On the u -invariant of a real function field

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Abstract We obtain a new upper bound on the dimensions of anisotropic quadratic torsion forms over a field that is an extension of finite transcendence degree of a real closed field.

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Let K be a field of characteristic different from 2. We refer to [5] and [9] for the basic concepts of the theory of quadratic forms over fields. By a *form* we always mean a regular quadratic form. Let WK denote the Witt ring of K and IK its fundamental ideal. For $n \in \mathbb{N}$ we set $I^n K = (IK)^n$ and write $I_t^n K$ for the torsion part of $I^n K$. We say that a quadratic form is *torsion* if it represents a torsion element in WK .

The u -invariant of K was defined in [1] as

$$u(K) = \sup \{ \dim(\varphi) \mid \varphi \text{ anisotropic torsion form over } K \} \in \mathbb{N} \cup \{\infty\}.$$

For nonreal fields this invariant has already been considered in [4]. Recall that K is *nonreal* if -1 is a sum of squares in K , *real* otherwise. If K is nonreal, then every quadratic form over K is torsion and $u(K)$ is the smallest integer n such that every quadratic form over K of dimension n is *universal* (i.e., it represents every element of K) if such an integer exists, otherwise $u(K) = \infty$. (This explains the choice of the letter u .)

We refer to [7, Chap. 8] for a survey on the u -invariant. One open problem is whether finiteness of the u -invariant is conserved under field extensions of finite transcendence

degree. For example, it is not known whether $u(\mathbb{Q}(X))$ is finite. Recently, in [8] it was proven that $u(K) = 8$ when K is a function field of a p -adic curve for a prime $p \neq 2$. For a field extension K/C of transcendence degree n of an algebraically closed field C , Tsen-Lang Theory yields that $u(K) \leq 2^n$. For extensions K/R of transcendence degree $n > 2$ of a real closed field R , it was shown in [2, Sect. 6] that $u(K) < 2^{n+2}$.

The aim of this article is to prove the following statement:

If $n \geq 2$ is such that $u(K(\sqrt{-1})) \leq 2^n$, then $u(K) \leq 2^{n+2} - 2n - 6$.

It applies in particular to the case where K is an extension of transcendence degree n of a real closed field and, for $n \geq 3$, it improves the bounds known so far for this case. For $n = 2$, one retrieves the observation from [1, Theorem 4.11] that $u(K(\sqrt{-1})) \leq 4$ implies $u(K) \leq 6$.

To obtain the result we will refine a method introduced in [1, Sect. 4] and sometimes referred to as the *Elman-Lam filtration of the u -invariant*. For a quadratic form φ over K and $n \in \mathbb{N}$, let $n \times \varphi$ denote the n -fold sum $\varphi \perp \dots \perp \varphi$. For $i \in \mathbb{N}$, let $\mathcal{S}_i(K)$ denote the set of anisotropic forms φ over K such that $2^i \times \varphi$ is hyperbolic, and put

$$u^i(K) = \sup\{\dim(\varphi) \mid \varphi \in \mathcal{S}_i(K)\} \in \mathbb{N} \cup \{\infty\}.$$

We thus have an increasing sequence $(\mathcal{S}_i(K))_{i \geq 0}$ of subsets of the set of all anisotropic forms over K and an increasing sequence $(u^i(K))_{i \geq 0}$ in $\mathbb{N} \cup \{\infty\}$. The set $\mathcal{S}_0(K)$ contains only the trivial quadratic form, so we have $u^0(K) = 0$. Moreover, since every torsion form is annihilated by some power of 2, the set $\cup_{i \geq 0} \mathcal{S}_i(K)$ consists of all anisotropic torsion forms over K , and therefore we have $u(K) = \sup_{i \geq 0} u^i(K)$. In [3], for any $k \geq 0$, fields K were constructed with $u^{k+1}(K) > u^k(K)$.

