Infinitary Properties of Valued and Ordered Vector Spaces

Author(s): Salma Kuhlmann

Source: *The Journal of Symbolic Logic*, Vol. 64, No. 1 (Mar., 1999), pp. 216-226

Published by: Association for Symbolic Logic

Stable URL: <http://www.jstor.org/stable/2586760>

Accessed: 24/11/2009 02:58

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at <http://www.jstor.org/page/info/about/policies/terms.jsp>. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at <http://www.jstor.org/action/showPublisher?publisherCode=asl>.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

Association for Symbolic Logic is collaborating with JSTOR to digitize, preserve and extend access to *The Journal of Symbolic Logic*.

<http://www.jstor.org>

INFINITARY PROPERTIES OF VALUED AND ORDERED VECTOR SPACES

SALMA KUHLMANN

§1. Introduction. The motivation of this work comes from two different directions: infinite abelian groups, and ordered algebraic structures. A challenging problem in both cases is that of classification. In the first case, it is known for example (cf. [KA]) that the classification of abelian torsion groups amounts to that of reduced p -groups. A milestone in the theory of infinite abelian groups was the classification up to isomorphism of countable reduced p -groups by numerical invariants called the *Ulm invariants* (given by Ulm in [U]). Ulm's theorem was later generalized by P. Hill to the class of totally projective groups. As to the second case, let us consider for instance the class of divisible ordered abelian groups. These may be viewed as ordered $\mathbb{Q}$ -vector spaces. Their theory being unstable, we cannot hope to classify them by numerical invariants. On the other hand, being o-minimal, the theory enjoys several good model theoretic properties (cf. [P-S]), so the search for some reasonable invariants is well motivated. The common denominator of the two cases, as well as of many others, is valuation theory. Indeed given an ordered vector space, one can consider it as a valued vector space, endowed with the natural valuation. Also, the socle $G[p]$ of a reduced abelian p -group G , endowed with the height function h_G , is a valued vector space over $\mathbb{F}_p$ (the prime field of characteristic p) with values in the ordinals (cf. [F]). Moreover, knowing the skeleton of $(G[p], h_G)$ is equivalent to knowing the Ulm invariants of G . So Ulm's theorem states that a countable reduced p -group is determined up to isomorphism by the skeleton of its socle. Thus, a natural question arises: does the structural study of a reduced p -group amount to that of its socle, when viewed as a valued $\mathbb{F}_p$ -vector space? It turns out that indeed several important classes of p -groups are classified by their socles (cf. [F]), however in general this is not the case (cf. [R]). On the other hand, it is often the case that invariants classifying countable models up to isomorphism serve to classify models of arbitrary cardinality, but only up to infinitary equivalence. In the case of torsion groups, back and forth arguments appear naturally and were first used by Kaplansky, later by P. Hill and C. Meggiben. A general result was given by J. Barwise and P. Eklof: (cf. [BA; Theorem 10]):

Two reduced p -groups are infinitarily equivalent if and only if they have the same Ulm invariants.

Received April 7, 1997; revised June 24, 1997.

Supported by a Deutsche Forschungsgemeinschaft Fellowship.

Another interesting result that may be viewed as a generalized Ulm theorem for valued vector spaces is due to R. Brown (cf. [BR]):

A valued vector space of countable dimension is determined up to isomorphism by its skeleton.

In his applications, Brown shows that his theorem holds for ordered vector spaces over $\mathbb{Q}$ (i.e. divisible ordered abelian groups) as well. In the same spirit, we have a theorem due to J. Erdős (cf. [E]):

A countable dimensional ordered $\mathbb{R}$ -vector space is determined up to isomorphism by its rank.

In [H1] and [H2], Hill introduces the Ulm invariants of a group relative to a subgroup, and uses them to generalize Ulm's theorem in the following way (cf. [H1; Theorem 4.1]):

Let G, G' be reduced p -groups, and A a common nice subgroup. Assume that G/A and G'/A are countable. Then G is isomorphic to G' over A if and only if G and G' have the same Ulm invariants relative to A .

The above mentioned theorems of Barwise – Eklof, Brown, Erdős and Hill are but a few instances of several scattered results which can be explained in a broader context. Each of them features some aspect of the general theory behind. The goal of this paper is to develop this theory systematically, and to prove a general classification theorem for valued vector spaces, which covers those results. Here we mean classification up to infinitary equivalence, rather than up to isomorphism. This is the appropriate setting: we have seen above, for example, that the Ulm invariants classify any p -group up to infinitary equivalence, whereas they classify only the totally projective ones up to isomorphism. The invariants we use for this classification are the skeletons. Actually, we do not need to fix the skeleton; we show a stronger result (cf. Theorem 3.1):

Let two valued vector spaces, both correct extensions of a common subspace, be given. Then they are infinitarily equivalent over that common subspace if and only if their skeletons are infinitarily equivalent over the skeleton of the common subspace.

Some explanation is required at this point. Of course, the theorem does not make sense unless we express valued vector spaces and their skeletons as structures in a first order language. This is done in Section 2, followed by several technical lemmas needed in the proofs of the main results. We work in a many sorted language, and one of the sorts is for the base field (which allows us to compare vector spaces over different base fields). If the extension we are considering is a **a fixed field extension** (i.e. the base field remains the same), then the extension is correct if and only if the subspace is nice. This brings us back to Hill's theorem, which illustrates the importance of considering equivalence over subspaces. It then appears that the natural extension of Hill's requirement about the relative Ulm invariants is just requiring isomorphism of the skeletons over the common subskeleton. But if the base field is not fixed in the extension, then niceness is not the right condition anymore. We then have to require correctness, and indeed the very reasonable algebraic properties of correct extensions (as developed in [K-K]) allow us to prove our Main Lemma, which is an embedding lemma.

