta = -1$ leads to a geometric Brownian motion.

Excess returns are defined as

$$r_{t,t+1} = \ln\left(S_{t+1} + D_{t+1}\right) - \ln S_t - r_f.$$

The following table summarizes the characteristics of the simulated excess returns.

Specification:	1	2	3	5	GBM
mean annualized volatility of monthly returns	0.158	0.158	0.249	0.164	0.128
mean annualized volatility of 4-year returns	0.158	0.158	0.225	0.162	0.128
mean autocorrelation (lag 1) of monthly returns	0.004	0.004	-0.001	-0.003	0
mean autocorrelation (lag 1) of 4-year returns	-0.012	-0.012	-0.133	-0.028	0
autocorrelation (lag 1) in monthly return volatility	-0.065	-0.064	0.902	0.862	0
autocorrelation (lag 4) in monthly return volatility	0.090	0.090	0.864	0.847	0

Table 4. Characteristics of excess returns and excess return volatility (dividend model)
The table shows the mean annualized volatility of monthly and 4-year-returns, the lag 1-serial correlation of these returns as well as lag 1- and lag 4-serial correlations in return volatility. For comparison we also show the theoretical values for a geometric Brownian motion (constant elasticity of the pricing kernel).

Table 4 illustrates that declining elasticity of the pricing kernel also leads in the infinite horizon case to higher volatility of excess returns. It also causes long-term reversals (negative serial correlation of 4-year excess returns). However, we find almost no positive serial correlation of monthly excess returns.

Even though all specifications generate excess volatility, only specifications 3 and 5 generate persistence in excess return volatility. For specifications 1 and 2, volatility is even negatively autocorrelated at lag 1. Notice that the results for specifications 1 and 2 differ very little. They are also very similar to those of the corresponding finite horizon specifications except for the persistence in volatility. Also notice that one would expect Specification 5 to have stronger effects since $|\delta|$ is the highest. Figure 6 illustrates these results. Since asset returns are time-independent, Figure 6 displays the asset

prices S_t as a function of the Dividend D_t only.

- insert Figure 6 here -

Figure 6 explains why we do not find higher volatility and stronger serial correlation for specification 5. The reason is that the asset price becomes a very steep function of dividends but only for relatively high dividend levels. Since we assume $D_0 = 1$ it is relatively unlikely that within 240 months dividends reach the interval where the asset price reacts strongly to changes in dividends. Therefore we do not observe such pronounced deviations from the geometric Brownian motion for specification 5. Table 4 would look differently if we started from another dividend level, for example $D_0 = 3$, even though asset returns are time-homogeneous. This confirms our conjecture that the time-dependence in the performance index model stems from the fixed terminal date T and the changing investment horizon $T - t$.

Figure 7 illustrates Proposition 7 that excess returns in an infinite horizon setting may be positively autocorrelated in some dividend range if the elasticity of the pricing kernel is declining. Figure 7 also shows the elasticity of the asset price with respect to the dividend. For specifications 3 and 5 this elasticity varies strongly.

- insert Figure 7 here -

This section has shown that excess returns in the dividend model have similar characteristics as in the performance index model. However, excess returns are time-independent in the dividend model and some effects observable in the performance index model are cushioned. This is not surprising, since in the infinite horizon setting we have a continuous dividend stream which generates less volatile income in the near future so that the effect of declining elasticity of the pricing kernel is mitigated.

3.4 Technical Analysis and Market Crashes

Figure 6 and Figure 7 show that also in the dividend model declining elasticity of the pricing kernel can explain stock market crashes. We see from these

figures for specifications 3 and 5 that there exist dividend intervals where the asset price reacts very strongly to changes in the level of dividends. Hence, the model predicts that even without a major drop in dividends the asset price might plunge. As argued before such a functional form of the asset price can also explain why support and resistance levels seem to exist. Hence, asset price functions $S_t(D_t)$ that are convex and concave, together with dividend regions where the asset price is a very steep function, may explain why technical analysis might be successfully applied and stock market crashes occur without significant fundamental news.

To conclude, this section has demonstrated for the dividend model and specific pricing kernels, that excess returns are predictable, they overreact and there is a rationale for stock market crashes without significant news.

4   Conclusion

This paper analyzes the impact of non-constant elasticity of the pricing kernel (non-constant relative risk aversion of the representative investor) on asset returns in a rational expectations model. If the pricing kernel has constant elasticity and the fundamental information process is governed by a geometric Brownian motion, then asset prices are also governed by a geometric Brownian motion. It is shown that declining elasticity of the pricing kernel can lead to short-term momentum, long-term reversals as well as high and persistent volatility of excess returns. Declining elasticity of the pricing kernel provides even a rationale for the successful use of chart analysis in an efficient market. If the pricing kernel has declining elasticity of a certain nature, then the asset price reaction to changes in fundamentals depends on the level of the fundamentals. In certain ranges the asset price reaction to changes in fundamentals is weak while in other ranges the reaction can be very strong. This implies that although we assume a rational and efficient market, the performance of fundamentals and asset returns are not linearly related. Hence, high asset returns do not necessarily signify strong fundamental growth. Also stock market crashes may occur without significant news. The analysis in this paper is based on two different settings, a finite horizon setting and an infinite horizon setting. The main results are the same in both settings.

We also derive analytical discrete time stochastic processes for both models. These processes provide new flexible alternatives to existing time-series

models since they are consistent with many empirical findings. In contrast to mainly empirically motivated time-series models, the proposed ones have a solid economic foundation. The numerical results are based on special forms of pricing kernels. Future research might use pricing kernels which are more complicated and more in line with the empirically observed shapes of pricing kernels.

