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Abstract: The propagation of singularities for the system of homogeneous thermoelasticity in one space dimension is studied. Linear and a class of semilinear Cauchy problems are considered.

AMS subject classification: 73 B 30, 35 B 65

1 Introduction

We consider the Cauchy problem for the one-dimensional system of thermoelasticity, both for the linear, homogeneous case and for a class of semilinear problems. We are interested in describing the propagation of singularities and the distribution of regular domains in the space-time region, respectively, if the initial data have different regularity in different parts of the real line.

The system of thermoelasticity is a hyperbolic-parabolic coupled system describing the elastic and the thermal behavior of an elastic medium. It is well known that with respect to the decay of solutions as time tends to infinity, and also with respect to the existence of global smooth solutions for small data in one dimension, the system behaves like a parabolic one, see for example [4, 5, 16, 11, 12, 18] or [9, 10, 17] and the references therein. It has been shown that large data for the quasilinear problem will lead to the development of singularities in finite time, see e.g. [3] and the references in [10]. On the other hand it was proved in [7] that solutions to the linear problem propagate singularities in the sense that the solutions do not show a smoothing effect, i.e., in general, the H^s-regularity of the initial data will not be improved.

Here we shall describe the behavior of the solutions for initial data which typically have the regularity

$$H^s(I\!R) \cap H^{s+1}(I\!R \setminus [a,b]),$$

while right-hand sides will behave differently on characteristic lines, see below. It will turn out that the behavior is dominated by the hyperbolic part (cf. [7]), and the characteristic lines are those from the wave equation. This will also hold for a class of semilinear problems. We shall describe the domains in the space-time area with H^s- or H^{s+1}-regularity precisely.

Our interest is devoted to the study of the following Cauchy problem in one-dimensional semilinear thermoelasticity

$$\left.\begin{array}{l} u_{tt} - \tau u_{xx} + \gamma\theta_x = f(u,\theta) \\ \theta_t - \kappa\theta_{xx} + \gamma u_{tx} = g(u), \\ u(t=0) = u_0,\ u_t(t=0) = u_1,\ \theta(t=0) = \theta_0, \end{array}\right\} \tag{1.1}$$

where $(u,\theta) = (u,\theta)(t,x)$, $t \geq 0$, $x \in I\!R$, represent the displacement and the temperature difference, respectively, $\tau, \kappa > 0$ and $\gamma \neq 0$ are constants, and f and g are smooth functions satisfying

$$f(0,0) = g(0) = 0. \tag{1.2}$$

Differentiating we obtain from (1.1) that u and θ satisfy the following fourth order equations

$$\left.\begin{array}{l} P(\partial)u = F, \\ P(\partial)\theta = G, \\ u(t=0) = u_0,\ u_t(t=0) = u_1,\ u_{tt}(t=0) = u_2, \\ \theta(t=0) = \theta_0,\ \theta_t(t=0) = \theta_1,\ \theta_{tt}(t=0) = \theta_2, \end{array}\right\} \tag{1.3}$$

where

$$P(\partial) = \partial_t^3 - \kappa\partial_t^2\partial_x^2 - (\tau+\gamma^2)\partial_t\partial_x^2 + \kappa\tau\partial_x^4, \tag{1.4}$$

with $\partial_t = \partial/\partial_t$, $\partial_x = \partial/\partial_x$, and

$$\left.\begin{array}{rcl} F &=& F(u,u_x,u_{xx},u_t,\theta,\theta_x,\theta_{xx},\theta_t) = \partial_t f - \kappa\partial_x^2 f - \gamma\partial_x g, \\ G &=& G(u,u_x,u_{xx},u_t,u_{tx},u_{tt},\theta,\theta_x,\theta_t,\theta_{tx}) = \partial_t^2 g - \tau\partial_x^2 g - \gamma\partial_t\partial_x f, \end{array}\right\} \tag{1.5}$$

F and G being quasilinear, and

$$\left.\begin{array}{rcl} u_2 &:=& f(u_0,\theta_0) + \tau u_0'' - \gamma\theta_0', \\ \theta_1 &:=& g(u_0) + \kappa\theta_0'' - \gamma u_1', \\ \theta_2 &:=& g'(u_0)u_1 + \kappa g'(u_0)u_0'' + \kappa g''(u_0)(u_0')^2 \\ && -\gamma f_1(u_0,\theta_0)u_0' - \gamma f_2(u_0,\theta_0)\theta_0' \\ && +\kappa(\kappa\theta_0'''' - \gamma u_1''') - \gamma(\tau u_0''' - \gamma\theta_0''). \end{array}\right\} \tag{1.6}$$

Here, a prime $'$ denotes differentiation with respect to a single variable and

$$f_1(u,\theta) := \frac{\partial f}{\partial u}(u,\theta), \quad f_2(u,\theta) := \frac{\partial f}{\partial \theta}(u,\theta).$$

The subject of this paper is to study the local existence and regularity of solutions (u, θ) to the semilinear problem (1.1) under the assumption that the initial data (u_0, u_1, θ_0) satisfy

$$\left.\begin{array}{l} u_0, \theta_0 \in H^s(I\!R) \cap H^{s+1}(I\!R \setminus [a,b]), \\ u_1 \in H^{s-1}(I\!R) \cap H^s(R \setminus [a,b]), \end{array}\right\} \tag{1.7}$$

where s will be in $I\!R_0^+$ and $0 < a < b < \infty$ are fixed. For this purpose, we first study the propagation of singularities for the linearized problem associated to (1.3) using Fourier analysis; at the same time, we obtain estimates for the solutions in a space of piecewise H^s-functions. Then, as usual, we use these estimates in an iteration scheme for the nonlinear problem. In order to formulate the main results, let us denote by I, II and III, respectively, the following three regions:

$$\begin{aligned} I &:= \{(x,t) | -\infty < x < a - \sqrt{\tau}t,\, 0 < t < \infty\} \\ &\cup \{(x,t) | \sqrt{\tau}t + b < x < \infty,\, 0 < t < \infty\} \\ &\cup \{(x,t) | b - \sqrt{\tau}t < x < \sqrt{\tau}t + a,\, \tfrac{b-a}{2\sqrt{\tau}} < t < \infty\}, \end{aligned}$$

$$\begin{aligned} II &:= \{(x,t) | a - \sqrt{\tau}t \le x < \sqrt{\tau}t + a,\, 0 < t < \tfrac{b-a}{2\sqrt{\tau}}\} \\ &\cup \{(x,t) | a - \sqrt{\tau}t \le x \le b - \sqrt{\tau}t,\, \tfrac{b-a}{2\sqrt{\tau}} \le t\}, \end{aligned}$$

$$\begin{aligned} III &:= \{(x,t) | b - \sqrt{\tau}t < x \le \sqrt{\tau}t + b,\, 0 < t < \tfrac{b-a}{2\sqrt{\tau}}\} \\ &\cup \{(x,t) | \sqrt{\tau}t + a \le x \le \sqrt{\tau}t + b,\, \tfrac{b-a}{2\sqrt{\tau}} \le t\}, \end{aligned}$$

which are illustrated in the following figure 1.1.

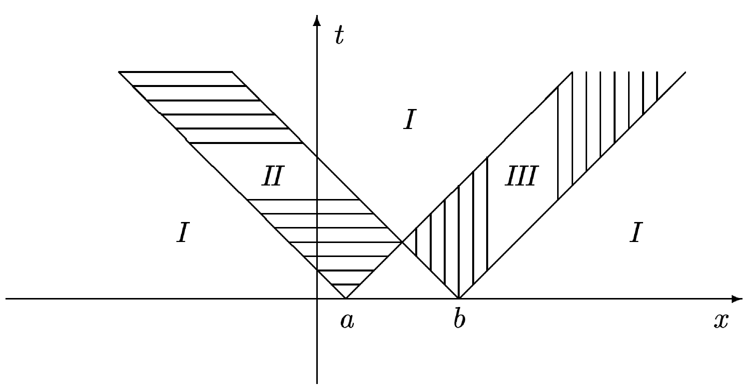

Figure 1.1: Areas of different regularity

For the linearized problem associated to (1.3), we have the following result:

Theorem 1.1 *Consider the linear Cauchy problem*

$$\left.\begin{array}{rcl} P(\partial)u & = & F(t,x) \\ u(t=0) & = & u_0,\ u_t(t=0)=u_1,\ u_{tt}(t=0)=u_2, \end{array}\right\} \tag{1.8}$$

and let $s \geq 4$ and $T > 0$ be fixed.

