Emergence of unstable modes for classical shock waves in isothermal ideal MHD

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HIGHLIGHTS

- MHD shock waves turn unstable through emergence of complex-conjugate mode pair.
- Critical manifolds are computed across which the stability change occurs.
- For slow shocks, the transition occurs also in the so-called parallel regime.
- Numerical method tracks invariant subspaces and preserves analyticity.

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ABSTRACT

This note studies classical magnetohydrodynamic shock waves in an inviscid fluidic plasma that is assumed to be a perfect conductor of heat as well as of electricity. For this mathematically prototypical material, it identifies, mainly numerically, two critical manifolds in parameter space, across which slow resp. fast MHD shock waves undergo emergence of a complex conjugate pair of unstable transverse modes. For slow shocks, this emergence occurs in a particularly interesting way already in the parallel case, in which it happens at the spectral value $\hat{\lambda} \equiv \lambda/|\omega| = 0$ and the critical manifold possesses a simple explicit algebraic representation. Results of refined numerical treatment show that within the set of non-parallel slow shocks the unstable mode pair emerges from two generically different spectral values $\hat{\lambda} = \pm i\gamma$. For fast shocks, the critical manifold does not intersect the parallel regime and the emergence within the set of non-parallel fast shocks again starts from two generically different spectral values.

0. Introduction

Magnetohydrodynamics (MHD) is the standard continuum description for self-consistent interactions of flows of charged/ionized gases or liquids with magnetic fields. Over the last several decades, it has been widely used to study interactions in many terrestrial and non-terrestrial contexts ranging from nuclear reactors over planetary dynamos to the solar wind and other space plasmas [1,2]. The nonlinear equations of magnetohydrodynamics, a kind of hybrid between the fundamental equations of fluid dynamics (Euler/Navier–Stokes) and those of electrodynamics (Maxwell) go back to Lundquist [3] and Alfvén [4]. Since these beginnings, research has paid attention to specific waves in plasmas, namely the magnetoacoustic waves, i.e., two kinds of generalized sound waves in which longitudinal variations of velocity, pressure and magnetic field interact, and Alfvén waves which consist of transverse variations of velocity and magnetic field [5].

While small-amplitude magnetoacoustic and Alfvén waves are waves of acoustic type, i.e., mathematically speaking, Fourier modes for the linearization of the MHD equations, there are a number of interesting situations, in which the nonlinear character of MHD leads to shock waves which dynamically form through “wave steepening”. Many planets are considered to have *magnetospheres* with interesting magneto-fluidic dynamics including shock waves; a famous large-scale “earthly” example is given by the *bow shock* that the solar wind forms around our planet [6]. Various types of MHD shock waves exist. Classical MHD shock waves are associated with the magnetoacoustic modes and, according to which one of the two they are associated with, classified as *slow* or *fast*. Besides these two types, there exist also *intermediate* shocks which roughly “interpolate” nonlinearly between the two magnetoacoustic modes and the Alfvén mode [7].

The equations of MHD express the conservation of mass, momentum, and magnetic flux. Different formulations exist according to whether (a) thermal effects are taken into account or not and (b) dissipative effects are taken into account or not. In some situations, heat flow is important (and an equation stating the conservation of energy is added), while many others are appropriately modeled

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as isothermal. In both cases the partial differential equations of MHD form a symmetrizable hyperbolic system if dissipation is not taken into account [8], and a mixed-type, namely hyperbolic-parabolic system, if it is accounted for [9]. Dissipative mechanisms are given by viscosity and electrical resistivity (as well as, in the non-isothermal context, heat conduction). Their combination leads to very interesting questions notably for shock waves (cf. [7, 10, 11]).

The present paper focusses on *ideal* isothermal MHD, i.e., dissipative mechanisms and temperature effects are not considered, and thus on a standard model of electromagnetism [12]. It studies the stability of classical (i.e., slow or fast), ideal (i.e., discontinuous) shock waves. In view of the abundance of MHD shock waves in nature and technology it is almost surprising that not very many studies have been devoted to this topic. In one space dimension, i.e., regarding merely longitudinal perturbations, the stability of slow and fast MHD shock waves was investigated and understood early on [5].¹ But for several space dimensions, i.e., including transverse perturbations, there are just a few contributions, mainly by Blokhin and Trakhinin [14–16]. A deep analytical result on hyperbolic free boundary value problems with interesting implications for MHD shock waves was proved by Métivier and Zumbrun [17].

The purpose of this paper is to study transitions from stability to instability for both fast and slow ideal MHD shock waves. How stability is described mathematically and what we do to locate and characterize this transition is described in the next section, together with a plan of the rest of the paper.

