Universität Konstanz

Multidimensional Contact Problems in Thermoelasticity

Jaime E. Muñoz Rivera
Reinhard Racke

Konstanzer Schriften in Mathematik und Informatik
Nr. 21, Dezember 1996

ISSN 1430–3558

Multidimensional contact problems in thermoelasticity*

Jaime E. Muñoz Rivera and Reinhard Racke

21/1996
Universität Konstanz
Fakultät für Mathematik und Informatik

Abstract: We consider dynamical resp. quasistatic thermoelastic contact problems in $\mathbb{R}^n$ modeling the evolution of temperature and displacement in an elastic body that may come into contact with a rigid foundation. The existence of solutions to these dynamical resp. quasistatic nonlinear problems and the exponential stability are investigated using a penalty method. Interior smoothing effects in the quasistatic case are also disussed.

AMS subject classification: 73 B 30, 35 Q 99

Keywords and phrases: quasistatic thermoelasticity, asymptotic behavior

1 Introduction

We consider dynamical resp. quasistatic thermoelastic contact problems which model the evolution of temperature and displacement in an elastic body that may come into contact with a rigid foundation. If $\Omega \subset \mathbb{R}^n \, (n \geq 2)$ denotes the reference configuration, we assume that the smooth boundary $\partial\Omega$ consists of three mutually disjoint parts $\Gamma_D, \Gamma_N, \Gamma_C$ such that $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N \cup \overline{\Gamma}_c$, and $\Gamma_D \neq \emptyset$. The body is held fixed on Γ_D, tractions are zero on Γ_N, and Γ_c is the part which may have contact with a rigid foundation. The temperature is held fixed on $\partial\Omega$. Then the dynamical initial boundary value problem for the displacement $u = u(t,x)$ and the temperature difference $\theta = \theta(t,x)$, where $t \geq 0$ and $x \in \overline{\Omega}$, to be considered is the following:

$$\varrho \partial_t^2 u_i - (C_{ijkl} u_{k,l})_{,j} + (m_{ij}\theta)_{,j} = 0, \qquad i = 1, \ldots, n, \tag{1.1}$$

$$\varrho c \partial_t \theta - (k_{ij}\theta_{,i})_{,j} + m_{ij}\partial_t u_{i,j} = 0, \tag{1.2}$$

$$u(t=0) = u_0, \quad u_t(t=0) = u_1, \qquad \theta(t=0) = \theta_0, \tag{1.3}$$

$$u_{|\Gamma_D} = 0, \qquad \sigma_{ij}\nu_{j_{|\Gamma_N}} = 0, \qquad \theta_{|\partial\Omega} = 0, \tag{1.4}$$

$$u_\nu \leq g, \qquad \sigma_\nu \leq 0, \qquad (u_\nu - g)\sigma_\nu = 0, \qquad \sigma_T = 0 \text{ on } \Gamma_c. \tag{1.5}$$

*Supported by a CNPq-KFAJ grant

Here ϱ and c denote the density and the heat capacity, respectively, and are assumed in the sequel to be equal to one without loss of generality. C_{ijkl}, m_{ij} and k_{ij} denote the components of the elasticity tensor, those of the thermal expansion tensor and those of the heat conduction tensor, respectively, and will satisfy

$$C_{ijkl} \in L^\infty(\Omega), \qquad C_{ijkl} = C_{jikl} = C_{klij}, \tag{1.6}$$

$$\exists \lambda_1 > 0 \qquad \forall \eta_{ij} = \eta_{ji} \qquad C_{ijkl}\eta_{ij}\eta_{kl} \geq \lambda_1 |\eta|^2, \tag{1.7}$$

$$m_{ij} \in W^{1,\infty}(\Omega), \qquad m_{ij} = m_{ji} \geq 0, \tag{1.8}$$

$$k_{ij} \in W^{1,\infty}(\Omega), \qquad k_{ij} = k_{ji}, \tag{1.9}$$

$$\exists \lambda_2 > 0 \qquad k_{ij}\xi_i\xi_j \geq \lambda_2 |\xi|^2. \tag{1.10}$$

The comma notation $_{,j}$ denotes the differentiation $\partial_j \equiv \partial/\partial x_j$ with respect to x_j as well as a subindex t will denote the differentiation $\partial_t \equiv \partial/\partial_t$. The initial values u_0, u_1 and θ_0 are prescribed with regularity to be made more precise in the final formulation of the system. The stress tensor is given by $\sigma = (\sigma_{ij})$

$$\sigma_{ij} = C_{ijkl}u_{k,l} - m_{ij}\theta.$$

Since on the boundary $\theta = 0$ we have there

$$\sigma_{ij} = C_{ijkl}u_{k,l}.$$

The unit normal vector in $x \in \partial\Omega$ is denoted by $\nu = \nu(x)$ and the normal component of u by u_ν:

$$u_\nu = u \cdot \nu.$$

The normal component σ_ν of the stress tensor is given by

$$\sigma_\nu = \sigma_{ij}\nu_i\nu_j$$

and the tangential part σ_T is

$$\sigma_T = \sigma\nu - \sigma_\nu \nu.$$

The function g describes the initial gap between the part Γ_c of the reference configuration and the rigid foundation and is assumed to satisfy

$$g \in H^{1/2}(\Gamma_c), \quad g \geq 0 \quad \text{a.e. on } \Gamma_c. \tag{1.11}$$

Hence the boundary conditions (1.5) describe Signorini's contact condition on Γ_c for a frictionless ($\sigma_T = 0$) contact.
The corresponding quasistatic system arises from (1.1)–(1.5) by omitting $\partial_t^2 u_i$ in (1.1) and prescribing only θ_0 in (1.3).

These thermoelastic contact problems, being nonlinear because of the contact boundary conditions (1.5), arise in applications such as the manufacturing of castings and pistons, see the paper of Shi & Shillor [13] for more details and references. The mathematical treatment of the dynamical problem (1.1)–(1.5) in one space dimension was discussed in a paper of Elliott & Tang [5], where more complicated boundary conditions are considered. Existence results in higher dimensions were announced by Figueiredo & Trabucho in [6]. They considered the case of constant contact which allows them to use only variational methods to solve the problem; this does not apply to the situation of the contact condition of Signorini's type considered here. We shall investigate the existence in $n \geq 1$ space dimensions for radially symmetrical situations following the approach of Kim [9] who discussed an obstacle problem for a wave equation. Using a penalty method, we obtain an existence result, and we also prove the exponential stability, which was proved for a special one-dimensional system by Muñoz Rivera & Lacerda Oliveira [11].

As a motivation for the stability results to be expected and as a tool to be used later, we also discuss the *linear* quasistatic system for classical boundary conditions like

$$u_{|\partial\Omega} = 0, \qquad \theta_{|\partial\Omega} = 0,$$

proving existence, exponential stability and smoothing.

For the quastistatic system, Shi & Shillor [13] proved the existence of a solution, while Ames & Payne [2] proved a uniqueness result, see also these papers for references on the one-dimensional case, where a series of papers has appeared in the last years.

We shall give a new existence proof using a penalty method. This will also allow us to make conclusions on the exponential stability of the system.

Finally, we shall prove that the quasistatic system has a smoothing effect in the interior of Ω, where u and θ are shown to be arbitrarily smooth, an effect which cannot be expected up to the boundary because of the mixed and contact boundary conditions.

We remark that we have assumed exterior forces and exterior heat supply to be zero only for simplicity.

The paper is organized as follows: In section 2 we prove an existence and exponential stability result for the dynamical system(1.1)–(1.5) in the case of radial symmetry using a penalty method. Section 3 studies the linear quasistatic system with classical Dirichlet or Neumann type boundary conditions proving existence, uniqueness and exponential stability. In section 4 the quasistatic contact problem is investigated, and we obtain an existence result and exponential stability using a penalty method. Finally, section 5 will provide the interior smoothing effect for the quasistatic contact problem.

Concerning the notation we remark that we use standard notations for Sobolev spaces and also the Einstein summation convention for repeated indices. $\langle \cdot, \cdot \rangle$ denotes the norm in $(L^2(\Omega))^n$; $\|\cdot\|$ the corresponding norm.

2 Existence for the dynamical contact problem

A dynamical thermoelastic contact problem was investigated by Elliott & Tang [5] in one space dimension following the approach of Kim [9] who used a penalty method for the treatment of

an obstacle problem for a wave equation. We also adopt this approach. The system character of our problem arising in multi-dimensional elasticity naturally leads to further problems in the estimates that have to be overcome by additional considerations compared to the two papers above. We shall discuss the case of radial symmetry.
We look for a solution to (1.1)–(1.5) in the following sense. Let

$$H^1_{\Gamma_D} := \{u \in (H^1(\Omega)^n| \quad u_{|\Gamma_D} = 0\}.$$

Definition 2.1 *(u,θ) is a* <u>*solution*</u> *to (1.1)–(1.5) for given $u_0 \in H^1_0(\Omega)$, $u_1 \in L^2(\Omega)$, $\theta_0 \in L^2(\Omega)$, if, for any $T > 0$,*

$$u \in L^\infty((0,T),\, H^1_{\Gamma_D}(\Omega)),\, u_t \in L^\infty((0,T),(L^2(\Omega))^n),$$

$$\theta \in L^\infty((0,T),L^2(\Omega)) \cap L^2((0,T),H^1_0(\Omega)) \tag{2.1}$$

$$u(t=0) = u_0, \quad u_t(t=0) = u_1, \quad \theta(t=0) = \theta_0, \tag{2.2}$$

$$u_\nu \leq g \text{ on } (0,T) \times \Gamma_C \qquad (a.e.), \tag{2.3}$$

and for all $w \in L^\infty((0,T),H^1_{\Gamma_D}(\Omega)) \cap W^{1,\infty}((0,T),(L^2(\Omega)^n)$ with $w_\nu \leq g$ on Γ_c the following inequality holds,

$$\begin{aligned}
&\langle u_t(T,\cdot), w(T,\cdot) - u(T,\cdot)\rangle - \langle u_1, w(0,\cdot) - u_0\rangle \\
- &\int_0^T \langle u_t, w_t\rangle dt + \int_0^T \{\langle u_t, u_t\rangle - \langle C_{ijkl}u_{k,l}, u_{i,j}\rangle\}\, dt \\
+ &\int_0^T \langle C_{ijkl}u_{k,l}, w_{i,j}\rangle dt - \int_0^T \langle m_{ij}\theta, w_{i,j} - u_{i,j}\rangle dt \geq 0,
\end{aligned} \tag{2.4}$$

and for all $z \in W^{1,2}((0,T),\, H^1_0(\Omega))$ the following equality holds,

$$\begin{aligned}
- &\int_0^T \langle \theta, z_t\rangle dt + \langle \theta(T,\cdot), z(T,\cdot)\rangle - \langle \theta_0, z(0)\rangle \\
+ &\int_0^T \langle k_{ij}\theta_{,i}, z_{,j}\rangle dt - \int_0^T \langle u_{i,j}, \partial_t(m_{ij}z)\rangle dt \\
+ &\langle m_{ij}u(T,\cdot)_{i,j},\, z(T,\cdot)\rangle - \langle m_{ij}u_{0i,j},\, z(0,\cdot)\rangle = 0.
\end{aligned} \tag{2.5}$$

In the sequel, we shall write $L^\infty(H^1_{\Gamma_D})$ instead of $L^\infty((0,T),\, H^1_{\Gamma_D}(\Omega))$, similarly for the other spaces; moreover we write $u(T)$ instead of $u(T,\cdot)$, and so on.
As a justification for the definition of a solution we remark that a smooth classical solution to

(1.1)–(1.5) obviously satisfies (2.5) and also (2.4), because multiplication of (1.1) by $u - w$ and partial integration yields that the left-hand side of (2.4) equals

$$\int_{\Gamma_c} \nu_j C_{ijkl} u_{k,l}(w_i - u_i) d\Gamma = \int_{\Gamma_c} \sigma_\nu(u)(w_\nu - u_\nu) d\Gamma$$
$$= \int_{\Gamma_c} \sigma_\nu(u)(w_\nu - g) d\Gamma \geq 0.$$

On the other hand, if (u, θ) is a solution in the sense of Definition 2.1, and if (u, θ) is smooth, then (1.1)–(1.5) follows in the classical sense in the following way:
Taking $w = u \pm h, \quad h \in (C_0^\infty((0,T) \times \Omega))^n$ and $z \in C_0^\infty((0,T) \times \Omega)$, we conclude from (2.4), (2.5) that the differential equations (1.1), (1.2) are satisfied in the distributional (and hence in the classical) sense. (1.3), (1.4) and

$$u_{\nu|\Gamma_c} \leq g,$$

are obvious.
Taking $w = u + h, \;\; h \in C^1([0,T) \times \overline{\Omega}), \;\; h_\nu \leq 0, \;\; h = h_\nu \cdot \nu$ on Γ_c, we conclude

$$\int_0^T \int_{\Gamma_c} \nu_j C_{ijkl} u_{k,l} h_i d\Gamma \, dt \geq 0$$

which implies

$$\int_0^T \int_{\Gamma_c} \sigma_\nu \cdot h_\nu d\Gamma \, dt \geq 0,$$

hence

$$\sigma_\nu \leq 0.$$

Taking $w = u \pm (u_\nu - g) \cdot \nu \xi, \qquad |\xi| \leq 1$, we conclude

$$\int_0^T \int_{\Gamma_C} \sigma_\nu (u_\nu - g) \xi \, d\Gamma \, dt = 0,$$

which implies

$$\sigma_\nu (u_\nu - g) = 0.$$

Finally, taking $w = u + h$, $h_\nu \leq 0$, $h \equiv h_\nu \cdot \nu + h_T$, we obtain

$$\int_0^T \int_{\Gamma_c} \sigma_\nu \cdot h_\nu d\Gamma \, dt + \int_0^T \int_{\Gamma_c}^{T} \sigma_T \cdot h_T d\Gamma \, dt \geq 0$$

which implies, choosing h_T appropriately,

$$\int_0^T \int_{\Gamma_C} \sigma_T h_T d\Gamma \, dt = 0$$

hence

$$\sigma_T = 0,$$

therefore (1.5) is also satisfied.

The existence of a solution will be proved for the radial symmetrical case by using a penalty method. For this purpose we consider the following approximating problem for a parameter $\varepsilon > 0$. This will be solved — in general, without restriction to radial symmetry, — and then *a priori* estimates are proved that will show the convergence, as $\varepsilon \downarrow 0$, of a subsequence to a solution of (1.1)–(1.5). The penalized problem is the following, first in classical notation. For given $u_0^\varepsilon, u_1^\varepsilon \in (H^2(\Omega))^n \cap (H_0^1(\Omega))^n$, $\theta_0^\varepsilon \in H^2(\Omega) \cap H_0^1(\Omega)$ find a solution $(u^\varepsilon, \theta^\varepsilon)$ to the following initial boundary value problem (2.6)–(2.10):

$$\partial_t^2 u_i^\varepsilon - (C_{ijkl} u_{k,l}^\varepsilon)_{,j} + (m_{ij}\theta^\varepsilon)_{,j} = 0, \tag{2.6}$$

$$\partial_t \theta^\varepsilon - (k_{ij}\theta_{,i}^\varepsilon)_{,j} + m_{ij}\partial_t u_{i,j}^\varepsilon = 0, \tag{2.7}$$

$$u^\varepsilon(t=0) = u_0^\varepsilon, \quad u_t^\varepsilon(t=0) = u_1^\varepsilon, \quad \theta^\varepsilon(t=0) = \theta_0^\varepsilon, \tag{2.8}$$

$$u_{|\Gamma_D}^\varepsilon = 0, \quad \sigma_{ij}^\varepsilon \nu_{j|\Gamma_N} = 0, \quad \theta_{|\partial\Omega}^\varepsilon = 0, \tag{2.9}$$

$$\sigma_\nu^\varepsilon = -\frac{1}{\varepsilon}(u_\nu^\varepsilon - g)^+ - \varepsilon\partial_t u_\nu^\varepsilon, \quad \sigma_T^\varepsilon = 0, \text{ on } \Gamma_c, \tag{2.10}$$

where $\sigma^\varepsilon = \sigma(u^\varepsilon)$ is the stress tensor corresponding to u^ε, and f^+ denotes the positive part of f. Let $\{w_j\}_{j=1}^\infty \subset H_{\Gamma_D}^1(\Omega)$ be an orthonormal basis in $(L^2(\Omega))^n$ and let $\{z_j\}_{j=1}^\infty \subset H_0^1(\Omega)$ be an orthonormal basis in $L^2(\Omega)$.