(1) If the initial data (u_0, u_1, u_2) satisfy

$u_0 \in H^s(I\!R) \cap H^{s+1}(I\!R \setminus [a,b])$, $u_1 \in H^{s-1}(I\!R) \cap H^s(I\!R \setminus [a,b])$,

$u_2 \in H^{s-3}(I\!R) \cap H^{s-2}(I\!R \setminus [a,b])$,

and if F satisfies

$$\left.\begin{array}{l} F \in L^2([0,T],\ H^{s-3}(I\!R)) \cap H^1([0,T], H^{s-4}(I\!R)), \\ (\partial_t + \sqrt{\tau}\partial_x)F \in L^2([0,T],\ H^{s-3}(I \cup III)) \\ (\partial_t - \sqrt{\tau}\partial_x)F \in L^2([0,T],\ H^{s-3}(I \cup II)), \end{array}\right\} \tag{1.9}$$

then the solution u to (1.8) has the following regularity

$$\left.\begin{array}{l} u \in \bigcap_{j=0}^{1} C^j([0,T],\ H^{s-j}(I\!R)) \cap C^2([0,T],\ H^{s-3}(I\!R)) \cap \bigcap_{j=2}^{3} H^j([0,T],\ H^{s+2-2j}(I\!R)), \\ (\partial_t + \sqrt{\tau}\partial_x)u \in L^2([0,T],\ H^s(I \cup III)), \\ (\partial_t - \sqrt{\tau}\partial_x)u \in L^2([0,T],\ H^s(I \cup II)). \end{array}\right\} \tag{1.10}$$

Moreover, there is a constant $c = c(T) > 0$ depending only upon T, such that the following estimates are valid:

$$\begin{array}{l} \|u\|_{\bigcap_{j=0}^{1} C^j([0,T],\, H^{s-j}(I\!R))} + \|u\|_{C^2([0,T],H^{s-3}(I\!R))} + \|u\|_{\bigcap_{j=2}^{3} H^j([0,T],\, H^{s+2-2j}(I\!R))} \\ \leq c(T) \left(\|u_0\|_s + \|u_1\|_{s-1} + \|u_2\|_{s-3} + \|F\|_{\bigcap_{j=0}^{1} H^j([0,T],H^{s-3-j}(I\!R))} \right) \end{array} \tag{1.11}$$

and

$$\begin{array}{rl} & \|(\partial_t + \sqrt{\tau}\partial_x)u\|_{L^2([0,T],H^s(I\cup III))} + \|(\partial_t - \sqrt{\tau}\partial_x)u\|_{L^2([0,T],H^s(I\cup II))} \\ \leq & c(T) \Big(\|u_0\|_s + \|u_1\|_{s-1} + \|u_2\|_{s-3} + \|u_0\|_{H^{s+1}(I\!R\setminus[a,b])} \\ + & \|u_1\|_{H^s(I\!R\setminus[a,b])} + \|u_2\|_{H^{s-2}(I\!R\setminus[a,b])} + \|F\|_{\bigcap_{j=0}^{1} H^j([0,T],H^{s-3-j}(I\!R))} \\ + & \|(\partial_t + \sqrt{\tau}\partial_x)F\|_{L^2([0,T],H^{s-3}(I\cup III))} + \|(\partial_t - \sqrt{\tau}\partial_x)F\|_{L^2([0,T],H^{s-3}(I\cup II))} \Big). \end{array} \tag{1.12}$$

(2) If, in addition to the above assumption, (u_2, F) also satisfy

$$u_2 \in H^{s-2}(I\!R), \quad F \in C^0([0,T], H^{s-3}(I\!R)), \tag{1.13}$$

the regularity in (1.10) improves to

$$\left.\begin{array}{l}(\partial_t + \sqrt{\tau}\partial_x)u \in C^0([0,T],\ H^s(I \cup III)) \\ (\partial_t - \sqrt{\tau}\partial_x)u \in C^0([0,T],\ H^s(I \cup II)),\end{array}\right\} \tag{1.14}$$

and we have an estimate similar to (1.12) with obvious modifications.

Now let us state the main result for the semilinear problem (1.1) as follows.

Theorem 1.2 *Let $s > 9/2$ be fixed and the assumption (1.7) as well as*

$$\kappa\theta_0'' - \gamma u_1' \in H^{s-1}(I\!R) \cap H^s(I\!R \setminus [a,b]) \tag{1.15}$$

be given. Then there is $T > 0$ such that there is a unique solution (u,θ) to (1.1) satisfying

$$\begin{aligned} u &\in \bigcap_{j=0}^{3} C^j([0,T],\ H^{s-j}(I\!R)) \cap H^4([0,T], H^{s-4}(I\!R)), \\ \theta &\in \bigcap_{j=0}^{1} C^j([0,T],\ H^{s-j}(I\!R)) \cap C^2([0,T], H^{s-3}(I\!R)) \\ &\quad \cap \bigcap_{j=2}^{3} H^j([0,T], H^{s+2-2j}(I\!R)). \end{aligned}$$

Moreover, we have

$$\left.\begin{array}{l}(\partial_t + \sqrt{\tau}\partial_x)\theta \in L^2([0,T],\ H^s(I \cup III)), \\ (\partial_t - \sqrt{\tau}\partial_x)\theta \in L^2([0,T],\ H^s(I \cup II)), \\ (\partial_t + \sqrt{\tau}\partial_x)^l u \in C^0([0,T],\ H^{s+1-l}(I \cup III)), \\ (\partial_t - \sqrt{\tau}\partial_x)^l u \in C^0([0,T],\ H^{s+1-l}(I \cup II)), \quad l = 1,2.\end{array}\right\} \tag{1.16}$$

Semilinear problems for special nonlinearities have been studied in [4], [6] with respect to global existence and blow-up, respectively, for H^s-data. It is interesting to compare the results above to those well-known for the purely hyperbolic case, i.e. $\theta = 0$, $\gamma = 0$, see for example [1], [2], [8], [13], [14], [15]. In our case the hyperbolic part is the predominating one leading to the same characteristic lines "$x \pm \sqrt{\tau}t = \text{const.}$" as in the purely hyperbolic case; nevertheless the parabolic impact is still present as can be seen in Theorem 1.1 looking at the regularity required for u_2, or in Theorem 1.2, observing that the range of l is restricted to $l = 1,2$.
In Section 2 we shall present Lemmata finally proving Theorem 1.1, while Section 3 will give the proof of Theorem 1.2 along a series of Lemmata. Finally, Section 3 will add some concluding remarks, e.g. on initial-*boundary* value problems.

Notation: We use standard notations for the Sobolev spaces $H^m(I\!R), m \in I\!R$, or the Banach spaces $L^2, H^j, C^j([0,T], H^s(I\!R)), j \in \mathbb{N}_0, s \in I\!R$. For any $\Omega \subset I\!R_0^+ \times I\!R$, let $\Omega_\tau := \Omega \cap \{t = \tau\}$; we define $L^2, H^j, C^j([0,T], H^s(\Omega_t))$ as the spaces of functions belonging to $L^2, H^j, C^j([T_1, T_2],$

$H^s([x_1, x_2])$ for any rectangle $[T_1, T_2] \times [x_1, x_2] \subset \overline{\Omega} \cap \{0 \leq t \leq T\}$, and we omit the index t of Ω_t for simplicity. Furthermore, in order to simplify the exposition, we introduce the following abbreviations:

$$H^s := H^s(I\!R), \qquad H^s_e := H^s(I\!R \setminus [a,b]),$$

$$L^2(H^s) := L^2([0,T], H^s),$$

if T is fixed, similarly for $H^s(H^s)$, $C^j(H^s)$;

$$L^2(H^s, I, III) := L^2([0,T], H^s(I \cup III))$$

similarly for $L^2(H^s, I, II)$, $C^0(H^s, I, III)$ and $C^0(H^s, I, II)$; the norm in H^s is denoted by $\|\cdot\|_s$, the norm in $L^2 = H^0$ by $\|\cdot\|$.

2 Linear problems

In this section we study the linear problem (1.8) and present the proof of Theorem 1.1. The problem (1.8) can be divided into four problems in each of which only one of u_0, u_1, u_2 and F is non-zero, we shall study these problems separately. First let us consider the case
$u_0 = u_1 = 0$, $F = 0$, i.e.

$$\left.\begin{array}{l} P(\partial)u = 0, \\ u(t=0) = u_t(t=0) = 0,\ u_{tt}(t=0) = u_2. \end{array}\right\} \tag{2.1}$$

Applying the Fourier transformation $\mathcal{F}$, we can express the solution u to (2.1) as

$$u(t,\cdot) = \mathcal{F}^{-1}_{\xi \to x}\left(\sum_{j=1}^{3} b_j^2(\xi) e^{-\beta_j(\xi)t} \hat{u}_2(\xi)\right), \tag{2.2}$$

where "^" denotes the Fourier transform, cf. [7], $\beta_j(\xi)$ $(j = 1,2,3)$ are the roots of the algebraic equation

$$-\beta^3 + \kappa\xi^2\beta^2 - (\tau + \gamma^2)\xi^2\beta + \kappa\tau\xi^4 = 0 \tag{2.3}$$

and

$$b_j^2(\xi) := 1 / \left(\prod_{l \neq j} (\beta_j(\xi) - \beta_l(\xi))\right), \quad j = 1,2,3. \tag{2.4}$$

Concerning the asymptotic behavior of β_j, it is known (cf. [9], [17]) that the following holds.

Lemma 2.1 *There are positive constants c, k_1, k_2 with $k_1 < k_2$ such that*

(1) If $|\xi| \le k_1$, we have

$$\left.\begin{array}{rcl} \beta_1(\xi) &=& \frac{\kappa\tau}{\tau+\gamma^2}\xi^2 + r_1(\xi), \\ \beta_2(\xi) &=& \overline{\beta_3(\xi)} = \frac{\kappa\gamma^2}{2(\tau+\gamma^2)} + i\sqrt{\tau+\gamma^2}\,\xi + r_2(\xi), \end{array}\right\} \tag{2.5}$$

where r_j $(j = 1, 2)$ is smooth and satisfies

$$|r_j(\xi)| \le c|\xi|^3. \tag{2.6}$$

(2) If $|\xi| \ge k_2$, we have

$$\left.\begin{array}{rcl} \beta_1(\xi) &=& \kappa\xi^2 - \frac{\gamma^2}{\kappa} - \frac{\alpha_1}{\kappa^3}\xi^{-2} + r_3(\xi), \\ \beta_2(\xi) &=& \overline{\beta_3(\xi)} = \frac{\gamma^2}{2\kappa} + \frac{\alpha_1}{2\kappa^3}\xi^{-2} + i(\sqrt{\tau}\xi + \frac{\alpha_2}{\kappa^2}\xi^{-1}) + r_4(\xi), \end{array}\right\} \tag{2.7}$$

where r_j $(j = 3, 4)$ is smooth and satisfies

$$|r_j(\xi)| \le c|\xi|^{-3}, \tag{2.8}$$

and

$$\alpha_1 := \gamma^2(\gamma^2 - \tau), \quad \alpha_2 := \frac{\gamma^2(4\tau - \gamma^2)}{8\sqrt{\tau}}.$$

Inserting these expansions into formula (2.4), we obtain by simple computations

Lemma 2.2 *(1) If $|\xi| \le k_1$, we have*

$$\left.\begin{array}{rcl} b_1^2(\xi) &=& \frac{1}{(\tau+\gamma^2)}\xi^{-2} + r_5(\xi), \\ b_2^2(\xi) &=& \overline{b_j^2(\xi)} = -\frac{1}{2(\tau+\gamma^2)}\xi^{-2} + i\frac{\kappa(2\tau-\gamma^2)}{4(\tau+\gamma^2)^{5/2}} + r_6(\xi), \end{array}\right\} \tag{2.9}$$

where r_j $(j = 5, 6)$ is smooth and bounded.