1. Shock waves in isothermal ideal MHD

In the equations of isothermal ideal MHD in two dimensional space,

$$\begin{aligned} 0 &= \rho_t + \operatorname{div}(\rho V) \\ 0 &= (\rho V)_t + \operatorname{div}\left(\rho V \otimes V + \left(p + \frac{1}{2}|B|^2\right)I - B \otimes B\right) \\ 0 &= B_t + \operatorname{div}(B \otimes V - V \otimes B), \end{aligned} \quad (1)$$

the dependent variables $\rho > 0$, $p > 0$, $V \in \mathbb{R}^2$ denote the fluid's density, pressure, and velocity. Pressure and density are related through the ideal gas law, $p = R\rho T$, with a constant temperature $T > 0$ ²; we write

$$p = \sigma^2 \rho \quad \text{with } \sigma = \sqrt{RT} \text{ the constant speed of sound.} \quad (2)$$

In addition to (1), the magnetic field $B \in \mathbb{R}^2$ satisfies

$$\operatorname{div} B = 0. \quad (3)$$

We abbreviate (1) as

$$U_t + F(U)_x + G(U)_y = 0 \quad (4)$$

with

$$U = \begin{pmatrix} \rho \\ \rho v_1 \\ \rho v_2 \\ b_1 \\ b_2 \end{pmatrix}, \quad F(U) = \begin{pmatrix} \rho v_1 \\ \rho v_1 v_1 + p + \frac{1}{2}(b_2^2 - b_1^2) \\ \rho v_1 v_2 - b_1 b_2 \\ 0 \\ b_2 v_1 - v_2 b_1 \end{pmatrix},$$

$$G(U) = \begin{pmatrix} \rho v_2 \\ \rho v_2 v_1 - b_2 b_1 \\ \rho v_2 v_2 + p + \frac{1}{2}(b_1^2 - b_2^2) \\ b_1 v_2 - v_1 b_2 \\ 0 \end{pmatrix}.$$

¹ For the dissipative case, stability studies even in one dimension are very recent; cf. [13].

² As noted above, isothermality is a physically appropriate standard assumption in many cases.

Using (3), we also write (1) as a symmetric hyperbolic system,³

$$\mathbf{D}(\tilde{U})\tilde{U}_t + \tilde{\mathbf{A}}(\tilde{U})\tilde{U}_x + \tilde{\mathbf{B}}(\tilde{U})\tilde{U}_y = 0 \quad (5)$$

with $\tilde{U} = (\rho, v_1, v_2, b_1, b_2)^\top$, $\mathbf{D}(\tilde{U}) = \operatorname{diag}(\sigma^2/\rho, \rho, \rho, 1, 1)$, and

$$\tilde{\mathbf{A}}(\tilde{U}) = \begin{pmatrix} \sigma^2 v_1/\rho & \sigma^2 & 0 & 0 & 0 \\ 0 & \rho v_1 & 0 & 0 & b_2 \\ 0 & 0 & \rho v_1 & 0 & -b_1 \\ 0 & b_2 & -b_1 & 0 & v_1 \end{pmatrix},$$

$$\tilde{\mathbf{B}}(\tilde{U}) = \begin{pmatrix} \sigma^2 v_2/\rho & 0 & \sigma^2 & 0 & 0 \\ 0 & \rho v_2 & 0 & -b_2 & 0 \\ \sigma^2 & 0 & \rho v_2 & b_1 & 0 \\ 0 & -b_2 & b_1 & v_2 & 0 \\ 0 & 0 & 0 & 0 & v_2 \end{pmatrix}.$$

Applying the chain rule, we rewrite (5) as

$$U_t + \mathbf{A}(U)U_x + \mathbf{B}(U)U_y = 0, \quad (6)$$

where

$$\mathbf{A} = \mathbf{T}\mathbf{D}^{-1}\tilde{\mathbf{A}}\mathbf{T}^{-1}, \quad \mathbf{B} = \mathbf{T}\mathbf{D}^{-1}\tilde{\mathbf{B}}\mathbf{T}^{-1} \quad (7)$$

with

$$\mathbf{T} = \frac{\partial U}{\partial \tilde{U}} = \begin{pmatrix} 1 & 0 & 0 \\ V & \rho I_2 & 0 \\ 0 & 0 & I_2 \end{pmatrix}. \quad (8)$$

Note that, as we have used (3) on the way from (1) to (5), the matrices $\mathbf{A}$ and $\mathbf{B}$ in (6) are *not* the Jacobians of the fluxes F and G .