The typical model we consider is a valued vector space, together with its skeleton. The reason for this choice is technical: it allows us to extend our results to valued vector spaces, endowed with extra predicates (cf. Section 4). It is mainly in view of applications to ordered vector spaces that we follow this course. In this way, we can prove Brown's theorem for ordered vector spaces over any ordered field, and recover Erdős' theorem. Another fruitful application to ordered fields was given in [K1]. There a criterion was derived for the infinitary equivalence of the ordered additive group of the field to its ordered multiplicative group of positive elements, yielding criteria for such a field to be exponential in the countable case.

§2. The theory of valued vector spaces with skeleton. Let K be a field, V a K -vector space and Γ a totally ordered set with ∞ as its last element. A surjective map

$$v : V \longrightarrow \Gamma$$

is a **valuation on V** (and $({}_K V, v)$ is a **valued vector space**) if for all $x, y \in V$ and $k \in K$, the following holds:

- (i) $v(x) = \infty$ if and only if $x = 0$,
- (ii) $v(x - y) \geq \min\{v(x), v(y)\}$ (ultrametric triangle law).
- (iii) $v(kx) = v(x)$ if $k \neq 0$.

We call $v(V) := \Gamma$ the **value set** or **rank** of $({}_K V, v)$. Let $\gamma \in \Gamma$ and put

$$\begin{aligned} V^\gamma &= \{x \in V; v(x) \geq \gamma\} \\ V_\gamma &= \{x \in V; v(x) > \gamma\}. \end{aligned}$$

Then V^γ, V_γ are K -subspaces (for $\gamma \neq \infty$) satisfying $V_\gamma \subset V^\gamma \subset V$. We put

$$B(V, \gamma) = V^\gamma / V_\gamma \quad B(V, \infty) = 0.$$

The K -vector space $B(V, \gamma)$ is called the **component corresponding to γ** . The **skeleton** of $({}_K V, v)$ is the system $S({}_K V, v) := (K, v(V), \{B(V, \gamma); \gamma \in v(V)\})$. We introduce a **coefficient map** π^V as follows. For every $\gamma \in \Gamma$, the **coefficient map corresponding to γ** is the canonical homomorphism

$$\pi^V(\gamma, -) : V^\gamma \longrightarrow B(V, \gamma) \quad \text{defined by} \quad \pi^V(\gamma, x) = x + V_\gamma.$$

Let (V_i, v_i) be valued K_i -vector spaces with value set Γ_i (for $i = 1, 2$). We say that they are **isomorphic** if there exists $h : K_1 \simeq K_2$ a field isomorphism, $f : V_1 \simeq V_2$ a vector space isomorphism, and $\varphi : \Gamma_1 \simeq \Gamma_2$ a chain isomorphism such that for all $x \in V_1$,

$$\varphi(v_1(x)) = v_2(f(x)).$$

Given two systems (for $i = 1, 2$) $S_i = (K_i, \Gamma_i, \{B_i(\gamma); \gamma \in \Gamma_i\})$ say that $S_1 \simeq S_2$ if and only if there exists $h : K_1 \simeq K_2$ and $\varphi : \Gamma_1 \simeq \Gamma_2$ as above, and for every $\gamma \in \Gamma_1$ a vector space isomorphism $\varphi_\gamma : B_1(\gamma) \simeq B_2(\varphi(\gamma))$. For the following easy result, see [K1; Lemma 2.2].

LEMMA 2.1. *Every isomorphism f of valued vector spaces induces canonically an isomorphism f_S of their skeletons.*

We wish to put the theory into a formal setting. We define the language and the theory of **valued vector spaces with skeleton**: L_{VS} is a four sorted language; one sort for the base field K , one for the vector space V , one for the rank Γ , and one for the components B . Further we need $<_\Gamma$ a binary predicate symbol, $-$, $^{-1}$, v , ζ unary function symbols, π , $+$, $\cdot$, $\cdot_{sc}$ binary function symbols, and constant symbols 0_K , 1_K , 0_V , ∞ .

The axioms of the theory T_{VS} in this language are the following:

- (A₀) The predicates K , V , Γ , B give a partition of the universe.
- (K) $(K, +, \cdot, -, ^{-1}, 0_K, 1_K)$ is a field.
- (Γ) $<_\Gamma$ is a total order of Γ with last element ∞ .
- (V) $(V, +, \cdot_{sc}, -, v, 0_V)$ is a valued K -vector space.
- (B) For all $\gamma \in \Gamma$:
 $(B(\gamma), -, +, \cdot_{sc}, \zeta(\gamma))$ is a K -vector space whose zero is $\zeta(\gamma)$.
- (RES) $\pi : \Gamma \times V \longrightarrow B$ is surjective, and $\forall \gamma \in \Gamma$, $\pi(\gamma, -)$ is a homomorphism.

Note that all those axioms except “ v is surjective”, and “ π is surjective” are universal, so preserved for substructures.

Let L_V denote the **language of valued vector spaces** (as a sublanguage of L_{VS}) and T_V the **theory of valued vector spaces** (obtained by forgetting the axioms concerning the components and the coefficient maps). Similarly let L_S and T_S denote the **language of skeletons** and the **theory of skeletons** respectively (obtained from L_{VS} in the obvious way).

We denote the models of T_{VS} , T_V and T_S by the symbols $\mathcal{M}$, $\mathbf{M}$, and M respectively.