(2) If $|\xi| \ge k_2$, we have

$$\left.\begin{array}{rcl} b_1^2(\xi) &=& \frac{1}{\kappa^2}\xi^{-4} + r_7(\xi), \\ b_2^2(\xi) &=& \overline{b_3^2(\xi)} = i\frac{1}{2\kappa\sqrt{\tau}}\xi^{-3} - \frac{1}{2\kappa^2}\xi^{-4} + r_8(\xi), \end{array}\right\} \tag{2.10}$$

where r_j $(j = 7, 8)$ is smooth and satisfies

$$|r_7(\xi)| \le c\xi^{-6}, \quad |r_8(\xi)| \le c|\xi|^{-5}. \tag{2.11}$$

For a fixed $k_0 \in (0, k_1)$ choose two functions $\chi_1, \chi_2 \in C_0^\infty(I\!R)$ satisfying

$$\left.\begin{array}{rcl} \chi_{1|[-k_0,k_0]} &=& 1, \ \operatorname{supp}\chi_1 \subset [-k_1, k_1], \\ \chi_{2|[-k_2,k_2]} &=& 1, \end{array}\right\} \tag{2.12}$$

for which the following identity obviously holds

$$\chi_1(\xi) + (1-\chi_1(\xi))\chi_2(\xi) + (1-\chi_2(\xi)) = 1, \;\; \xi \in I\!R. \tag{2.13}$$

Thus, the solution u given in (2.2) can be decomposed into

$$u(t,x) = \sum_{j=1}^{3} u^{(j)}(t,x) \tag{2.14}$$

where

$$u^{(1)}(t,x) := \mathcal{F}^{-1}_{\xi\to x}\left(\sum_{j=1}^{3} b_j^2(\xi)e^{-\beta_j(\xi)t}\chi_1(\xi)\hat{u}_2(\xi)\right), \tag{2.15}$$

$$u^{(2)}(t,x) := \mathcal{F}^{-1}_{\xi\to x}\left(\sum_{j=1}^{3} b_j^2(\xi)e^{-\beta_j(\xi)t}(1-\chi_1(\xi))\chi_2(\xi)\hat{u}_2(\xi)\right) \tag{2.16}$$

and

$$u^{(3)}(t,x) = \sum_{j=1}^{3} v_j(t,x) \tag{2.17}$$

with

$$v_j(t,x) := \mathcal{F}^{-1}_{\xi\to x}(b_j^2(\xi)e^{-\beta_j(\xi)t}(1-\chi_2(\xi))\hat{u}_2(\xi)). \tag{2.18}$$

The different terms are investigated separately. At first we shall prove

Lemma 2.3 *For any $u \in L^2$ we have*

$$u^{(1)},\, u^{(2)} \in C^k([0,\infty), H^s) \tag{2.19}$$

for any $k \in \mathbb{N}$, $s \geq 0$. Moreover, for any $T > 0$ there is $c > 0$ such that

$$\|u^{(1)}, u^{(2)}\|_{C^k(H^s)} \leq c\|u_2\|. \tag{2.20}$$

PROOF: We shall verify (2.19), the estimate (2.20) will be obvious from this proof using the mapping properties of the Fourier transform. The assertion is immediately clear for $u^{(2)}$ by using the fact that b_j^2 and β_j are smooth on the support of $(1-\chi_1(\xi))\chi_2(\xi)$. Concerning $u^{(1)}$ we exploit Lemmata 2.1 and 2.2 to obtain that on the support of χ_1 we have

$$\left.\begin{array}{rcl} b_1^2(\xi)e^{-\beta_1(\xi)t} &=& \frac{1}{(\tau+\gamma^2)}\xi^{-2} + R_1(t,\xi), \\ b_2^2(\xi)e^{-\beta_2(\xi)t} + b_3^2(\xi)e^{-\beta_3(\xi)t} &=& -\frac{1}{(\tau+\gamma^2)}\xi^{-2} + R_2(t,\xi) \end{array}\right\} \tag{2.21}$$

where R_j $(j=1,2)$ is smooth in $t \geq 0$ and bounded in $\xi \in \operatorname{supp}\chi_1$. Inserting (2.21) into (2.15) it follows

$$u^{(1)} \in C^0([0,\infty), H^s)$$

for any $s \geq 0$ provided $u_2 \in L^2$.

By differentiating (2.15) with respect to t we obtain

$$u^{(1)} \in C^1([0,\infty), H^s)$$

in the same way. Successively, we conclude (2.19).

Q.E.D.

For the term v_1 given in (2.18), we get

Lemma 2.4 *For any $u_2 \in H^{s-3}$, with $s \geq 3$, we have*

$$v_1 \in \bigcap_{j=0}^{3} H^j(H^{s+2-2j}) \tag{2.22}$$

for any $T > 0$, and the estimate

$$\|v_1\|_{\bigcap_{j=0}^{3} H^j(H^{s+2-2j})} \leq c\|u_2\|_{s-3} \tag{2.23}$$

holds with a constant $c = c(T) > 0$.

Proof: As above, it suffices to prove (2.22). Using Lemmata 2.1 and 2.2 we know that on the support of $(1 - \chi_2(\xi))$

$$b_1^2(\xi)e^{-\beta_1(\xi)t} = e^{-(\kappa\xi^2 - \frac{\gamma^2}{\kappa})t}\left(\frac{1}{\kappa^2}\xi^{-4} + r(t,\xi)\right), \tag{2.24}$$

where r is smooth and satisfies

$$|\partial_t^l r(t,\xi)| \leq c(l,T)\xi^{-6} \tag{2.25}$$

for $l \geq 0$, $t \in [0,T]$.

Substituting (2.24) into (2.18) for $j = 1$, it follows

$$\begin{aligned} v_1(t,x) &= \mathcal{F}^{-1}_{\xi\to x}(e^{-(\kappa\xi^2-\frac{\gamma^2}{\kappa})t}\left(\tfrac{1}{\kappa^2}\xi^{-4} + r(t,\xi)\right)(1-\chi_2(\xi))\hat{u}_2(\xi)) \\ &\equiv \tfrac{1}{\kappa^2}e^{\frac{\gamma^2}{\kappa}t}\tilde{v}_1(t,x) + R(t,x), \end{aligned} \tag{2.26}$$

where

$$\tilde{v}_1(t,x) := \mathcal{F}^{-1}_{\xi\to x}(e^{-\kappa\xi^2 t}\xi^{-4}(1-\chi_2(\xi))\hat{u}_2(\xi))$$

satisfies

$$\left.\begin{aligned} (\partial_t - \kappa\partial_x^2)\tilde{v}_1 &= 0 \\ \tilde{v}_1(0,x) &= \mathcal{F}^{-1}_{\xi\to x}(\xi^{-4}(1-\chi_2(\xi))\hat{u}_2(\xi)). \end{aligned}\right\} \tag{2.27}$$

Obviously, if $u_2 \in H^{s-3}$ then

$$\tilde{v}_1(0,\cdot) \in H^{s+1}(I\!R). \tag{2.28}$$

The classical theory for heat equations as in (2.27) yields

$$\tilde{v}_1 \in \bigcap_{j=0}^{3} H^j(H^{s+2-2j}), \tag{2.29}$$

for any $T > 0$ and $s \geq 3$.

On the other hand, using (2.25) we get

$$R \in \bigcap_{j=0}^{3} C^j(H^{s+3-2j}). \tag{2.30}$$

Combining (2.29) and (2.30) the conclusion (2.22) follows.

Q.E.D.

Lemma 2.5 *If $u_2 \in H^{s-3}$, $s \in I\!R$ arbitrary, we have*

$$(v_2, v_3) \in \bigcap_{j=0}^{2} C^j([0,\infty), H^{s-j}) \tag{2.31}$$

and the estimate

$$\|v_2, v_3\|_{\bigcap_{j=0}^{2} C^j(H^{s-j})} \leq c(T)\|u_2\|_{s-3} \tag{2.32}$$

holds for any $T > 0$.

Proof: We study the term v_2, the term v_3 can be discussed similarly. Using Lemmata 2.1 and 2.2, we obtain on the support of $1 - \chi_2$

$$b_2^2(\xi)e^{-\beta_2(\xi)t} = e^{-(\frac{\gamma^2}{2\kappa}+i\sqrt{\tau}\xi)t}\left(\frac{i\xi^{-3}}{2\kappa\sqrt{\tau}} + r(t,\xi)\right), \tag{2.33}$$

where r is smooth and satisfies

$$|\partial_t^l r(t,\xi)| \leq c_l(T)\xi^{-4} \tag{2.34}$$

for all $l \geq 0$, $t \in [0,T]$. Substituting (2.33) into (2.18) it follows

$$v_2(t,x) = e^{-\frac{\gamma^2}{2\kappa}t}\mathcal{F}^{-1}\left(e^{-i\sqrt{\tau}\xi t}(1-\chi_2(\xi))\hat{u}_2(\xi)\left(\frac{i\xi^{-3}}{2\kappa\sqrt{\tau}} + r(t,\xi)\right)\right)$$

which immediately implies (2.31) when $u_2 \in H^{s-3}$.

Q.E.D.