Definition 2.2 *$(u^\varepsilon, \theta^\varepsilon)$ is a <u>solution</u> to (2.6)–(2.10) for $u_0^\varepsilon, u_1^\varepsilon \in (H^2(\Omega) \cap H_0^1(\Omega))^n$, $\theta_0^\varepsilon \in H^2(\Omega) \cap H_0^1(\Omega)$, if for any $T > 0$,*

$$u^\varepsilon \in W^{2,\infty}(L^2) \cap W^{1,\infty}(H_{\Gamma_D}^1), \quad \theta^\varepsilon \in W^{1,\infty}(L^2) \cap W^{1,2}(H_0^1), \tag{2.11}$$

$$u^\varepsilon(t=0) = u_0^\varepsilon, \quad u_t^\varepsilon(t=0) = u_1^\varepsilon, \quad \theta^\varepsilon(t=0) = \theta_0^\varepsilon, \tag{2.12}$$

and for all $p \in \mathbb{N}$ and for a.e. $t \in [0,T]$:

$$\begin{aligned}&\langle \partial_t^2 u^\varepsilon, w_p\rangle + \langle C_{ijkl}u_{k,l}^\varepsilon, w_{pi,j}\rangle - \langle m_{ij}\theta^\varepsilon, w_{pi,j}\rangle = \\ &-\frac{1}{\varepsilon}\int_{\Gamma_c}(u_\nu^\varepsilon - g)^+ w_{p\nu}\, d\Gamma\, dt - \varepsilon\int_{\Gamma_c}\partial_t u_\nu^\varepsilon w_{p\nu} d\Gamma,\end{aligned} \tag{2.13}$$

$$\langle \partial_t\theta^\varepsilon, z_p\rangle + \langle k_{ij}\theta_{,i}^\varepsilon, z_{p,j}\rangle + \langle m_{ij}\partial_t u_{i,j}^\varepsilon, z_p\rangle = 0. \tag{2.14}$$

It is again easy to see that a smooth classical solution to (2.6)–(2.10) is a solution (in the sense of Definition 2.2), and a solution (in the sense of Definition 2.2), which is smooth, is a classical solution, compare the considerations following Definition 2.1.

Theorem 2.3 *For given $u_0^\varepsilon, u_1^\varepsilon \in (H^2(\Omega)\cap H_0^1(\Omega))^n$, $\theta_0^\varepsilon \in H^2(\Omega)\cap H_0^1(\Omega)$, there is a solution $(u^\varepsilon, \theta^\varepsilon)$ to the penalized problem (2.6)–(2.10).*

Remark: There is no assumption on radial symmetry here.

Proof: A Galerkin method will be the appropriate tool as in [9], [5]. Defining v and ψ by

$$v := u^\varepsilon - u_0^\varepsilon - tu_1^\varepsilon,$$

$$\psi := \theta^\varepsilon - \theta_0^\varepsilon,$$

then (v, ψ) should satisfy, for $i = 1, \ldots, n$,

$$\begin{aligned} & \partial_t^2 v_i - (C_{ijkl}v_{k,l})_{,j} + (m_{ij}\psi)_{,j} \\ = \;& (C_{ijkl}u^\varepsilon_{ok,l} + tu^\varepsilon_{1k,l})_{,j} - (m_{ij}\theta^\varepsilon_o)_{,j} =: f_i, \end{aligned} \tag{2.15}$$

$$\partial_t\psi - (k_{ij}\psi^\varepsilon_{,i})_{,j} + m_{ij}\partial_t v_{i,j} = (k_{ij}\theta^\varepsilon_{o,i})_{,j} - m_{ij}u^\varepsilon_{1i,j} =: g, \tag{2.16}$$

$$v(t=0) = 0, \quad v_t(t=0) = 0, \quad \psi(t=0) = 0, \tag{2.17}$$

as well as the boundary conditions (2.9), (2.10) with $(u^\varepsilon, \theta^\varepsilon)$ being replaced by (v, ψ), all to be understood in the sense of Definition 2.2. We make the ansatz

$$v^m(t,x) = \sum_{p=1}^m a_{mp}(t)w_p(x), \quad m \in \mathbb{N}, \tag{2.18}$$

$$\psi^m(t,x) = \sum_{p=1}^m b_{mp}(t)z_p(x), \tag{2.19}$$

where $(a_{mp})_{p=1}^m$, $(b_{mp})_{p=1}^m$ satisfy

$$\begin{aligned} & \langle \partial_t^2 v^m, w_p\rangle + \langle C_{ijkl}v^m_{k,l}, w_{pi,j}\rangle - \langle m_{ij}\psi^m, w_{pi,j}\rangle \\ & = \langle f, w_p\rangle - \frac{1}{\varepsilon}\int\limits_{\Gamma_c}(v^m_\nu - g)^+ w_{p\nu}d\Gamma - \varepsilon\int\limits_{\Gamma_c}\partial_t v^m_\nu w_{p\nu}d\Gamma, \end{aligned} \tag{2.20}$$

for $p = 1, \ldots, m$, as well as

$$\langle \partial_t\psi^m, z_p\rangle + \langle k_{ij}\psi^m_i, z_{p,j}\rangle + \langle m_{ij}\partial_t v^m_{i,j}, z_p\rangle = \langle g, z_p\rangle, \tag{2.21}$$

$$a_{mp}(0) = 0, \quad \frac{d}{dt}a_{mp}(0) = 0, \quad b_{mp}(0) = 0. \tag{2.22}$$

The equations (2.20), (2.21) are a system of nonlinear ordinary differential equation for a_{mp} and b_{mp}, $p = 1, \ldots, m$, with prescribed initial values given in (2.22). Since the (only) nonlinearity in (2.20) is Lipschitz continuous, there is a smooth solution $\{a_{mp}, b_{mp}\}_{p=1}^m$ in $[0, T]$ for any $T > 0$.

Now we derive *a priori* estimates for (v^m, ψ^m). Multiplication of (2.20) by $\frac{d}{dt}a_{mp}(t)$ and (2.21) by $b_{mp}(t)$ and summation over p yields

$$\frac{1}{2}\frac{d}{dt}\|\partial_t v^m\|^2 + \frac{1}{2}\frac{d}{dt}\langle C_{ijkl}v^m_{k,l}, v^m_{i,j}\rangle - \langle m_{ij}\psi^m, \partial_t v^m_{i,j}\rangle$$
$$= \langle f, \partial_t v^m\rangle - \frac{1}{\varepsilon}\int\limits_{\Gamma_c}(v^m_\nu - g)^+\partial_t v^m_\nu d\Gamma - \varepsilon\int\limits_{\Gamma_c}|\partial_t v^m_\nu|^2 d\Gamma, \tag{2.23}$$

$$\frac{1}{2}\frac{d}{dt}\|\psi^m\|^2 + \langle k_{ij}\psi^m_{,i}, \psi^m_{,j}\rangle + \langle m_{ij}\partial_t v^m_{i,j}, \psi^m\rangle = \langle g, \psi^m\rangle, \tag{2.24}$$

or

$$\frac{1}{2}\frac{d}{dt}\{\|\partial_t v^m\|^2 + \langle C_{ijkl}v^m_{k,l}, v^m_{i,j}\rangle + \|\psi^m\|^2 + \frac{1}{\varepsilon}\int\limits_{\Gamma_c}|(v^m_\nu - g)^+|^2 d\Gamma\}$$
$$+\langle k_{ij}\psi^m_{,j}, \psi^m_{,j}\rangle$$
$$= \langle f, \partial_t v^m\rangle - \varepsilon\int\limits_{\Gamma_C}|\partial_t v^m_\nu|^2 d\Gamma + \langle g, \psi^m\rangle. \tag{2.25}$$

Integrating (2.25) with respect to $t \in [0, T]$, we conclude that there is a positive constant

$$c = c(T, \|u^\varepsilon_0\|_{H^2}, \|u^\varepsilon_1\|_{H^2}, \|\theta^\varepsilon_0\|_{H^2}) \tag{2.26}$$

not depending on m, and depending on ε only in the way indicated in (2.26), such that for any $t \in [0, T]$ the following estimates (2.27)–(2.31) hold:

$$\|\partial_t v^m(t)\| \le c, \tag{2.27}$$

$$\langle C_{ijkl}v^m_{k,l}, v^m_{i,j}\rangle(t) \le c, \tag{2.28}$$

$$\int\limits_0^t \langle k_{ij}\psi^m_{,i}, \psi^m_{,j}\rangle(s)ds \le c, \tag{2.29}$$

$$\frac{1}{\varepsilon}\int\limits_{\Gamma_c}|(v^m_\nu - g)^+|^2 d\Gamma \le c, \tag{2.30}$$

$$\varepsilon\int\limits_0^t\int\limits_{\Gamma_c}|\partial_t v^m_\nu|^2 d\Gamma\, ds \le c. \tag{2.31}$$

Therefore, using Korn's inequality, we obtain that

$$(v^m)_m \text{ is bounded in } W^{1,\infty}(L^2) \cap L^\infty(H^1_{\Gamma_D}), \tag{2.32}$$

$$(\psi^m)_m \text{ is bounded in } L^\infty(L^2) \cap L^2(H_0^1). \tag{2.33}$$

Next, the boundedness of $\partial_t^2 v^m$ in $L^2(L^2)$ will be proved. The estimate will be uniform only in $m \in \mathbb{N}$, not in $\varepsilon > 0$ (cf. (2.26)).

Differentiation of (2.20), (2.21) with respect to t, then multiplication by $\frac{d^2}{dt^2} a_{mp}(t)$ and $\frac{d}{dt} b_{mp}(t)$, respectively, and summation leads to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\|\partial_t^2 v^m\|^2 + \langle C_{ijkl}\partial_t v_{k,l}^m, \partial_t v_{i,j}^m\rangle + \|\partial_t \psi^m\|^2\right\} + \langle k_{ij}\partial_t\psi_{,i}^m, \partial_t\psi_{,j}^m\rangle \\
&= \langle \partial_t f, \partial_t^2 v^m\rangle - \frac{1}{\varepsilon}\int_{\Gamma_c} (\partial_t (v_\nu^m - g)^+)\partial_t^2 v_\nu^m d\Gamma - \varepsilon \int_{\Gamma_c} |\partial_t^2 v_\nu^m|^2 d\Gamma \\
&\leq \|\partial_t f\|\,\|\partial_t^2 v^m\| + \frac{1}{2\varepsilon^3}\int_{\Gamma_c} |\partial_t (v_\nu^m - g)^+|^2 d\Gamma - \frac{\varepsilon}{2}\int_{\Gamma_c} |\partial_t^2 v_\nu^m| d\Gamma.
\end{aligned} \tag{2.34}$$

Using (2.31) we conclude, by Gronwall's inequality,

$$\|\partial_t^2 v^m(t)\| \leq c_1(\varepsilon), \tag{2.35}$$

$$\langle C_{ijkl}\partial_t v_{k,l}^m(t), \partial_t v_{i,j}^m(t)\rangle \leq c_1(\varepsilon), \tag{2.36}$$

$$\|\partial_t \psi^m(t)\| \leq c_1(\varepsilon), \tag{2.37}$$

$$\int_0^t \langle k_{ij}\partial_t\psi_{,i}^m,\, \partial_t\psi_{,j}^m\rangle(s)ds \leq c_1(\varepsilon), \tag{2.38}$$

where the positive constant $c_1(\varepsilon)$ is independent of m_0 and $t \in [0,T]$, $T > 0$, but may depend on ε. Thus we have that, for fixed $\varepsilon > 0$,

$$(\partial_t^2 v^m)_m \text{ is bounded in } L^\infty(L^2), \tag{2.39}$$

$$(\partial_t v^m)_m \text{ is bounded in } L^\infty(H_{\Gamma_D}^1), \tag{2.40}$$

$$(\partial_t \psi^m)_m \text{ is bounded in } L^\infty(L^2) \cap L^2(H_0^1). \tag{2.41}$$

It follows from (2.32), (2.33) and (2.39)–(2.41) that for fixed ε there is a subsequence, again denoted by $((v^m, \psi^m))_m$, and (v, ψ) such that, as $m \to \infty$,

$$v^m \overset{*}{\rightharpoonup} v \quad \text{in } W^{2,\infty}(L^2) \cap W^{1,\infty}(H_{\Gamma_D}^1), \tag{2.42}$$

$$\psi^m \overset{*}{\rightharpoonup} \psi \quad \text{in } W^{1,\infty}(L^2), \tag{2.43}$$

$$\psi^m \rightharpoonup \psi \quad \text{in } W^{1,2}(H_0^1). \tag{2.44}$$

With the help of Lemma 1.4 from [9] (essentially Gagliardo-Nirenberg type estimates) the convergence

$$(v_\nu^m - g)^+ \to (v_\nu - g)^+ \text{ in } C^0([0,T], L^2(\Gamma_c)) \tag{2.45}$$

as well as

$$\partial_t v^m \to \partial_t v \text{ in } C^0([0,T], L^2(\partial\Omega)) \tag{2.46}$$

follow.
Using (2.42)–(2.46) and letting $m \to \infty$ in (2.20), (2.21), we conclude that

$$u^\varepsilon := v + u_0^\varepsilon + tu_1^\varepsilon, \quad \theta^\varepsilon := \psi + \theta_0^\varepsilon$$

satisfy (2.11)–(2.14).

Q.E.D.

Now we turn to the original problem (1.1)–(1.5) and show that a subsequence of $(u^\varepsilon, \theta^\varepsilon)$ where $(u^\varepsilon, \theta^\varepsilon)$ solves a penalized problem for $\varepsilon > 0$ according to Theorem 2.3, converges to a solution of (1.1)–(1.5).

Theorem 2.4 *For given $u_0 \in H_0^1(\Omega), u_1 \in (L^2(\Omega))^3,\ \theta_0 \in L^2(\Omega)$ there is a solution (u,θ) to (1.1)–(1.5) in the case of radial symmetry.*

Radial symmetry means that the domain Ω is radially symmetrical, i.e. invariant under transformations of the special orthogonal group $SO(3)$ if $n = 3$ resp. $O(2)$ if $n = 2$. The typical examples are balls or annular domains. Radial symmetry of (u,θ) means

$$\forall \mathcal{A} \in SO(3) \text{ resp. } O(2)\ \forall x \in \Omega : u(\mathcal{A}x) = \mathcal{A}u(x), \quad \theta(\mathcal{A}x) = \theta(x),$$

or, equivalently (see [7]), with $r := |x|$:
There exists a function $w : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ such that $u(x) = xw(r)$, and
there exists a function $\psi : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ such that $\theta(x) = \psi(r)$.
Of course, for the existence of radially symmetrical solutions to the penalized, and later for the original, problem, the coefficients have to satisfy invariance conditions, too. As an example we consider the homogeneous, isotropic case, where, in particular,

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}), \tag{2.47}$$

and the differential equations (1.1),(1.2) turn into

$$u_{tt} - \mu\Delta u - (\mu + \lambda)\nabla \operatorname{div} u + \bar{m}\nabla\theta = 0, \tag{2.48}$$
$$c\theta_t - \lambda_2\Delta\theta + \bar{m} \operatorname{div} u_t = 0, \tag{2.49}$$

where μ and λ are the Lamé moduli satisfying $\mu > 0$ and $2\mu + n\lambda > 0$, $\rho, c, \kappa > 0$ and $m \neq 0$ are constants.
We also notice that in this case

$$\sigma_{\nu_{|\partial\Omega}} = \lambda \operatorname{div} u + 2\mu u_{i,k}\nu_i\nu_k,$$

and it is easy to see that the boundary conditions (2.9),(2.10) allow for radially symmetrical solutions, i.e. if the initial data are radially symmetrical, then the solution according to Theorem 2.3 will have the same symmetry; namely, if (u,θ) is the solution, then also (v,ξ) with $v(x) :=$

$\mathcal{A}^{-1}u(\mathcal{A}x)$, $\xi(x) := \theta(\mathcal{A}x)$ will be a solution to the same initial data and boundary conditions for any (special) orthogonal $\mathcal{A}$. Hence, by uniqueness, $(u,\theta) = (v,\xi)$.