A prototypical ideal MHD shock wave is of the form

$$U(t, x, y) = \begin{cases} U^- = (\rho^-, \rho^- V^-, H^-), & (x, y) \cdot N < st, \\ U^+ = (\rho^+, \rho^+ V^+, H^+), & (x, y) \cdot N > st, \end{cases} \quad (9)$$

where $N = (N_1, N_2) \in S^1$ is the direction of propagation and s the speed of the shock wave. Function (9) being a weak solution of (1) is equivalent to the Rankine-Hugoniot conditions

$$-s(U^+ - U^-) + N_1(F(U^+) - F(U^-)) + N_2(G(U^+) - G(U^-)) = 0.$$

According to the theories of Blokhin [19] and Majda [20, 21] on the persistence of shock fronts, the local-in-time stability behavior of a planar discontinuous Lax shock (9) is governed by its Lopatinski determinant,

$$\Delta : S_+ = (\overline{\mathbb{C}_+} \times \mathbb{R}) \setminus \{(0, 0)\} \rightarrow \mathbb{C}, \quad \text{where } \mathbb{C}_+ = (0, \infty) \times i\mathbb{R}, \quad (10)$$

a function $\Delta(\lambda, \omega)$ of the spectral parameter λ and a transverse wave number ω that we will spell out in the next section. One distinguishes *uniform stability*, corresponding to

$$\Delta^{-1}(0) = \emptyset,$$

from *neutral stability* or *strong instability*, which correspond to

$$\emptyset \neq \Delta^{-1}(0) \subset i\mathbb{R} \times \mathbb{R} \quad \text{or} \quad \emptyset \neq \Delta^{-1}(0) \cap (\mathbb{C}_+ \times \mathbb{R}), \quad (11)$$

respectively.

As we detail soon below, system (1) admits *slow* and *fast* shock waves. In this paper we show that both in the family of all slow MHD shock waves for (1) and that of all fast MHD shock waves for (1), there are transitions from uniform stability to strong instability. In either case, the regime of stability and that of instability are separated by a codimension-one hypersurface in parameter space.

³ This issue has an interesting history, cf. [8, 18].

We compute significant portions of these surfaces and demonstrate the actual emergence of unstable modes across them.

The idea of this paper follows earlier work by Filippova [22], Blokhin and Trakhinin [14], Trakhinin [15], who have found and described these phenomena in the non-isothermal case. While the non-isothermal case has wider applications, the isothermal case is not just a special subcase; isothermality changes the physics. We present results on the isothermal case not only for completeness, but mainly since they are partially more explicit and also since they are partially qualitatively different.

The plan of this paper is as follows. Section 2 first introduces a parametrization of all MHD shock waves for (1) by the components of their relative flux, (19)–(22), and recapitulates the notions of slow and fast shock waves and, for both classes, a distinction between parallel and non-parallel shocks. It then precisely defines the Lopatinski determinant for slow and fast shock waves. In Section 3, we study the transition from stability to instability in the particularly interesting case of parallel slow shocks. In apparent contrast with the non-isothermal case [22], the corresponding separatrix can be given an explicit algebraic description; this is very useful not only for our present study, but, we think, also for other possible investigations.⁴ In Section 4, we numerically study what happens to the separatrix, and to the associated emergence of unstable modes, upon perturbing away from the parallel case, still within the family of slow shocks. We find that the phenomenon persists (while our explicit description does not seem to), and the behavior of the critical Fourier–Laplace frequency reflects the associated symmetry breaking. Section 5 presents our findings on fast shock waves. In their class, all parallel shocks are stable, and the separatrix locating the stability transition in parameter space is two-sheeted.

2. Slow and fast, parallel and non-parallel shock waves

Differing from gas dynamics, MHD admits a variety of types of shock waves. In this paper, we are interested in *Lax* shocks.⁵ A crucial property for the stability of an ideal shock wave (9) is the total number d of its outgoing characteristics; it is called a Lax shock if $d = 4$; cf. [24].⁶ In this section we depict types of MHD Lax shocks.

Due to rotational and Galilean invariance we henceforth assume without loss of generality that

$$N = (1, 0) \quad \text{and} \quad s = 0;$$

i. e., we exclusively consider shock waves of the form

$$U(t, x, y) = \begin{cases} U^- = (\rho^-, \rho^- V^-, H^-), & x < 0, \\ U^+ = (\rho^+, \rho^+ V^+, H^+), & x > 0, \end{cases} \quad (12)$$

and the Rankine–Hugoniot conditions read

$$F(U^-) = F(U^+). \quad (13)$$

We start the discussion by noting that for waves (12), as for any solutions of (1) whose spatial dependence is only via x , the divergence-free condition (3) reduces to

$$b_1 = a, \quad a \text{ any constant.} \quad (14)$$

We assume (14) and simply write b, v, w instead of b_2, v_1, v_2 . In all calculations that follow we assume w. l. o. g. that the speed σ of sound is 1.