REMARK 2.2. (1) Let $\mathcal{M}$ a model of T_{VS} , canonically, we can “restrict” $\mathcal{M}$ to a model $V(\mathcal{M})$ of T_V , and to a model $S(\mathcal{M})$ of T_S . Also, if $f : \mathcal{M}_1 \xrightarrow{\sim} \mathcal{M}_2$ is an isomorphism of models of T_{VS} , canonically we can “restrict” f to an isomorphism $f_V : V(\mathcal{M}_1) \xrightarrow{\sim} V(\mathcal{M}_2)$ of the corresponding models of T_V , and to an isomorphism $f_S : S(\mathcal{M}_1) \xrightarrow{\sim} S(\mathcal{M}_2)$ of the corresponding models of T_S . We say that f is a **lifting** of g (respectively of σ) if $f_V = g$ (respectively $f_S = \sigma$).

(2) Fix an arbitrary field K_0 . Consider on the one hand the class of K_0 -valued vector spaces, and on the other hand the class of models of T_V for which $K = K_0$. Then there is a bijective correspondence between those two classes. Moreover, every isomorphism of valued K_0 -vector spaces determines an isomorphism over K_0 of the corresponding models of T_V (more precisely, it determines an isomorphism fixing the common substructure generated by the set of parameters K_0), and vice-versa. Similarly, we have a bijective correspondence between the class of skeletons of valued K_0 -vector spaces on the one hand, and that of models of T_S for which $K = K_0$, and every isomorphism of skeletons determines an isomorphism over K_0 of the corresponding models of T_S .

Let L be an arbitrary language and $\mathcal{A}, \mathcal{B}$ L -structures such that $\mathcal{A} \subset \mathcal{B}$. Recall that $\mathcal{B}$ is said to be finitely generated over $\mathcal{A}$ if there exist $b_1, \dots, b_n \in B$ such that $\{b_1, \dots, b_n\} \cup A$ generates the L -structure $\mathcal{B}$. If $A \subset B$ we denote by $\langle A \rangle_{\mathcal{B}}$ the substructure of $\mathcal{B}$ generated by A . Recall that ${}_K V$ denotes a K -vector space V . Now let $\{x_i; i \in I\} \subset V$. We denote by ${}_K^V \langle \{x_i; i \in I\} \rangle$ the K -subspace of V generated by the x_i ’s (by convention, ${}_K^V \langle \emptyset \rangle = 0$).

Let K_0 be a subfield of K . If V is a K -vector space, then V is in a canonical manner a K_0 -vector space. We write ${}_K V$ and ${}_{K_0} V$ to distinguish these two structures. We say that ${}_{K_0} V_0 \subset {}_K V$ is an **extension** if ${}_{K_0} V_0$ is a K_0 -subspace of ${}_{K_0} V$, and that ${}_K V$ is **finitely generated** over ${}_{K_0} V_0$ if K is finitely generated over K_0 as a field and V is finitely generated over ${}_K \langle V_0 \rangle$ as a vector space. The following lemma is straightforward.

LEMMA 2.3. (1) Let $\mathbf{M}$ and $\mathbf{M}_0$ be models of T_V such that $\mathbf{M}_0 \subset \mathbf{M}$. Then $\mathbf{M}$ is finitely generated over $\mathbf{M}_0$ if and only if ${}_K V$ is finitely generated over ${}_{K_0} V_0$.

(2) Let M and M_0 be models of T_S such that $M_0 \subset M$. Then M is finitely generated over M_0 if and only if K is finitely generated over K_0 as a field, and there exists a finite $\Gamma' \subset \Gamma$, such that for every $\gamma \in \Gamma \setminus \Gamma'$, $B(\gamma) = {}_K^{B(\gamma)} \langle B_0(\gamma) \rangle$ and for every $\gamma \in \Gamma'$, $B(\gamma)$ is finitely generated over ${}_K^{B(\gamma)} \langle B_0(\gamma) \rangle$ as a vector space.

It then follows easily that for $\mathcal{M}$ and $\mathcal{M}_0$ models of T_{VS} with $\mathcal{M}_0 \subset \mathcal{M}$, we have $\mathcal{M}$ is finitely generated over $\mathcal{M}_0$ if and only if ${}_K V$ is finitely generated over ${}_{K_0} V_0$.

Lemma 2.3 says that the larger skeleton is finitely generated over the smaller one if for all but finitely many values, the larger components are just the smaller ones “blown up” to the larger field. Moreover, at the finitely many exceptional values, we allow only a finite set of generators. The lemma does not tell us that if an extension of T_V models is finitely generated, then the same holds for the corresponding skeleton extension. In fact, this will not be true in general. So we need a further assumption. Let V be a valued K -vector space. Recall that $\mathcal{B} \subset V$ is **K -valuation independent** if $0 \notin \mathcal{B}$ and

$$v\left(\sum_{x \in \mathcal{B}} k_x x\right) = \min_{\{x \in \mathcal{B}; k_x \neq 0\}} \{v(x)\}$$

holds for all $k_x \in K$ such that $k_x = 0$ except for a finite number of x 's. Now suppose that ${}_{K_0} V_0 \subset {}_K V$. Say that ${}_{K_0} V_0$ is **valuation disjoint** from K in ${}_K V$ if K_0 -valuation independent elements in V_0 remain K -valuation independent in V . In case of valuation disjoint extensions, it was shown in [K-K; Proposition 3.10] that taking components “commutes” with “blowing up”. So we can prove

LEMMA 2.4. Suppose $\mathbf{M}, \mathbf{M}_0 \models T_V$ and that ${}_{K_0} V_0$ is valuation disjoint from K in ${}_K V$. If $\mathbf{M}$ is finitely generated over $\mathbf{M}_0$, then $S(\mathbf{M})$ is finitely generated over $S(\mathbf{M}_0)$.