PROOF of Theorem 2.4: Let $\varepsilon > 0$ and let $u_0^\varepsilon, u_1^\varepsilon \in (H^2(\Omega)\cap H_0^1(\Omega))^n$, $\theta_0^\varepsilon \in H^2(\Omega)\cap H_0^1(\Omega)$ be radially symmetrical such that, as $\varepsilon \downarrow 0$,

$$u_0^\varepsilon \to u_1^\varepsilon \text{ in } (H_0^1(\Omega))^n, \tag{2.50}$$

$$u_1^\varepsilon \to u_1 \text{ in } (L^2(\Omega))^n, \tag{2.51}$$

$$\theta_0^\varepsilon \to \theta_0 \text{ in } L^2(\Omega). \tag{2.52}$$

Let $(u^\varepsilon, \theta^\varepsilon)$ be the solution to (2.6)–(2.10) according to Theorem 2.3, and corresponding to the initial values $(u_0^\varepsilon, u_1^\varepsilon, \theta_0^\varepsilon)$. By the known regularity of $(u^\varepsilon, \theta^\varepsilon)$ we can substitute w_p by $\partial_t u^\varepsilon$ in (2.13), and z_p by θ^ε in (2.14), respectively, and we obtain (cf. (2.25))

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\{\|\partial_t u^\varepsilon\|^2 + \langle C_{ijkl}u^\varepsilon_{k,l}, u^\varepsilon_{i,j}\rangle + \|\theta^\varepsilon\|^2 + \frac{1}{\varepsilon}\int_{\Gamma_c} |(u^\varepsilon_\nu - g)^+|^2\, d\Gamma + \langle k_{ij}\theta^\varepsilon_{,i}, \theta^\varepsilon_{,j}\rangle\\ &= -\varepsilon \int_{\Gamma_c} |\partial_t u^\varepsilon_\nu|^2\, d\Gamma.\end{aligned} \tag{2.53}$$

Hence, there is a positive constant

$$c_2 = c_2(T, \|u_0\|_{H^1}, \|u_1\|_{L^2}, \|\theta_0\|_{L^2}) \tag{2.54}$$

which is independent of ε, such that for any $t \in [0,T]$ we get

$$\|\partial_t u^\varepsilon\| \le c_2, \tag{2.55}$$

$$\langle C_{ijkl}u^\varepsilon_{k,l}, u^\varepsilon_{i,j}\rangle \le c_2, \tag{2.56}$$

$$\int_0^t \langle k_{ij}\theta^\varepsilon_{,i}, \theta^\varepsilon_{,j}\rangle(s)ds \le c_2, \tag{2.57}$$

$$\frac{1}{\varepsilon}\int_{\Gamma_c} |(u^\varepsilon_\nu - g)^+|^2 d\Gamma \le c_2 \tag{2.58}$$

$$\varepsilon \int_0^t \int_{\Gamma_c} |\partial_t u^\varepsilon_\nu|^2 d\Gamma\, ds \le c_2. \tag{2.59}$$

The estimates (2.55)–(2.59) imply the existence of a subsequence, again denoted by $(u^\varepsilon, \theta^\varepsilon)$, and of (u,θ) such that, as $\varepsilon \downarrow 0$,

$$u^\varepsilon \stackrel{*}{\rightharpoonup} u \quad \text{in } L^\infty(H^1_{\Gamma_D}) \cap W^{1,\infty}(L^2), \tag{2.60}$$

$$\theta^\varepsilon \stackrel{*}{\rightharpoonup} \theta \quad \text{in } L^\infty(L^2), \tag{2.61}$$

$$\theta^\varepsilon \rightharpoonup \theta \quad \text{in } L^2(H_0^1). \tag{2.62}$$

Using Lemma 1.4 from [9] again, we conclude

$$u^\varepsilon \to u \text{ in } C^0(L^2). \tag{2.63}$$

The equations (2.13), (2.14) imply for $(u^\varepsilon, \theta^\varepsilon)$

$$\begin{aligned} &\int_0^T \{\langle \partial_t^2 u^\varepsilon, w - u^\varepsilon\rangle + \langle C_{ijkl} u^\varepsilon_{k,l}, w_{i,j} - u^\varepsilon_{i,j}\rangle \\ - \quad & \langle m_{ij}\theta^\varepsilon, w_{i,j} - u^\varepsilon_{i,j}\rangle\} dt \\ = \quad & -\frac{1}{\varepsilon}\int_0^T \int_{\Gamma_c} (u^\varepsilon_\nu - g)^+ (w_\nu - u^\varepsilon_\nu) d\Gamma\, dt - \varepsilon \int_0^T \int_{\Gamma_c} \partial_t u^\varepsilon_\nu (w_\nu - u^\varepsilon_\nu) d\Gamma\, dt, \end{aligned} \tag{2.64}$$

for any $w \in L^\infty(H^1_{\Gamma_D}) \cap W^{1,\infty}(L^2)$ such that $w_\nu \le g$, and

$$\int_0^T \Big\{\langle \partial_t \theta^\varepsilon, z_t\rangle + \langle k_{ij}\theta^\varepsilon_{,i}, z_{,j}\rangle + \langle m_{ij}\partial_t u^\varepsilon_{i,j}, z\rangle\Big\} dt = 0, \tag{2.65}$$

for any $z \in W^{1,2}(H^1_0)$.
Integration by parts in (2.64) yields

$$\begin{aligned} & \langle u^\varepsilon_t(T), w(T) - u^\varepsilon(T)\rangle - \langle u^\varepsilon_1, w(0) - u^\varepsilon_0\rangle \\ & - \int_0^T \langle u^\varepsilon_t, w_t\rangle dt + \int_0^T \langle u^\varepsilon_t, u^\varepsilon_t\rangle - \langle C_{ijkl} u^\varepsilon_{k,l}, u^\varepsilon_{i,j}\rangle dt \\ + \quad & \int_0^T \langle C_{ijkl} u^\varepsilon_{k,l}, w_{i,j}\rangle dt - \int_0^T \langle m_{ij}\theta^\varepsilon, w_{i,j} - u^\varepsilon_{i,j}\rangle dt \\ & \ge -\varepsilon \int_0^T \int_{\Gamma_c} \partial_t u^\varepsilon_\nu w_\nu d\Gamma\, dt + \varepsilon \int_0^T \int_{\Gamma_c} \partial_t u^\varepsilon_\nu u^\varepsilon_\nu d\Gamma\, dt \\ = \quad & -\varepsilon \int_0^T \int_{\Gamma_c} \partial_t u^\varepsilon_\nu w_\nu\, d\Gamma\, dt + \frac{\varepsilon}{2}\left\{\int_{\Gamma_c} |u^\varepsilon_\nu(T)|^2 d\Gamma - \int_{\Gamma_c} |u^\varepsilon_{0\nu}|^2 d\Gamma\right\} \end{aligned} \tag{2.66}$$

where we used

$$\begin{aligned} & -\frac{1}{\varepsilon}\int_0^T \int_{\Gamma_c} (u^\varepsilon_\nu - g)^+ (w_\nu - u^\varepsilon_\nu) d\Gamma\, dt \\ = \quad & -\frac{1}{\varepsilon}\int_0^T \int_{\Gamma_c} (u^\epsilon_\nu - g)^+ (w_\nu - g) d\Gamma\, dt + \frac{1}{\varepsilon}\int_0^T \int_{\Gamma_c} |(u^\varepsilon_\nu - g)^+|^2 d\Gamma\, dt \\ \ge \quad & 0. \end{aligned}$$

Integration by parts in (2.65) leads to

$$
\begin{aligned}
& -\int_0^T \langle \theta^\varepsilon, z_t\rangle dt + \langle \theta^\varepsilon(T), z(T)\rangle - \langle \theta_0^\varepsilon, z(0)\rangle \\
& + \int_0^T \langle k_{ij}\theta^\varepsilon_{,i}, z_{,j}\rangle dt - \int_0^T \langle u^\varepsilon_{i,j}, \partial_t(m_{ij}z)\rangle dt + \langle m_{ij}u^\varepsilon(T)_{i,j}, z(T)\rangle - \langle m_{ij}u^\varepsilon_{oi,j}, z(0)\rangle \\
= \;& 0.
\end{aligned} \tag{2.67}
$$

Using (2.60)–(2.62) we conclude from (2.67), as $\varepsilon \downarrow 0$, that (u,θ) satisfy (2.5).
It remains to justify (2.3) and (2.4). Notice that u satisfies (2.3) because

$$
\|(u_\nu - g)^+\|_{L^2(\Gamma_c)} = \lim_{\varepsilon\downarrow 0} \|(u^\varepsilon_\nu - g)^+\|_{L^2(\Gamma_c)} = 0
$$

by (2.63),(2.58).
We claim that the following inequality holds:

$$
\int_0^T \langle u_t, u_t\rangle - \langle C_{ijkl}u_{k,l}, u_{i,j}\rangle dt \geq \limsup_{\varepsilon\downarrow 0} \int_0^T \langle u^\varepsilon_t, u^\varepsilon_t\rangle - \langle C_{ijkl}u^\varepsilon_{k,l}, u^\varepsilon_{i,j}\rangle dt. \tag{2.68}
$$

The convergence

$$
|u^\varepsilon_t|^2 - C_{ijkl}u^\varepsilon_{k,l}u^\varepsilon_{i,j} \to |u_t|^2 - C_{ijkl}u_{k,l}u_{i,j} \tag{2.69}
$$

in $\mathcal{D}'((0,T)\times\Omega)$, i.e. in the sense of distributions, follows from the theory of compensated compactness, see Corollary 4.3 of [3]; take

$$
v^i := (\partial_t u^\varepsilon_i, -(C_{ijkl}u^\varepsilon_{k,l})_{j=1,\dots,n}), \quad w^i := (\partial_t u^\varepsilon_i, (u^\varepsilon_{i,j})_{j=1,\dots,n})
$$

and summation over $i = 1,\dots,n$, and observe

$$
-(m_{ij}\theta^\varepsilon)_{,j} \in L^2(L^2).
$$

The differential equations (2.6), (2.7) for $(u^\varepsilon, \theta^\varepsilon)$ are satisfied in the distributional sense. Since

$$
\theta^\varepsilon_t \in L^\infty(L^2) \text{ and } \partial_t u^\varepsilon_{i,j} \in L^\infty(L^2)
$$

we have

$$
(k_{ij}\theta^\varepsilon_{,i})_{,j} \in L^\infty(L^2),
$$

hence

$$
\theta^\varepsilon \in L^\infty(H^2 \cap H^1_0). \tag{2.70}
$$

Moreover,

$$
\partial_t^2 u^\varepsilon \in L^\infty(L^2)
$$

and (2.69) imply

$$
(C_{ijkl}u^\varepsilon_{k,l})_{,j} \in L^\infty(L^2)
$$

from which we conclude interior regularity in the following sense:

$$\forall \delta > 0: \quad u^\varepsilon \in L^\infty((0,T), H^2(\Omega_\delta)), \tag{2.71}$$

where

$$\Omega_\delta := \{x \in \Omega |\, \mathrm{dist}\,(x, \partial\Omega) > \delta\}, \quad 0 < \delta \le \delta_0$$

(δ_0 fixed).

Remark: The H^2-regularity up to $\partial\Omega$ cannot be expected because of the mixed boundary conditions for u^ε.

Let Ω_δ as above, $S_\delta := \partial\Omega_\delta$, and let $h \in (C^1(\overline{\Omega}))^n$ such that

$$h(x) = \nu(x) = \text{ exterior normal in } x \in S_\delta,$$

if $\mathrm{dist}(x, \partial\Omega) = \delta \le \delta_0/2$,

$$h(x) = 0$$

if $\mathrm{dist}(x, \partial\Omega) \ge \delta_0$.

Writing (u, θ) instead of $(u^\varepsilon, \theta^\varepsilon)$ for simplicity, we multiply the first differential equation (2.6) by $h_k u_{i,k}$ and integrate.

$$\begin{aligned}\int_0^T \int_{\Omega_\delta} \partial_t^2 u_i h_k u_{i,k} dx\, dt &= \int_{\Omega_\delta} \partial_t u_i(T) h_k u_{i,k}(T) dx - \int_{\Omega_\delta} u_{1i} h_k u_{0i,k} dx \\ &\quad - \frac{1}{2} \int_0^T \int_{S_\delta} |\partial_t u|^2 d\Gamma\, dt + \frac{1}{2} \int_0^T \int_{\Omega_\delta} h_{k,k} |\partial_t u|^2 dx\, dt. \end{aligned}\tag{2.72}$$

With

$$\tilde{\sigma}_{ij} := \sigma_{ij} + m_{ij}\theta \tag{2.73}$$

we obtain

$$\begin{aligned} - \int_0^T \int_{\Omega_\delta} \sigma_{ij,j} h_k u_{i,k} dx\, dt &= - \int_0^T \int_{\Omega_\delta} \tilde{\sigma}_{ij,j} h_k u_{i,k} dx\, dt - \int_0^T \int_{\Omega_\delta} (m_{ij}\theta)_{,j} h_k u_{i,k} dx\, dt \\ &= - \int_0^T \int_{S_\delta} \nu_j \tilde{\sigma}_{ij} \nu_k u_{i,k} d\Gamma\, dt + \int_0^T \int_{\Omega_\delta} \tilde{\sigma}_{ij} h_{k,j} u_{i,k} dx\, dt \\ &\quad + \int_0^T \int_{\Omega_\delta} \tilde{\sigma}_{ij} h_k (u_{i,k})_{,j} dx\, dt - \int_0^T \int_{\Omega_\delta} (m_{ij}\theta)_{,j} h_k u_{i,k} dx\, dt. \end{aligned}\tag{2.74}$$

Since

$$\begin{aligned} \tilde{\sigma}_{ij} h_k (u_{i,k})_{,j} &= \tilde{\sigma}_{ij} h_k (u_{i,j})_{,k} = C_{rsij} u_{i,j} h_k (u_{r,s})_{,k} \\ &= (C_{ijrs} u_{r,s})_{,k} h_k u_{i,j} - (C_{ijrs})_{,k} u_{r,s} h_k u_{i,j} \\ &= (\tilde{\sigma}_{ij})_{,k} h_k u_{i,j} - (C_{ijrs})_{,k} u_{r,s} h_k u_{i,j}, \end{aligned}$$

we have

$$\tilde{\sigma}_{ij}h_k(u_{i,k})_{,j} = \frac{1}{2}h_k(\tilde{\sigma}_{ij}u_{i,j})_{,k} - \frac{1}{2}(C_{ijrs})_{,k}u_{r,s}h_k u_{i,j}.$$

Inserting this into (2.74), we conclude

$$\begin{aligned}
-\int_0^T\int_{\Omega_\delta}\sigma_{ij,j}h_k u_{i,k}dx\,dt &= -\int_0^T\int_{S_\delta}\nu_j\tilde{\sigma}_{ij}\nu_k u_{i,k}d\Gamma\,dt + \int_0^T\int_{\Omega_\delta}\tilde{\sigma}_{ij}h_{k,j}u_{i,k}dx\,dt \\
&\quad -\frac{1}{2}\int_0^T\int_{\Omega_\delta}\tilde{\sigma}_{ij}u_{i,j}h_{k,k}dx\,dt + \frac{1}{2}\int_0^T\int_{S_\delta}\tilde{\sigma}_{ij}u_{i,j}d\Gamma\,dt \\
&\quad -\frac{1}{2}\int_0^T\int_{\Omega_\delta}(C_{ijrs})_{,k}u_{r,s}h_k u_{i,j} - \int_0^T\int_{\Omega_\delta}(m_{ij}\theta)_{,j}h_k u_{i,k}dx\,dt. \quad (2.75)
\end{aligned}$$