Appendix

Proof of Corollary 1

First, we give the proof for declining elasticity. Since elasticity is positive, it necessarily converges to a lower bound. Since $\frac{\partial}{\partial I_t}\left(\frac{v^{DE}(t,I_t)}{I_t}\right) \geq 0$ and $\frac{v^{DE}(t,I_t)}{I_t} \leq 1$ it follows from the Theorem of Bolzano-Weierstrass that

$$\lim_{I_t \to \infty}\left(\frac{v^{DE}(t,I_t)}{I_t}\right) = c \ ,$$

where c is some positive constant with $c \leq 1$ and $F_t = v^{DE}(t,I_t)$. Since $\lim_{I_t \to \infty} v^{DE}(t,I_t) = \infty$ it follows from the rule of L'Hopital that

$$c = \lim_{I_t \to \infty}\left(\frac{v^{DE}(t,I_t)}{I_t}\right) = \lim_{I_t \to \infty}\left(\frac{\partial}{\partial I_t}v^{DE}(t,I_t)\right) \ .$$

Hence, the elasticity of the forward price with respect to I_t converges to 1, i.e.

$$\lim_{I_t \to \infty}\left(\frac{\partial}{\partial I_t}v^{DE}(t,I_t)\frac{I_t}{v^{DE}(t,I_t)}\right) = 1 \ .$$

For $I_t < \infty$ we have already seen that the elasticity is higher than 1. Hence,

$$\lim_{I_t \to \infty}\left(\Sigma_F^{DE}(t)\right) = \sigma_I \ ,$$

while $\Sigma_F^{DE}(t) > \sigma_I \ \forall I_t < \infty$.

Consider increasing elasticity converging to an upper bound. Since $\frac{\partial}{\partial I_t}\left(\frac{v^{IE}(t,I_t)}{I_t}\right) \leq 0$ and $\frac{v^{IE}(t,I_t)}{I_t} > 0$ it follows from the Theorem of Bolzano-Weierstrass that

$$\lim_{I_t \to \infty}\left(\frac{v^{IE}(t,I_t)}{I_t}\right) = c \ ,$$

where c is some positive constant with $c > 0$ and $F_t = v^{IE}(t, I_t)$. Note that because I_T is lognormally distributed, $\frac{v^{IE}(t, I_t)}{I_t} \leq 0$ would imply arbitrage possibilities. Since $\lim_{I_t \to \infty} v^{IE}(t, I_t) = \infty$ it follows from the rule of L'Hopital that

$$c = \lim_{I_t \to \infty} \left(\frac{v^{IE}(t, I_t)}{I_t} \right) = \lim_{I_t \to \infty} \left(\frac{\partial}{\partial I_t} v^{IE}(t, I_t) \right) .$$

Hence, the elasticity of the forward price with respect to I_t converges to 1. For $I_t < \infty$ we have already seen that the elasticity is smaller than 1. Hence,

$$\lim_{I_t \to \infty} \left(\Sigma_F^{IE}(t) \right) = \sigma_I ,$$

while $\Sigma_F^{IE}(t) < \sigma_I \; \forall I_t < \infty$. ∎

Proof of Proposition 7

Let $CER_{t,\tau} \equiv \int_t^\tau (dS_s / S_s) + \int_t^\tau (D_s / S_s - r_f) \, ds$, then using equation (8) we obtain

$$Cov\left(CER_{t,\tau}, \mu_V(\tau) - r_f \middle| \mathcal{F}_t \right) = Cov\left(\int_t^\tau \frac{dS_s}{S_s} + \int_t^\tau \frac{D_s}{S_s} ds, \int_t^\tau d\mu_V(s) \middle| \mathcal{F}_t \right)$$

$$= E\left(\left[\int_t^\tau \frac{dS_s}{S_s} + \int_t^\tau \frac{D_s}{S_s} ds - E\left(\int_t^\tau \frac{dS_s}{S_s} + \int_t^\tau \frac{D_s}{S_s} ds \middle| \mathcal{F}_t \right) \right] \right.$$
$$\left. \times \left[\int_t^\tau d\mu_V(s) - E\left(\int_t^\tau d\mu_V(s) \middle| \mathcal{F}_t \right) \right] \middle| \mathcal{F}_t \right)$$

$$= E\left(\left(\int_t^\tau \eta_s^{V,D} \sigma_D dW_s \right) \left(\int_t^\tau \left\{ \eta_s^{V,D} \frac{\partial \eta_s^{\Phi,D}}{\partial D_s} + \eta_s^{\Phi,D} \frac{\partial \eta_s^{V,D}}{\partial D_s} \right\} \sigma_D^3 D_s dW_s \right) \middle| \mathcal{F}_t \right)$$

$$= \int_t^\tau E\left(\underbrace{\left\{ \eta_s^{V,D} \frac{\partial \eta_s^{\Phi,D}}{\partial D_s} + \eta_s^{\Phi,D} \frac{\partial \eta_s^{V,D}}{\partial D_s} \right\}}_{= \frac{\partial \mu_s / \partial D_s}{\sigma_I^2}} \sigma_D^4 D_s \eta_s^{V,D} \middle| \mathcal{F}_t \right) ds.$$

The elasticities $\eta_s^{V,D}$ and $\eta_s^{\Phi,D}$ are positive. The instantaneous volatility of

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the total return index, $\Sigma_V(s)$, satisfies $\Sigma_V(s) = \eta_s^{V,D}\sigma_D$. Therefore $\frac{\partial \Sigma_V(s)}{\partial D_s} = \frac{\partial \eta_s^{V,D}}{\partial D_s}\sigma_D$. Hence $Cov\left(CER_{t,\tau}, \mu_V(\tau) - r_f \middle| \mathcal{F}_t\right) < [>]\, 0$ if the elasticity of the pricing kernel is declining [increasing] and the instantaneous volatility is declining [increasing] in D_s. ∎