Lemma 2.6 *For any $s \in I\!R$, $k \geq 1$ and $u_2 \in H^{s-3} \cap H_e^{s+k-3}$ we have*

$$\left.\begin{array}{l} (\partial_t + \sqrt{\tau}\partial_x)^l(v_2,v_3) \in C^0([0,\infty), H^s, I, III), \\ (\partial_t - \sqrt{\tau}\partial_x)^l(v_2,v_3) \in C^0([0,\infty), H^s, I, II) \end{array}\right\} \tag{2.35}$$

for any $l \in \{1,\dots,k\}$; moreover, for any $T>0$, the following estimate holds:

$$\begin{aligned}&\sum_{l=1}^{k}\left(\|(\partial_t+\sqrt{\tau}\partial_x)^l(v_2,v_3)\|_{C^0(H^s,I,III)}+\|(\partial_t-\sqrt{\tau}\partial_x)^l(v_2,v_3)\|_{C^0(H^s,I,II)}\right)\\&\qquad\le c(T)\left(\|u_2\|_{s-3}+\|u_2\|_{H_e^{s+k-3}}\right).\end{aligned}\tag{2.36}$$

Proof: Consider v_2; we shall obtain an even stronger result than claimed in (2.35), (2.36). From (2.18) we have

$$(\partial_t+\sqrt{\tau}\partial_x)^l v_2=\mathcal{F}^{-1}(b_2^2(\xi)e^{-\beta_2(\xi)t}(-\beta_2(\xi)+i\sqrt{\tau}\xi)^l(1-\chi_2(\xi)\hat{u}_2(\xi)).\tag{2.37}$$

On the support of $1-\chi_2$ we have

$$b_2^2(\xi)e^{-\beta_2(\xi)t}(-\beta_2(\xi)+i\sqrt{\tau}\xi)^l=e^{-(\frac{\gamma^2}{2\kappa}+i\sqrt{\tau}\xi)t}\left(c\xi^{-3}+\sum_{j=4}^{k'}c_j(t)\xi^{-j}+r_1(t,\xi)\right)\tag{2.38}$$

for any $k'\ge 3$, where c_j and r_1 are smooth and

$$|\partial_t^\alpha r_1(t,\xi)|\le c_\alpha(T)|\xi|^{-k'-1}\tag{2.39}$$

for any $\alpha\ge 0$, $t\in[0,T]$.

Substituting (2.38) into (2.37) gives

$$(\partial_t+\sqrt{\tau}\partial_x)^l v_2\in C^0([0,\infty),H^s),\tag{2.40}$$

if $u_2\in H^{s-3}$, and the estimate

$$\sum_{l=1}^{k}\|(\partial_t+\sqrt{\tau}\partial_x)^l v_2\|_{C^0(H^s)}\le c(T)\|u_2\|_{s-3}.\tag{2.41}$$

In the same way we obtain

$$(\partial_t-\sqrt{\tau}\partial_x)^l v_2=\mathcal{F}^{-1}(b_2^2(\xi)e^{-\beta_2(\xi)t}(-\beta_2(\xi)-i\sqrt{\tau}\xi)^l(1-\chi_2(\xi))\hat{u}_2(\xi)),\tag{2.42}$$

and on the support of $1-\chi_2$

$$b_2^2(\xi)e^{-\beta_2(\xi)t}(-\beta_2(\xi)-i\sqrt{\tau}\xi)^l=e^{-(\frac{\gamma^2}{2\kappa}+i\sqrt{\tau}\xi)t}(c\xi^{l-3}+\sum_{j=4-l}^{k'}\tilde{c}_j(t)\xi^{-j}+r_2(t,\xi)),\tag{2.43}$$

where $\tilde{c}_j$ and r_2 have the same properties as c_j and r_1 in (2.39), respectively. Plugging (2.43) into (2.42) it follows

$$\begin{aligned}(\partial_t-\sqrt{\tau}\partial_x)^l v_2(t,x)&=e^{-\frac{\gamma^2}{2\kappa}t}\mathcal{F}^{-1}(e^{-i\sqrt{\tau}\xi t}(c\xi^{l-3}+\sum_{j=4-l}^{k'}\tilde{c}_j(t)\xi^{-j}+r_2(t,\xi))(1-\chi_2(\xi))\hat{u}_2(\xi))\\&\equiv v_2^{(0)}(t,x)+\sum_{j=4-l}^{k'}v_2^{(j)}(t,x)+R(t,x)\end{aligned}\tag{2.44}$$

with obvious notations. Clearly, when $u_2 \in H_e^{s+k-3} \cap H^{s-3}$, we have

$$v_2^{(0)}, \sum_{j=4-l}^{k'} v_2^{(j)} \in C^0([0,\infty), H^{s+k-l}, I, II). \tag{2.45}$$

and R satisfying

$$R \in C^0([0,\infty), H^{s+k-l}) \tag{2.46}$$

by setting $k' \geq k-l+2$. Thus, we obtain

$$(\partial_t - \sqrt{\tau}\partial_x)^l v_2 \in C^0([0,\infty), H^{s+k-l}, I, II)) \tag{2.47}$$

and the estimate

$$\sum_{l=1}^{k} \|(\partial_t - \sqrt{\tau}\partial_x)^l v_2\|_{C^0(H^{s+k-l},I,II))} \leq c(T)(\|u_2\|_{s-3} + \|u_2\|_{H_e^{s+k-3}}). \tag{2.48}$$

From (2.40), (2.41), (2.47) and (2.48) we conclude the assertions (2.35), (2.36) for v_2.

Q.E.D.

Now we can state a complete result for the problem (2.1) as follows.

Proposition 2.7 *For any $s \geq 3$ and $u_2 \in H^{s-3} \cap H_e^{s-2}$ the solution u of (2.1) satisfies*

$$\left.\begin{aligned} u \in \bigcap_{j=0}^{1} C^j(H^{s-j}) \cap C^2(H^{s-3}) \cap \bigcap_{j=2}^{3} H^j(H^{s+2-2j}), \\ (\partial_t + \sqrt{\tau}\partial_x)u \in L^2(H^s, I, III), \\ (\partial_t - \sqrt{\tau}\partial_x)u \in L^2(H^s, I, II), \end{aligned}\right\} \tag{2.49}$$

and the following estimates hold:

$$\|u\|_{\bigcap_{j=0}^{1} C^j(H^{s-j}) \cap C^2(H^{s-3})} + \|u\|_{\bigcap_{j=1}^{3} H^j(H^{s+2-2j})} \leq c(T)\|u_2\|_{s-3} \tag{2.50}$$

and

$$\|(\partial_t + \sqrt{\tau}\partial_x)u\|_{L^2(H^s,I,III)} + \|(\partial_t - \sqrt{\tau}\partial_x)u\|_{L^2(H^s,I,II)} \leq c(T)(\|u_2\|_{s-3} + \|u_2\|_{H_e^{s-2}}), \tag{2.51}$$

for any $T > 0$, where $c(T) > 0$ only depends on T.

PROOF: From Lemma 2.4 we have

$$v_1 \in C^0([0,\infty), H^{s+1}) \cap C^1([0,\infty), H^{s-1}) \cap C^2([0,\infty), H^{s-3}) \tag{2.52}$$

if $u_2 \in H^{s-3}$ with $s \geq 3$.

Combining the Lemmata 2.3, 2.4, 2.5 and (2.52), and using the equation (2.1), the first line of (2.49) follows. On the other hand. Lemma 2.4 implies

$$v_1 \in H^1(H^s) \cap L^2(H^{s+2}). \tag{2.53}$$

The Lemmata 2.3, 2.6 together with (2.53) yield the remaining claims in (2.49).

Q.E.D.

Remark 2.8 *From (2.53) we know that when $u_2 \in H^{s-3} \cap H_e^{s-2}$ the result (2.49) can* not *be improved to*

$$\left.\begin{array}{l}(\partial_t + \sqrt{\tau}\partial_x)u \in C^0(H^s, I, III), \\ (\partial_t - \sqrt{\tau}\partial_x)u \in C^0(H^s, I, II)\end{array}\right\} \tag{2.54}$$

due to the heat conduction part; in particular, we have in general

$$u \notin C^1(H^s(I)) \cap C^0(H^{s+1}(I)). \tag{2.55}$$

Similarly, when the assumption for u_2 is strengthened to $u_2 \in H^{s-3} \cap H_e^{s+k-3}$ for a fixed integer $k \geq 2$, we can not *get more regularity of $(\partial_t \pm \partial_x)^l u$ in the regions I, II and III than expressed in (2.49) for any $l \in \{2, \ldots, k\}$. These two phenomenona are typically different from the situation for purely hyperbolic equations (cf. [13], [14], [15]).*

Let us now consider the linear Cauchy problem (1.8) with (u_2, F) vanishing:

$$\left.\begin{array}{l}P(\partial)u = 0 \\ u(t=0) = u_0,\ u_t(t=0) = u_1,\ u_{tt}(t=0) = 0.\end{array}\right\} \tag{2.56}$$

It is easy to see that the solution u of (2.56) can be represented as

$$u(t,x) = \sum_{j=1}^{3} \mathcal{F}^{-1}_{\xi \to x}(e^{-\beta_j(\xi)t}(b_j^0(\xi)\hat{u}_0(\xi) + b_j^1(\xi)\hat{u}_1(\xi))) \tag{2.57}$$

where, for $j = 1, 2, 3$,

$$b_j^0 := \prod_{l \neq j} \beta_l \Big/ \left(\prod_{l \neq j}(\beta_j - \beta_l)\right), \quad b_j^1 := \sum_{l \neq j} \beta_l \Big/ \left(\prod_{l \neq j}(\beta_j - \beta_l)\right). \tag{2.58}$$

In the same way as Proposition 2.7 we can establish a result for problem (2.56) as follows.