Following [5, 11, 24, 25], two states

$$\begin{aligned} U^- &= (\rho^-, \rho^- v^-, \rho^- w^-, a, b^-) \quad \text{and} \\ U^+ &= (\rho^+, \rho^+ v^+, \rho^+ w^+, a, b^+) \end{aligned} \quad (15)$$

that satisfy the Rankine–Hugoniot conditions (13) constitute a

$$\text{slow Lax shock iff } 0 < \rho^+(v^+)^2 < \rho^-(v^-)^2 < a^2 \quad (16)$$

and a

$$\text{fast Lax shock iff } a^2 < \rho^+(v^+)^2 < \rho^-(v^-)^2. \quad (17)$$

Two states (15) do satisfy the Rankine–Hugoniot conditions (13) if and only if the two quadruples (ρ^-, v^-, w^-, b^-) and (ρ^+, v^+, w^+, b^+) have coinciding images under the mapping

$$\begin{pmatrix} \rho \\ v \\ w \\ b \end{pmatrix} \mapsto \begin{pmatrix} \rho v \\ \rho v^2 + \rho + \frac{1}{2}b^2 \\ \rho v w - ab \\ bv - aw \end{pmatrix} \quad (18)$$

that F induces by omitting its fourth, trivial component, in other words if both quadruples satisfy the four equations

$$\rho v = m \quad (19)$$

$$\rho v^2 + \rho + \frac{1}{2}b^2 = j \quad (20)$$

$$vb - aw = c \quad (21)$$

$$mw - ab = d \quad (22)$$

for the same values of the four parameters $m, j, c, d \in \mathbb{R}$. As simple arguments⁷ show, we lose no generality in assuming that

$$m > 0, \quad d = 0, \quad \text{and } \rho v^2 \neq a^2. \quad (23)$$

Using (22) in (21) and inserting the result and (19) in (20) then yields

$$\begin{aligned} g^{amc}(v) &\equiv m \frac{1+v^2}{v} + \frac{1}{2} \left(\frac{mc}{mv - a^2} \right)^2 = j, \text{ to be solved for} \\ v &\in (0, a^2/m) \cup (a^2/m, \infty). \end{aligned} \quad (24)$$

As for every solution v of (24), relations (19), (21), (22) provide unique associated values for ρ, w and b , it suffices to discuss (24). One distinguishes two cases.

$c = 0$: *parallel shocks* (magnetic field parallel to direction of propagation). In this case, (24) has two solutions

$$v^\pm = \frac{1}{2} \left(\frac{j}{m} \mp \sqrt{\left(\frac{j}{m} \right)^2 - 4} \right) \quad \text{if } \frac{j}{m} > 2, \quad \text{with } 0 < v^+ < v^-.$$

The corresponding states (15) constitute a

$$\text{slow parallel shock iff } mv^- < a^2 \quad (25)$$

and a

$$\text{fast parallel shock iff } a^2 < mv^+. \quad (26)$$

The fact that the value of a has no influence on the ρ, v, w, b components of parallel shocks is easily understood by noticing that they have $b = w = 0$ and thus are purely gas dynamical.

$c \neq 0$: *non-parallel shocks* (magnetic field not parallel to direction of propagation). In this case, g^{amc} tends to ∞ not only for $v \searrow 0$ and $v \nearrow \infty$, but also for $v \rightarrow a^2/m$. Thus for every

$$j > j_{\min}^s(a, m, c) = \min_{(0, a^2/m)} g^{amc},$$

⁴ For instance, studies including also dissipative effects; Zumbun has announced such a study [23].

⁵ MHD has also interesting non-Lax shocks. See [11] and references therein.

⁶ For a system with n equations, d would have to be equal to $n - 1$.

⁷ Reversing, shifting, scaling, and the observation that cases with $m = 0$ or $\rho v^2 = a^2$ give no Lax shocks.

(24) has two solutions

$$v_s^+(a, m, c, j) < v_s^-(a, m, c, j) < a^2/m$$

that constitute a slow shock. Similarly, for every

$$j > j_{\min}^f(a, m, c) = \min_{(a^2/m, \infty)} g^{amc},$$

there are two solutions

$$a^2/m < v_f^+(a, m, c, j) < v_f^-(a, m, c, j)$$

that define a fast shock.

According to [20], the Lopatinski determinant (10) of any Lax shock (12) is given by

$$\Delta(\lambda, \omega) = \det(R^-(\lambda, \omega), J(\lambda, \omega), R^+(\lambda, \omega)) \quad (27)$$

with

$$J(\lambda, \omega) := \lambda(U^+ - U^-) + i\omega(G(U^+) - G(U^-)), \text{ "jump vector",}$$

$$R^-(\lambda, \omega), \text{ base of the stable space of } \mathbf{L}^- := (\lambda \mathbf{I} + i\omega \mathbf{B}^-)(\mathbf{A}^-)^{-1},$$