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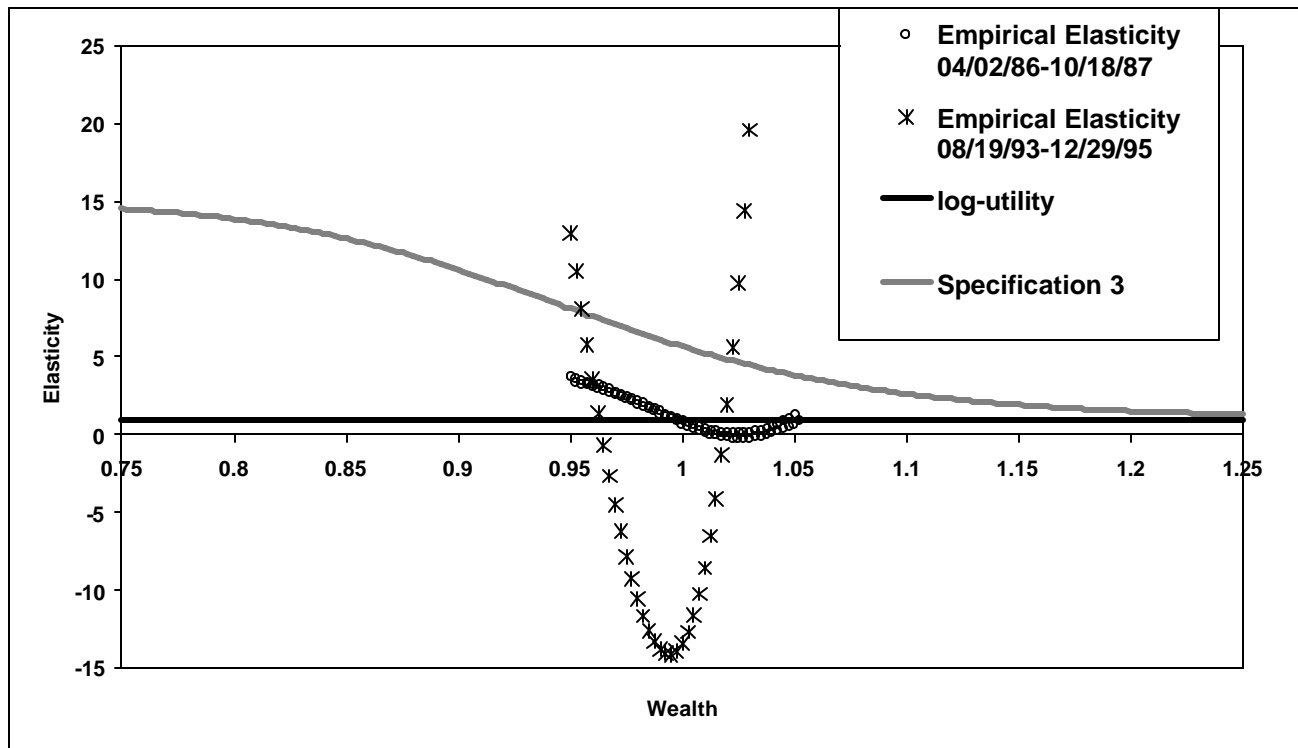
Figure 1: Comparison of the elasticity of pricing kernels
The elasticity of the pricing kernel specification 3 is shown and compared to the elasticity of the benchmark pricing kernel (logarithmic utility) as well as to the elasticity of 2 empirical pricing kernels (Jackwerth, 2000, p. 442). The empirical pricing kernels correspond to different time periods, one