Proposition 2.9 *For any $s \geq 2$ and $u_0 \in H^s \cap H_e^{s+1}$ and $u_1 \in H^{s-1} \cap H_e^s$, the solution u of (2.56) satisfies*

$$\left.\begin{array}{l}u \in \bigcap_{j=0}^{2} C^j(H^{s-j}) \cap H^3(H^{s-3}), \\ (\partial_t + \sqrt{\tau}\partial_x)u \in C^0(H^s, I, III) \\ (\partial_t - \sqrt{\tau}\partial_x)u \in C^0(H^s, I, II)\end{array}\right\} \tag{2.59}$$

and the estimates

$$\|u\|_{\bigcap_{j=0}^{2} C^j(H^{s-j})} + \|u\|_{H^3(H^{s-3})} \leq c(T)\left(\|u_0\|_s + \|u_1\|_{s-1}\right), \tag{2.60}$$

and

$$\left.\begin{array}{l}\|(\partial_t+\sqrt{\tau}\partial_x)u\|_{C^0(H^s,I,III)}+\|(\partial_t-\sqrt{\tau}\partial_x)u\|_{C^0(H^s,I,II)}\\ \le c(T)(\|u_0\|_s+\|u_1\|_{s-1}+\|u_0\|_{H_e^{s+1}}+\|u_1\|_{H_e^s}).\end{array}\right\} \tag{2.61}$$

Next we consider the last case for problem (1.8), i.e.

$$\left.\begin{array}{l}P(\partial)u=F(t,x),\\ u(t=0)=u_t(t=0)=u_{tt}(t=0)=0.\end{array}\right\} \tag{2.62}$$

Obviously, the solution u of (2.62) can be represented as

$$u(t,x)=\mathcal{F}^{-1}_{\xi\to x}\left(\int_0^t\sum_{j=1}^3 b_j^2(\xi)e^{-\beta_j(\xi)(t-t_1)}\hat{F}(t_1,\xi)dt_1\right), \tag{2.63}$$

where $\hat{F}(t,\cdot)$ denotes the Fourier transform of $F(t,\cdot)$, compare (2.2). Using the identity (2.13) we can decompose u into

$$u(t,x)=\sum_{j=1}^3 u^{(j)}(t,x), \tag{2.64}$$

where

$$u^{(1)}(t,x):=\mathcal{F}^{-1}\left(\int_0^t\sum_{j=1}^3 b_j^2(\xi)e^{-\beta_j(\xi)(t-t_1)}\chi_1(\xi)\hat{F}(t_1,\xi)dt_1\right), \tag{2.65}$$

$$u^{(2)}(t,x):=\mathcal{F}^{-1}\left(\int_0^t\sum_{j=1}^3 b_j^2(\xi)e^{-\beta_j(\xi)(t-t_1)}(1-\chi_1(\xi))\chi_2(\xi)\hat{F}(t_1,\xi)dt_1\right), \tag{2.66}$$

and

$$u^{(3)}(t,x):=\sum_{j=1}^3 v_j(t,x) \tag{2.67}$$

with

$$v_j(t,x):=\mathcal{F}^{-1}\left(\int_0^t b_j^2(\xi)e^{-\beta_j(\xi)(t-t_1)}(1-\chi_2(\xi))\hat{F}(t_1,\xi)dt_1\right). \tag{2.68}$$

In the same way as in Lemma 2.3, we have

Lemma 2.10 *For any integer $k\ge 0$ and*

$$F\in\bigcap_{j=0}^k C^j(L^2)\qquad\left(\bigcap_{j=0}^k H^j(L^2)\ \textit{resp.}\right)$$

$u^{(1)},u^{(2)}$ *satisfy*

$$(u^{(1)},u^{(2)})\in\bigcap_{j=0}^{k+1} C^j(H^s)\qquad\left(\bigcap_{j=0}^{k+1} H^j(H^s)\ \textit{resp.}\right)$$

for any $s \geq 0$ and $T > 0$; moreover, the estimate

$$\begin{array}{rcl} \|u^{(1)}, u^{(2)}\|_{\bigcap\limits_{j=0}^{k+1} C^j(H^s)} & \leq & c(T)\|F\|_{\bigcap\limits_{j=0}^{k} C^j(L^2)} \\ \left(\|u^{(1)}, u^{(2)}\|_{\bigcap\limits_{j=0}^{k+1} H^{j+1}(H^s)} \right. & \leq & \left. c(T)\|F\|_{\bigcap\limits_{j=0}^{k} H^j(H^s)} \quad \textit{resp.} \right) \end{array} \tag{2.69}$$

is valid.

For the term v_1 we get

Lemma 2.11 *For any $s \geq 2m-2$, $m \in \mathbb{N}$, and $F \in \bigcap\limits_{j=0}^{m} H^j(H^{s-3-2j})$ we have $v_1 \in \bigcap\limits_{j=0}^{m+1} H^j(H^{s+3-2j})$; moreover, v_1 satisfies*

$$\|v_1\|_{\bigcap\limits_{j=0}^{m+1} H^j(H^{s+3-2j})} \leq c(T)\|F\|_{\bigcap\limits_{j=0}^{m} H^j(H^{s-3-2j})}. \tag{2.70}$$

Proof: Again, it suffices to prove the regularity, the estimate (2.70) will then be obvious. Employing the expansion (2.24) for (2.68) with $j=1$, it follows

$$\begin{array}{rcl} v_1(t,x) & = & \mathcal{F}^{-1}\left(\int\limits_0^t e^{-(\kappa\xi^2 - \frac{\gamma^2}{\kappa})(t-t_1)} \left(\frac{1}{\kappa^2}\xi^{-4} + r(t-t_1,\xi) \right) (1-\chi_2(\xi))\hat{F}(t_1,\xi)dt_1 \right) \\ & \equiv & \frac{1}{\kappa^2} e^{\frac{\gamma^2}{\kappa}t} \tilde{v}_1(t,x) + R(t,x) \end{array} \tag{2.71}$$

where $\tilde{v}_1(t,x) := \mathcal{F}^{-1}\left(\int\limits_0^t e^{-\kappa\xi^2(t-t_1)}\xi^{-4}(1-\chi_2(\xi))e^{-\frac{t^2}{\kappa}t_1}\hat{F}(t_1,\xi)dt_1 \right)$ satisfies

$$\left. \begin{array}{rcl} (\partial_t - \kappa\partial_x^2)\tilde{v}_1 & = & \mathcal{F}^{-1}(\xi^{-4}(1-\chi_2(\xi))\hat{F}(t,\xi))e^{-\frac{\gamma^2}{\kappa}t}, \\ \tilde{v}_1(t=0) & = & 0 \end{array} \right\} \tag{2.72}$$

which immediately implies

$$\tilde{v}_1 \in \bigcap_{j=0}^{m+1} H^j(H^{s+3-2j}) \tag{2.73}$$

if $F \in \bigcap\limits_{j=0}^{m} H^j(H^{s-3-2j})$, with $s \geq 2m-2$.
Using the property (2.52) of $r = r(t,\xi)$ in the term $R = R(t,x)$, it is easy to deduce

$$R \in \bigcap_{j=0}^{m+1} H^j(H^{s+3-2j}). \tag{2.74}$$

From (2.73), (2.74) and (2.71) the conclusion follows.

Q.E.D.

Lemma 2.12 *For any $s \in I\!R$, $m \in \mathbb{N}$ and*

$$F \in \bigcap_{j=0}^{m} H^j(H^{s-3-j}) \tag{2.75}$$

we have

$$(v_2, v_3) \in \bigcap_{j=0}^{m} C^j(H^{s-j}) \cap H^{m+1}(H^{s-m-1}), \tag{2.76}$$

and the estimate

$$\|v_2, v_3\|_{\bigcap_{j=0}^{m} C^j(H^{s-j}) \cap H^{m+1}(H^{s-m-1})} \le c(T) \|F\|_{\bigcap_{j=0}^{m} H^s(H^{s-3-j})}. \tag{2.77}$$

PROOF: Using the expansion (2.33) in (2.68) with $j = 2$, we get

$$v_2 = \mathcal{F}^{-1}\left(\int_0^t e^{-(\frac{\gamma^2}{2\kappa}+i\sqrt{\tau}\xi)(t-t_1)}(\frac{i\xi^{-3}}{2\kappa\sqrt{\tau}} + r(t-t_1,\xi))(1-\chi_2(\xi))\hat{F}(t_1,\xi)dt_1\right)$$

which implies

$$v_2 \in \bigcap_{j=0}^{m} C^j(H^{s-j}) \cap H^{m+1}(H^{s-m-1})$$

if (2.75) is valid, by using the fact that

$$|\partial_t^l r(t,\xi)| \le c_l(T)\xi^{-4}, \quad l \ge 0, \quad t \in [0,T].$$

The result for v_3 can be verified similarly.

Q.E.D.

Lemma 2.13 *(1) For any $s \in I\!R$ and F satisfying*

$$\left.\begin{array}{l} F \in L^2(H^{s-3}) \cap H^1(H^{s-4}), \\ (\partial_t + \sqrt{\tau}\partial_x)F \in L^2(H^{s-3}, I, III), \\ (\partial_t - \sqrt{\tau}\partial_x)F \in L^2(H^{s-3}, I, II), \end{array}\right\} \tag{2.78}$$

the functions v_2, v_3 given in (2.68) satisfy

$$\left.\begin{array}{l} (\partial_t + \sqrt{\tau}\partial_x)(v_2, v_3) \in L^2(H^s, I, III), \\ (\partial_t - \sqrt{\tau}\partial_x)(v_2, v_3) \in L^2(H^s, I, II). \end{array}\right\} \tag{2.79}$$

Moreover, we have the estimate

$$\left.\begin{array}{l} \|(\partial_t + \sqrt{\tau}\partial_x)(v_2, v_3)\|_{L^2(H^s,I,III)} + \|(\partial_t - \sqrt{\tau}\partial_x)(v_2, v_3)\|_{L^2(H^s,I,II)} \\ \quad \le c(T)(\|F\|_{\bigcap_{j=0}^{1} H^j(H^{s-3-j})} + \|(\partial_t + \sqrt{\tau}\partial_x)F\|_{L^2(H^{s-3},I,III)} \\ \qquad + \|(\partial_t - \sqrt{\tau}\partial_x)F\|_{L^2(H^{s-3},I,II)}). \end{array}\right\} \tag{2.80}$$

(2) If we additionally assume

$$F \in C^0(H^{s-3}) \tag{2.81}$$

then the result (2.79) is strengthened to

$$\left.\begin{array}{l}(\partial_t + \sqrt{\tau}\partial_x)(v_2, v_3) \in C^0(H^s, I, III),\\ (\partial_t - \sqrt{\tau}\partial_x)(v_2, v_3) \in C^0(H^s, I, II),\end{array}\right\} \tag{2.82}$$

and an estimate similar to (2.80) holds with obvious modifications.