$$R^+(\lambda, \omega), \text{ base of the unstable space of } \mathbf{L}^+ := (\lambda \mathbf{I} + i\omega \mathbf{B}^+)(\mathbf{A}^+)^{-1},$$

where $\mathbf{A}^\pm, \mathbf{B}^\pm$ denote $\mathbf{A}(U^\pm), \mathbf{B}(U^\pm)$. The theory of hyperbolic initial-boundary value problems [20,26] implies that $R^\pm$ are well-defined bundles of constant dimension. To be precise, it is on $\mathbb{C}_+ \times \mathbb{R} = S_+ \setminus (i\mathbb{R} \times \mathbb{R})$ that the Lopatinski matrices $\mathbf{L}^\pm$ have constantly trivial neutral spaces and thus "consistent splitting", i.e., stable and unstable spaces of constant dimensions, so that in particular

$$d^- = \dim \text{span } R^-(\lambda, \omega) \text{ and}$$

$$d^+ = d - d^- = \dim \text{span } R^+(\lambda, \omega)$$

are constant; for points $(\lambda, \omega) \in S_+$ with purely imaginary values of λ , the $R^\pm(\lambda, \omega)$ are defined as limits from the interior of S_+ [26]. From the one-dimensional 'Lax counting' of characteristic speeds [5,24], we know that

$$d^- = 1 \text{ and } d^+ = 3 \text{ for slow MHD shocks,} \quad (28)$$

while

$$d^- = 0 \text{ and } d^+ = 4 \text{ for fast MHD shocks.} \quad (29)$$

The Lopatinski determinant Δ being degree-one homogeneous in (λ, ω) , we from now on restrict the transverse wave number to

$$\omega = \pm 1.$$

To avoid abundant notation, we also fix from now, again by scaling without loss of generality,

$$\rho^- = 1 \quad (30)$$

and use the two parameters ρ^+, c instead of the three parameters j, m, c . For parallel shocks, our choice (30) implies

$$v^- = \sqrt{\rho^+}, \quad v^+ = 1/\sqrt{\rho^+}. \quad (31)$$

3. Transition to instability for parallel slow shocks

Harmless as it may seem, the following is a key observation.

Theorem 1. For slow parallel shocks in (1), (2), (3) with $\rho^- = 1$,

$$\Delta(0, \pm 1) = 0 \text{ if } \rho^+ = \frac{a^2 + 2}{a^2 + 1}. \quad (32)$$

Proof. Using (16) and (31) the slow parallel Lax shock condition for $\rho^- = 1$ is

$$1 < \rho^+ < a^2. \quad (33)$$

We set $(\lambda, \omega) = (0, \pm 1)$ and investigate the spectral properties of $\mathbf{L}^-$ and $\mathbf{L}^+$. Straightforward manipulations show that the eigenvalues of $\mathbf{L}^-$ read

$$\mu_1^- = \mu_2^- = \mu_3^- = 0, \quad \mu_4^- = \sqrt{\frac{a^2(\rho^+ - 1) + \rho^+}{(a^2 - \rho^+)(\rho^+ - 1)}},$$

$$\mu_5^- = -\sqrt{\frac{a^2(\rho^+ - 1) + \rho^+}{(a^2 - \rho^+)(\rho^+ - 1)}}.$$

Due to (33) eigenvalue μ_5 is stable with corresponding eigenvector

$$w_5^- = \begin{pmatrix} \frac{1}{\sqrt{\rho^+}} \\ -i\sqrt{(a^2 - \rho^+)}\left(\frac{a^2}{\rho^+} - \frac{1}{1 - \rho^+}\right) \\ a \\ 0 \end{pmatrix}.$$

In case of $\mathbf{L}^+$ the eigenvalues are

$$\mu_1^+ = \mu_2^+ = \mu_3^+ = 0, \quad \mu_4^+ = \sqrt{\frac{a^2(\rho^+ - 1) - \rho^+}{(a^2 - 1)(\rho^+ - 1)}},$$

$$\mu_5^+ = -\sqrt{\frac{a^2(\rho^+ - 1) - \rho^+}{(a^2 - 1)(\rho^+ - 1)}}.$$

We now assume $a^2(\rho^+ - 1) < \rho^+$, since some straightforward but tedious manipulations in the case $a^2(\rho^+ - 1) \geq \rho^+$ show $\Delta(0, \pm 1) \neq 0$. Hence, μ_4^+ and μ_5^+ are purely imaginary. We now apply the techniques presented in Lancaster [27]. Due to Lancaster μ_1^+, μ_2^+ and μ_3^+ depend smoothly on $\lambda = \alpha + i\beta$ near 0 and we can compute

$$\left. \frac{d}{d\alpha} \text{Re}(\mu_i^+(\alpha, \beta)) \right|_{(\alpha, \beta)=0} > 0, \quad i = 1, 2, 3.$$