for the pre-crash period 04/02/86 to 10/18/87 and one for the post-crash period 08/19/93 to 12/29/95.
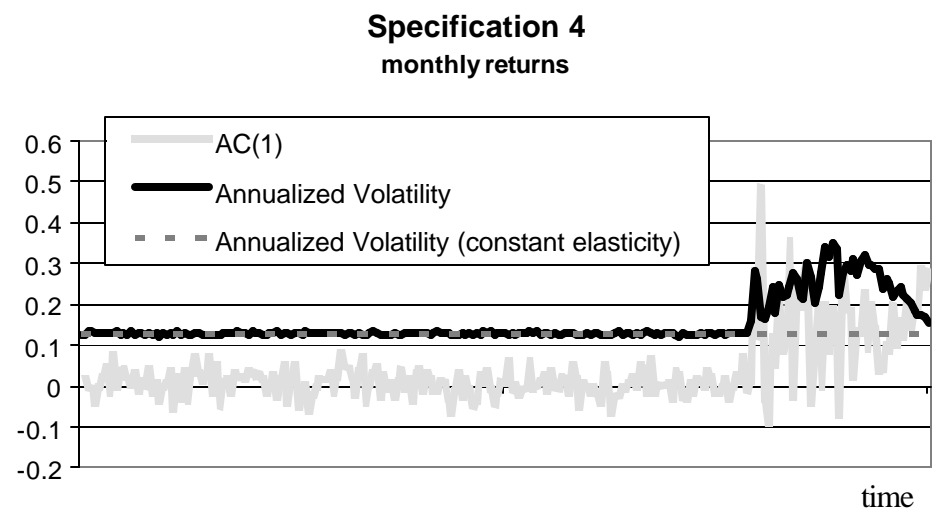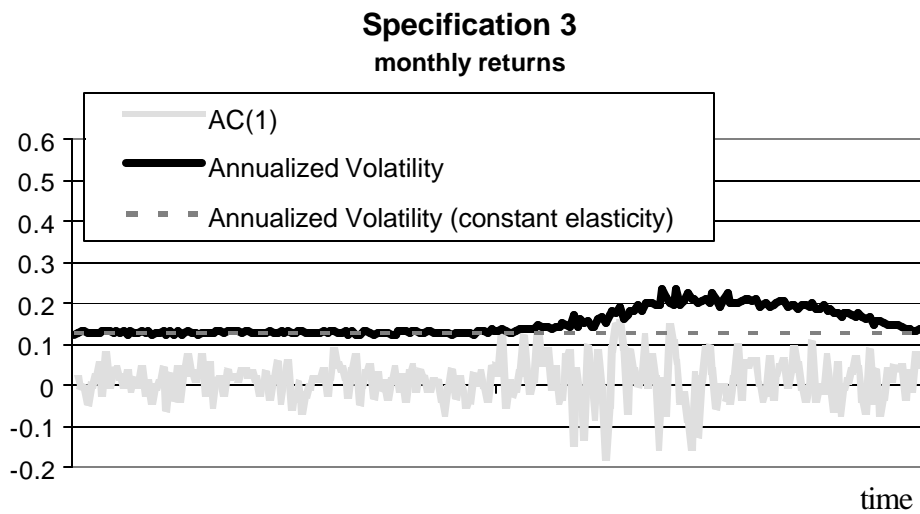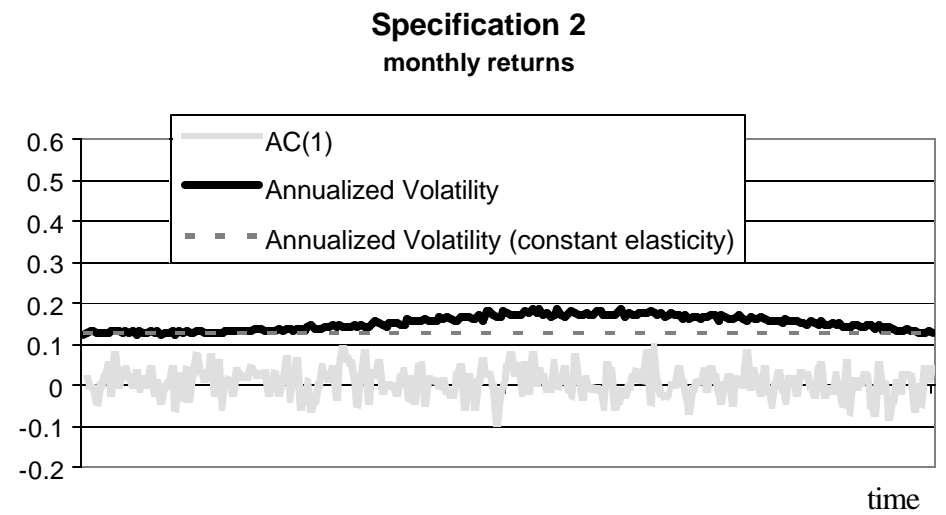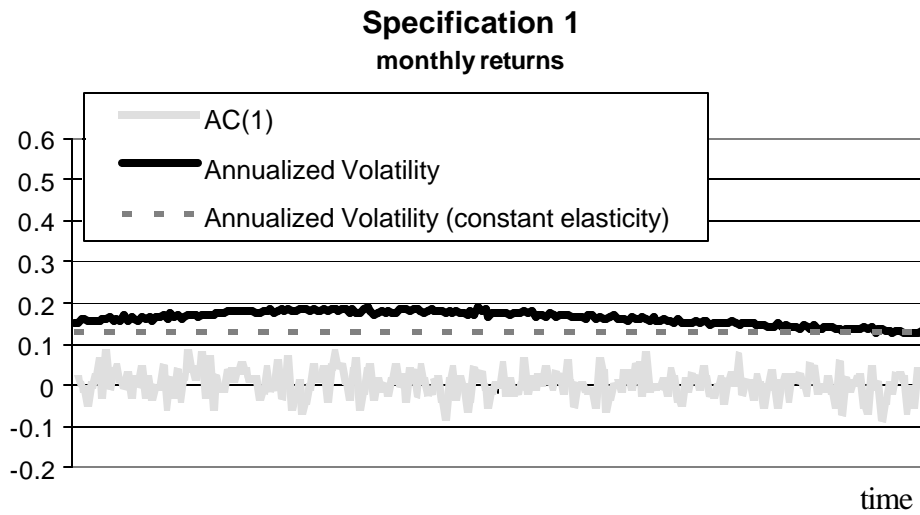
Figure 2: Serial correlation and volatility of monthly excess returns (performance index model)
The figure shows the autocorrelation of lag 1 for monthly excess returns and the annualized volatility of monthly excess returns over time.