PROOF. Set $W = {}_K \langle V_0 \rangle$. By Lemma 2.3, we have that K is finitely generated over K_0 and $\dim(V/W)$ is finite; hence

$$\dim(V/W) \geq \sum_{\gamma \in v(V)} \dim(B(V, \gamma)/B(W, \gamma))$$

(cf. [K1; Proposition 2.10]). In particular, this sum has finite support, i.e. there exists a finite subset $\Gamma' \subset v(V)$ such that for every $\gamma \notin \Gamma'$, we have that $\dim(B(V, \gamma)/B(W, \gamma)) = 0$. Moreover, for every $\gamma \in \Gamma'$, $\dim(B(V, \gamma)/B(W, \gamma))$ is finite. On the other hand, ${}_{K_0} V_0$ being valuation disjoint from K in ${}_K V$, we have for every $\gamma \in v(V)$

$$B(W, \gamma) = {}_K^{B(V, \gamma)} \langle B(V_0, \gamma) \rangle.$$

Hence by Lemma 2.3, $S(\mathbf{M})$ is finitely generated over $S(\mathbf{M}_0)$. -1

Recall that the subspace V_0 of V is **nice in V** if for all $x \in V$, the subset $v(x + V_0) = \{v(x + z); z \in V_0\}$ of $v(V)$ admits a maximum (cf. [F] and [K2] for more on nice subspaces). We say that ${}_K V_0 \subset {}_K V$ is a **correct extension** if V_0 is valuation disjoint from K in V and ${}_K \langle V_0 \rangle$ is nice in V . Let $\mathbf{M}$ and $\mathbf{M}_0$ be models of T_V such that $\mathbf{M}_0 \subset \mathbf{M}$. We say that $\mathbf{M}$ is a correct extension of $\mathbf{M}_0$ if and only if ${}_K V_0 \subset {}_K V$ is a correct extension. We close this section by the following result which is a mere reformulation in our formal language of the corresponding algebraic result given in [K-K; Corollary 4.6]. It states that for correct extensions, embeddings of the skeletons lift to embeddings of the valued vector spaces.

LEMMA 2.5 (Main Lemma). *Let $\mathcal{M}_0, \mathcal{M}_1$ and $\mathcal{M}_2$ be models of T_{VS} such that $\mathcal{M}_0 \subset \mathcal{M}_i$ for $i = 1, 2$, $V(\mathcal{M}_1)$ is a correct extension of $V(\mathcal{M}_0)$ and $\mathcal{M}_1$ is finitely generated over $\mathcal{M}_0$. Suppose that there is an embedding σ of $S(\mathcal{M}_1)$ in $S(\mathcal{M}_2)$ over $S(\mathcal{M}_0)$. Then there exists an embedding f of $\mathcal{M}_1$ in $\mathcal{M}_2$ over $\mathcal{M}_0$ such that $f_S = \sigma$ (i.e. f is a lifting of σ).*

§3. $L_{\infty\omega}$ -equivalence. Let L be a language, $\mathcal{A}, \mathcal{B}$ L -structures and $\mathcal{A}_0$ a common substructure. We refer the reader to [BA] or [PO] for the definitions of the infinitary language $L_{\infty\omega}$, of infinitary equivalence of $\mathcal{A}$ and $\mathcal{B}$ over $\mathcal{A}_0$ (that we write “ $\mathcal{A} \equiv_{\infty\omega} \mathcal{B}$ over $\mathcal{A}_0$ ”), of local isomorphism, of karpian family, as well as for the main known facts concerning these notions. We write

$$I : \mathcal{A} \underset{I}{\simeq} \mathcal{B} \text{ (over } \mathcal{A}_0\text{)}$$

if I is a nonempty karpian family of isomorphisms (over $\mathcal{A}_0$) of $\mathcal{A}$ -substructures onto $\mathcal{B}$ -substructures.

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be models of T_{VS} and $F : \mathcal{M}_1 \underset{I}{\simeq} \mathcal{M}_2$, $G : V(\mathcal{M}_1) \underset{I}{\simeq} V(\mathcal{M}_2)$, and $\Sigma : S(\mathcal{M}_1) \underset{I}{\simeq} S(\mathcal{M}_2)$. We say that G (respectively, Σ) **lifts to F** if and only if for every $f \in F$ there exists $g \in G$ (respectively, $\sigma \in \Sigma$) such that f is a lifting of a restriction of g (respectively, of σ).

For the rest of this section, suppose $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2 \models T_{VS}$ such that $\mathcal{M}_0 \subset \mathcal{M}_i$, $i = 1, 2$. Our aim now is to prove the following

THEOREM 3.1. *Suppose that $V(\mathcal{M}_i)$ is a correct extension of $V(\mathcal{M}_0)$, $i = 1, 2$. Then the following are equivalent:*

- (1) $\mathcal{M}_1 \equiv_{\infty\omega} \mathcal{M}_2$ over $\mathcal{M}_0$ in L_{VS} .
- (2) $V(\mathcal{M}_1) \equiv_{\infty\omega} V(\mathcal{M}_2)$ over $V(\mathcal{M}_0)$ in L_V .
- (3) $S(\mathcal{M}_1) \equiv_{\infty\omega} S(\mathcal{M}_2)$ over $S(\mathcal{M}_0)$ in L_S .