Thus μ_1^+, μ_2^+ and μ_3^+ are unstable with eigenvectors

$$w_1^+ = \begin{pmatrix} \sqrt{\rho^+} \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad w_2^+ = \begin{pmatrix} a(\rho^+ - 1) \\ 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \quad w_3^+ = \begin{pmatrix} 0 \\ 0 \\ -a\sqrt{\rho^+} \\ 0 \\ 1 \end{pmatrix}.$$

This shows that for $(\lambda, \omega) = (0, \pm 1)$ one can take the bases $R^-(0, \pm 1) = (w_5^-)$ and $R^+(0, \pm 1) = (w_1^+, w_2^+, w_3^+)$. Together with $J(0, \pm 1) = (0, 0, \pm i, 0, 0)$, this yields

$$\Delta(0, \pm 1) = \pm 2i[\rho^+(a^2 + 1) - (a^2 + 2)]$$

which leads to $\Delta(0, \pm 1) = 0$ if and only if $\rho^+(a^2 + 1) - (a^2 + 2) = 0$. $\square$

We numerically investigate the stability situation for slow parallel Lax shocks notably with (a, ρ^+) near the critical parameter manifold

$$M_s^{*,0} = \left\{ (a, \rho^+); \rho^+ = \frac{a^2 + 2}{a^2 + 1} \right\}.$$

To find or exclude zeros of $\Delta(\cdot, \pm 1)$ in $\mathbb{C}^+$, we use the argument principle and compute the winding number

$$N = \frac{1}{2\pi i} \int_{\Delta(\Gamma, \pm 1)} \frac{1}{w} dw$$

Fig. 1. Stability domains for slow parallel Lax shocks. Slow shocks exist only to the right of the black curve. The red curve is $M_s^{*,0}$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

for boundaries Γ of appropriate large semidisks, finding one zero in $\overline{\mathbb{C}_+}$, indeed lying in $\mathbb{C}_+$, for $\rho^+ > \frac{a^2+2}{a^2+1}$, and no zero in case of $\rho^+ < \frac{a^2+2}{a^2+1}$. Fig. 1 displays the domains of stability and instability thus identified.

In the winding-number calculations it is crucial to reflect the analyticity of the Lopatinski determinant Δ in the spectral parameter $\lambda \in \mathbb{C}^+$ by properly continuing the bases of the stable and unstable subspaces of $R^\pm(\lambda, \pm 1)$ in λ (cf. Section 6 for details).

In the unstable case $\rho^+ > \frac{a^2+2}{a^2+1}$, we also calculate the unstable eigenvalue λ^* , which is always real for parallel shocks. Fig. 2 shows its graph as a function of the parameters.

4. Symmetry breaking: non-parallel slow shocks

The situation of parallel shocks is degenerate as it possesses a reflectional symmetry in the transverse (y -)direction. For the Lopatinski determinant this symmetry means that $\Delta(\lambda, -\omega)$ vanishes exactly if $\Delta(\lambda, \omega)$ does. Perturbing the parameter c away from 0 breaks this symmetry, and the zero of Δ that we found, for $c = 0$ at $\lambda = 0$, moves along the imaginary axis.

We write Ω_s for the set of all parameter triples (a, ρ^+, c) that permit a (then unique) slow shock, and

$$\Delta_s^{a, \rho^+, c} : \overline{\mathbb{C}_+} \times \{1\} \rightarrow \mathbb{C}, \quad (a, \rho^+, c) \in \Omega_s,$$

for the corresponding Lopatinski determinant.

Numerical Observation 1. There are two smooth functions R and γ with

$$R = R(a, c), \text{ even in } c \text{ and } R(a, 0) = \frac{a^2 + 2}{a^2 + 1}, \quad \text{and}$$

$$\gamma = \gamma(a, c), \text{ odd in } c \text{ and non-trivial,}$$

both defined on a certain open domain $\Upsilon_s \subset (0, \infty) \times \mathbb{R}$, such that

$$(a, R(a, c), c) \in \Omega_s \quad \text{and} \quad \Delta^{a, R(a, c), c}(i\gamma(a, c), 1) = 0$$

for all $(a, c) \in \Upsilon_s$. (See Fig. 3.)

Like already its $c = 0$ section $M_s^{*,0}$ (cf. Fig. 1), the whole manifold

$$M_s^* = \{(a, R(a, c), c) : (a, c) \in \Upsilon_s\}$$

plays the role of a separatrix: Slow shocks defined by (a, c, ρ^+) are unstable for $\rho^+ > R(a, c)$ and stable if $\rho^+ < R(a, c)$.