The interest of this filtration lies in the possibility of relating the values $u^i(K)$ ($i \geq 1$) to each other and comparing $u^1(K)$ with $u(K(\sqrt{-1}))$. An argument from [1, Sect. 2] shows that any quadratic form has an orthogonal decomposition $\sigma \perp \tau$ such that $2 \times \sigma$ is anisotropic and $2 \times \tau$ is hyperbolic. Applying this for $i > 1$ to the forms in $\mathcal{S}_i(K)$ readily yields the inequality $u^i(K) \leq u^1(K) + \frac{1}{2}u^{i-1}(K)$. Hence, $u^n(K) \leq 2u^1(K)$ holds for all $n \geq 1$, and therefore $u(K) \leq 2u^1(K)$.

In [2, Theorem 6.2] it was shown that $u(K) < 4u(K(\sqrt{-1}))$ whenever $u(K(\sqrt{-1}))$ is finite. The proof in [2, p. 38] is quickly rearranged to obtain the following.

Theorem 1 (Elman-Lam) *Assume that -1 is not a square in K . Then $u^1(K) \leq 2(u(K(\sqrt{-1})) - 1)$ and $u(K) \leq 4(u(K(\sqrt{-1})) - 1)$.*

Proof Let s_* denote the quadratic form transfer for the quadratic extension $K(\sqrt{-1})/K$ given by the K -linear map $s : K(\sqrt{-1}) \rightarrow K$ with $s(1) = 0$ and $s(\sqrt{-1}) = 1$ (cf. [5, Chap. VII, Sect. 1]). Given $\varphi \in \mathcal{S}_1(K)$, it follows from [5, Chap. VII, Theorem 3.4] that $\varphi = s_*(\gamma)$ for a form γ over $K(\sqrt{-1})$, and as φ is anisotropic, γ does not represent 1 over $K(\sqrt{-1})$. Hence, γ is not universal over $K(\sqrt{-1})$ and therefore $\frac{1}{2} \dim(\varphi) = \dim(\gamma) \leq u(K(\sqrt{-1})) - 1$. This proves the first inequality. As $u(K) \leq 2u^1(K)$, the second inequality follows from the first.

In order to improve the estimate $u(K) \leq 2u^1(K)$ in certain cases, we introduce a second sequence of field invariants. For any positive integer i we define $\mathcal{S}'_i(K) = \{\varphi \in \mathcal{S}_i(K) \mid 2^{i-1} \times \varphi \text{ anisotropic}\}$ and

$$w^i(K) = \sup \{\dim(\varphi) \mid \varphi \in \mathcal{S}'_i(K)\} \in \mathbb{N} \cup \{\infty\}.$$

Obviously, $w^i(K) \leq u^i(K)$ holds for any $i \geq 1$, and $\mathcal{S}_1(K) = \mathcal{S}'_1(K)$ yields that $w^1(K) = u^1(K)$. For any $i > 1$, if $\varphi \in \mathcal{S}'_i(K)$, then $2 \times \varphi \in \mathcal{S}'_{i-1}(K)$, which shows the inequality $w^i(K) \leq \frac{1}{2}w^{i-1}(K)$. Thus $(w^i(K))_{i \geq 1}$ is a decreasing sequence in $\mathbb{N} \cup \{\infty\}$.

When saying that an element of $\mathbb{N} \cup \{\infty\}$ is *even*, we allow it to be ∞ . Recall that the *level* of K , denoted by $s(K)$, is the (additive) order of the form $\langle 1, 1 \rangle$ in the Witt ring WK (cf. [5, Chap. XI, Sect. 2]).