It is easy to see that (1) implies (2) and (3), even without the assumption that $V(\mathcal{M}_i)$ is a correct extension of $V(\mathcal{M}_0)$. Indeed take $F : \mathcal{M}_1 \underset{I}{\simeq} \mathcal{M}_2$ over $\mathcal{M}_0$ and set

$$F_V = \{f_V; f \in F\} \text{ and } F_S = \{f_S; f \in F\}.$$

Then $F_V : V(\mathcal{M}_1) \underset{I}{\simeq} V(\mathcal{M}_2)$ over $V(\mathcal{M}_0)$ and $F_S : S(\mathcal{M}_1) \underset{I}{\simeq} S(\mathcal{M}_2)$ over $S(\mathcal{M}_0)$.

The implication (2) implies (1) also holds in general, as seen from the next

LEMMA 3.2. *Every karpian family $G \neq \emptyset$ of isomorphisms over $V(\mathcal{M}_0)$ of $V(\mathcal{M}_1)$ -substructures which are finitely generated over $V(\mathcal{M}_0)$ onto $V(\mathcal{M}_2)$ -substructures*

which are finitely generated over $V(\mathcal{M}_0)$ lifts to a karpian family $F \neq \emptyset$ of local isomorphisms over $\mathcal{M}_0$.

PROOF. Without loss of generality, we may assume that the domain and range of the functions in G are submodels which are finitely generated over $V(\mathcal{M}_0)$ (indeed, just by making v surjective, we can enlarge any substructure to a submodel). Further we can suppose that $\text{id} : V(\mathcal{M}_0) \rightarrow V(\mathcal{M}_0)$ is an element of G . Let $g \in G$, $\text{dom } g = \mathbf{M}'_1$, $\text{im } g = \mathbf{M}'_2$. Let $\mathcal{M}'_i$ be the submodel of $\mathcal{M}_i$ generated by $K'_i \cup V'_i$ for $i = 1, 2$. Let now $\tilde{g} : \mathcal{M}'_1 \rightarrow \mathcal{M}'_2$ be the isomorphism obtained from g and satisfying that $\tilde{g}_V = g$. Then $\tilde{G} = \{\tilde{g}; g \in G\}$ is a family of local isomorphisms over $\mathcal{M}_0$ of $\mathcal{M}_1$ -submodels onto $\mathcal{M}_2$ -submodels and $\text{id} : \mathcal{M}_0 \rightarrow \mathcal{M}_0$ is in $\tilde{G}$, therefore $\tilde{G}$ is nonempty. Let us show that $\tilde{G}$ has the forth property:

Let $\tilde{g} \in \tilde{G}$, $\tilde{g} : \mathcal{M}'_1 \rightarrow \mathcal{M}'_2$ and $a \in |\mathcal{M}_1|$. As already observed, there exists a submodel $\mathcal{M}''_1$ of $\mathcal{M}_1$ containing $\langle |\mathcal{M}'_1| \cup \{a\} \rangle_{\mathcal{M}_1}$ and finitely generated over $\langle |\mathcal{M}'_1| \cup \{a\} \rangle_{\mathcal{M}_1}$. By Lemma 2.3, $V(\mathcal{M}''_1)$ is finitely generated over $V(\mathcal{M}'_1) = \text{dom } g$. So let $g' \in G$ be such that $g \subset g'$ and $V(\mathcal{M}''_1) \subset \text{dom } g'$, then $\tilde{g} \subset \tilde{g}'$ and $a \in \text{dom } \tilde{g}'$.

The back property is shown similarly. $\dashv$

The result (3) implies (1) in Theorem 3.1 is the most difficult to establish. For this implication, the assumption that $V(\mathcal{M}_i)$ is a correct extension of $V(\mathcal{M}_0)$ is necessary. We shall need the following result, which was proved in [K-K] (Propositions 3.16 and 3.18).

PROPOSITION 3.3. Suppose that ${}_F U \subset {}_K V$ is a correct extension.

(1) If ${}_F U \subset {}_{K'} V' \subset {}_K V$, then ${}_F U \subset {}_{K'} V'$ is also a correct extension.

(2) If ${}_{K'} V' \subset {}_K V$ is finitely generated over ${}_F U$, then there exists ${}_{K''} V''$, finitely generated over ${}_F U$, such that ${}_{K'} V' \subset {}_{K''} V'' \subset {}_K V$ and ${}_{K''} V'' \subset {}_K V$ is a correct extension.

THEOREM 3.4. Suppose that $V(\mathcal{M}_i)$ is a correct extension of $V(\mathcal{M}_0)$, $i = 1, 2$. Then every karpian family $\Sigma \neq \emptyset$ of isomorphisms over $S(\mathcal{M}_0)$ of $S(\mathcal{M}_1)$ -substructures which are finitely generated over $S(\mathcal{M}_0)$ onto $S(\mathcal{M}_2)$ -substructures which are finitely generated over $S(\mathcal{M}_0)$ lifts to a karpian family $F \neq \emptyset$ of local isomorphisms over $\mathcal{M}_0$.

PROOF. Let Σ be as in the hypothesis of the theorem. As above, we may suppose that the domains and ranges of the local isomorphisms in Σ are submodels, and that $\text{id} : S(\mathcal{M}_0) \rightarrow S(\mathcal{M}_0)$ is an element of Σ . Set

$F = \{f; f : \mathcal{M}'_1 \xrightarrow{\sim} \mathcal{M}'_2 \text{ over } \mathcal{M}_0, \text{ where } \mathcal{M}'_i \text{ is a } \mathcal{M}_i\text{-submodel which is finitely generated over } \mathcal{M}_0 \text{ such that } V(\mathcal{M}_i) \text{ is a correct extension of } V(\mathcal{M}'_i) \text{ for } i = 1, 2, \text{ and there exists } \sigma \in \Sigma \text{ such that } f_S \subset \sigma\}$.