The equations (2.6),(2.72),(2.75) imply

$$\begin{aligned}
\int_0^T\int_{S_\delta}|u_t|^2 d\Gamma\,dt - \int_0^T\int_{S_\delta}C_{ijkl}u_{k,l}u_{i,j}d\Gamma\,dt &= 2\int_{\Omega_\delta}\{\partial_t u_i(T)h_k u_{i,k}(T) - u_{1i}h_k u_{0i,k}\}\,dx \\
&\quad +2\int_0^T\int_{\Omega_\delta}\Big\{h_{k,k}|u_t|^2 + \tilde{\sigma}_{ij}h_{k,j}u_{i,j} \\
&\quad -\frac{1}{2}(C_{ijrs})_{,k}u_{r,s}h_k u_{i,j} - (m_{ij}\theta)_{,j}h_k u_{i,k}\Big\}\,dx\,dt. \\
&\quad -2\int_0^T\int_{\Omega_\delta}\nu_j\tilde{\sigma}_{ij}\nu_k u_{i,k}d\Gamma\,dt \\
&= C_\delta^1 \underbrace{-2\int_0^T\int_{\Omega_\delta}\nu_j\tilde{\sigma}_{ij}\nu_k u_{i,k}d\Gamma\,dt}_{=:R_\delta}, \quad (2.76)
\end{aligned}$$

for some C_δ^1 which satisfies

$$\exists M = M(\|u_0\|_{H^1}, \|u_1\|_{L^2}, \|\theta_0\|_{L^2}) \quad \forall 0<\delta\le\delta_0 : |C_\delta^1| \le M \quad (2.77)$$

according to (2.55)–(2.57). In particular, M is independent of ε.
In order to get control of the boundary integral R_δ, we multiply (2.6) by $h_m\tilde{\sigma}_{im}$ and obtain (cp. Lemma 2.3 in [7])

$$\begin{aligned}
\int_{\Omega_\delta}\partial_t^2 u_i h_m\tilde{\sigma}_{im}dx &= \frac{d}{dt}\int_{\Omega_\delta}\partial_t u_i h_m\tilde{\sigma}_{im}dx - \int_{\Omega_\delta}h_m C_{imps}\partial_t u_i\partial_s\partial_t u_p dx \\
&= \frac{d}{dt}\int_{\Omega_\delta}\partial_t u_i h_m\tilde{\sigma}_{im}dx - \frac{1}{2}\int_{\Omega_\delta}h_m C_{imps}\partial_s(\partial_t u_i\partial_t u_p)dx \\
&\quad +\frac{1}{2}\int_{\Omega_\delta}h_m C_{imps}\Big[\partial_s\partial_t u_i\partial_t u_p - \partial_t u_i\partial_s\partial_t u_p\Big]dx. \quad (2.78)
\end{aligned}$$

Now we shall use — for the first time — the radial symmetry, i.e.

$$u_i(t,x) = x_i w(t,r), \quad r = |x|,$$

for some function w, in particular,

$$\operatorname{rot} u = 0,$$

which implies

$$\begin{aligned}
\left[\partial_s\partial_t u_i \partial_t u_p - \partial_t u_i \partial_s \partial_t u_p\right] &= (\partial_t u_p \partial_i - \partial_t u_i \partial_p)\,\partial_t u_s \\
&= \partial_t w\,(x_p\partial_i - x_i\partial_p)\,x_s\partial_t w \\
&\equiv G, \qquad (2.79)
\end{aligned}$$

and G contains at most first-order derivatives (w_t) of u because

$$(x_p\partial_i - x_i\partial_p)\,\partial_t w = \left(\frac{x_p x_i}{r} - \frac{x_i x_p}{r}\right)(\partial/\partial_r)\partial_t w = 0.$$

Combining (2.78) and (2.79) we get

$$\begin{aligned}
\int\limits_{\Omega_\delta} \partial_t^2 u_i h_m \tilde{\sigma}_{im}\,dx &= \frac{d}{dt}\int\limits_{\Omega_\delta} \partial_t u_i h_m \tilde{\sigma}_{im}\,dx - \frac{1}{2}\int\limits_{S_\delta} C_{imps}\nu_m \partial_t u_i \nu_s \partial_t u_p\,d\Gamma \\
&\quad + \frac{1}{2}\int\limits_{\Omega_\delta} (h_m C_{imps})_{,s}\partial_t u_i \partial_t u_p\,dx + \frac{1}{2}\int\limits_{\Omega_\delta} h_m C_{imps} G\,dx. \qquad (2.80)
\end{aligned}$$

Moreover,

$$\begin{aligned}
-\int\limits_{\Omega_\delta} \tilde{\sigma}_{ij,j} h_m \tilde{\sigma}_{im}\,dx &= -\int\limits_{S_\delta} \nu_j \tilde{\sigma}_{ij}\nu_m\tilde{\sigma}_{im}\,d\Gamma + \int\limits_{\Omega_\delta} \tilde{\sigma}_{ij} h_{m,j}\tilde{\sigma}_{im}\,dx + \int\limits_{\Omega_\delta} \tilde{\sigma}_{ij} h_m \tilde{\sigma}_{im,j}\,dx \\
&= -\int\limits_{S_\delta} |\tilde{\sigma}\cdot\nu|^2 d\Gamma + \int\limits_{\Omega_\delta} \tilde{\sigma}_{ij} h_{m,j}\tilde{\sigma}_{im}\,dx + \int\limits_{\Omega_\delta} \tilde{\sigma}_{ij} h_j \tilde{\sigma}_{im,m}\,dx \\
&\quad + \int\limits_{\Omega_\delta} \tilde{\sigma}_{ij}\left\{h_m\partial_j - h_j\partial_m\right\}\tilde{\sigma}_{im,m}\,dx.
\end{aligned}$$

Observing that for the radial symmetrical case

$$h_k = x_k/r,$$

the last integral vanishes, and we obtain

$$-\int\limits_{\Omega_\delta} \tilde{\sigma}_{ij,j} h_m \tilde{\sigma}_{im}\,dx = -\frac{1}{2}\int\limits_{S_\delta} |\tilde{\sigma}\cdot\nu|^2 d\Gamma + \frac{1}{2}\int\limits_{\Omega_\delta} \tilde{\sigma}_{ij} h_{m,j}\tilde{\sigma}_{im}\,dx. \qquad (2.81)$$

Combining (2.6),(2.80),(2.81), and integrating with respect to t yields

$$\int_0^T\int\limits_{S_\delta} |\tilde{\sigma}\cdot\nu|^2 d\Gamma\,dt + \int_0^T\int\limits_{S_\delta} \underbrace{C_{imps}\nu_m\partial_t u_i \nu_s \partial_t u_p}_{\geq 0 \quad \text{by (1.7)}}\,d\Gamma\,dt = C_\delta^2, \qquad (2.82)$$

where C_δ^2 satisfies the same estimate as C_δ^1 in (2.77).
The boundary integral R_δ can now be estimated by

$$|R_\delta| \leq \beta^{-1}\int_0^T\int_{S_\delta}|\tilde\sigma\cdot\nu|^2 d\Gamma\, dt + \beta\int_0^T\int_{S_\delta}|\nabla u|^2 d\Gamma\, dt$$
$$\leq \beta^{-1}|C_\delta^2| + \beta\int_0^T\int_{S_\delta}|\nabla u|^2 d\Gamma\, dt, \tag{2.83}$$

where $\beta > 0$ is still arbitrary.
For $\delta > 0$ let $\varrho_\delta \in C^\infty(\mathbb{R})$, ϱ_δ increasing, $\varrho_\delta(s) = 0$ if $s \leq \delta/2$, $\varrho_\delta(s) = 1$ if $s \geq \delta$, and let

$$\psi_\delta(x) := \varrho_\delta(\mathrm{dist}\,(x, \partial\Omega)).$$

Then ψ_δ is constant on S_δ for each δ. Integrating (2.76) with respect to $\delta \in [0, \delta^*]$, $0 < \delta^* \leq \delta_0/2$, after multiplication with $(1 - \psi_\delta(x))$, and using (2.83), we obtain

$$\left|\int_0^T\int_{U_{\delta^*}}(1-\psi_\delta)\left(|u_t^\varepsilon|^2 - C_{ijkl}u_{k,l}^\varepsilon u_{i,j}^\varepsilon\right)dx\, dt\right| \leq \delta^* M + \beta^{-1}\delta^* M + \beta M,$$

where

$$U_{\delta^*} := \{x \in \Omega |\, \mathrm{dist}\,(x, \partial\Omega) < \delta_*\} = \Omega \setminus \overline{\Omega}_{\delta^*}.$$

As a consequence, we have for any $\tau > 0$ the existence of a $\bar\delta = \bar\delta(\tau)$ (independent of ε) such that

$$\left|\int_0^T\int_{U_{\delta^*}}(1-\psi_\delta)\left(|u_t^\varepsilon|^2 - C_{ijkl}u_{k,l}^\varepsilon u_{i,j}^\varepsilon\right)dx\, dt\right| \leq \tau, \tag{2.84}$$

if $\delta^* \leq \bar\delta(\tau)$. (Choose $\beta = \beta(\tau) := \tau/(3M)$, $\bar\delta := \tau/(3M)\min\{1, \beta\} = \tau/(3M)\min\{1, \tau/(3M)\}$.)
Writing $1 = \psi_\delta + (1 - \psi_\delta)$ we conclude by (2.69) and (2.84) the claimed inequality (2.68).
Now, the desired inequality (2.4) follows as $\varepsilon \downarrow 0$ from (2.66), using (2.60)–(2.63) and (2.68).

Q.E.D.

Finally, we shall prove the exponential decay for the radially symmetrical, homogeneous case.
We consider the system (2.48), (2.49), now defined over the set:

$$\Omega := B(r_1, 0) \setminus \overline{B(r_0, 0)},$$

where we are denoting by $B(r, 0)$ the ball of radius r with center in 0, i.e. Ω is an annular region.
First we treat the penalized problem, and we assume the following boundary conditions:

$$u = 0, \quad \frac{\partial\theta}{\partial\nu} = 0, \quad \text{on} \quad \partial B(r_0, 0) = \Gamma_D, \tag{2.85}$$

and the contact conditions on $\partial B(r_1, 0) = \Gamma_c$ which in our case can be written as:

$$2\mu\frac{\partial u}{\partial\nu}\cdot\nu + \lambda \mathrm{div} u = -\frac{1}{\varepsilon}(u_\nu - g)^+ \quad \text{on} \quad \Gamma_c, \tag{2.86}$$
$$\theta = 0 \quad \text{on} \quad \Gamma_c. \tag{2.87}$$

Observe that the boundary condition for θ in (2.85) is different from the Dirichlet boundary condition studied before, but the corresponding existence theorems, both for the penalized and for the original contact problem, hold as well, the proof carries over almost literally, noticing, for example, that the Dirichlet boundary condition is still given on Γ_c.
For simplicity in our notations we will drop the upper index ε. Since u, θ are radially symmetrical solutions we have

$$\begin{aligned} u_i(t,x) &= x_i w(t,r), \quad r = |x|, \\ \theta(t,x) &= \psi(t,r), \end{aligned}$$

for some w and ψ. Then the system (2.48), (2.49) is equivalent to

$$w_{tt} - (2\mu+\lambda)w_{rr} - (2\mu+\lambda)\frac{n+1}{r}w_r + \frac{\bar{m}}{r}\theta_r = 0, \tag{2.88}$$

$$c\theta_t - \lambda_2\theta_{rr} - \lambda_2(n-1)\frac{\theta_r}{r} + \bar{m}nw_t + \bar{m}rw_{rt} = 0, \tag{2.89}$$

where w and θ satisfy

$$w(r_0) = 0, \quad \theta_r(r_0) = 0,$$

among other boundary conditions. Note that there exist positive constants C_0, C_1 (which do not depend on ε) satisfying

$$C_0\int_\Omega |\nabla u|^2\,dx \le \int_{r_0}^1 |w_r|^2\,dr \le C_1\int_\Omega |\nabla u|^2\,dx,$$

$$C_0\int_\Omega |u_t|^2\,dx \le \int_{r_0}^1 |w_t|^2\,dr \le C_1\int_\Omega |u_t|^2\,dx.$$

To show the exponential decay we shall exploit the following two Lemmas.

Lemma 2.5 *Let q be any $C^2([r_0,1])$-function such that $q(r_0) = 0$ and $q(1) = 1$, let $\alpha > 0$ and let f be a function in $W^{1,2}(L^2)$. Then for any solution $\varphi \in L^2(H^1)\cap W^{1,2}(H^1)\cap W^{2,2}(L^2)$ of*

$$\varphi_{tt} - \alpha\varphi_{rr} = f, \tag{2.90}$$

we have that

$$\begin{aligned} -\frac{d}{dt}\left\{\int_{r_0}^1 q(x)\varphi_t\varphi_r\,dr\right\} &= -\frac{1}{2}\left\{|\varphi_t(1,t)|^2 + \alpha|\varphi_r(1,t)|^2\right\} \\ &\quad +\frac{1}{2}\int_{r_0}^1 q'(r)\left\{|\varphi_t|^2 + \alpha|\varphi_r|^2\right\}dr - \int_{r_0}^1 q(r)\varphi_r f\,dr. \end{aligned}$$

Proof: To get the above equality, multiply equation (2.90) by $q\varphi_r$ and make an integration by parts.

Q.E.D.

Let us introduce the function ψ by

$$\psi(t,r) := \int_{r_0}^r \theta(t,\rho)\,d\rho.$$

Then ψ satisfies the following equation

$$c\psi_t - \lambda_2\psi_{rr} - \lambda_2(n-1)\int_{r_0}^r \frac{\theta_r}{r}\,d\rho + \bar{m}n\int_{r_0}^r w_t\,d\rho - \bar{m}\int_{r_0}^r w_t\,d\rho + \bar{m}rw_t = 0. \tag{2.91}$$

With these notations we have

Lemma 2.6 *Let q be any $C^2([r_0,1])$-function such that $q(r_0)=0$ and $q(1)=1$. Then, for $\delta>0$ small enough, we have*

$$\begin{aligned}\frac{d}{dt}\left\{\int_{r_0}^1 \psi w_t\,dr-\delta\int_{r_0}^1 q(r)w_tw_r\,dr\right\} \;\le\; & -\frac{\bar m}{4}\int_{r_0}^1|w_t|^2\,dr\\ & -\frac{\delta}{4}\left\{|w_t(1,t)|^2+(2\mu+\lambda)|w_r(1,t)|^2\right\}\\ & +\delta\int_{r_0}^1|w_r|^2\,dr+C\int_{r_0}^1|\theta_x|^2\,dr.\end{aligned}$$

PROOF: Multiplying equation (2.91) by w_t we get

$$\begin{aligned}\frac{d}{dt}\int_{r_0}^1 c\psi w_t\,dr &= \int_{r_0}^1 c\psi_t w_t\,dr+\int_{r_0}^1 c\psi w_{tt}\,dr\\ &= \lambda_2\int_{r_0}^1\psi_{rr}w_t\,dr+(n-1)\lambda_2\int_{r_0}^1\left(\int_{r_0}^r\frac{\theta_r}{r}\,d\rho\right)w_t\,dr\\ &\quad -\bar m(n-1)\int_{r_0}^1\left(\int_{r_0}^r w_t\,d\rho\right)w_t\,dr-\bar m\int_{r_0}^1 r|w_t|^2\,dr+\int_{r_0}^1 c\psi w_{tt}\,dr\\ &= \lambda_2\int_{r_0}^1\theta_r w_t\,dr+(n-1)\lambda_2\int_{r_0}^1\int_{r_0}^r\frac{\theta_r}{r}\,d\rho w_t\,dr\\ &\quad -\frac{\bar m(n-1)}{2}\left(\int_{r_0}^1 w_t\,d\rho\right)^2-\bar m\int_{r_0}^1 r|w_t|^2\,dr-(2\mu+\lambda)c\int_{r_0}^1\theta w_r\,dr\\ &\quad +(2\mu+\lambda)(n+1)\int_{r_0}^1\psi\frac{w_r}{r}\,d\rho-\bar m c\int_{r_0}^1\psi\frac{\theta_r}{r}\,dr+c(2\mu+\lambda)\psi(1,t)w_r(1,t).\end{aligned}$$

Using Sobolev's embedding theorem, we conclude from the above identity

$$\frac{d}{dt}\int_{r_0}^1\psi w_t\,dr\le-\frac{\bar m}{2}\int_{r_0}^1 r|w_t|^2\,d\rho+\delta\int_{r_0}^1|w_r|^2\,dr+C_\delta\int_{r_0}^1|\theta_r|^2\,dr+\frac{\delta}{4}(2\mu+\lambda)|w_r(1,t)|^2.$$

Since $w\in L^2(H^2)\cap W^{1,2}(H^1)$, $\theta\in W^{1,2}(H^1)$ we can use Lemma 2.5 for $\varphi=w$, $\alpha=2\mu+\lambda$ and $f=-(2\mu+\lambda)n\frac{w_r}{r}+\bar m\frac{\theta_r}{r}\in W^{1,2}(L^2)$, which proves our conclusion.