PROOF: It suffices to justify (2.79) and (2.82) for v_2. From (2.68) we have

$$\begin{aligned}(\partial_t \pm \sqrt{\tau}\partial_x)v_2 &= \mathcal{F}^{-1}(b_2^2(\xi)(1-\chi_2(\xi))\hat{F}(t,\xi))\\ &+\mathcal{F}^{-1}\left(\int_0^t b_2^2(\xi)(-\beta_2(\xi) \pm i\sqrt{\tau}\xi)e^{-\beta_2(\xi)(t-t_1)}(1-\chi_2(\xi))\hat{F}(t_1,\xi)dt_1\right),\end{aligned} \tag{2.83}$$

and

$$\begin{aligned}(\partial_t \pm \sqrt{\tau}\partial_x)v_2 &= \mathcal{F}^{-1}(b_2^2(\xi)e^{-\beta_2(\xi)t}(1-\chi_2(\xi))\hat{F}(t,\xi)\\ &+\mathcal{F}^{-1}\left(\int_0^t b_2^2(\xi)e^{-\beta_2(\xi)(t-t_1)}(1-\chi_2(\xi))\mathcal{F}((\partial_t \pm \sqrt{\tau}\partial_x)F(t_1,\cdot))(\xi)dt_1\right)\end{aligned} \tag{2.84}$$

To study $(\partial_t + \sqrt{\tau}\partial_x)v_2$ we use the formula (2.83). Exploiting the expansion (2.38) for (2.83), it follows

$$\begin{aligned}(\partial_t + \sqrt{\tau}\partial_x)v_2 &= \mathcal{F}^{-1}\left(\left(\frac{i\xi^{-3}}{2\kappa\sqrt{\tau}} + r_1(\xi)\right)(1-\chi_2(\xi))\hat{F}(t,\xi)\right)\\ &+\mathcal{F}^{-1}\left(\int_0^t e^{-(\frac{\gamma^2}{2\kappa}+i\sqrt{\tau}\xi)(t-t_1)}(c\xi^{-3} + r_2(t-t_1,\xi))(1-\chi_2(\xi))\hat{F}(t_1,\xi)dt_1\right),\end{aligned} \tag{2.85}$$

where

$$|r_1(\xi)| \le \tilde{c}\xi^{-4} \text{ and } |r_2(t,\xi)| \le c(T)\xi^{-4} \tag{2.86}$$

for $\xi \in \mathrm{supp}(1-\chi_2(\xi))$ and $t \in [0,T]$. From (2.85) we obtain

$$(\partial_t + \sqrt{\tau}\partial_x)v_2 \in L^2(H^s) \text{ resp. } C^0(H^s) \tag{2.87}$$

when $F \in L^2(H^{s-3})$ resp. $F \in C^0(H^{s-3})$.

To study $(\partial_t - \sqrt{\tau}\partial_x)v_2$ we use the formula (2.84). Exploiting (2.38) again for (2.84), it follows

$$\begin{aligned}(\partial_t - \sqrt{\tau}\partial_x)v_2 &= \mathcal{F}^{-1}(e^{-(\frac{\gamma^2}{2\kappa}+i\sqrt{\tau}\xi)t}\left(\frac{i\xi^{-3}}{2\kappa\sqrt{\tau}} + r(t,\xi)\right)(1-\chi_2(\xi))\hat{F}(0,\xi))\\ &\quad+\mathcal{F}^{-1}(\int_0^t e^{-(\frac{\gamma^2}{2\kappa}+i\sqrt{\tau}\xi)(t-t_1)}(\frac{i\xi^{-3}}{2\kappa\sqrt{\tau}} + r(t-t_1,\xi))(1-\chi_2(\xi))\\ &\quad\mathcal{F}((\partial_t - \sqrt{\tau}\partial_x)F(t_1,\cdot))(\xi)dt_1)\\ &\equiv v_2^{(1)} + v_2^{(2)}\end{aligned} \tag{2.88}$$

with obvious notations. From (2.78) we conclude

$$F(0,\cdot) \in H^{s-4} \cap H_e^{s-3},$$

and using this for $v_2^{(1)}$, we get

$$v_2^{(1)} \in C^0(H^s, I, II). \tag{2.89}$$

On the other hand, using (2.78) for $v_2^{(2)}$ yields

$$v_2^{(2)} \in C^0(H^s, I, II), \tag{2.90}$$

hence (2.88)–(2.90) imply

$$(\partial_t - \sqrt{\tau}\partial_x)v_2 \in C^0(H^s, I, II). \tag{2.91}$$

Combining (2.87) and (2.91) the conclusions (2.79) and (2.82) for v_2 follow immediately.

Q.E.D.

Now we can summarize the previous lemmata to the following result for the problem (2.62).

Proposition 2.14 *(1) For any $s \geq 4$ and F satisfying (2.78), the solution u of (2.62) satisfies*

$$\left.\begin{array}{l} u \in \bigcap_{j=0}^{1} C^j(H^{s-j}) \cap C^2(H^{s-3}) \cap \bigcap_{j=2}^{3} H^j(H^{s+2-2j}), \\ (\partial_t + \sqrt{\tau}\partial_x)u \in L^2(H^s, I, III), \\ (\partial_t - \sqrt{\tau}\partial_x)u \in L^2(H^s, I, II). \end{array}\right\} \tag{2.92}$$

Moreover, we have the following two estimates:

$$\begin{aligned} &\|u\|_{\bigcap_{j=0}^{1} C^j(H^{s-j})} + \|u\|_{C^2(H^{s-3})} + \|u\|_{\bigcap_{j=2}^{3} H^j(H^{s+2-2j})} \\ &\leq c(T)\left(\|F\|_{L^2(H^{s-3})} + \|F\|_{H^1(H^{s-4})}\right), \end{aligned} \tag{2.93}$$

and

$$\begin{aligned} &\|(\partial_t + \sqrt{\tau}\partial_x)u\|_{L^2(H^s,I,III)} + \|(\partial_t - \sqrt{\tau}\partial_x)u\|_{L^2(H^s,I,III)} \\ &\leq c(T)(\|F\|_{\bigcap_{j=0}^{1} H^j(H^{s-3-j})} + \|(\partial_t + \sqrt{\tau}\partial_x)F\|_{L^2(H^{s-3},I,III)} \\ &+\|(\partial_t - \sqrt{\tau}\partial_x)F\|_{L^2(H^{s-3},I,II)}). \end{aligned} \tag{2.94}$$

(2) If additionally the assumption (2.81) holds, then the last two lines in (2.92) can be strengthened as in (2.82) with an estimate similar to (2.94) being valid with obvious modifications.

Proof: With the notations given in (2.64)–(2.68) we know from the Lemmata 2.10, 2.11 that, under the assumption

$$F \in \bigcap_{j=0}^{1} H^j(H^{s-3-j}), \tag{2.95}$$

we have

$$u^{(1)} + u^{(2)} + v_1 \in \bigcap_{j=0}^{2} H^j(H^{s+3-2j}) \tag{2.96}$$

which implies

$$u^{(1)} + u^{(2)} + v_1 \in C^0(H^{s+2}) \cap C^1(H^s). \tag{2.97}$$

Using Lemma 2.12, we have

$$v_2 + v_3 \in \bigcap_{j=0}^{1} C^j(H^{s-j}) \cap H^2(H^{s-2}). \tag{2.98}$$

Combining (2.96)–(2.98) it follows, assuming (2.95) that the solution u of (2.62) satisfies

$$u \in \bigcap_{j=0}^{1} C^j(H^{s-j}) \cap H^2(H^{s-2}). \tag{2.99}$$

Obviously, using the equation $P(\partial)u = 0$, this implies

$$u \in H^3(H^{s-4}). \tag{2.100}$$

The relations (2.99), (2.100) prove the first statement in (2.92).
The results in Lemma 2.13 and (2.96), (2.97) imply the regularity of $(\partial_t \pm \sqrt{\tau}\partial_x)u$ in the regions I, II and III. The estimates (2.93) and (2.94) are now simple consequences of our discussion.

Q.E.D.

Finally in this section, we are able to present the
PROOF OF THEOREM 1.1: Combining the propositions 2.7, 2.9 and 2.14, the results claimed in (1) follow. Furthermore, using (2.50), Proposition 2.9 and Proposition 2.14 (2), the assertion (1.14) is immediately deduced.

Q.E.D.