$F \neq \emptyset$ since $\text{id} : \mathcal{M}_0 \rightarrow \mathcal{M}_0$ is in F . Let us show that F is a karpian family. Indeed, F has the forth property:

Let $f' \in F$, say $f' : \mathcal{M}'_1 \xrightarrow{\sim} \mathcal{M}'_2$, and $a \in |\mathcal{M}_1|$. Let $\mathcal{M}_a$ be a submodel of $\mathcal{M}_1$ such that $\langle |\mathcal{M}'_1| \cup \{a\} \rangle_{\mathcal{M}_1} \subset \mathcal{M}_a$ and $\mathcal{M}_a$ is finitely generated over $\mathcal{M}'_1$. Then ${}_{K_a} V_a$ is finitely generated over ${}_{K'_1} V'_1$. So by hypothesis and Proposition 3.3, there exists ${}_{K''_1} V''_1$ finitely generated over ${}_{K_a} V_a$, and such that ${}_{K''_1} V''_1 \subset {}_{K_1} V_1$ is a correct extension. Let $\mathcal{M}''_1$ be the submodel generated by $K''_1 \cup V''_1$, then $\mathcal{M}''_1$ is finitely generated

over $\mathcal{M}'_1$ and $V(\mathcal{M}_1)$ is a correct extension of $V(\mathcal{M}'_1)$. Moreover, since $V(\mathcal{M}_1)$ is a correct extension of $V(\mathcal{M}'_1)$, then by Proposition 3.3, $V(\mathcal{M}'_1)$ is a correct extension of $V(\mathcal{M}_1)$. So by Lemma 2.4, $S(\mathcal{M}'_1)$ is finitely generated over $S(\mathcal{M}_1)$.

Let therefore $\sigma \in \Sigma$ such that $S(\mathcal{M}'_1) \subset \text{dom } \sigma$ and $f'_S \subset \sigma$. Identify now $\mathcal{M}'_1$ and $\mathcal{M}'_2$ by f' and call it $\mathcal{M}'_0$, and $S(\mathcal{M}'_1)$ and $S(\mathcal{M}'_2)$ by f'_S and call it $S(\mathcal{M}'_0)$. Then σ is an embedding of $S(\mathcal{M}'_1)$ in $S(\mathcal{M}_2)$ over $S(\mathcal{M}'_0)$. All hypothesis of Lemma 2.5 are realized, therefore we have an embedding f'' of $\mathcal{M}'_1$ in $\mathcal{M}_2$ over $\mathcal{M}'_0$ such that $f''_S = \sigma$. This means that we have an embedding f'' of $\mathcal{M}'_1$ in $\mathcal{M}_2$ extending f' and such that $f''_S = \sigma$.

It remains to verify that if $f''(\mathcal{M}'_1) = \mathcal{M}''_2$, then $V(\mathcal{M}_2)$ is a correct extension of $V(\mathcal{M}''_2)$. But $\mathcal{M}''_2$ being finitely generated over $\mathcal{M}'_2$, we have that V''_2 is finitely generated over $V''_{K'_2} \langle V'_2 \rangle$. Consequently, $V''_2 \langle V''_2 \rangle$ is finitely generated over $V''_{K'_2} \langle V'_2 \rangle$. Since a finitely generated extension of a nice subspace is again nice (cf. [K2; Corollary 3.7]), it follows that $V''_2 \langle V''_2 \rangle$ is nice in $K_2 V_2$.

So it remains to show that $K''_2 V''_2$ is valuation disjoint from K_2 in $K_2 V_2$. Now $K''_1 V''_1$ is valuation disjoint from K_1 in $K_1 V_1$. Equivalently, for every $\gamma \in \Gamma''_1$, $B''_1(\gamma)$ is linearly disjoint from K_1 in $B_1(\gamma)$ (as shown in [K-K; Proposition 3.12]). On the other hand $\sigma : S(\mathcal{M}'_1) \rightarrow S(\mathcal{M}'_2)$ is an isomorphism. In particular, $\sigma : K''_1 \rightarrow K''_2$ is a field isomorphism, $\sigma : \Gamma''_1 \rightarrow \Gamma''_2$ is a chain isomorphism, and for every $\gamma \in \Gamma''_1$, $\sigma : B''_1(\gamma) \rightarrow B''_2(\sigma(\gamma))$ is a vector space isomorphism (modulo an identification of K''_1 and K''_2 by σ).

We use this to show that $B''_2(\sigma(\gamma))$ is linearly disjoint from K_2 in $B_2(\sigma(\gamma))$, for every $\gamma \in \Gamma''_1$ (which will establish the desired result, again by [K-K; Proposition 3.12]).

Let $\mathcal{B}_\gamma$ be a K''_1 -basis of $B''_1(\gamma)$; then $\mathcal{B}_\gamma$ remains K_1 -independent. Clearly, $\sigma(\mathcal{B}_\gamma)$ is a K''_2 -basis of $B''_2(\sigma(\gamma))$. We show that it remains K_2 -independent (by [K-K; Corollary 3.3], the existence of such a basis implies that $B''_2(\sigma(\gamma))$ is linearly disjoint from K_2 in $B_2(\sigma(\gamma))$ as required). If not, there exist $k_1, \dots, k_n \in K_2$ and $b_1, \dots, b_n \in \mathcal{B}_\gamma$ such that $\sum_{i=1}^n k_i \sigma(b_i) = 0$. Since $\sigma \in \Sigma$, there exists $\tau \in \Sigma$ such that $\sigma \subset \tau$ and $k_1, \dots, k_n \in \text{im } \tau$. Then

$$\sum_{i=1}^n \tau^{-1}(k_i) b_i = \tau^{-1} \left(\sum_{i=1}^n k_i \sigma(b_i) \right) = 0,$$

which contradicts that $\mathcal{B}_\gamma$ is K_1 -independent.