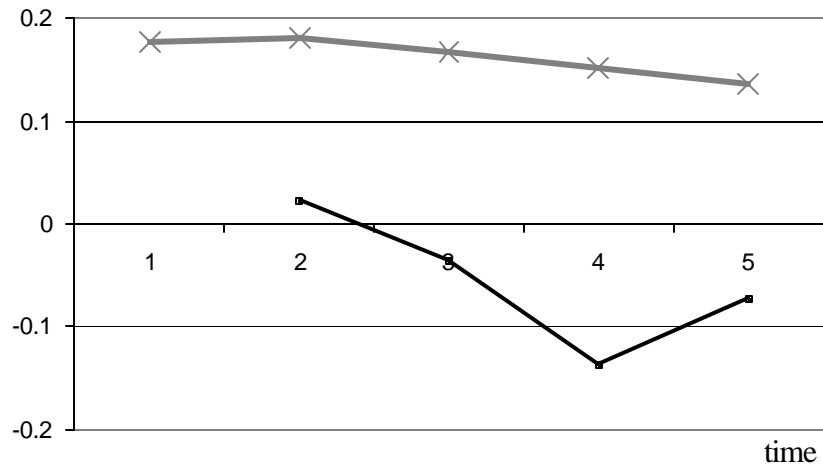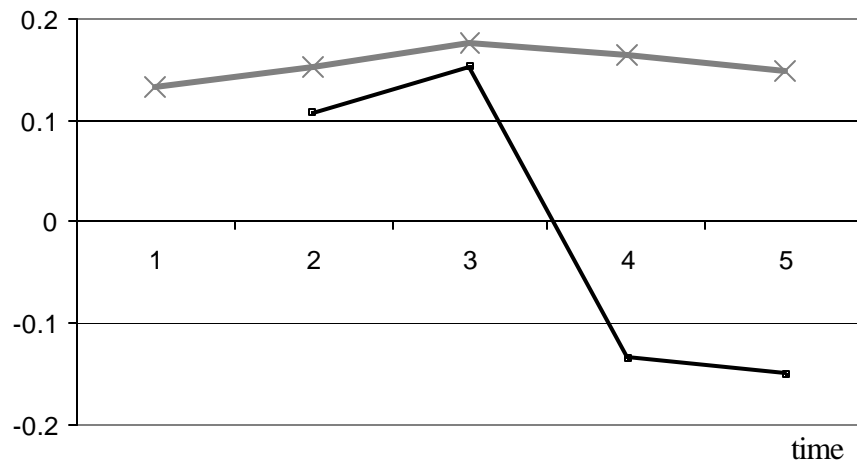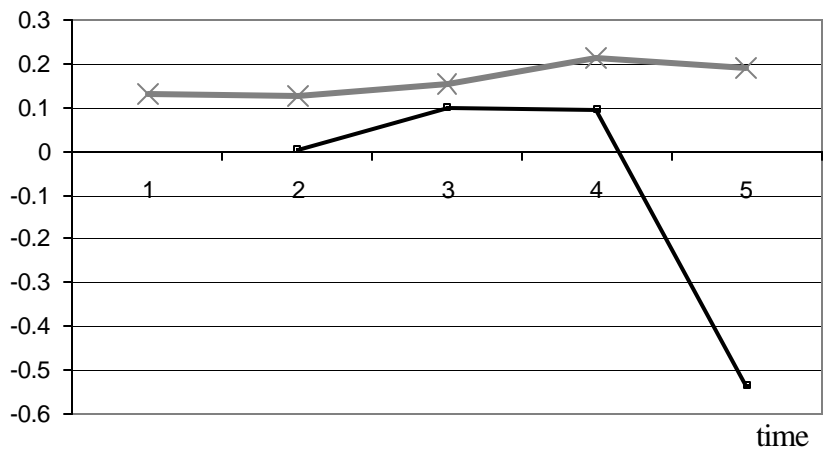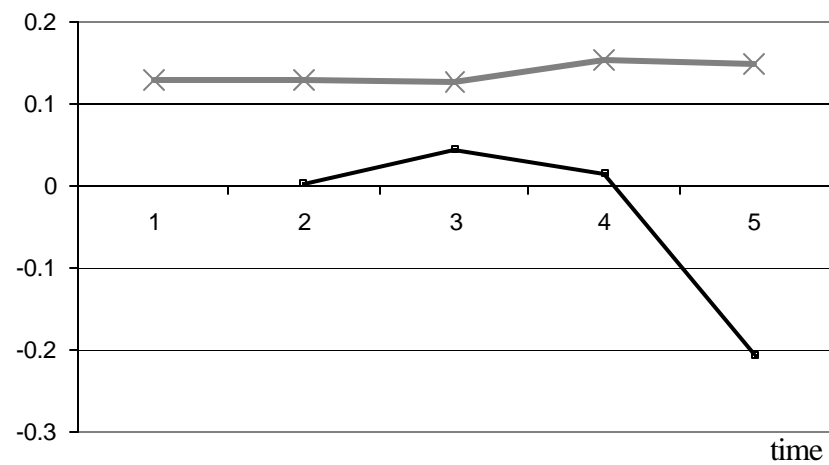
Figure 3: Serial correlation and volatility of 4-year excess returns (performance index model)
The figure shows the autocorrelation of lag 1 for 4-year excess returns (black) and the annualized volatility of 4-year excess returns (grey) over time. Since we simulate 20 years of data, there are five 4-year periods.
