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Nonlinear Evolution Equations in Thermoelasticity

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1 Introduction

In this note we report on different thermoelastic systems. They are models for the description of elastic, heat conductive media. Depending on the choice of the elastic part we will consider a coupled parabolic–weakly parabolic system (e.g. a thermoelastic plate), or a coupled parabolic–2^{nd}-order hyperbolic system (e.g. a thermoelastic membrane) with or without viscous effects. We are interested in both the linearized and the nonlinear system, looking for a description of the asymptotic behavior of smooth solutions, for smoothing effects of the system, and, specifically for the nonlinear system, for the global existence in time of solutions.

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For hyperbolic systems it is well known that in many cases locally existing smooth solutions tend to develop singularities in finite time. The basic problem for the systems in question here is whether and in which way the added damping by heat conduction and/or viscosity will assure the global existence of solutions.

In Section 2 we consider model equations for thermoelastic plates, the simplest linear example being

$$u_{tt} + \Delta^2 u + \delta\Delta\theta = 0 \tag{1.1}$$

$$\theta_t - \Delta\theta - \delta\Delta u_t = 0, \tag{1.2}$$

where $\delta \neq 0,\ t \geq 0,\ x \in \Omega \subset \mathbb{R}^n$, and $u = u(t,x)$ and $\theta = \theta(t,x)$ are the deflection and the temperature difference, respectively; the boundary $\partial\Omega$ is assumed to be smooth. Additionally, one has initial conditions

$$u(t=0) = u_0, \quad u_t(t=0) = u_1, \quad \theta(t=0) = \theta_0, \tag{1.3}$$

and the boundary conditions

$$u = \Delta u = 0, \quad \theta = 0, \quad \text{on } \partial\Omega. \tag{1.4}$$

More generally, we shall consider nonlinear coupled systems with nonlocal nonlinearities in a separable Hilbert space H of the following type,

$$u_{tt} + M([u,\theta])A^2 u + N([u,\theta])A\theta = 0, \tag{1.5}$$

$$\theta_t + Q([u,\theta])A\theta - N([u,\theta])Au_t = 0, \tag{1.6}$$

where $M, N, Q : \mathbb{R}^5 \to \mathbb{R}$ are smooth, M, N^2, Q strictly positive, and

$$[u,\theta] := (\|u_t\|^2,\ \|A^{1/2}u\|^2,\ \|A^{1/2}u_t\|^2,\ \|Au\|^2,\ \|A^{1/2}\theta\|^2),$$

$\|\cdot\|$ denotes the norm in H, $A : D(A) \subset H \to H$ is a non-negative, self-adjoint operator,

$$u = (u^1, \ldots, u^k), \quad \theta = (\theta^1, \ldots, \theta^l) : [0,\infty) \to H^{k(l)},$$

A being applied to each component.

Solutions will satisfy the initial condition (1.3) and the abstract "boundary condition"

$$u(t) \in D(A^2), \quad \theta(t) \in D(A), \quad t \geq 0. \tag{1.7}$$

The local well-posedness and the global existence of small solutions as well as the smoothing and decay properties will be investigated. For a class of nonlinearities large solutions can be obtained. Moreover, we look at

$$u_{tt} + M(\|A^{1/2}u\|^2, \|\theta\|^2)Au + N(\|A^{1/2}u\|^2, \|\theta\|^2)A^\beta\theta = 0, \tag{1.8}$$

$$R(\|A^{1/2}u\|^2, \|\theta\|^2)\theta_t + Q(\|A^{1/2}u\|^2, \|\theta\|^2)A^\alpha\theta - N(\|A^{1/2}u\|^2, \|\theta\|^2)A^\beta u_t = 0, \tag{1.9}$$

where $\alpha, \beta \geq 0$.

Replacing A by $A^{1/2}$, we recover for $\alpha = \beta = 1/2$ the system (1.5), (1.6) again, respectively (1.1), (1.2) with $H = L^2(\Omega)$, $D(A^{1/2}) = H^2(\Omega) \cap H_0^1(\Omega)$, $A^{1/2}v = -\Delta v$, $M = R = Q = 1$, $N = -\delta$. For $\alpha = 0$, $\beta = 1/2$ we get equations of purely viscoelastic type:

$$u_{tt} + \gamma A u - \int_0^t k(t,\tau) A u(\tau) d\tau = 0,$$

where $\gamma > 0$, and k is a kernel which is not necessarily of convolution type.