Proposition 1 *Let $n \geq 1$. If $s(K) \geq 2^n$, then $u^n(K)$ and $w^n(K)$ are even. If $s(K) < 2^n$, then $u(K) = u^n(K)$ and $w^i(K) = 0$ for $i > n$.*

Proof Assume that $s(K) \geq 2^n$. Then $2^n \times \langle 1 \rangle$ is not hyperbolic. Since every zero divisor in WK is contained in IK (cf. [5, Chap. VIII, Corollary 8.5]), it follows that $\mathcal{S}_n(K)$ contains only forms of even dimension. In particular, $u^n(K)$ and $w^n(K)$ are even. Assume now that $s(K) < 2^n$. Then $2^n \times \langle 1 \rangle$ is hyperbolic, so $\mathcal{S}_n(K)$ consists of all anisotropic forms over K and $\mathcal{S}'_i(K)$ contains only the zero form for $i > n$. Hence, $u(K) = u^n(K)$ and $w^i(K) = 0$ for $i > n$.

Lemma *Let $n \geq 0$. For every form φ over K there is a form ψ over K such that $2^n \times \psi$ is the anisotropic part of $2^n \times \varphi$.*

Proof This follows from [5, Chap. X, Theorem 4.11] applied to the Pfister form $2^n \times \langle 1 \rangle$ and the anisotropic part of $2^n \times \varphi$.

Proposition 2 *If $n > 1$, then $u^n(K) \leq u^{n-1}(K) + w^n(K)$.*

Proof Let $n > 1$. Let $\varphi \in \mathcal{S}_n(K)$. It follows from the Lemma that the anisotropic part of $2^{n-1} \times \varphi$ is equal to $2^{n-1} \times \gamma$ for a form γ over K . Then $\gamma \in \mathcal{S}'_n(K)$, so $\dim(\gamma) \leq w^n(K)$. Let ψ be the anisotropic part of $\varphi \perp -\gamma$. Then $2^{n-1} \times \psi$ is hyperbolic, so $\psi \in \mathcal{S}_{n-1}(K)$ and $\dim(\psi) \leq u^{n-1}(K)$. Since φ is anisotropic and Witt equivalent to $\psi \perp \gamma$, we have $\dim(\varphi) \leq \dim(\psi) + \dim(\gamma) \leq u^{n-1}(K) + w^n(K)$. This shows that $u^n(K) \leq u^{n-1}(K) + w^n(K)$.

Corollary *For any $n \geq 1$, we have $u^n(K) \leq \sum_{i=1}^n w^i(K)$.*

Proof Since $u^1(K) = w^1(K)$, this follows from Proposition 2.

Theorem 2 *Let $n \geq 2$ be such that $I_t^{n+1}K = 0$. Then*

$$u(K) \leq u^{n-2}(K) + w^{n-1}(K) \leq \sum_{i=1}^{n-1} w^i(K).$$

Moreover, $w^{n-1}(K)$ is either even or equal to 1.

Proof Suppose that ρ is a form over K such that $2^{n-2} \times \rho$ is anisotropic and represents an element of $I_t^n K$. Then for any form γ over K with $\dim(\gamma) = 2$, it follows that $2^{n-2} \times \rho \otimes \gamma$ represents an element of $I_t^{n+1} K = 0$ and therefore is hyperbolic. This shows in particular that $\rho \in \mathcal{S}'_{n-1}(K)$ and thus $\dim(\rho) \leq w^{n-1}(K)$, and that either $2^{n-2} \times \rho$ is universal or the zero form. Moreover, since all zero divisors in WK are contained in IK , it follows that ρ has even dimension.