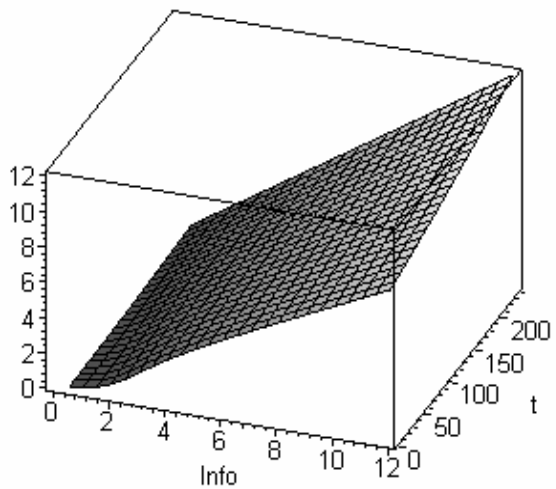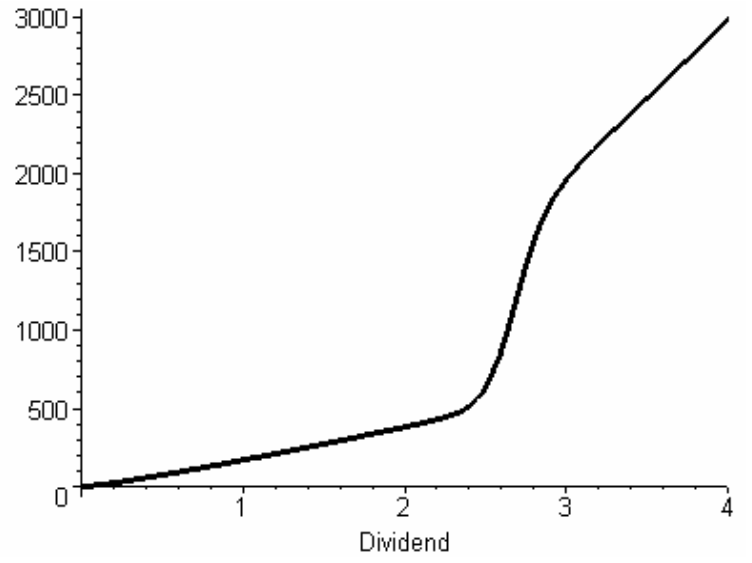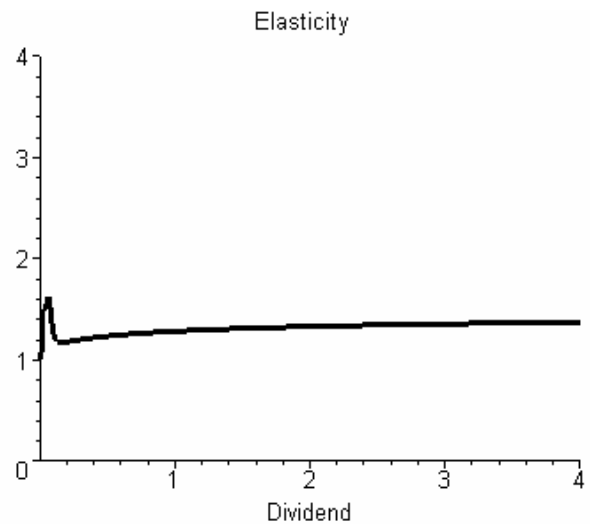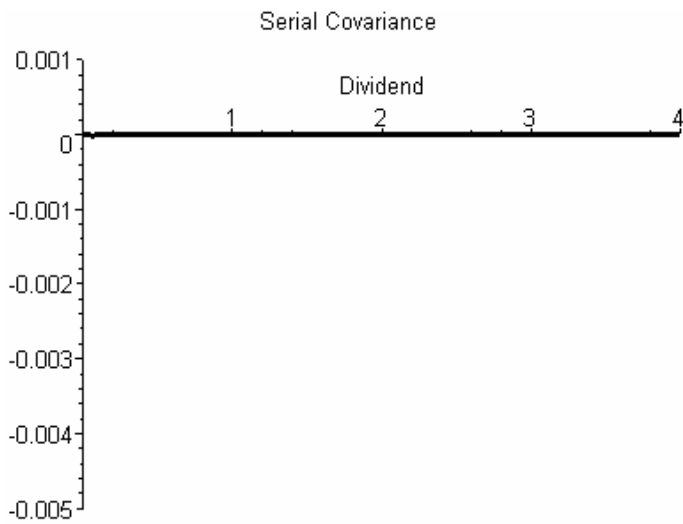
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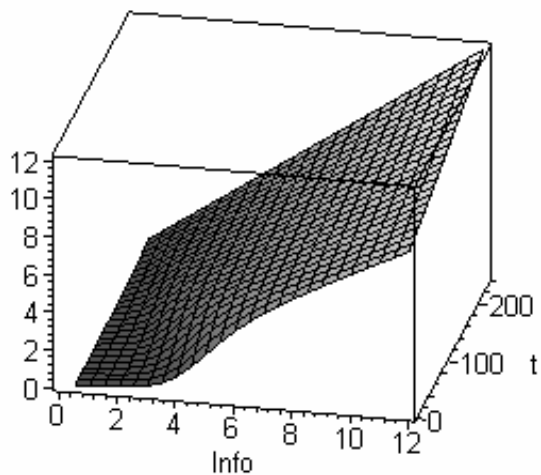
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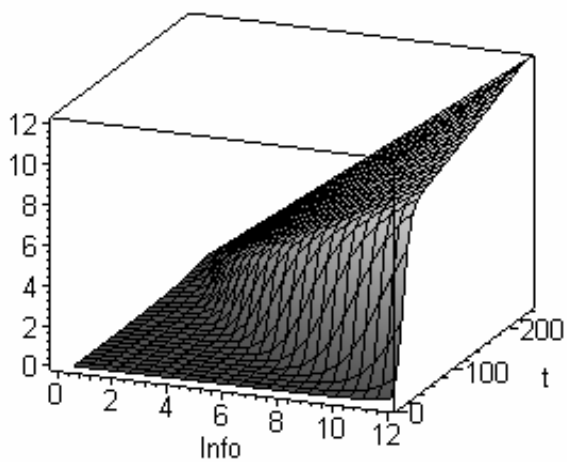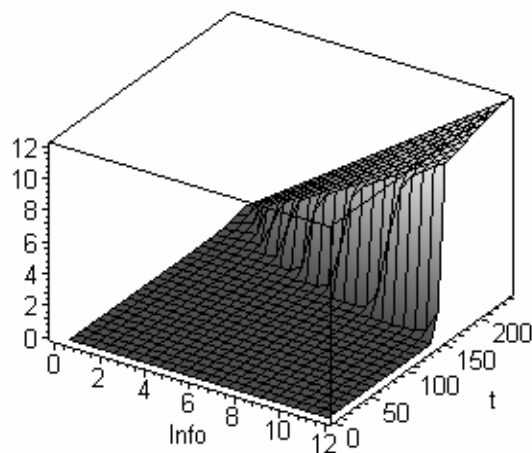
Specification 3