3 Semilinear problems in thermoelasticity

The purpose of this section is to study the semilinear problems (1.1) and (1.3), and to prove Theorem 1.2. In the remainder of this paper, $s > 9/2$ is a fixed real number.
To solve the nonlinear problem (1.1), we use the following iteration scheme:

$$\left.\begin{array}{l} u_{tt}^{\nu+1} - \tau u_{xx}^{\nu+1} + \gamma\theta_x^{\nu+1} = f(u^\nu, \theta^\nu), \\ \theta_t^{\nu+1} - \kappa\theta_{xx}^{\nu+1} + \gamma u_{xt}^{\nu+1} = g(u^\nu), \\ u^{\nu+1}(t=0) = u_0, \quad u_t^{\nu+1}(t=0) = u_1, \quad \theta^{\nu+1}(t=0) = \theta_0, \end{array}\right\} \tag{3.1}$$

with the iteration starting point $(u^0, \theta^0) := (0,0)$. Formally, the scheme (3.1) gives rise to an iteration scheme for the nonlinear problem (1.3) as follows.

$$\left.\begin{array}{rcl} P(\partial)u^{\nu+1} &=& F^\nu, \\ P(\partial)\theta^{\nu+1} &=& G^\nu, \\ u^{\nu+1}(t=0) &=& u_0, \quad u_t^{\nu+1}(t=0) = u_1, \quad u_{tt}^{\nu+1}(t=0) = u_2, \\ \theta^{\nu+1}(t=0) &=& \theta_0, \quad \theta_t^{\nu+1}(t=0) = \theta_0, \quad \theta_{tt}^{\nu+1}(t=0) = \theta_2, \end{array}\right\} \tag{3.2}$$

where F^ν and G^ν are determined by (u^ν, θ^ν) in the same manner as in (1.5), and u_2, θ_1, θ_2 are given in (1.6).

For the initial data (u_0, u_1, θ_0) in (1.1) we suppose

$$\left.\begin{array}{l} u_0, \theta_0 \in H^s \cap H_e^{s+1}, \\ u_1, \kappa\theta_0'' - \gamma u_1' \in H^{s-1} \cap H_e^s, \end{array}\right\} \tag{3.3}$$

which obviously implies

$$\left.\begin{array}{l} u_2 \in H^{s-2} \cap H_e^{s-1}, \\ \theta_1 \in H^{s-1} \cap H_e^s, \quad \theta_2 \in H^{s-3} \cap H_e^{s-2}. \end{array}\right\} \tag{3.4}$$

For the iteration scheme (3.1), at first, we have

Lemma 3.1 *For any $T > 0$, there are a constant $c(T) > 0$ and a smooth positive, increasing function $a(\cdot)$ such that for all $\nu \in \mathbb{N}$, $t \in [0,T]$:*

$$\begin{array}{rl} & \|u^{\nu+1}, \theta^{\nu+1}\|^2_{\bigcap_{j=0}^{1} C^j([0,t],H^{s-j})} + \|u^{\nu+1}, \theta^{\nu+1}\|^2_{C^2([0,t],H^{s-3})} \\ & + \int_0^t \sum_{j=2}^{3} \|\partial_t^j(u^{\nu+1}, \theta^{\nu+1})(t_1)\|^2_{s+2-2j} dt_1 \\ \leq & c(T)(\|u_0, \theta_0\|_s^2 + \|u_1\|_{s-1}^2 + \|\kappa\theta_0'' - \gamma u_1'\|_{s-1}^2 \\ & + a(M^\nu) \int_0^t \sum_{j=0}^{2} \|\partial_t^j(\theta^\nu, u^\nu)(t_1)\|^2_{s-1-j} dt_1 \\ & + a(M^\nu) a(M^{\nu-1}) \int_0^t \|\partial_t(u^{\nu-1}, \theta^{\nu-1})(t_1)\|^2_{s-4} dt_1), \end{array} \tag{3.5}$$

where

$$M^\nu := \|u^\nu, \theta^\nu\|_{\bigcap_{j=0}^{1} C^j([0,T],H^{s-2-j})}.$$

PROOF: Applying (1.11) in the iteration scheme (3.2) we have the following estimate

$$\begin{aligned} & \|u^{\nu+1},\theta^{\nu+1}\|^2_{\bigcap_{j=0}^{1} C^j([0,t],H^{s-j})} + \|u^{\nu+1},\theta^{\nu+1}\|^2_{C^2([0,t],H^{s-3})} \\ & \quad + \int_0^t \sum_{j=2}^{3} \|\partial_t(u^{\nu+1},\theta^{\nu+1})(t_1)\|^2_{s+2-2j} dt_1 \\ & \leq \; c(T)\Big(\|u_0,\theta_0\|^2_s + \|u_1,\theta_1\|^2_{s-1} + \|u_2,\theta_2\|^2_{s-3} \\ & \quad + \int_0^t \sum_{j=0}^{1} \|\partial_t^j(F^\nu,G^\nu)(t_1)\|^2_{s-3-j} dt_1\Big) \end{aligned} \tag{3.6}$$

for all $T>0$, $t\in[0,T]$.

From the expression for F,G given in (1.5), we obtain

$$\begin{aligned} & \sum_{j=0}^{1} \|\partial_t^j(F^\nu,G^\nu)(t_1)\|^2_{s-3-j} \\ & \leq a(M^\nu)\left(\sum_{j=0}^{3} \|\partial_t^j u^\nu(t_1)\|^2_{s-1-j} + \sum_{j=0}^{2} \|\partial_t^j \theta^\nu(t_1)\|^2_{s-1-j}\right). \end{aligned} \tag{3.7}$$

On the other hand, from the first equation in (3.1), we deduce

$$u^\nu_{tt} = f(u^{\nu-1},\theta^{\nu-1}) + \tau u^\nu_{xx} - \gamma\theta^\nu_x$$

which implies

$$\begin{aligned} \|\partial_t^3 u^\nu(t_1)\|^2_{s-4} \; & \leq \; a(M^{\nu-1})(\|\partial_t u^{\nu-1}(t_1)\|^2_{s-4} + \|\partial_t\theta^{\nu-1}(t_1)\|^2_{s-4}) \\ & \quad + c_0(\|\partial_t u^\nu(t_1)\|^2_{s-2} + \|\partial_t\theta^\nu(t_1)\|^2_{s-3}) \end{aligned} \tag{3.8}$$

with $c_0 := \max\{\tau,|\gamma|\}$.

Substituting (3.7) and (3.8) into (3.6), the estimate (3.5) follows.

Q.E.D.

As simple consequences of (3.5) we shall obtain the following two lemmata.

Lemma 3.2 *There is $T_1>0$ such that the sequence $\{u^\nu,\theta^\nu\}_\nu$ is bounded in*

$$\bigcap_{j=0}^{1} C^j([0,T_1],H^{s-j}) \cap C^2([0,T_1],H^{s-3}) \cap \bigcap_{j=2}^{3} H^j([0,T_1],H^{s+2-2j}).$$

PROOF: Fix any $T_0>0$ and choose K as

$$K := c(T_0)\;\; (\|u_0,\theta_0\|^2_s + \|u_1,\kappa\theta_0'' - \gamma u_1'\|^2_{s-1} + 1). \tag{3.9}$$

Since $(u^0,\theta^0)=(0,0)$, we conclude from (3.5)

$$\begin{aligned} & \|u^1,\theta^1\|^2_{\bigcap_{j=0}^{1} C^j([0,T_0],H^{s-j})} + \|u^1,\theta^1\|^2_{C^2([0,T_0],H^{s-3})} \\ & \quad + \int_0^{T_0} \sum_{j=2}^{3} \|\partial_t^j(u^1,\theta^1)(t_1)\|^2_{s+2-2j} dt_1 \leq K. \end{aligned} \tag{3.10}$$

Choose $T_1 > 0$ small enough, such that

$$c(T_1)(\|u_0, \theta_0\|_s^2 + \|(u_1, \kappa\theta_0'' - \gamma u_1'\|_{s-1}^2 + a(K)KT_1 + a^2(K)KT_1) \leq K. \tag{3.11}$$

Then, using (3.5), (3.10) and induction on ν, we conclude for each ν

$$\begin{aligned} &\|u^\nu, \theta^\nu\|^2_{\bigcap_{j=0}^{1} C^j([0,T_1],H^{s-j})} + \|u^\nu, \theta^\nu\|^2_{C^2([0,T],H^{s-3})} \\ &+ \int_0^{T_1} \sum_{j=2}^{3} \|\partial_t^j (u^\nu, \theta^\nu)(t_1)\|^2_{s+2-2j} dt_1 \leq K. \end{aligned}$$

Q.E.D.

Lemma 3.3 *There is $T_2 \in (0, T_1]$ such that $\{u^\nu, \theta^\nu\}_\nu$ is convergent in $\bigcap_{j=0}^{1} C^j([0,T_2], H^{s-j}) \cap C^2([0,T_2], H^{s-3}) \cap \bigcap_{j=2}^{3} H^j([0,T_2], H^{s+2-2j})$.*

Proof: For the iteration scheme (3.1), we can establish the following estimate in a similar manner as in Lemma 3.1:

$$\begin{aligned} &\|u^{\nu+1} - u^\nu, \theta^{\nu+1} - \theta^\nu\|^2_{\bigcap_{j=0}^{1} C^j([0,T],H^{s-j}) \cap C^2([0,T],H^{s-3})} \\ &+ \int_0^T \sum_{j=2}^{3} \|\partial_t (u^{\nu+1} - u^\nu, \theta^{\nu+1} - \theta^\nu)(t_1)\|^2_{s+2-2j} dt_1 \\ \leq\ & C_1 (\int_0^T \sum_{j=0}^{2} \|\partial_t (u^\nu - u^{\nu-1}, \theta^\nu - \theta^{\nu-1})(t_1)\|^2_{s+1-j} \\ &+ \int_0^T \|\partial_t (u^{\nu-1} - u^{\nu-2}, \theta^{\nu-1} - \theta^{\nu-2})(t_1)\|^2_{s-4} dt_1) \end{aligned} \tag{3.12}$$

for any $\nu \geq 2$ and $T \in (0, T_1]$, where c_1 is independent of T. It is easy to conclude the desired result from (3.12).

Q.E.D.