Numerical Observation 2. There are an $\varepsilon > 0$ and a smooth function $\eta + i\zeta : \Upsilon_s \times [0, \varepsilon] \rightarrow \mathbb{C}$ with

$$\eta(a, c, 0) = 0, \quad \zeta(a, c, 0) = \gamma(a, c) \quad \text{and}$$

$$\eta(a, c, \tau) > 0 \quad \text{for } \tau > 0,$$

such that

$$\Delta^{a+R(a, c), c}(\eta(a, c, \tau) + i\zeta(a, c, \tau), 1) = 0$$

for all $\tau \in [0, \varepsilon]$. I.e., for $\tau > 0$, $\lambda^* = \eta + i\zeta$ is an unstable eigenvalue for $\omega = 1$. (See Fig. 4.)

5. Transition to instability for fast shocks

In marked contrast to the situation for non-isothermal gases [15], we find that *parallel* fast shocks are always uniformly stable. However, for non-parallel fast shocks there is a stability transition.

We write Ω_f for the set of all parameter triples (a, ρ^+, c) that permit a (then unique) fast shock, and

$$\Delta_f^{a, \rho^+, c} : \overline{\mathbb{C}_+} \times \{1\} \rightarrow \mathbb{C}, \quad (a, \rho^+, c) \in \Omega_f,$$

for the corresponding Lopatinski determinant.

Numerical Observation 3. There are smooth functions C and δ with

$$C = C(a, \rho) > 0 \quad \text{and} \quad \delta = \delta(a, \rho), \quad (34)$$

Fig. 2. The unstable eigenvalue for slow parallel Lax shocks.

Fig. 3. Left: The critical manifold M_s^* . The red curve corresponds to $c = 0$ and thus to the red curve in Fig. 1. The blue curves correspond to $c = 0.01$ and $c = -0.01$. Right: Shift of the eigenvalue $\lambda = i\gamma$ along the imaginary axis. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

defined on a certain open domain $\Upsilon_f \subset (0, \infty)^2$, such that

$$(a, \rho, \pm C(a, \rho)) \in \Omega_f \quad \text{and} \quad \Delta^{a, \rho, \pm C(a, \rho)}(\pm i\delta(a, \rho), 1) = 0$$

for all $(a, \rho) \in \Upsilon_f$. (See Fig. 5).

The separatrix

$$M_f^* = \{(a, \rho, \pm C(a, \rho)) : (a, \rho) \in \Upsilon_f\}$$

is two-sheeted. Shock waves defined by (a, ρ^+, c) are unstable for $|c| > C(a, \rho^+)$ and stable if $|c| < C(a, \rho^+)$.

Numerical Observation 4. There are an $\varepsilon > 0$ and a smooth function $\kappa + i\chi : \Upsilon_f \times [0, \varepsilon] \rightarrow \mathbb{C}$ with

$$\kappa(a, \rho, 0) = 0, \quad \chi(a, \rho, 0) = \delta(a, \rho) \quad \text{and}$$

$$\kappa(a, \rho, \tau) > 0 \quad \text{for} \quad \tau > 0,$$

such that

$$\Delta^{a, \rho, C(a, \rho) + \tau}(\kappa(a, c, \tau) + i\chi(a, c, \tau), 1) = 0$$

for all $\tau \in [0, \varepsilon]$. I.e., for $\tau > 0$, $\lambda^* = \kappa + i\chi$ is an unstable eigenvalue for $\omega = 1$.

As plots demonstrating this observation are qualitatively similar to those in Fig. 4, we refrain from displaying them.

6. Numerical and implementation issues

We discuss here some details which are relevant for the numerics and the implementation of the algorithms. The main challenges are (i) to make the problem of following zeros of the Lopatinski determinant Δ in their dependence on the spectral parameter λ and the real-valued parameters occurring in the Rankine–Hugoniot conditions amenable to existing continuation techniques, and (ii) to properly capture the analyticity of Δ in λ in the calculations of the winding number.

6.1. Continuation of zeros of the Lopatinski determinant

For applying parameter continuation to our problem $\Delta^{a, \rho^+, c}(\alpha + i\beta, \pm 1) = 0$ with respect to the real parameters α, β, a, m and c , we rewrite the problem as

$$F(X) = 0, \quad F : \mathbb{R}^3 \rightarrow \mathbb{R}^2$$

with

$$F(X) := \begin{pmatrix} \operatorname{Re} \left(\Delta^{a, \rho^+, c}(\alpha + i\beta, \pm 1) \right) \\ \operatorname{Im} \left(\Delta^{a, \rho^+, c}(\alpha + i\beta, \pm 1) \right) \end{pmatrix},$$

where X contains three of the five parameters, for example β, a and m , and the other two are fixed. We use the Matlab package MatCont [28] to compute branches $X = X(s)$ with

$$F(X(s)) = 0, \quad s \in I, \quad I \text{ some interval } \subset \mathbb{R}.$$

For proper use in MatCont, the function F must be C^k -smooth ($k \geq 1$), that means the underlying bases of $R^\pm(\alpha + i\beta, \pm 1)$ have to be C^k in s . We use the computation of invariant subspaces (CIS) algorithm of Dieci and Friedman [29] to ensure this smoothness. Advantageously, the CIS algorithm needs the spectral information of the stable and unstable subspaces $R^\pm$ only once at the initial point, as it continues invariant subspaces of a matrix with respect to parameters as long as the dimensions of the subspaces do not break down. This property of the algorithm is particularly helpful for values of λ near/slightly across the imaginary axis, where some spatial eigenvalues have real parts that vanish/change sign.