The back property is shown by a symmetrical argument. ⊥

Thus, we have established Theorem 3.1.

Recall that two countably generated structures are $L_{\infty\omega}$ -equivalent if and only if they are isomorphic (cf. [BA; Theorem 2]). We now want to apply our above results to models of T_V . Let $\mathbf{M}_i \models T_V$ with $|\mathbf{M}_i| = K_i \cup V_i \cup \Gamma_i$ for $i = 0, 1, 2$, and $\mathbf{M}_0 \subset \mathbf{M}_i$ for $i = 1, 2$.

COROLLARY 3.5. *Suppose that $\mathbf{M}_i$ is a correct extension of $\mathbf{M}_0$ for $i = 1, 2$. Then*

$$\mathbf{M}_1 \equiv_{\infty\omega} \mathbf{M}_2 \text{ over } \mathbf{M}_0 \text{ if and only if } S(\mathbf{M}_1) \equiv_{\infty\omega} S(\mathbf{M}_2) \text{ over } S(\mathbf{M}_0).$$

If moreover $\mathbf{M}_i$ is countably generated over $\mathbf{M}_0$ for $i = 1, 2$, then

$$\mathbf{M}_1 \simeq \mathbf{M}_2 \text{ over } \mathbf{M}_0 \text{ if and only if } S(\mathbf{M}_1) \simeq S(\mathbf{M}_2) \text{ over } S(\mathbf{M}_0).$$

PROOF. The first assertion follows immediately from Theorem 3.1. For the second one, it suffices to observe that if $\mathbf{M}_i$ is countably generated over $\mathbf{M}_0$, then $S(\mathbf{M}_i)$ is countably generated over $S(\mathbf{M}_0)$. One establishes this fact by a simple argument using Lemma 2.4. $\dashv$

Note that by Lemma 2.3, $\mathbf{M}_i$ is countably generated over $\mathbf{M}_0$ if and only if K_i is countably generated over K_0 and V_i is countably generated over V_0 .

Applying Corollary 3.5 with the hypothesis: $\mathbf{M}_0 \subset \mathbf{M}_i$ is a fixed field extension for $i = 1, 2$, we obtain (using Remark 2.2 (2))

COROLLARY 3.6. *Let V_0, V_1, V_2 valued K -vector spaces and suppose V_0 is nice in V_i for $i = 1, 2$. Then*

$$V_1 \equiv_{\infty\omega} V_2 \text{ over } V_0 \text{ (as valued } K\text{-vector spaces)}$$

if and only if

$$S(V_1) \equiv_{\infty\omega} S(V_2) \text{ (over } S(V_0)).$$

If moreover $\dim(V_i/V_0) \leq \aleph_0$ for $i = 1, 2$, then

$$V_1 \simeq V_2 \text{ (over } V_0) \text{ if and only if } S(V_1) \simeq S(V_2) \text{ (over } S(V_0)).$$

Note that this corollary covers several known results. Indeed, setting $V_0 = 0$ in the second assertion yields that a countable dimensional valued vector space is determined up to isomorphism by its skeleton (cf. R. Brown [BR]). In the context of p -groups, the second assertion yields an extension of Zippin's theorem due to P. Hill. Also setting again $V_0 = 0$ in the first assertion yields a theorem due to Barwise and Eklof (cf. [BA; Theorem 10]).

We refer readers interested in applications of the theory of valued vector spaces to p -groups to L. Fuchs [F].

§4. Application to ordered vector spaces. We now apply the above results to ordered vector spaces over ordered fields. We view them as valued vector spaces, by endowing them with their natural valuation. In this case, the skeletons we obtain are ordered skeletons, that is, the components are themselves ordered vector spaces. Isomorphisms of ordered vector spaces and isomorphisms of ordered skeletons are then defined in the obvious way (see [K1; Section 2] for more details). Note that an isomorphism f of ordered vector spaces preserves the natural valuation, and that the induced f_S is an isomorphism of ordered skeletons. The converse was proved in [K1] (Proposition 2.19):

LEMMA 4.1. *Assume that f is an isomorphism preserving the natural valuation. If the induced f_S is an isomorphism of ordered skeletons, then f is an isomorphism of ordered vector spaces.*

We can formalize these notions by taking new predicate symbols $<_K$, $<_V$ and $<_B$. In $L_V \cup \{<_K, <_V\}$, the **theory of ordered vector spaces with the natural valuation** is basically given by the

- (1) axioms expressing that $(K, <_K)$ is an ordered field,
- (2) axioms expressing that $(V, <_V)$ is an ordered K -vector space and the
- (3) axiom expressing that v is indeed the natural valuation:
 $(v(x) = v(y)) \iff \exists k \in K (|y| <_V k|x| \wedge |x| <_V k|y|).$

Similarly, in $L_S \cup \{<_K, <_B\}$, the **theory of ordered skeletons** has the:

- (1) axioms expressing that $(K, <_K)$ is an ordered field, and the
- (2) axioms expressing that: for all $\gamma \in \Gamma$, $(B(\gamma), <_B)$ is an ordered K -vector space.