Specification 4

Figure 4: Forward prices as function of the information and time (performance index model)
The figure shows the forward prices as function of the information (Info) and time (t).
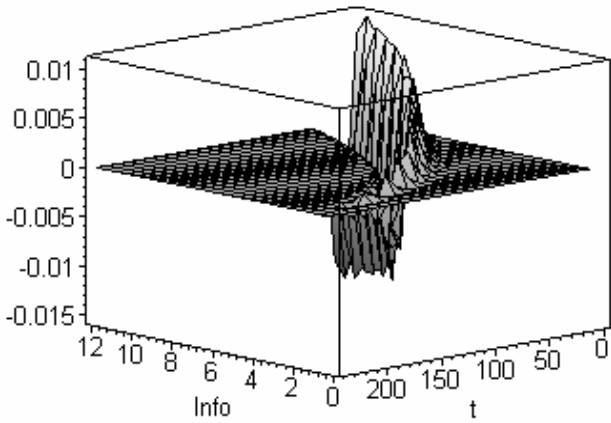
Serial covariance of
asset returns
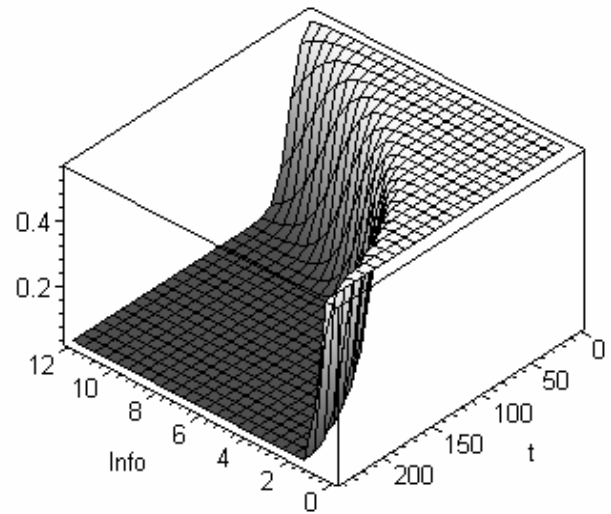
Sharpe ratio
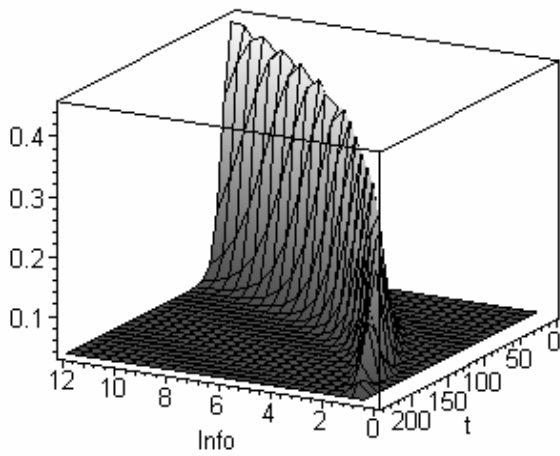
Volatility of asset returns
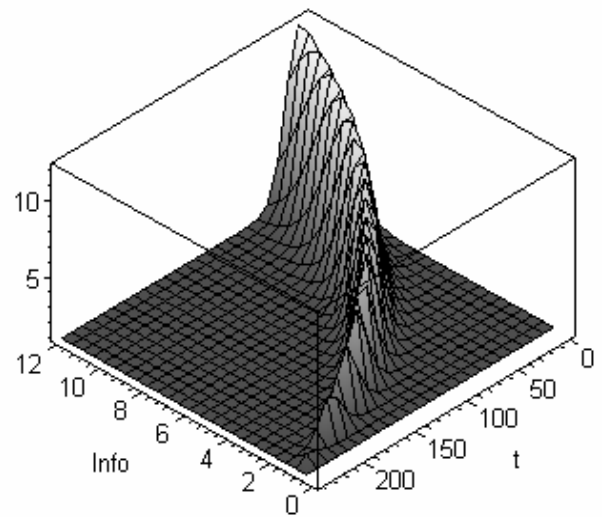
Elasticity of the asset price with
respect to the information

Figure 5: Serial covariance, Sharpe ratio, volatility and the elasticity of the asset price for Specification 3 (performance index model)