As the matrices $L^\pm$ are complex valued, we have to modify the algorithm in [29] at appropriate places.

6.2. Initialization procedure

To initialize the described combination of MatCont with CIS, we use Muller's method (see for example Press et al. [30]). The advantage of this method is that no numerical differentiation of Δ is needed.

6.3. Computation of winding numbers

For the evaluation of contour integrals

$$N = \frac{1}{2\pi i} \int_{\Delta(\Gamma, \pm 1)} \frac{1}{w} dw,$$

we utilize an algorithm of Kravanja and van Barel [31]. The algorithm is based on integration techniques with orthogonal polynomials and works derivative free. It does assume an analytic representation of $\Delta(\Gamma, \pm 1)$ along the contour Γ .

6.4. Capturing the analyticity of the Lopatinski determinant

The essential requirement for properly capturing an analytic representation of the Lopatinski determinant Δ numerically is to compute analytic bases $R^\pm(\lambda, \omega)$ of the (un)stable invariant spaces of $L^\pm(\lambda, \omega)$. As a by-product of this approach to the computation of connecting orbits in dynamical systems, Beyn has given a method that allows to turn an arbitrary basis $E(\mu)$ of the (un)stable invariant space $\Sigma(\mu)$ of a parameter-dependent matrix $M(\mu)$ into

Fig. 4. Unstable modes: Real part η (left) and imaginary part ζ (right) of the eigenvalue for $c = 0.01$ (top row), $c = 0$ (middle row), and $-c = 0.01$ (bottom row). Red and blue curves correspond to red and blue curves in Figs. 1 through 3. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

another basis $R(\mu)$ which has the same regularity with respect to the parameter μ as M [32]. The method is a nice application of the implicit function theorem; see Lemma C in the appendix of [32] for details. In our context, the method ensures the analyticity of the bases $R^\pm(\lambda, \omega)$. Note that the approach needs the spectral splitting of the eigen bundles of the Lopatinski matrices $L^\pm$ at any λ and is thus restricted to $\lambda \in \mathbb{C}_+$.

7. Conclusion

Classical magnetohydrodynamic shock waves have been investigated regarding their stability in the case of an inviscid fluid that is a perfect conductor of heat, thus isothermal, and of electricity. As stability in this perfect-fluid context means the Kreiss–Lopatinski condition in non-uniform (neutral stability) or uniform version (strong stability), we have tested the Lopatinski determinant for

zeros in the right half of the complex spectral-parameter plane or notably on the imaginary axis.

Similarly to previous studies [14, 15, 22] on non-isothermal fluids, our results show stability in certain parameter regimes and instability in others, for both classical species of magnetohydrodynamic shock waves, fast and slow. In contrast to those previous studies, we carefully investigated the transition from stability to instability that occurs with growing amplitude, and computed critical manifolds in the parameter space of the Rankine–Hugoniot jump conditions. In all cases that we investigated numerically, we found the emergence of a complex conjugate pair of unstable modes from a pair of imaginary zeros of the Lopatinski determinant. An interesting difference from the non-isothermal case consists in the fact that, according to our observations, the isothermal case has no regime of neutral stability.

Well-conditioned computational access to the phenomenon was enabled by an explicit analytic formula we found to describe

Fig. 5. Left: Local portion of the critical manifold M_J^* . Right: Shift of the eigenvalue $\lambda = i\delta$ along the imaginary axis.

a section of one of the critical manifolds that corresponds to the coplanar case of so-called parallel shocks. We obtained the first systematic results on slow shocks by continuation from that section, before we then managed to extend the investigations to regimes far away from the coplanar case, and thus successfully treat slow shocks. Regarding computational methods, we have not created anything fundamentally new, but are certain that our combination of refined techniques proposed by Beyn [19] and Dieci and Friedman [29] with existing continuation software [28], Kravanja and van Barel's contour-integral evaluation method [31] and Muller's simple derivative-free zero finder [30] can be recommended for this kind of studies.

In ongoing research, we investigate the global topography of the critical manifolds. We also hope to carry over the techniques presented in this paper to the non-isothermal case and to relativistic magnetohydrodynamic shock waves.

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