Q.E.D.

Now we are able to prove the exponential decay for the penalized problem associated to the dynamical contact case.

Theorem 2.7 *Under the above conditions, and with u_0 such that $u_0\cdot\nu\le g$, the energy $\mathcal{E}^\varepsilon=\mathcal{E}^\varepsilon(t)$ to the penalized system (2.48), (2.49) defined by*

$$\mathcal{E}^\varepsilon(t):=\frac{1}{2}\int_\Omega|u_t^\varepsilon|^2+C_{ijkl}u_{k,l}^\varepsilon u_{i,j}^\varepsilon+|\theta^\varepsilon|^2\,dx+\frac{1}{2\varepsilon}\int_{\Gamma_c}|(u_\nu^\varepsilon-g)^+|^2\,d\Gamma_c$$

decays exponentially as time goes to infinity, i.e.

$$\exists\, C>0,\ \gamma>0,\ \forall\, t\ge 0:\mathcal{E}^\varepsilon(t)\le C\,\mathrm{e}^{-\gamma t},\tag{2.92}$$

where γ and C are independent of ε.

PROOF: Multiplying as usual equation (2.48) by u_t and equation (2.49) by θ we get, dropping ε for simplicity,

$$\frac{d}{dt}\mathcal{E}(t) = -\int_\Omega |\nabla\theta|^2\, dx.$$

Multiplying equation (2.48) by u and integrating by parts over Ω we get

$$\begin{aligned}\frac{d}{dt}\left\{\int_\Omega uu_t\, dx\right\} &= \int_\Omega |u_t|^2\, dx - \int_\Omega \mu|\nabla u|^2 + (\mu+\lambda)|\mathrm{div} u|^2\, dx \\ &+\bar{m}\int_\Omega \nabla\theta u\, dx - \frac{1}{2\varepsilon}\int_{\Gamma_c} |(u_\nu - g)^+|^2\, d\Gamma_c,\end{aligned}$$

from where it follows that

$$\begin{aligned}\frac{d}{dt}\left\{\int_\Omega uu_t\, dx\right\} &\leq \int_\Omega |u_t|^2\, dx - \frac{1}{2}\int_\Omega C_{ijkl}u_{k,l}u_{i,j}\, dx \\ &-\frac{1}{2\varepsilon}\int_{\Gamma_c} |(u_\nu - g)^+|^2\, d\Gamma_c + C_1\int_\Omega |\nabla\theta|^2\, dx,\end{aligned}$$

where C_1, and C_2, C_3 below, are positive constants. Using Lemma 2.6 and choosing η and δ small enough we get

$$\frac{d}{dt}\left\{\int_{r_0}^1 \psi w_t\, dr - \delta\int_{r_0}^1 q(x)w_t w_r\, dr + \eta\int_\Omega uu_t\, dx\right\} \leq$$

$$-\kappa_0\left\{\int_\Omega |u_t|^2 + |\nabla u|^2 + \frac{1}{2\varepsilon}\int_{\Gamma_c} |(u_\nu - g)^+|^2\, d\Gamma_c\right\} + C\int_\Omega |\nabla\theta|^2\, dx.$$

So taking N large enough we obtain

$$\frac{d}{dt}\left\{N\mathcal{E}(t) + \int_{r_0}^1 \psi w_t\, dr - \delta\int_{r_0}^1 q(x)w_t w_r\, dr + \eta\int_\Omega uu_t\, dx\right\} \leq$$

$$-\kappa_0\left\{\int_\Omega |u_t|^2 + |\nabla u|^2 + |\nabla\theta|^2\, dx + \frac{1}{2\varepsilon}\int_{\Gamma_c} |(u_\nu - g)^+|^2\, d\Gamma_c\right\}$$

for some $\kappa_0 > 0$ being independent of ε. Since we also have the relation

$$\left|\int_{r_0}^1 \psi w_t\, dr - \delta\int_{r_0}^1 q(x)w_t w_r\, dr + \eta\int_\Omega uu_t\, dx\right| \leq C_2\mathcal{E}(t),$$

we conclude, taking N large enough, that

$$\mathcal{L}(t) := N\mathcal{E}(t) + \int_{r_0}^1 \psi w_t\, dr - \delta\int_{r_0}^1 q(x)w_t w_r\, dr + \eta\int_\Omega uu_t\, dx$$

satisfies

$$\frac{d}{dt}\mathcal{L}(t) \leq -\gamma\mathcal{L}(t)$$

for some positive constant γ which is independent of ε. So we conclude

$$\mathcal{L}(t) \leq \mathcal{L}(0)e^{-\gamma t}$$

and hence

$$\mathcal{E}(t) \leq C_3\mathcal{E}(0)e^{-\gamma t}.$$

By our choice of the initial data, the right hand side of the above inequality is bounded, which proves the Theorem.

Q.E.D.

Using the lower semicontinuity of norms, when $\varepsilon \to 0$ we finally obtain that the energy associated to the original contact problem also decays exponentially, that is we have proved

Theorem 2.8 *The energy $\mathcal{E} = \mathcal{E}(t)$ to the system (1.1), (1.2), (1.3), (1.5), (2.85), (2.86),*

$$\mathcal{E}(t) := \frac{1}{2}\int_\Omega |u_t|^2 + C_{ijkl}u_{k,l}u_{i,j} + |\theta|^2\, dx$$

decays exponentially as time goes to infinity, i.e.

$$\exists\, C > 0,\ \gamma > 0,\ \forall\, t \geq 0 : \mathcal{E}(t) \leq C\mathcal{E}(0)\, \mathrm{e}^{-\gamma t}. \tag{2.93}$$

3 Linear quasistatic thermoelasticity

The linear quasistatic problem arises from the full dynamical problem (1.1)–(1.5) by neglecting the acceleration term $\rho\partial_t^2 u_i$ in (1.1), and, consequently, prescribing only θ_0 in (1.3), and replacing the boundary conditions for u by Dirichlet type boundary conditions (see the remarks below for other linear boundary conditions). That is, we consider the following linear elliptic-parabolic initial boundary value problem:

$$-\,(C_{ijkl}u_{k,l})_{,j} + (m_{ij}\theta)_{,j} \;=\; 0, \quad i = 1, \ldots, n, \tag{3.1}$$

$$\partial_t\theta - (k_{ij}\theta_{,i})_{,j} + m_{ij}\partial_t u_{i,j} \;=\; 0, \tag{3.2}$$

$$\theta(t=0) \;=\; \theta_0, \tag{3.3}$$

$$u_{/\partial\Omega} = 0, \qquad \theta_{/\partial\Omega} = 0. \tag{3.4}$$

According to the paper of Shi & Xu [14], the known results on well-posedness are restricted to one-dimensional situations, see Day [4], or to homogeneous and isotropic media for Ω being equal to the unit disk on $\mathbb{R}^2$; see [14], where methods from the theory of complex functions are basic ingredients of the proofs.

Here, we discuss the general case of a bounded C^2-reference configuration Ω in any space dimension, allowing the medium to be anisotropic and non-homogeneous (see below for weakening the C^2-assumption on the boundary). We consider Dirichlet boundary conditions for the displacement vector and for the temperature, but it will be pointed out at the end of this section how other boundary conditions can be dealt with. In passing, we note that the questions in [14] concerning well-posedness are answered. The analyticity of solutions is not touched in the general situation.

The well-posedness of the problem will be obtained, i.e. the unique existence of a solution and the continuous dependence on the data is proved. Moreover, the exponential decay to equilibrium is obtained as a by-product.

All coefficients are assumed to be smooth functions, e.g. $C^1(\bar{\Omega})$ should be an appropriate regularity class for Theorem 3.1.(i)). In the homogeneous case all coefficients would be independent of x, in the specific homogeneous and isotropic case the equations (3.1), (3.2) reduce to

$$- (\lambda + \mu)\nabla \mathrm{div}\, u - \mu \Delta u + \bar{m} \nabla \theta \;=\; 0, \tag{3.5}$$

$$\partial_t \theta - \lambda_2 \Delta \theta + \bar{m}\, \mathrm{div}\, \partial_t u \;=\; 0. \tag{3.6}$$

Here λ and μ are the Lamé constants, $\bar{m} > 0$ is the coupling (interaction) constant and λ_2 is a positive constant representing the heat conductivity constant. The equations (3.5), (3.6) are essentially those considered for the unit disk in $\mathbb{R}^2$ in [14].
We look for solutions

$$u \in C^0([0,\infty), (H^3(\Omega) \cap H_0^1(\Omega))^3), \tag{3.7}$$

$$\theta \in C^1([0,\infty), L^2(\Omega)) \cap C^0([0,\infty), H^2(\Omega) \cap H_0^1(\Omega)) \tag{3.8}$$

to the elliptic-parabolic problem (3.1)–(3.4). We will obtain such solutions via the natural decoupling arising from (3.1),(3.2).

Theorem 3.1 *(i) Let $\theta_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. Then there exists a unique solution (u, θ) of (3.1)–(3.4) with*

$$u \in C^0([0,\infty), (H^3(\Omega) \cap H_0^1(\Omega))^3),$$
$$\theta \in C^1([0,\infty), L^2(\Omega)) \cap C^0([0,\infty), H^2(\Omega) \cap H_0^1(\Omega)).$$

θ and u decay to zero exponentially as $t \to \infty$, i.e.

$$\exists d_1, d_2, d_3 > 0 \;\; \forall t \geq 0 : \qquad \|\theta(t,\cdot)\| \leq d_1 e^{-d_2 t} \|\theta_0\|,$$
$$\|u(t)\|_{H^1} \leq d_3 e^{-d_2 t} \|\theta_0\|.$$

(ii) If $\bar{m} := \|(m_{ij})_{ij}\|_{L^\infty}$ is sufficiently small, and if $\partial\Omega$ is smooth, and additionally $\theta_0 \in H^{2(m-1)}(\Omega)$, $m \geq 2$, satisfies the usual compatibility conditions, then

$$u \in \bigcap_{j=0}^{m} C^j([0,\infty), (H^{2(m-j)+1}(\Omega))^3),$$

$$\theta \in \bigcap_{j=0}^{m} C^j([0,\infty), H^{2(m-j)}(\Omega)).$$

The higher derivatives decay to zero exponentially, i.e.

$$\exists d_4, d_5 > 0 \;\; \forall t \geq 0 : \qquad \|\theta(t,\cdot)\|_{H^{2(m-1)}} \leq d_4 e^{-d_2 t} \|\theta_0\|_{H^{2(m-1)}},$$
$$\|u(t)\|_{H^{2m-1}} \leq d_5 e^{-d_2 t} \|\theta_0\|_{H^{2(m-1)}}.$$

We distinguish between d_1 and d_3 because the dependence on the coefficients is different, see the remarks below. We note that *a posteriori* the acceleration u_{tt} goes to zero as $t \to \infty$ and hence remains small for all times.

Proof of Theorem 3.1:

The natural decoupling arising from (3.1), (3.2) is obtained using the elasticity operator

$$E = -\left((\partial_j C_{ijkl} \partial_l\right)_{ik},$$

formally first, which is well-defined by

$$E : D(E) \subset (L^2(\Omega))^n \to (L^2(\Omega))^n,$$

$$\begin{aligned} D(E) &:= \{u \in (H_0^1))^n | \, Eu \in (L^2(\Omega))^n\} \\ &= (H^2(\Omega) \cap H_0^1(\Omega))^n. \end{aligned}$$

The latter equality follows from elliptic regularity theory. E is an elliptic, self-adjoint, positive operator, cf. [10], and

$$E^{-1} : (L^2(\Omega))^n \to D(E)$$

is continuous as well as

$$E^{-1} : H^{-1}(\Omega) \to (H_0^1(\Omega))^n,$$

where $H^{-1}(\Omega)$ is the dual space to $(H_0^1(\Omega))^n$.
These mapping properties are known, see [10] for the former; the latter follows from the representation theorem of Lax&Milgram for coercive forms on $(H_0^1(\Omega))^n$).

The system (3.1), (3.2) is then equivalent to

$$u = -E^{-1}(\nabla' M'\theta), \tag{3.9}$$

$$\partial_t\theta - (k_{ij}\theta_{,i})_{,j} - M\nabla E^{-1}\nabla' M'\partial_t\theta = 0, \tag{3.10}$$

where

$$M = (m_{ij})_{ij}.$$

Remark: This natural ansatz is also discussed in [1], where an elliptic-parabolic problem for a *scalar* u is studied in a general framework for nonlinear equations. The corresponding operator E is the Laplace operator $-\Delta$, the solutions live in L^p-spaces, with $p > 2$.

An appropriate solution θ of the decoupled equation (3.10), together with initial and boundary conditions given in (3.3), (3.4), will solve the problem.

Theorem 3.2 *Let $\theta_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. Then there exists a unique solution*

$$\theta \in C^1(([0,\infty), L^2(\Omega)) \cap C^0([0,\infty), H^2(\Omega) \cap H_0^1(\Omega))$$

to (3.10), (3.3), (3.4). θ decays exponentially to zero as $t \to \infty$, i.e.

$$\exists d_1, d_2 > 0 \;\; \forall t \geq 0 : \|\theta(t,\cdot)\| \leq d_1 e^{-d_2 t}\|\theta_0\|.$$

If $\bar{m}$ $(= \|(m_{ij})_{ij}|_{L^\infty})$ is small with respect to the norm of E^{-1}, and if $\partial\Omega$ is smooth, and additionally $\theta_0 \in H^{2(m-1)}(\Omega)$, $m \geq 2$, then

$$\theta \in \bigcap_{j=0}^{m} C^j([0,\infty), H^{2(m-j)}(\Omega))$$

and

$$\exists d_4 > 0 \;\; \forall t \geq 0 : \|\theta(t,\cdot)\|_{H^{2(m-1)}} \leq d_4 e^{-d_2 t}\|\theta_0\|_{H^{2(m-1)}}.$$

By (3.9) we obtain the complete unique solution (u,θ) of the original problem in the class described in (3.7), (3.8), yielding the decay of u too, which will prove Theorem 3.1.