Suppose now that ψ is a torsion form over K with $\dim(\psi) > 2$ such that $2^{n-2} \times \psi$ is anisotropic. Let d denote the discriminant (signed determinant) of ψ . Set $\beta = \langle 1, -d \rangle$ if ψ has even dimension and $\beta = \langle d \rangle$ otherwise. Then $\psi \perp -\beta$ represents an element of $I_t^2 K$ and $2^{n-2} \times (\psi \perp -\beta)$ is not hyperbolic. By the Lemma, there is a form ρ over K such that $2^{n-2} \times \rho$ is the anisotropic part of $2^{n-2} \times (\psi \perp -\beta)$. Then $2^{n-2} \times \rho$ represents a nonzero element of $I_t^n K$. Hence, by the above argument, $\dim(\rho)$ is even, $\dim(\rho) \leq w^{n-1}(K)$, and $2^{n-2} \times \rho$ is universal. In particular, $2^{n-2} \times (\rho \perp \beta)$ is isotropic. As $2^{n-2} \times \psi$ is the anisotropic part of $2^{n-2} \times (\rho \perp \beta)$, we obtain that $\dim(\psi) < \dim(\rho) + \dim(\beta)$. Therefore $\dim(\psi) \leq \dim(\rho) \leq w^{n-1}(K)$. Using this for all $\psi \in \mathcal{S}'_{n-1}(K)$ shows that $w^{n-1}(K)$ is even or equal to 1.

If $w^{n-1}(K) < 2$, then for $i \geq n$ we have $w^i(K) = 0$ and $u^i(K) = u^{i-1}(K)$ by Proposition 2, thus $u(K) = u^{n-1}(K) \leq u^{n-2}(K) + w^{n-1}(K)$. Assume now that $w^{n-1}(K) \geq 2$. Let φ be an arbitrary anisotropic torsion form over K . By the Lemma, there exists a form ψ over K such that $2^{n-2} \times \psi$ is the anisotropic part of $2^{n-2} \times \varphi$. Then ψ is torsion and the above arguments show that $\dim(\psi) \leq w^{n-1}(K)$. Since the anisotropic part of $\varphi \perp -\psi$ is an element of $\mathcal{S}_{n-2}(K)$ and since φ is the anisotropic part of $(\varphi \perp -\psi) \perp \psi$, it follows that $\dim(\varphi) \leq u^{n-2}(K) + w^{n-1}(K)$. This shows the first inequality in the statement. The second inequality is clear from the Corollary.

Theorem 3 *Let $n \geq 2$ be such that $u(K(\sqrt{-1})) \leq 2^n$. Then*

$$u(K) \leq 2^{n+2} - 2n - 6.$$

Proof The hypothesis implies that $I_t^{n+1} K = 0$ (cf. [5, Chap. XI, (6.27)]). By Theorem 1, we have $w^1(K) = u^1(K) < 2u(K(\sqrt{-1})) \leq 2^{n+1}$. Recall that $w^i(K) \leq \frac{1}{2}w^{i-1}(K)$ for $i > 1$. We thus have $w^i(K) < 2^{n-i+2}$ for all $i \geq 1$. If $w^i(K)$ is even for every $i \leq n-1$, then using Theorem 2 we conclude that $u(K) \leq \sum_{i=1}^{n-1} w^i(K) \leq \sum_{i=1}^{n-1} (2^{n-i+2} - 2) = 2^{n+2} - 2n - 6$. Assume now that $w^m(K)$ is odd for some $m \leq n-1$. By Proposition 1, then $s(K) = 2^{m-1}$, $u(K) = u^m(K)$, and, for $1 \leq i < m$, $w^i(K)$ is even and thus $w^i(K) \leq 2^{n-i+2} - 2$. Moreover, if $m = n-1$, then $w^m(K) = 1$, by Theorem 2. Using the Corollary, we now obtain the chain of inequalities $u(K) = u^m(K) \leq \sum_{i=1}^m w^i(K) \leq 1 + \sum_{i=1}^{n-2} (2^{n-i+2} - 2) \leq 2^{n+1} - 2n - 3$. As $n \geq 2$, we have $2^{n+1} \geq 8$ and thus even $u(K) \leq 2^{n+2} - 2n - 11$.

It follows from [6] that there exists a field K such that $u(K) = 6$ and $u(K(\sqrt{-1})) = 4$. For $n = 3$ Theorem 3 states that $u(K(\sqrt{-1})) \leq 8$ implies $u(K) \leq 20$. It is therefore interesting to ask whether a field K with $u(K) = 20$ and $u(K(\sqrt{-1})) = 8$ does exist.

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