The figure shows the instantaneous serial covariance of asset returns, the instantaneous Sharpe ratio, the instantaneous volatility of asset returns and the elasticity of the asset price with respect to the information (Info) as functions of the information and time (t) for specification 3. The instantaneous serial covariance, $\mathrm{cov}_t\left(CER_{t,t}, \boldsymbol{m}(\boldsymbol{t})\right)$ with $\boldsymbol{t} \to t,$ is the cross variation between the expected excess return and the cumulated excess return.
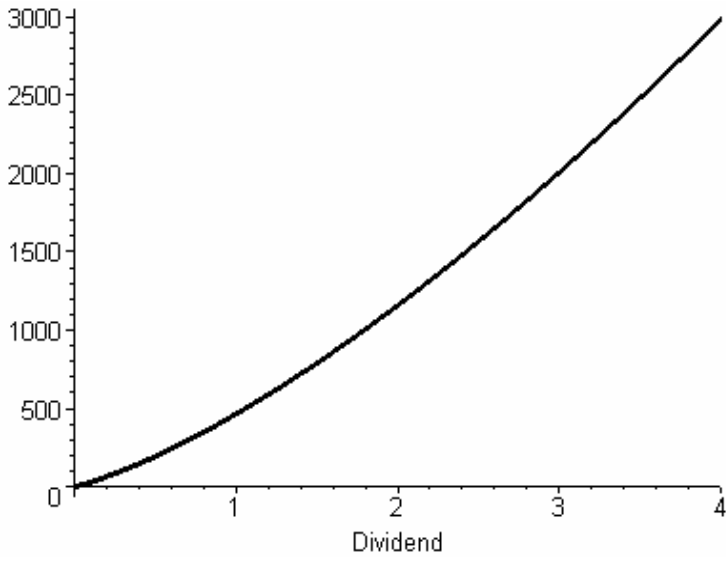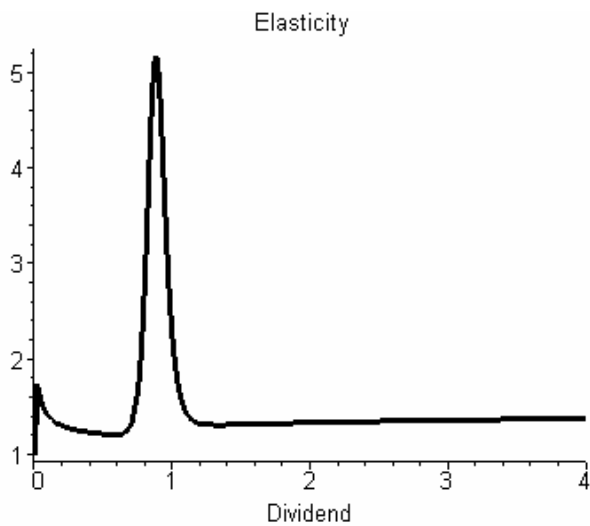
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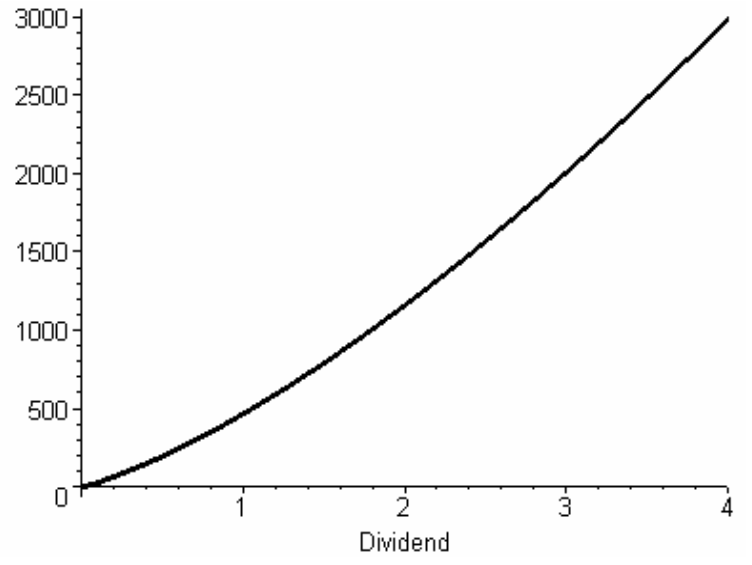
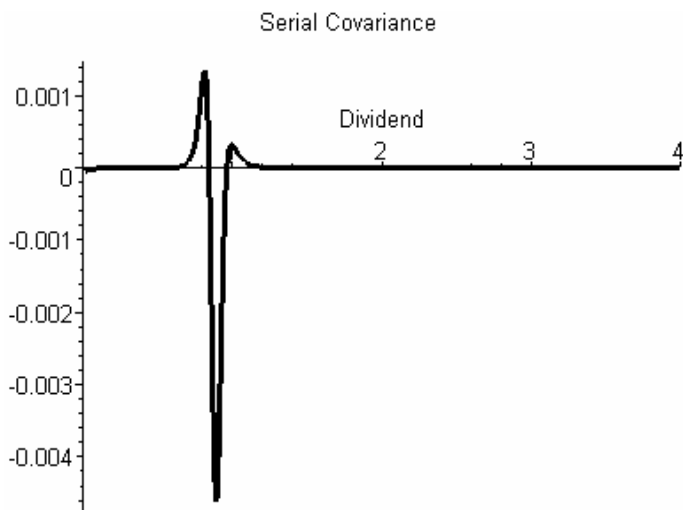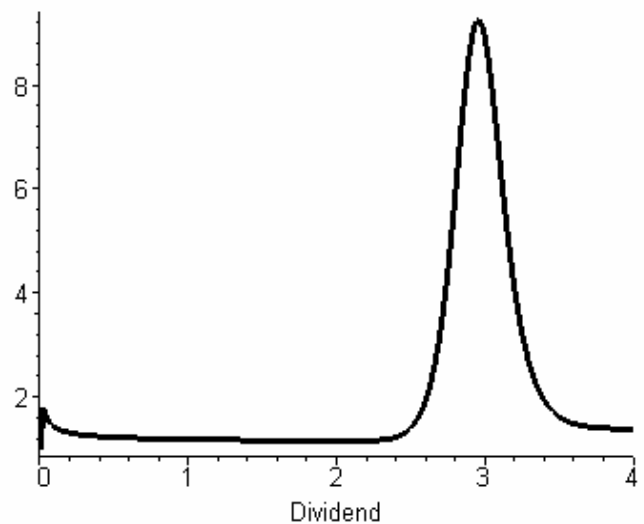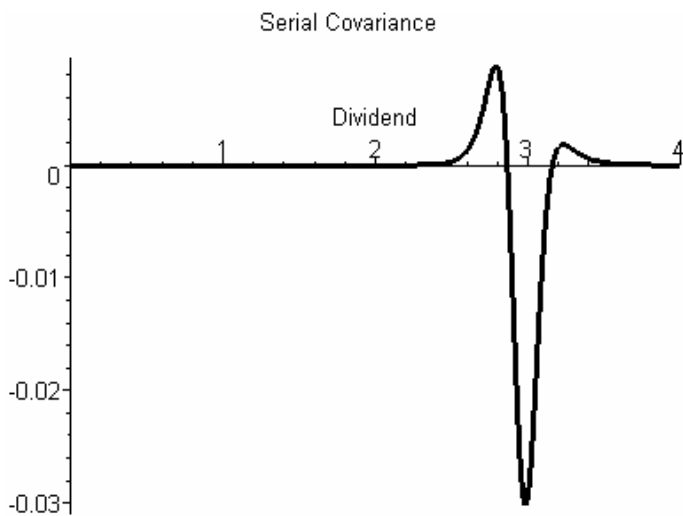Specification 2
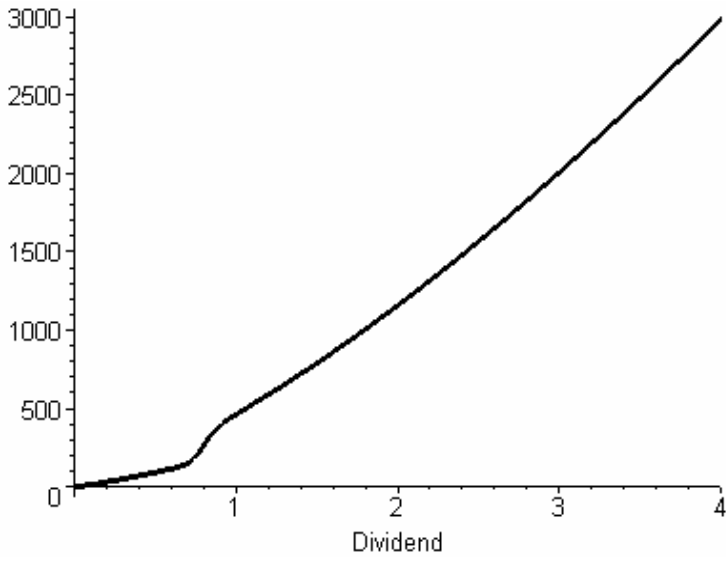
Specification 3

Specification 5

Figure 6: Asset price as a function of the dividend (dividend model)
The figure shows the spot prices as a function of the Dividend.

Figure 7: Serial covariance of excess returns and the elasticity of the asset price with respect to the dividend (dividend model)

The figure shows the instantaneous serial covariance (instantaneous cross variation between the expected excess return and the cumulated excess return) and the elasticity of the asset price with respect to the dividend as functions of the dividend for specifications 1, 3 and 5.