PROOF of Theorem 3.2:

The equation (3.10) is equivalent to

$$G(\theta_t) - (k_{ij}\theta_{,i})_{,j} = 0, \tag{3.11}$$

where $G : L^2(\Omega) \to L^2(\Omega)$ is given by

$$Gv := v - M\nabla E^{-1}\nabla' M' v.$$

Since for $v, w \in L^2(\Omega)$ we have

$$\begin{aligned} \langle Gv, w\rangle &= \langle v, w\rangle + \langle\langle E^{-1}\nabla' M' v, \nabla' M' w\rangle\rangle \\ &= \langle v, Gw\rangle, \end{aligned}$$

where $\langle\langle g, h\rangle\rangle$ denotes the dual pairing for $g \in H_0^1(\Omega), h \in H^{-1}(\Omega)$ (= dual space of $H_0^1(\Omega)$ here), we conclude that G is a positive, self-adjoint operator,

$$\langle Gv, v\rangle_{L^2} \geq \|v\|_{L^2}^2. \tag{3.12}$$

Thus $G^{-1} : L^2(\Omega) \to L^2(\Omega)$ is a well-defined continuous operator, and (3.11) is equivalent to

$$\theta_t - G^{-1}(k_{ij}\theta_{,i})_{,j} = 0. \tag{3.13}$$

This evolution equation can be solved uniquely for an initial value given in (3.3) in the following Hilbert space. Let $\mathcal{H} := L^2(\Omega)$ equipped with the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ defined by

$$\langle v, w\rangle_{\mathcal{H}} := \langle Gv, w\rangle.$$

Let $A : D(A) \subset \mathcal{H} \to \mathcal{H}, \quad D(A) := H^2(\Omega) \cap H_0^1(\Omega), \quad Av := -G^{-1}(k_{ij}v_{,i})_{,j}$.
A is densely defined and symmetric, $\langle Av, w\rangle_{\mathcal{H}} = \langle -\nabla' L\nabla v, w\rangle = \langle v, -\nabla' L \nabla w\rangle = \langle v, Aw\rangle_{\mathcal{H}}$, and $w \in D(A^*)$, A^* denoting the adjoint operator, implies

$$\exists f \in \mathcal{H} \ \forall v \in D(A) : \langle -(k_{ij}v_{,i})_{,j}, w\rangle = \langle v, Gf\rangle,$$

hence, by the self-adjointness of the realization of $\partial_j k_{ij} \partial_i$ on $D(A)$ in $L^2(\Omega)$, we conclude $w \in D(A)$. Therefore A is self-adjoint, and our problem reads as

$$\theta_t + A\theta = 0, \tag{3.14}$$

with initial condition given by (3.3). We apply the spectral theorem for self-adjoint operators (or semigroup theory) and obtain a unique solution θ satisfying

$$\theta \in C^1([0,\infty), \mathcal{H}) \cap C^0([0,\infty), D(A)),$$

provided $\theta_0 \in \ D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. This proves the unique existence of a solution (u,θ) in the desired class described in (i).
Since

$$\langle Av, v\rangle_{\mathcal{H}} = \langle k_{ij}v_{,i}, v_{,j}\rangle \geq \lambda_2 \|\nabla v\|_{L^2}^2 \geq l_1 c_p \|v\|^2 \geq \lambda_2 c_p / k \|v\|_{\mathcal{H}}^2, \tag{3.15}$$

where c_p ist the Poincaré constant and k is the norm of G in $L^2(\Omega)$, which implies that A is positive. The inverse A^{-1} is compact, because $A^{-1} = -(\partial_j k_{ij} \partial_i)^{-1} G$ with compact $(\partial_j k_{ij} \partial_i)^{-1}$ and continuous G. Therefore, the spectrum of A is its discrete spectrum, a subset of $(0, \infty)$ with only accumulation point ∞ and smallest eigenvalue denoted by d_2. The spectral theorem for self-adjoint operators implies

$$\theta(t) = \int_{d_2}^{\infty} e^{-\lambda t} dP_\lambda \theta_0,$$

$(P_\lambda)_{\lambda \in \mathbb{R}}$ being the spectral resolution for A.
If $d_1 = d_1(E^{-1}, (m_{ij}))$ denotes the square root of the norm of G in $L^2(\Omega)$, we conclude

$$\|\theta(t, \cdot)\| \leq \|\theta(t, \cdot)\|_{\mathcal{H}} \leq e^{-d_2 t} \|\theta_0\|_{\mathcal{H}} \leq d_1 e^{-d_2 t} \|\theta_0\|.$$

This completes the proof of Theorem 3.2.(i), and thus of Theorem 3.1.(i) too.
The proof of (ii) immediately follows from the following Lemma 3.3 by differentiating the equation for θ with respect to t. Q.E.D.

Lemma 3.3 *Let $s \in \mathbb{N}_0$.*
(i) $G : H^s(\Omega) \longrightarrow H^s(\Omega)$ is continuous.
(ii) A has the usual elliptic regularity property, i.e. $v \in H^2(\Omega) \cap H_0^1(\Omega)$ and $Av \in H^s(\Omega)$ imply $v \in H^{s+2}(\Omega)$ and $\|v\|_{H^{s+2}} \leq c_s \|Av\|_{H^s}$.
(iii) If $\bar{m}$ $(= \|(m_{ij})_{ij}\|_{L^\infty})$ is small with respect to the norm of E^{-1}, then G is a homeomorphism of $H^s(\Omega)$.

Proof: (i): Using the boundedness of $E^{-1} : H^{s-1}(\Omega) \longrightarrow H^{s+1}(\Omega)$, cf. [8], the assertion is obvious.
(ii) is a consequence of (i) and the elliptic regularity of $\partial_j k_{ij} \partial_i$.
(iii): $G = (Id + B)$ with the bounded operator $B := M \nabla E^{-1} \nabla' M'$, and the inverse of G is given by a Neumann series if $\|B\|_{H^s \to H^s} < 1$; this is fulfilled, if $\bar{m}^2 < c_s(E^{-1})$, where $c_s(E^{-1})$ is essentially the norm of $E^{-1} : H^{s-1}(\Omega) \longrightarrow H^{s+1}(\Omega)$. Q.E.D.

Remarks:

1. The smallness assumption on $\bar{m}$ is satisfied e.g. for homogeneous, isotropic media, cf. [10].

2. A generates an analytic semigroup.

3. In the homogeneous, isotropic case we have $d_1 \leq 1 + \tilde{d}_1(E^{-1}) \bar{m}^2$.

4. A lower bound for d_2 is given by $\lambda_2 c_p / k$ according to (3.15). The constant d_1 depends on $\bar{m}$. The decay rate d_2 for u in

$$\|u(t)\| \leq d_3 e^{-d_2 t} \|\theta_0\|$$

is bounded from below as $\bar{m}$ vanishes, while $d_3 \to \infty$ as $\bar{m}$ vanishes.

5. The C^2-condition on Ω in Theorem 3.1.(i) and Theorem 3.2.(i) can be dropped if we replace

$$(H^2(\Omega) \cap H_0^1(\Omega))^n \quad \text{by} \quad D(E) = \{v \in (H_0^1(\Omega))^n | EV \in (L^2(\Omega))^n)\}$$

and

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega) \quad \text{by} \quad D(-\partial_j k_{ij} \partial_i) := \{v \in H_0^1(\Omega) | (k_{ij} v_i)_j \in L^2(\Omega)\}$$

throughout the text, i.e. in this case the boundary of Ω may be *arbitrary.*

The Dirichlet (u)-Dirichlet (θ) boundary conditions (3.4) have been chosen for simplicity of the presentation, other boundary conditions such as Neumann-Neumann, Neumann-Dirichlet or Dirichlet-Neumann can be dealt with similarly, for example Neumann-Neumann,

$$\sigma_{ij} \nu_{j\,/\partial\Omega} = 0, \ \ \nu_j k_{ij} \theta_{i\,/\partial\Omega} = 0,$$

There are nontrivial solutions in the null space of the corresponding elasticity operator E (cf. [10], [12]), as well as in the null space of the corresponding realization of $\partial_j k_{ij} \partial_i$ (consisting of the constant functions). In order to carry over the arguments from the proof of Theorem 3.1, one concentrates on initial values θ_0 from the orthogonal complement of the constant functions, i.e. on initial values with mean value $\int_\Omega \theta_0$ zero. Observe that $M\nabla\theta$ lies in the orthogonal complement of the null space of E, and that a Poincaré inequality holds. We do not go into further details and end up with the remark that Ω has to satisfy the restricted cone property (cf. [12], [10]).

As the proof of Theorem 3.1 shows, it would be possible to deal with exterior forces and exterior heat supply, i.e. with non-zero right-hand sides f and g in (3.1) and (3.2), respectively.

4 Quasistatic contact problems — existence and stability

The quasistatic contact problem arises from the full dynamical problem (1.1)–(1.5) by neglecting the acceleration term $\varrho\partial_t^2 u_i$ in (1.1), and, consequently, prescribing only θ_0 in (1.3). Therefore, we consider the following problem

$$-(C_{ijkl} u_{k,l})_{,j} + (m_{ij}\theta)_{,j} = 0, \qquad i = 1, \ldots, n, \tag{4.1}$$

$$\partial_t \theta - (k_{ij}\theta_{,i})_{,j} + m_{ij} \partial_t u_{i,j} = 0, \tag{4.2}$$

$$\theta(t=0) = \theta_0, \tag{4.3}$$

$$u_{|\Gamma_D} = 0, \qquad \sigma_{ij}\nu_{j_{|\Gamma_N}} = 0, \qquad \theta_{|\partial\Omega} = 0, \tag{4.4}$$

$$u_\nu \le g, \qquad \sigma_\nu \le 0, \qquad (u_\nu - g)\sigma_\nu = 0, \qquad \sigma_T = 0, \quad \text{on } \Gamma_c. \tag{4.5}$$

Shi & Shillor [13] gave an existence proof, provided

$$\overline{m} := \sup_{x,i,j} |m_{ij}(x)| \quad \text{is small enough.} \tag{4.6}$$

Ames & Payne [2] proved a uniqueness and continuous dependence result.
We shall prove an existence result using a penalty method, which will also allow us to prove the exponential stability. The condition (4.6) will also be required.

Definition 4.1 *(u,θ) is a* <u>solution</u> *to (4.1)–(4.5) for given $\theta_0 \in H^1_0(\Omega)$, if, for any $T>0$,*

$$u \in L^2(H^1_{\Gamma_D}), ((C_{ijkl}u_{k,l})_{,j})_{i=1,\dots,n} \in L^2(L^2), u_t \in L^2(H^1), \tag{4.7}$$

$$\theta \in L^2(H^2 \cap H^1_0) \cap C^0(H^1_0), \theta_t \in L^2(L^2), \tag{4.8}$$

$$\theta(t=0) = \theta_0, \tag{4.9}$$

$$u_\nu \le g \text{ on } (0,T)\times\Gamma_c \text{ (a.e.)}, \tag{4.10}$$

and for all $w \in L^\infty(H^1_{\Gamma_D})$ with $w_\nu \le g$ on Γ_c the following inequality holds,

$$\int_0^T \langle C_{ijkl}u_{k,l}, w_{i,j}\rangle dt - \int_0^T \langle C_{ijkl}u_{k,l}, u_{i,j}\rangle dt$$
$$- \int_0^T \langle m_{ij}\theta, \ w_{i,j} - u_{i,j}\rangle dt \ge 0, \tag{4.11}$$

and for all $z \in L^2(H^1_0)$ the following equality holds,

$$\int_0^T \langle \theta_t, z\rangle + \langle k_{ij}\psi_i, z_{,j}\rangle + \langle m_{ij}\partial_t u_{i,j}, z\rangle dt = 0. \tag{4.12}$$

A solution will be obtained by studying an associated penalized problem for the parameter $\varepsilon > 0$, then proving *a priori* estimates and finally letting ε tend to zero. The advantage will be that finally the exponential stability can be proved. The penalized problem to be solved first is given as follows:

$$-(C_{ijkl}u^\varepsilon_{k,l})_{,j} + (m_{ij}\theta^\varepsilon)_{,j} = 0, \qquad i = 1,\dots,n, \tag{4.13}$$

$$\partial_t\theta^\varepsilon - (k_{ij}\theta^\varepsilon_{,i})_{,j} + m_{ij}\partial_t u^\varepsilon_{i,j} = 0, \tag{4.14}$$

$$\theta^\varepsilon(t=0) = \theta_0 \in H^1_0(\Omega), \tag{4.15}$$

$$u^\varepsilon \in L^2(H^1_{\Gamma_D}), ((C_{ijkl}u^\varepsilon_{k,l})_{,j})_{i=1,\dots,n} \in L^2(L^2), u^\varepsilon_t \in L^2(H^1), \tag{4.16}$$

$$\theta^\varepsilon \in L^2(H^2 \cap H^1_0), \theta^\varepsilon_t \in L^2(L^2), \tag{4.17}$$

$$u^\varepsilon_{|\Gamma_D} = 0, \quad \nu_j C_{ijkl}u^\varepsilon_{k,l_{|\Gamma_N}} = 0, \quad \sigma_{T|\Gamma_c} = 0, \tag{4.18}$$

$$\sigma_\nu(u^\varepsilon)_{|\Gamma_c} = -\frac{1}{\varepsilon}(u^\varepsilon_\nu - g)^+{}_{|\Gamma_c}. \tag{4.19}$$

A solution of this penalized problem, for fixed $\varepsilon > 0$, will be obtained by a fixed point argument. For this purpose let $\Gamma > 0$ and

$$\begin{aligned} W^\Gamma \ := \ & \{u \in L^2(H^1) \mid ((C_{ijkl}u_{k,l})_{,j})_{i=1,\dots,n} \in L^2(L^2), u_t \in L^2(H^1), \\ & \|u_t\|_{L^2(H^1)} \le \Gamma, \ \|((C_{ijkl}u_{k,l})_{,j})_{i=1,\dots,n}\|_{L^2(L^2)} \le \Gamma\}. \end{aligned}$$

Lemma 4.2 *W^Γ is a Banach space with norm $\|u\|_{W^\Gamma} := \|u\|_{L^2(H^1)}$.*

The Proof follows from the completeness of $L^2(H^1)$ with respect to the norm $\|\cdot\|_{W^\Gamma}$ and the weak compactness of balls in $L^2(L^2)$ resp. $L^2(H^1)$.

Q.E.D.

For $v \in W^\Gamma$ we define

$$u = Sv,$$

and hence the map S as follows. Let θ be the solution to the parabolic equation

$$\theta_t - (k_{ij}\theta_{,i})_{,j} = -m_{ij}\partial_t v_{i,j}, \tag{4.20}$$

with initial value

$$\theta(t=0) = \theta_0. \tag{4.21}$$

Then there exists a solution θ with the regularity

$$\theta \in L^2(H^2 \cap H_0^1) \cap C^0(H_0^1), \quad \theta_t \in L^2(L^2). \tag{4.22}$$

Now let u be the solution to the penalized problem

$$-(C_{ijkl}u_{k,l})_{,j} + (m_{ij}\theta)_{,j} = 0, \qquad i = 1, \ldots, n, \tag{4.23}$$

$$u_{|\Gamma_D} = 0, \qquad \sigma_{ij}\nu_{j|\Gamma_N} = 0, \tag{4.24}$$

$$\sigma_\nu(u)_{|\Gamma_c} = -\frac{1}{\varepsilon}(u_{\nu|\Gamma_c} - g)^+, \qquad \sigma_T = 0, \quad \text{on } \Gamma_c. \tag{4.25}$$

A solution u to (4.23)–(4.25) is obtained by minimizing

$$J(u) := \langle C_{ijkl}u_{k,l}, u_{i,j}\rangle - 2\langle (m_{ij}\theta)_{,j}, u_i\rangle + \frac{1}{\varepsilon}\int_{\Gamma_C} |(u_\nu - g)^+|^2 d\Gamma$$

on

$$\{w \in (H^1(\Omega))^n |\ w = 0 \text{ on } \Gamma_D\},$$

where t is regarded as a parameter.

Theorem 4.3 *If $\Gamma = \Gamma(\theta_0)$ is chosen large enough, and if $\bar{m}$ is small enough, where the required smallness is independent of ε, then*

$$S : W^\Gamma \to W^\Gamma$$

and S is a contraction mapping.