Below, let $\mathbf{M}_i$ denote an ordered vector space, with $|\mathbf{M}_i| = K_i \cup V_i \cup \Gamma_i$, for $i = 0, 1, 2$ and $\mathbf{M}_0 \subset \mathbf{M}_i$, for $i = 1, 2$. Applying Theorem 3.1 we get

COROLLARY 4.2. *Suppose that $\mathbf{M}_i$ is a correct extension of $\mathbf{M}_0$ for $i = 1, 2$. Then $\mathbf{M}_1 \equiv_{\infty\omega} \mathbf{M}_2$ over $\mathbf{M}_0$ (as ordered vector spaces) if and only if $S(\mathbf{M}_1) \equiv_{\infty\omega} S(\mathbf{M}_2)$ over $S(\mathbf{M}_0)$ (as ordered skeletons). If moreover $\mathbf{M}_i$ is countably generated over $\mathbf{M}_0$ for $i = 1, 2$, then $\mathbf{M}_1 \simeq \mathbf{M}_2$ over $\mathbf{M}_0$ if and only if $S(\mathbf{M}_1) \simeq S(\mathbf{M}_2)$ over $S(\mathbf{M}_0)$.*

Indeed by Lemma 4.1, lifting a partial isomorphism of ordered skeletons will produce a partial isomorphism of the corresponding ordered vector spaces. On the other hand, the observation before that lemma says that a partial isomorphism of ordered vector spaces induces a partial isomorphism of the corresponding ordered skeletons.

Let us apply this last corollary with the assumption: $\mathbf{M}_0 \subset \mathbf{M}_i$ is a fixed field extension for $i = 1, 2$. Observe that in this situation (where $K_1 = K_2 = K_0$), $\mathbf{M}_1 \equiv_{\infty\omega} \mathbf{M}_2$ over K_0 if and only if $V_1 \equiv_{\infty\omega} V_2$ (as ordered K_0 -vector spaces).

COROLLARY 4.3. *Let V_1, V_2 be ordered K -vector spaces over an ordered field K . Then*

$$V_1 \equiv_{\infty\omega} V_2 \text{ (as ordered } K\text{-vector spaces)}$$

if and only if

$$S(V_1) \equiv_{\infty\omega} S(V_2) \text{ (over } K \text{ in } L_S(<_K, <_B)).$$

If moreover $\dim_K V_i \leq \aleph_0$ for $i = 1, 2$, then

$$V_1 \simeq V_2 \text{ (as ordered } K\text{-vector spaces)}$$

if and only if

$$S(V_1) \simeq S(V_2) \text{ (as ordered skeletons).}$$

The second assertion in the above corollary is the analogue to Brown's theorem (cf. [BR]) for ordered vector spaces.

We also get the following generalization of a theorem due to J. Erdős (cf. [E]):

COROLLARY 4.4. *Let V_1, V_2 be ordered $\mathbb{R}$ -vector spaces. Then*

$$V_1 \equiv_{\infty\omega} V_2 \text{ as ordered } \mathbb{R}\text{-vector spaces}$$

if and only if

$$\Gamma_1 \equiv_{\infty\omega} \Gamma_2 \text{ as ordered sets.}$$

PROOF. It suffices to observe that in the ordered skeleton of an ordered $\mathbb{R}$ -vector space, all $\mathbb{R}$ -archimedean components are isomorphic to $\mathbb{R}$. $\dashv$

REFERENCES

[BA] J. BARWISE, *Back and forth through infinitary logic*, **Studies in model theory** (M. Morley, editor), Mathematical Association of America, Buffalo, NY, pp. 5–34.

- [B-E] J. BARWISE and P.C. EKLOF, *Infinitary properties of abelian torsion groups*, *Annals of Mathematical Logic* (1970), pp. 25–68.
- [BR] R. BROWN, *Valued vector spaces of countable dimension*, *Universitatis Debreceniensis. Institutum Mathematicum. Publicationes Mathematicae*, vol. 18 (1971), pp. 149–151.
- [E] J. ERDÖS, *On the structure of ordered real vector spaces*, *Universitatis Debreceniensis. Institutum Mathematicum. Publicationes Mathematicae*, vol. 4 (1955–56), pp. 334–343.
- [F] L. FUCHS, *Abelian p -groups and mixed groups*, Montréal, 1980.
- [H1] P. HILL, *On the classification of abelian groups*, photocopied manuscript, 1967.
- [H2] ———, *On the classification of abelian groups*, *Abelian groups and modules, Proceedings of the conference in Udine 1984*, CISM Courses Lectures, Springer, 1984, pp. 1–16.
- [KA] I. KAPLANSKY, *Infinite abelian groups*, Ann Arbor, 1954.
- [K-K] F.-V. KUHLMANN and S. KUHLMANN, *Ax–Kochen–Ershov principles for valued and ordered vector spaces*, *Ordered algebraic structures* (W.C. Holland and J. Martinez, editors), Kluwer Academic Publishers, 1997, pp. 237–259.
- [K1] S. KUHLMANN, *On the structure of nonarchimedean exponential fields I*, *Archive for Mathematical Logic*, vol. 34 (1995), pp. 145–182.
- [K2] ———, *Valuation bases for extensions of valued vector spaces*, *Forum Mathematicum*, vol. 8 (1996), pp. 723–735.
- [P-S] A. PILLAY and C. STEINHORN, *Definable sets in ordered structures I*, *Transactions of the American Mathematical Society*, vol. 295 (1986), pp. 565–592.
- [PO] B. POIZAT, *Cours de théorie des modèles*, Villeurbanne, 1985.
- [R] F. RICHMAN, *Extensions of p -bounded groups*, *Archiv der Mathematik*, vol. 21 (1970), pp. 449–454.
- [U] H. ULM, *Zur Theorie der abzählbar-unendlichen abelschen Gruppen*, *Mathematische Annalen*, vol. 107 (1933), pp. 774–803.

THE FIELDS INSTITUTE
222, COLLEGE STREET
TORONTO, ONTARIO,
CANADA M5T 3J1

E-mail: skuhlman@fields.utoronto.ca