Combining Lemma 3.2 and Lemma 3.3, we obtain

Proposition 3.4 *If (u_0, u_1, θ_0) satisfy*

$$u_0, \theta_0 \in H^s \text{ and } u_1, \theta_0'' - \gamma u_1' \in H^{s-1}, \tag{3.13}$$

then there is a unique solution (u, θ) to (1.1) in the space

$$C^0([0,T_2], H^s) \cap C^1([0,T_2], H^{s-1}) \cap C^2([0,T_2], H^{s-3}) \cap \bigcap_{j=2}^{3} H^j([0,T_2], H^{s+2-2j}) \tag{3.14}$$

with $T_2 > 0$ given in Lemma 3.2.

Let us now study the further regularity of u. From $(u,\theta) \in \bigcap_{j=0}^{1} C^j([0,T_2],H^{s-j})$ we deduce

$$f(u,\theta) \in C^0([0,T_2],H^s) \cap C^1([0,T_2],H^{s-1}) \tag{3.15}$$

which implies

$$u_{tt} = f(u,\theta) + \tau u_{xx} - \gamma\theta_x \in C^0([0,T_2],H^{s-2}) \cap C^1([0,T_2],H^{s-3}). \tag{3.16}$$

Similarly, the property $(u,\theta) \in H^2([0,T_2],H^{s-2})$ yields

$$u_{tt} \in H^2([0,T_2],H^{s-4}). \tag{3.17}$$

Combining (3.15)–(3.17) it follows

Proposition 3.5 *The solution u obtained in Proposition 3.4 satisfies*

$$u \in \bigcap_{j=0}^{3} C^j([0,T_2],H^{s-j}) \cap H^4([0,T_2],H^{s-4}). \tag{3.18}$$

The final result concerns the special regularity of (u,θ) in the regions I, II and III under the assumption (3.3).

Proposition 3.6 *Under the assumption 3.3, the solution (u,θ) of problem (1.1) satisfies*

$$\left.\begin{array}{l}(\partial_t + \sqrt{\tau}\partial_x)\theta \in L^2([0,T_2],H^s,I,III),\\ (\partial_t - \sqrt{\tau}\partial_x)\theta \in L^2([0,T_2],H^s,I,II),\\ (\partial_t + \sqrt{\tau}\partial_x)^l u \in C^0([0,T_2],H^{s+1-l},I,III),\\ (\partial_t - \sqrt{\tau}\partial_x)^l u \in C^0([0,T_2],H^{s+1-l},I,II),\ l=1,2.\end{array}\right\} \tag{3.19}$$

PROOF:

(1) At first we prove (3.19) for the case $l=1$. Formally, using Theorem 1.1 for problem (1.3) we have

$$\begin{aligned}&\|(\partial_t + \sqrt{\tau}\partial_x)u\|_{C^0(H^s,I,III)} + \|(\partial_t - \sqrt{\tau}\partial_x)u\|_{C^0(H^s,I,II)}\\ &\quad +\|(\partial_t + \sqrt{\tau}\partial_x)\theta\|_{L^2(H^s,I,III)} + \|(\partial_t - \sqrt{\tau}\partial_x)\theta\|_{L^2(H^s,I,II)}\\ &\leq c(T)(\|u_0,\theta_0\|_s + \|u_1,\kappa\theta_0'' - \gamma u_1'\|_{s-1} + \|u_0,\theta_0\|_{H_e^{s+1}}\\ &\quad +\|u_1,\kappa\theta_0'' - \gamma u_1'\|_{H_e^s} + \|F\|_{C^0(H^{s-3})\cap H^1(H^{s-4})}\\ &\quad +\|G\|_{\bigcap_{j=0}^{1} H^j(H^{s-3-j})} + \|(\partial_t + \sqrt{\tau}\partial_x)(F,G)\|_{L^2(H^{s-3},I,III)}\\ &\quad +\|(\partial_t - \sqrt{\tau}\partial_x)(F,G)\|_{L^2(H^{s-3},I,II)}).\end{aligned} \tag{3.20}$$

The specific form of (F,G) given in (1.5) implies

$$\begin{aligned}\|F\|_{C^0(H^{s-3})\cap H^1(H^{s-4})} &\leq c(\|u,\theta\|_{C^0(H^{s-1})}+\|u,\theta\|_{C^1(H^{s-3})}\\ &\quad +\|u,\theta\|_{\bigcap_{j=1}^{2} H^j(H^{s-2j})}),\end{aligned} \tag{3.21}$$

$$\|G\|_{\bigcap_{j=0}^{1} H^j(H^{s-3-j})} \leq c(\|u\|_{\bigcap_{j=0}^{3} H^j(H^{s-1-j})}+\|\theta\|_{\bigcap_{j=0}^{2} H^j(H^{s-1-j})}), \tag{3.22}$$

and

$$\begin{aligned}\|(\partial_t \pm \sqrt{\tau}\partial_x)(F,G)\|_{L^2(H^{s-3})} &\leq \|F,G\|_{\bigcap_{j=0}^{1} H^j(H^{s-2-j})}\\ &\leq c(\|u\|_{\bigcap_{j=0}^{3} H^j(H^{s-j})}+\|\theta\|_{\bigcap_{j=0}^{2} H^j(H^{s-j})})\end{aligned} \tag{3.23}$$

where $c>0$ is a constant.

For fixed $T \in (0,T_2]$, using the Propositions 3.4 and 3.5, the right-hand sides of (3.21)–(3.23) are finite when (3.3) is valid, hence also that in (3.20) which implies (3.19) for $l=1$.

(2) Let $\overline{f}(t,x) := f(u(t,x),\theta(t,x)) - \gamma\theta_x(t,x)$, where (u,θ) is still the solution to (1.1). From (3.19) for $l=1$, we have

$$\left.\begin{aligned}(\partial_t+\sqrt{\tau}\partial_x)\overline{f} \in L^2(H^{s-1},I,III),\\ (\partial_t-\sqrt{\tau}\partial_x)\overline{f} \in L^2(H^{s-1},I,II).\end{aligned}\right\} \tag{3.24}$$

Using the usual d'Alembert formula for the problem

$$\left.\begin{aligned}&u_{tt}-\tau u_{xx}=\overline{f}(t,x),\\ &u(t=0)=u_0, \quad u_t(t=0)=u_1,\end{aligned}\right\} \tag{3.25}$$

we get

$$\begin{aligned}u(t,x) &= \tfrac{1}{2}\{u_0(x+\sqrt{\tau}t)+u_0(x-\sqrt{\tau}t)+\tfrac{1}{\sqrt{\tau}}\int_{x-\sqrt{\tau}t}^{x+\sqrt{\tau}t} u_1(\xi)d\xi\\ &\quad +\tfrac{1}{\sqrt{\tau}}\int_0^t \int_{x-\sqrt{\tau}(t-t_1)}^{x+\sqrt{\tau}(t-t_1)} \overline{f}(t_1,\xi)d\xi dt_1\}\end{aligned}$$

which implies

$$\begin{aligned}(\partial_t+\sqrt{\tau}\partial_x)^2u(t,x) &= 2\tau u_0''(x+\sqrt{\tau}t)+2\sqrt{\tau}u_1'(x+\sqrt{\tau}t)\\ &\quad +\overline{f}(0,x+\sqrt{\tau}t)+\int_0^t(\partial_t+\sqrt{\tau}\partial_x)\overline{f}(t_1,x+\sqrt{\tau}(t-t_1)dt_1,\end{aligned} \tag{3.26}$$

and hence belongs to

$$C^0([0,T_2],H^{s-1},I,III)$$

because of (3.24) and the fact that

$$\overline{f} = f(u,\theta) - \gamma\theta_x \in C^0([0,T_2], H^{s-1}).$$

Similarly, we obtain

$$(\partial_t - \sqrt{\tau}\partial_x)^2 u \in C^0([0,T_2], H^{s-1}, I, II).$$

Q.E.D.

As a simple consequence of (3.19) we get

Corollary 3.7 *In the region I we have*

$$u \in \bigcap_{j=0}^{2} C^j([0,T_2], H^{s+1-j}, I). \tag{3.27}$$

The Proof of Theorem 1.2 is now given by a combination of the Propositions 3.4, 3.5 and 3.6.

Q.E.D.

We remark that an additional independence of g on θ cannot be treated in the regularity class considered in Theorem 1.2.

4 Concluding remarks

We have discussed and described the propagation of singularities for the Cauchy problem of linear thermoelasticity as well as for a class of semilinear Cauchy problems in one space dimension. It turned out that the characteristic picture is dominated by the hyperbolic part with exactly the same characteristic lines. Nevertheless the parabolic part induced by the heat conduction not only presents technical difficulties, but influences the kind of results obtainable; although the heat equation itself has a well-known smoothing effect, in thermoelasticity it does not have a smoothing effect. Instead, it prevents the analysis of being exactly the same as for wave equations. By the infinite propagation speed and having the real line as domain of dependence it has through the coupling a deregularizing effect, visible in Theorem 1.2 in the restriction $l = 1, 2$ in (1.16). On the other hand, a certain smoothing effect can be seen in Theorem 1.1 looking at the regularity needed for u_2.

Finally we remark that most arguments rely on a careful analysis in Fourier space and the use of optimal regularity results for heat equations which can be carried over to boundary value problems — in principle —, replacing H^s by $D(A^{s/2})$, where A is the self-adjoint Laplace operator for Dirichlet or Neumann boundary conditions.

Examples:

1. $\Omega = (0, \infty)$, $u_{|\partial\Omega} = 0$, $\theta_{x_{|\partial\Omega}} = 0$.

   Either extend θ symmetrically and u anti-symmetrically to all of $\mathbb{R}$ and use the results above, or use the Fourier sine resp. Fourier cosine transformation instead of the Fourier transformation, as done in [5].

2. $\Omega = (0, 1)$, $u_{|\partial\Omega} = 0$, $\theta_{x_{|\partial\Omega}} = 0$.

   Use Fourier sine resp. cosine series expansions.

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