Proof: Let $v \in W^\Gamma$ and $u := Sv$. By the definition of u we immediately obtain

$$\|u\|_{L^2(H^1)} \le c_1\bar{m}\|\theta\|_{L^2(L^2)}, \tag{4.26}$$

$$\begin{aligned} \|((C_{ijkl}u_{k,l})_{,j})_{i=1,\ldots,n}\|^2_{L^2(L^2)} &\le c_2^2\bar{m}^2\|\theta\|_{L^2(H^1)} \\ &\le c_2^2\bar{m}^2\|\theta_0\|^2 + c_3^2\bar{m}^4\|v_t\|^2_{L^2(L^2)} \\ &\le c_2^2\bar{m}^2\|\theta_0\|^2 + c_3^2\bar{m}^4\Gamma^2. \end{aligned} \tag{4.27}$$

where, in this proof, $c_1, c_2, \ldots$ will denote constants that do neither depend on $\bar{m}$ nor on ε. In order to see that $u \in W^\Gamma$, we have to estimate u_t. Let $h \neq 0$ and

$$v^h(t,\cdot) := (v(t+h,\cdot)-v(t,\cdot))/h,\ u^h(t,\cdot) := (u(t+h,\cdot)-u(t,\cdot))/h,\ \theta^h(t,\cdot) := (\theta(t+h,\cdot)-\theta(t,\cdot))/h.$$

From the obvious differential equations for u^h and θ^h we obtain

$$\int_\Omega C_{ijkl} u^h_{k,l} u^h_{i,j} dx - \int_{\Gamma_c} \nu_j C_{ijkl} u^h_{k,l} u^h_i d\Gamma = \int_\Omega \theta^h m_{ij} u^h_{i,j} dx. \tag{4.28}$$

For the boundary term, we obtain, using the notation

$$u^1(t,\cdot) := u(t+h,\cdot), \quad u^2(t,\cdot) := u(t,\cdot),$$

$$\begin{aligned}
\int_{\Gamma_c} \nu_j C_{ijkl} u^h_{k,l} u^h_i d\Gamma &= \int_{\Gamma_c} \sigma^1_\nu (u^1_\nu - g) + \sigma^2_\nu (u^2_\nu - g) d\Gamma - \int_{\Gamma_c} \sigma^1_\nu (u^2_\nu - g) + \sigma^2_\nu (u^1_\nu - g) d\Gamma \\
&= -\frac{1}{\varepsilon} \int_{\Gamma_c} |(u^1_\nu - g)^+|^2 + |(u^2_\nu - g)^+|^2 d\Gamma + \frac{1}{\varepsilon} \int_{\Gamma_c} (u^1_\nu - g)^+ (u^2_\nu - g) + (u^2_\nu - g)^+ (u^1_\nu - g) d\Gamma \\
&= \int_{\Gamma_c \cap \Gamma^+} \ldots d\Gamma + \int_{\Gamma_c \setminus \Gamma^+} \ldots d\Gamma,
\end{aligned} \tag{4.29}$$

where

$$\Gamma^+ := \{x \in \Gamma_c |\ u^1_\nu > g \text{ and } u^2_\nu > g\}.$$

$$\int_{\Gamma_c \cap \Gamma^+} \ldots d\Gamma = -\frac{1}{\varepsilon} \int_{\Gamma^+} (u^1_\nu - u^2_\nu)^2 d\Gamma \leq 0, \tag{4.30}$$

$$\begin{aligned}
\int_{\Gamma_c \setminus \Gamma^+} \ldots d\Gamma &= -\frac{1}{\varepsilon} \int_{\Gamma_c \setminus \Gamma^+} |(u^1_\nu - g)^+|^2 + |(u^2_\nu - g)^+|^2 d\Gamma \\
&+ \frac{1}{\varepsilon} \int_{\Gamma_c \setminus \Gamma^+} (u^1_\nu - g)^+ (u^2_\nu - g) + (u^2_\nu - g)^+ (u^1_\nu - g) d\Gamma \\
&\leq -\frac{1}{\varepsilon} \int_{\Gamma_c \setminus \Gamma^+} |(u^1_\nu - g)^+|^2 + |(u^2_\nu - g)^+|^2 d\Gamma \\
&\leq 0.
\end{aligned} \tag{4.31}$$

The estimates (4.29)–(4.31) imply

$$\int_{\Gamma_c} \nu_j C_{ijkl} u^h_{k,l} u^h_i d\Gamma \leq 0. \tag{4.32}$$

Combining (4.28) and (4.32) we get the estimate

$$\begin{aligned}
\|u^h\|^2_{L^2(H^1)} &\leq c_4^2 \bar{m}^2 \|\theta^h\|^2_{L^2(L^2)} \\
&\leq c_4^2 \bar{m}^2 \|\theta_t\|^2_{L^2(L^2)} \\
&\leq c_4^2 \bar{m}^2 \|\theta_0\|^2 + c_5^2 \bar{m}^4 \|v_t\|^2_{L^2(H^1)},
\end{aligned} \tag{4.33}$$

which implies

$$u_t \in L^2(H^1)$$

and

$$\begin{aligned} \|u_t\|^2_{L^2(H^1)} &\le c_4^2 \bar{m}^2 \|\theta_0\|^2 + c_5^2 \bar{m}^4 \|v_t\|^2_{L^2(H^1)} \\ &\le c_4^2 \bar{m}^2 \|\theta_0\|^2 + c_5^2 \bar{m}^4 \Gamma^2. \end{aligned} \tag{4.34}$$

Choosing $\bar{m}$ and Γ such that

$$\bar{m}^2 (c_3^2 + c_5^2) \le \frac{1}{2}, \tag{4.35}$$

$$\Gamma^2 = \Gamma^2(\theta_0) = 2\|\theta_0\|^2 \max\{c_2^2, c_4^2\}, \tag{4.36}$$

we conclude from (4.27) and (4.34) that S maps W^Γ into itself.
Now we prove the contraction property. For this purpose let

$$v^j \in W^\Gamma, \quad u^j := Sv^j, \qquad j = 1, 2.$$

Let θ^j denote the solution to (4.20), (4.21) with respect to v^j. Replacing u^h by $u^{12} := u^1 - u^2$ and so on, we can derive analogous estimates to those given in (4.28)–(4.33), and conclude

$$\begin{aligned} \|u^{12}\|^2_{L^2(H^1)} &\le c_4^2 \bar{m}^2 \|\theta^{12}\|^2_{L^2(L^2)} \\ &\le c_5^2 \bar{m}^4 \|v^{12}\|^2_{L^2(H^1)}, \end{aligned} \tag{4.37}$$

which proves that S is a contraction, if

$$c_5 \bar{m}^2 < 1. \tag{4.38}$$

Q.E.D.

The unique fixed point u of S together with the associated θ is then the desired solution to the penalized problem, i.e. we have proved

Theorem 4.4 *For given $\varepsilon > 0$ and $\theta_0 \in H_0^1(\Omega)$ there is a unique solution $(u^\varepsilon, \theta^\varepsilon)$ to the penalized problem (4.13)–(4.19), provided $\bar{m}$ is small enough.*

From the estimates in the proof of Theorem 4.3, we conclude the validity of the following lemma.

Lemma 4.5 *$(\theta^\varepsilon)_\varepsilon$ is bounded in $W^{1,2}(L^2) \cap L^2(H^2 \cap H_0^1) \cap C^0(H_0^1)$, $(u^\varepsilon)_\varepsilon$ is bounded in $W^{1,2}(H^1)$, and $((C_{ijkl}u^\varepsilon_{k,l})_{,j})_{i=1,\dots,n}$ is bounded in $L^2(L^2)$.*

Now we can prove the existence of a solution to the quasistatic contact problem.

Theorem 4.6 *For given $\theta_0 \in H_0^1(\Omega)$ there exists a solution (u, θ) to (4.1)–(4.5) in the sense of Definition 4.1, provided $\bar{m}$ is small enough.*

Remark: The solution is unique, which follows from the result of Ames & Payne [2].
PROOF: Let $(u^\varepsilon, \theta^\varepsilon)$ be the solution to the penalized problem according to Theorem 4.4. From

Lemma 4.5 we conclude that there exists a subsequence, again denoted by $(u^\varepsilon, \theta^\varepsilon)$, and (u, θ) such that

$$\theta^\varepsilon \stackrel{*}{\rightharpoonup} \theta \quad \text{in } L^\infty(H_0^1), \tag{4.39}$$

$$\theta^\varepsilon \rightharpoonup \theta \quad \text{in } W^{1,2}(L^2) \cap L^2(H^2 \cap H_0^1), \tag{4.40}$$

$$u^\varepsilon \rightharpoonup u \quad \text{in } W^{1,2}(H^1), \tag{4.41}$$

$$((C_{ijkl}u^\varepsilon_{k,l})_{,j})_{i=1,\ldots,n} \rightharpoonup ((C_{ijkl}u_{k,l})_{,j})_{i=1,\ldots,n} \quad \text{in } L^2(L^2). \tag{4.42}$$

It shall be proved that (u, θ) is a solution to (4.1)–(4.5). It only remains to justify the relations (4.11) and (4.12). Since $(u^\varepsilon, \theta^\varepsilon)$ satifies (4.12), we can use (4.39)–(4.42) to see that also (u, θ) satisfies (4.12).

For $w \in L^\infty(H^1_{\Gamma_D})$, $w_\nu \le g$ on Γ_c, we have

$$\begin{aligned}
&\int_0^T \langle C_{ijkl}u^\varepsilon_{k,l}, w_{i,j} - u^\varepsilon_{i,j}\rangle dt - \int_0^T \langle m_{ij}\psi^\varepsilon_t, w_{i,j} - u^\varepsilon_{i,j}\rangle dt \\
= \quad & \int_0^T \int_{\Gamma_c} \sigma^\varepsilon_\nu (w_\nu - u^\varepsilon_\nu) d\Gamma\, dt \\
= \quad & -\frac{1}{\varepsilon} \int_0^T \int_{\Gamma_c} (u^\varepsilon_\nu - g)^+ (w_\nu - u^\varepsilon_\nu) d\Gamma\, dt \\
= \quad & -\frac{1}{\varepsilon} \int_0^T \int_{\Gamma_c} (u^\varepsilon_\nu - g)^+ (w_\nu - g) d\Gamma\, dt + \frac{1}{\varepsilon} \int_0^T \int_{\Gamma_c} (u^\varepsilon_\nu - g)^+ (u^\varepsilon_\nu - g) d\Gamma\, dt \\
\ge \quad & 0.
\end{aligned} \tag{4.43}$$

Using the lower semi-continuity of norms and (4.39)–(4.42), we obtain

$$\int_0^T \langle -C_{ijkl}u_{k,l} + m_{ij}\theta, u_{i,j}\rangle dt \ge \limsup_{\varepsilon \downarrow 0} \int_0^T \langle -C_{ijkl}u^\varepsilon_{k,l} + m_{ij}\theta^\varepsilon, u^\varepsilon_{i,j}\rangle dt. \tag{4.44}$$

From (4.43) and (4.44), we conclude the validity of (4.11).

Q.E.D.

Finally, we want to describe the exponential stability.

Let $u^\varepsilon, \theta^\varepsilon$ denote the solution to the penalized problem as given in Theorem 4.4. Similarly as in [2] we change variables from θ^ε to ψ^ε, where

$$\psi^\varepsilon(t, x) := \int_0^t \theta^\varepsilon(s, x) ds + \psi^\varepsilon_0(x). \tag{4.45}$$

Here, $\psi^\varepsilon_0 \in H^2(\Omega) \cap H^1_0(\Omega)$ is defined as solution to

$$(k_{ij}\psi_{0,i})_{,j} = \theta_0 + m_{ij}u^\varepsilon_i(t = 0)_{,j}, \tag{4.46}$$

where $u_i^\varepsilon(t=0)$ is defined by θ_0 via (4.13). Then $(u^\varepsilon, \psi^\varepsilon)$ satisfies

$$\psi_t^\varepsilon - (k_{ij}\psi_{,i})_{,j} + m_{ij}u_{i,j}^\varepsilon = 0. \tag{4.47}$$

Multiplying (4.47) by ψ_t we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega k_{ij}\psi_{,i}^\varepsilon\psi_{,j}^\varepsilon dx = -\int_\Omega |\psi_t^\varepsilon|^2 dx - \int_\Omega m_{ij}u_{i,j}^\varepsilon\psi_t^\varepsilon dx. \tag{4.48}$$

From (4.13) we obtain

$$\begin{aligned} -\int_\Omega m_{ij}u_{i,j}^\varepsilon\psi_t^\varepsilon dx &= -\int_\Omega C_{ijkl}u_{k,l}^\varepsilon u_{i,j}^\varepsilon dx + \int_{\Gamma_c}\nu_j C_{ijkl}u_{k,l}^\varepsilon u_i^\varepsilon d\Gamma \\ &= -\int_\Omega C_{ijkl}u_{k,l}^\varepsilon u_{i,j}^\varepsilon dx - \frac{1}{\varepsilon}\int_{\Gamma_c}|(u_\nu^\varepsilon - g)^+|^2 d\Gamma. \end{aligned} \tag{4.49}$$

From (4.48), (4.49) we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega k_{ij}\psi_{,i}^\varepsilon\psi_{,j}^\varepsilon dx \le -\int_\Omega |\psi_t^\varepsilon|^2 dx - \int_\Omega C_{ijkl}u_{k,l}^\varepsilon u_{i,j}^\varepsilon dx - \frac{1}{\varepsilon}\int_{\Gamma_c}|(u_\nu^\varepsilon - g)^+|^2 d\Gamma. \tag{4.50}$$

On the other hand we obtain from (4.13), (4.14)

$$-\int_\Omega (C_{ijkl}u_{k,l}^\varepsilon)_{,j}\partial_t u_i^\varepsilon dx + \int_\Omega \theta_t^\varepsilon\theta^\varepsilon dx - \int_\Omega (k_{ij}\theta_{,i}^\varepsilon)_{,j}\theta^\varepsilon dx = 0,$$

which implies

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left\{\int_\Omega C_{ijkl}u_{k,l}^\varepsilon u_{i,j}^\varepsilon + |\theta^\varepsilon|^2 dx\right\} &= -\int_\Omega k_{,j}\theta_{,i}^\varepsilon\theta_{,j}^\varepsilon dx + \int_{\Gamma_c}\sigma_\nu^\varepsilon\partial_t(u_\nu^\varepsilon - g)d\Gamma \\ &= -\int_\Omega k_{ij}\theta_{,i}^\varepsilon\theta_{,j}^\varepsilon dx - \frac{1}{\varepsilon}\int_{\Gamma_c}(u_\nu - g)^+\partial_t(u_\nu - g)d\Gamma, \end{aligned}$$

where

$$\frac{1}{2}\frac{d}{dt}\left\{\int_\Omega C_{ijkl}u_{k,l}^\varepsilon u_{i,j}^\varepsilon + |\theta^\varepsilon|^2 dx + \frac{1}{\varepsilon}\int_{\Gamma_c}|(u_\nu^\varepsilon - g)^+|^2 d\Gamma\right\} \le -\int_\Omega k_{ij}\theta_{,i}^\varepsilon\theta_{,j}^\varepsilon dx. \tag{4.51}$$

Combining (4.50) and (4.51) we obtain

$$\begin{aligned} &\frac{d}{dt}\frac{1}{2}\left\{\int_\Omega C_{ijkl}u_{k,l}^\varepsilon u_{i,j}^\varepsilon + |\theta^\varepsilon|^2 + k_{ij}\psi_{,i}\psi_{,j}^\varepsilon\right\} dx + \frac{1}{\varepsilon}\int_{\Gamma_c}|u_\nu^\varepsilon - g)^+|^2 d\Gamma\} \\ \le\; & -\left\{\int_\Omega C_{ijkl}u_{k,l}^\varepsilon u_{i,j}^\varepsilon + |\theta^\varepsilon|^2 + k_{i,j}\theta_{,i}^\varepsilon\theta_{,j}^\varepsilon dx + \frac{1}{\varepsilon}\int_{\Gamma_c}|(u_\nu - g)^+|^2 d\Gamma\right\}. \end{aligned} \tag{4.52}$$

Multiplying (4.14) by $\psi\varepsilon$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\psi^\varepsilon|^2 dx &= -\int_\Omega k_{ij}\psi^\varepsilon_{,i}\psi^\varepsilon_{,j}dx - \int_\Omega m_{ij}u^\varepsilon_{i,j}\psi^\varepsilon dx \\ &\leq -\frac{1}{2}\int_\Omega k_{ij}\psi^\varepsilon_{,i}\psi^\varepsilon_{,j}dx + c_1\int_\Omega C_{ijkl}u^\varepsilon_{k,l}u^\varepsilon_{i,j}dx, \end{aligned}\tag{4.53}$$

where the constant $c_1 > 0$ is independent of ε.
Let

$$\begin{aligned}E_\varepsilon(t) \;:=\; & \frac{1}{2}\left\{\int_\Omega C_{ijkl}u^\varepsilon_{k,l}u^\varepsilon_{i,j} + |\theta^\varepsilon|^2 + k_{ij}\psi^\varepsilon_{,i}\psi^\varepsilon_{,j}dx \right. \\ & \left. + \frac{1}{\varepsilon}\int_{\Gamma_c} |(u^\varepsilon_\nu - g)^+|^2 d\Gamma + \frac{1}{2c_1}\int_\Omega |\psi^\varepsilon|^2 dx\right\}\end{aligned}\tag{4.54}$$

Then the estimates (4.52) and (4.53), together with Poincaré's estimate applied to ψ^ε, imply

$$\frac{d}{dt}E_\varepsilon(t) \leq -\beta E_\varepsilon(t), \quad t \geq 0, \tag{4.55}$$

for some constant $\beta > 0$ which is independent of ε. Hence we have proved:

Theorem 4.7 *For fixed $\varepsilon > 0$ the "energy" E_ε, defined for solutions $(u^\varepsilon, \psi^\varepsilon)$ to the penalized problem (4.13)-(4.19) in (4.54), decays to zero exponentially, i.e.*

$$\exists \beta > 0 \quad \forall t \geq 0 \quad E_\varepsilon(t) \leq E_\varepsilon(0)e^{-\beta t}, \tag{4.56}$$

and β is independent of ε.

If

$$E(u^\varepsilon, \psi^\varepsilon)(t) := E_\varepsilon(t) - \frac{1}{2\varepsilon}\int_{\Gamma_C} |(u^\varepsilon_\nu - g)^+|^2 d\Gamma,$$

then obviously

$$E(u^\varepsilon, \psi^\varepsilon)(t) \leq E_\varepsilon(0)e^{-\beta t}.$$

Lemma 4.8
(i)

$$\exists \alpha_1 > 0\, \forall \varepsilon > 0 \,:\, E(u^\varepsilon, \psi^\varepsilon)(0) \leq \alpha_1 \|\theta_0\|^2, \tag{4.57}$$

(ii)

$$\exists \alpha_2 > 0 \quad \forall \varepsilon > 0 \quad \forall t \geq 0 : \frac{1}{\varepsilon}\int_{\Gamma_c} |(u^\varepsilon_\nu - g)^+|^2 d\Gamma \leq \alpha_2 \|\theta_0\|^2. \tag{4.58}$$

PROOF:

$$-C_{ijkl}u^{\varepsilon}_{k,l}(0) = -(m_{,j}\theta_0)_{,j}, \quad i = 1, \ldots, n.$$

This implies

$$\int\limits_{\Omega} C_{ijkl}u^{\varepsilon}_{k,l}(0)u^{\varepsilon}_{i,j}(0)dx + \frac{1}{\varepsilon}\int\limits_{\Gamma_c} |(u^{\varepsilon}_{\nu}(0)-g)^+|^2 d\Gamma \leq -\int\limits_{\Omega} m_{ij}\theta_0 u_{i,j}(0)$$
$$\leq c_1\|\theta_0\|^2 + \frac{1}{2}\int\limits_{\Omega} C_{ijkl}u^{\varepsilon}_{k,l}(0)u^{\varepsilon}_{i,j}(0)dx,$$

where $c_1 > 0$ is a constant. Therefore

$$\int\limits_{\Omega} C_{ijkl}u^{\varepsilon}_{k,l}(0)u^{\varepsilon}_{i,j}(0)dx + \frac{1}{\varepsilon}\int\limits_{\Gamma_c} |(u^{\varepsilon}_{\nu}(0)-g)^+|^2 \leq 2c_1\|\theta_0\|^2. \tag{4.59}$$

$$\int\limits_{\Omega} |\psi^{\varepsilon}(0)|^2 dx + \int\limits_{\Omega} k_{ij}\psi^{\varepsilon}_{,i}(0)\psi^{\varepsilon}_{,j}(0)dx \leq c_1\|\psi^{\varepsilon}_0\|^2_{H^1}$$
$$\leq c_1(\|\theta_0\|^2 + \|m_{ij}u_{i,j}(0)\|^2 \quad \text{(by (4.46))}$$
$$\leq c_1\|\theta_0\|^2 \quad \text{(by (4.59))}. \tag{4.60}$$

We use c_1 to denote various constants being independent of ε.
The estimates (4.59), (4.60) prove (4.57). From (4.51) and (4.59) we conclude for $t \geq 0$

$$\frac{1}{\varepsilon}\int\limits_{\Gamma_c} |(u^{\varepsilon}_{\nu}(t)-g)|^2 d\Gamma \leq \alpha_2\|\theta_0\|^2, \tag{4.61}$$

for some $\alpha_2 > 0$, hence (4.58) is proved.

Q.E.D.

Corollary 4.9 $\exists \beta_1, \beta_2 > 0 \quad \forall t \geq 0 \quad \forall \varepsilon > 0 : E(u^{\varepsilon}, \psi^{\varepsilon})(t) \leq \beta_1\|\theta_0\|^2 e^{\beta_2} t.$

Now let (u, ψ) be the solution to the quasistatic problem obtained from $(u^{\varepsilon}, \psi^{\varepsilon})_{\varepsilon}$ as $\varepsilon \downarrow 0$ (Theorem 4.5). Let the "energy" of (u, ψ) be defined by

$$E(t) := \int\limits_{\Omega} C_{ijkl}u_{k,l}u_{i,j} + |\psi_t|^2 + k_{ij}\psi_{,i}\psi_{,j}dx. \tag{4.62}$$

Then the exponential stability is expressed in the next theorem.

Theorem 4.10

$$\forall t \geq 0 : E(t) \leq \beta_1\|\theta_0\|^2 e^{-\beta_2 t}, \tag{4.63}$$

(β_1, β_2 from Corollary 4.8).

PROOF: Corollary 4.8 and the weak lower semi-continuity of the norm yield the proof.

Q.E.D.

5 Smoothing in the interior for the quasistatic contact problem

The linear, quasistatic problem with Dirichlet boundary conditions has smoothing properties like those known for the solutions to heat equations, in particular, the solution $(u,\theta)(t,x)$ is infinitely smooth in t and x as soon as $t>0$, no matter how smooth the initial data is (cf. the remarks in section 3). This behavior cannot be expected in general for the quasistatic contact problem because of the mixed boundary conditions for u. We shall prove that the solution (u,θ) to the quasistatic thermoelastic contact problem (4.1)–(4.5) is infinitely smooth with respect to x in the interior of Ω, if $t>0$.
Naturally, we assume in this section that all coefficients are C^∞-smooth. We also assume that (u,θ) is a solution to (1.1)–(1.5) as given in the previous section. Let $\varphi\in C_0^\infty(\Omega)$ such that $\operatorname{supp}\varphi\subset\Omega_{\delta_0/2}$ for some $\delta_0>0$, with $\Omega_{\delta_0/2}$ open such that

$$\Omega_{\delta_0/2}=\{x\in\Omega|\operatorname{dist}(x,\partial\Omega)>\delta_0/2\}.$$

Also let

$$\varphi=1 \text{ in } \Omega_{3\delta_0/4}.$$

Let

$$(\tilde u,\tilde\theta):=(\varphi u,\varphi\theta).$$

Then $(\tilde u,\tilde\theta)$ satisfy

$$-(C_{ijkl}\tilde u_{k,l})_{,j}+(m_{ij}\tilde\theta)_{,j}=R_i(u,\nabla u,\theta),\quad i=1,\dots,n, \tag{5.1}$$

$$\tilde\theta_t-(k_{ij}\tilde\theta_{,i})_{,j}+m_{ij}\partial_t\tilde u_{i,j}=Q(u_t,\theta,\nabla\theta), \tag{5.2}$$

$$\tilde\theta(t=0)=\tilde\theta_0:=\varphi\theta_0, \tag{5.3}$$

$$\operatorname{supp}\tilde u,\ \operatorname{supp}\tilde\theta\subset\subset\Omega. \tag{5.4}$$

Here R_i and Q are given by

$$R_i\ =\ -\varphi_{,l,j}C_{ijkl}u_k-\varphi_{,l}(C_{ijkl}u_k)_{,j}-\varphi_{,j}C_{ijkl}u_{k,l}+\varphi_{,j}m_{ij}\theta, \tag{5.5}$$

$$Q=-\varphi_{,i,j}k_{ij}\theta-\varphi_{,i}(k_{ij}\theta)_{,j}-\varphi_{,j}\theta_{,i}+\varphi_{,j}m_{ij}\partial_t u_i. \tag{5.6}$$

In particular,

$$\operatorname{supp}R_i\cup\operatorname{supp}Q\subset\operatorname{supp}\nabla\varphi\subset\subset\Omega.$$

Let $r_1>0$ such that $\Omega\subset\subset B:=B(0,r_1)$.
Let (w,ϑ) solve in $(0,\infty)\times B$:

$$-(C_{ijkl}w_{k,l})_{,j}+(m_{ij}\vartheta)_{,j}=0,\quad i=1,\dots,n, \tag{5.7}$$

$$\vartheta_t-(k_{ij}\vartheta_{,i})_{,j}+m_{ij}\partial_t w_{i,j}=0, \tag{5.8}$$

$$\vartheta(t=0)=\tilde{\theta}_0, \tag{5.9}$$

$$w_{|\partial B}=0, \quad \vartheta_{|\partial B}=0. \tag{5.10}$$

The linear, quasistatic problem (5.7)–(5.10) can be solved as shown in Section 2 assuming

$$\theta_0 \in H^2(\Omega)\cap H_0^1(\Omega), \tag{5.11}$$

and (w,ϑ) is smooth in B as $t>0$.
Let (v,ϱ) be the solution in $(0,\infty)\times B$ to

$$-(C_{ijkl}v_{k,l})_{,j}+(m_{ij}\varrho)_{,j}=R_i, \qquad i=1,\dots,n, \tag{5.12}$$

$$\varrho_t-(k_{ij}\varrho_{,i})_{,j}+m_{ij}\partial_t v_{i,j}=Q, \tag{5.13}$$

$$\varrho(t=0)=0, \tag{5.14}$$

$$v_{|\partial B}=0, \quad \varrho_{|\partial B}=0. \tag{5.15}$$

The right-hand sides R_i, Q satisfy

$$R=(R_1,\dots,R_n)\in W^{1,2}(L^2), \quad Q\in L^2(H^1). \tag{5.16}$$

The system (5.12)–(5.15) can be transformed into a parabolic equation for ϱ, essentially, as in Section 2:

$$v=-E^{-1}\nabla' M\theta+E^{-1}R, \quad v_{|\partial B}=0, \tag{5.17}$$

where

$$M=(m_{ij})_{ij}, \quad E=(-\partial_j C_{ijkl}\partial_l)_i,$$

hence

$$\varrho_t+A\varrho=F, \tag{5.18}$$

$$\varrho(t=0)=0, \tag{5.19}$$

$$\varrho_{|\partial B}=0, \tag{5.20}$$

where

$$A\varrho=-(\mathrm{Id}-M\nabla E^{-1}\nabla' M')^{-1}(k_{ij}\varrho_{,i})_{,j}, \tag{5.21}$$

$$F=Q-M\nabla\partial_t E^{-1}R. \tag{5.22}$$

The regularity of R and Q given in (5.16) as well as the mapping properties of the operator E (cf. Section 2) imply

$$F \in L^2(H^1). \tag{5.23}$$

This regularity of F implies for the (existing) solution ϱ to (5.18)–(5.20):

$$\varrho \in L^2(H^3(\Omega_{\delta_0/2}), \quad \varrho_t \in L^2(H^1(\Omega_{\delta_0/2})) \tag{5.24}$$

by the usual regularity for parabolic equations, see for example p.11 in [15], applied to $\partial_j \varrho$, smoothly cut off in $\Omega_{\delta_0/2}$.
By (5.17) and (5.24) we conclude

$$v \in W^{1,2}(H^2(\Omega_{\delta_0/2})). \tag{5.25}$$

By uniqueness we have in $(0, \infty) \times B$

$$\tilde{u} = w + v, \quad \tilde{\theta} = \vartheta + \varrho.$$

Since (w, ϑ) is smooth for $t > 0$, we conclude from (5.24), (5.25) and the fact that

$$u = \tilde{u}, \quad \theta = \tilde{\theta} \text{ in } \Omega_{3\delta_0/4} \tag{5.26}$$

that, for any $\tau > 0$,

$$u \in W^{1,2}((\tau,\infty), H^2(\Omega_{3\delta_0/4}), \quad \theta \in L^2((\tau,\infty), H^3(\Omega_{3\delta_0/4})), \; \theta_t \in L^2((\tau,\infty), H^1(\Omega_{3\delta_0/4})), \tag{5.27}$$

i.e. more regularity for (u, θ).
This implies (cp. (5.16))

$$R \in W^{1,2}((\tau,\infty), \; H^1(\Omega_{3\delta_0/4}), \; Q \in L^2((0,\infty), H^2(\Omega_{3\delta_0/4})), \tag{5.28}$$

i.e. higher regularity for F in (5.18).
Proceeding by induction ($\delta_0 > 0$ chosen appropriately) we have proved

Theorem 5.1 *Let $\theta_0 \in H_0^1(\Omega)$. Then the solution (u, θ) to (4.1)–(4.5) satisfies*

$$\forall \tau > 0 \;\; \forall \Omega' \subset\subset \Omega \;\; \forall m \in \mathbb{N} : u \in W^{1,2}((\tau,\infty), H^m(\Omega')), \;\; \theta \in W^{1,2}((\tau,\infty), H^m(\Omega')). \tag{5.29}$$

This interior smoothing effect in x carries over to smoothing in t, which can be seen by differentiating the equation for $(\tilde{u}, \tilde{\theta})$ with respect to t and applying the same arguments again. As a corollary we have

Corollary 5.2 *Let $\theta_0, (u, \theta)$ as in Theorem 5.1. Then*

$$\forall \tau > 0 \;\; \forall \Omega' \subset\subset \Omega : \quad u \in C^\infty((\tau,\infty) \times \Omega'), \;\; \theta \in C^\infty((\tau,\infty) \times \Omega)). \tag{5.30}$$

References

[1] H. Amann, *Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems*, Manuscript, 1993.

[2] K.A. Ames, L.E. Payne, *Uniqueness and continuous dependence of solutions to a multidimensional thermoelastic contact problem*, J. Elasticity, 34 (1994), pp. 139–148.

[3] B. Dacorogna, *Weak continuity and weak lower semicontinuity of non-linear functionals*, Lec. Notes Math., 922, 1982.

[4] W.A. Day, *Heat conduction within linear thermoelasticity*, Springer Tracts Nat. Philos. 30, Springer-Verlag, New York et al., 1985.

[5] C.M. Elliott, Q. Tang, *A dynamic contact problem in thermoelasticity*, Nonlinear Anal., T.M.A., 23 (1994), pp. 883–898.

[6] I. Figueiredo, L. Trabucho, *Some existence results for contact and friction problems in thermoelasticity and in thermoviscoelasticity*, In: "Asymptotic Methods For Elastic Structures", Proceedings of the International Conf., Lisbon (1993), P.G. Ciarlet et al. (eds.), de Gruyter (1995), pp. 223–235.

[7] S. Jiang, J.E. Muñoz Rivera, R. Racke, *Asymptotic stability and global existence in thermoelasticity with symmetry*, Quart. Appl. Math., to appear.

[8] S. Jiang, R. Racke, *On some quasilinear hyperbolic-parabolic initial boundary value problems*, Math. Meth. Appl. Sci., 12 (1990), pp. 315–339.

[9] J.U. Kim, *A boundary thin obstacle problem for a wave equation*, Comm. PDE, 14 (1989), pp. 1011–1026.

[10] R. Leis, *Initial boundary value problems in mathematical physics.* B.G. Teubner-Verlag, Stuttgart; John Wiley & Sons, Chichester et al., 1986.

[11] J.E. Muñoz Rivera, M. de Lacerda Oliveira, *Exponential stability for a contact problem in thermoelasticity*, IMA J. Appl. Math., to appear.

[12] R. Racke *On the time-asymptotic behaviour of solutions in thermoelasticity*, Proc. Roy. Soc. Edinburgh, 107A (1987), pp. 289–298.

[13] P. Shi, M. Shillor, *Existence of a solution to the n dimensional problem of thermoelastic contact*, Comm. PDE 17 (1992), pp. 1597–1618.

[14] P. Shi, Y. Xu, *Decoupling of the quasistatic system of thermoelasticity on the unit disk*, J. Elasticity, 31 (1993), pp. 209-218.

[15] S. Zheng *Nonlinear parabolic equations and hyperbolic-parabolic coupled systems*, Pitman Monographs Surv. Pure Appl. Math. 76, Longman; John Wiley & Sons, New York, 1995.

Jaime E. Muñoz Rivera, Department of Research and Development, National Laboratory for Scientific Computation, Rua Lauro Müller 455, 22290 Rio de Janeiro, RJ, Brasil
e-mail: rivera@server01.lncc.br

Reinhard Racke, Faculty of Mathematics and Computer Science, University of Constance, P.O. Box 5560, 78434 Konstanz, Germany
e-mail: racke@dirichlet.mathe.uni-konstanz.de