In Section 3 we discuss classical 2^{nd}-order (non-)linear thermoelasticity. For one-dimensional models, where the equations are of the following type,

$$\begin{aligned} u_{tt} - \tilde{S}(u,\theta)_x &= f_1, \\ (\theta + T_0) N(u_x,\theta)_t - Q(u_x,\theta,\theta_x)_x &= f_2, \end{aligned}$$

u describing the displacement, and θ describing the temperature difference, the picture is more or less complete. In two- or threedimensional situations already the linearized systems lead to complex answers concerning the decay of solutions, even for the simplest homogeneous, isotropic case, where the equations look like

$$\begin{aligned} U_{tt} - ((2\mu + \lambda)\nabla\nabla' - \mu\nabla \times \nabla \times) U + \gamma \nabla \theta &= 0, \\ \delta\theta_t - \beta\Delta\theta + \gamma\nabla' U_t &= 0. \end{aligned}$$

The behavior strongly depends on the geometry of the domain; for bounded domains there are cases where the solution decays to zero exponentially, other cases where it does not decay at all or arbitrarily slowly.

Reminding the fact that the decay of solutions to the linearized equations often is an essential tool for deriving *a priori* estimates for the nonlinear system, it will become clear that there exist only few results on global existence for nonlinear systems in more than one space dimension, as for the Cauchy problem ($\Omega = \mathbb{R}^3$) and for radially symmetrical bounded domains.

We shall give a survey on results and some details concerning the recently obtained result for symmetrical configurations.

Finally, in section 4, we consider a one-dimensional model in thermoviscoelasticity, the equations being of the form

$$\begin{aligned} w_{tt} - (f_1(w_x)\vartheta + f_2(w_x))_x - \mu w_{txx} &= 0, \\ \vartheta_t - \vartheta f_1(w_x) w_{tx} - \mu w_{tx}^2 - k\vartheta_{xx} &= 0, \end{aligned}$$

where w is the displacement, ϑ is the absolute temperature, $k > 0, f_1, f_2$ are appropriate functions, and $\mu > 0$ is the viscosity.

For a bounded reference configuration we shall have a global solution

$$(w_x, w_t, \vartheta) \in L^\infty \times W^{1,\infty} \times H^1$$

to an associated initial boundary value problem for arbitrarily large data.

The notation will be as follows:
$L^p(\Omega), W^{m,p}(\Omega), W_0^{m,p}(\Omega),\ H^m_{(0)}(\Omega) \equiv W^{m,2}_{(0)}(\Omega),\ 1 \le p \le \infty, m \in \mathbb{N}_0,\ \Omega$ a domain in $\mathbb{R}^n$, denote the usual Sobolev spaces with norm $\|\cdot\|_p$ and $\|\cdot\|_{m,p}$, respectively, cf. [1]. $\langle\cdot,\cdot\rangle$: inner product in $L^2(\Omega)$ or any other Hilbert space, $\|\cdot\|$ denoting the corresponding norm.
$C^k(\Omega),\ C_0^\infty(\Omega)$: classical function spaces, $k \in \mathbb{N}_0 \cup \{\infty\}$.
$\partial_j = \frac{\partial}{\partial x_j},\ \nabla$: gradient, $\partial_t = \frac{\partial}{\partial t},\ D = (\partial_t, \partial_1, \ldots, \partial_n)$. A subscript t or x denotes ∂_t or $\frac{\partial}{\partial x}$, respectively. Δ denotes the Laplace operator.
$L^p(I, B)$ denotes the space of strongly measurable functions from $I \subset \mathbb{R}$ into a Banach space B, the p-th power of which are integrable (essentially bounded if $p = \infty$); analogously: $C^k(I, B),\ k \in N_0 \cup \{\infty\}$.

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2 Thermoelastic plates with non-local nonlinearities

2.1 Smoothing and small solutions

We consider the following nonlinear coupled system in a separable Hilbert space H:

$$u_{tt} + M([u,\theta])A^2u + N([u,\theta])A\theta = 0, \tag{2.1}$$

$$\theta_t + Q([u,\theta])A\theta - N([u,\theta])Au_t = 0, \tag{2.2}$$

$$u(t=0) = u_0 \qquad u_t(t=0) = u_1, \qquad \theta(t=0) = \theta_0, \tag{2.3}$$

$$u(t) \in D(A^2)^{l_1}, \qquad \theta(t) \in D(A)^{l_2}, \quad t \ge 0. \tag{2.4}$$

Here, $A : D(A) \subset H \to H$ is a non-negative, self-adjoint, linear operator, $u = (u^1, \ldots, u^{l_1})$, $\theta = (\theta^1, \ldots, \theta^{l_2}) : [0,\infty) \to H^{l_1(l_2)}$; A is applied to each component. $M, N, Q : \mathbb{R}^5 \to \mathbb{R}$ are smooth, M, N^2, Q are strictly positive and

$$[u,\theta] = (\|u_t\|^2, \|A^{1/2}u\|^2, \|A^{1/2}u_t\|^2, \|Au\|^2, \|A^{1/2}\theta\|^2),$$

where $\|\cdot\|$ denotes the norm in H.

In the simplest case, where $H = L^2(\Omega)$ for a domain $\Omega \subset \mathbb{R}^n$, $l_1 = l_2 = 1$, $M = Q = 1$, $N = -\delta \neq 0$, $A =$ realization of $-\Delta$ with Dirichlet boundary conditions, we obtain a model of a thermoelastic plate,

$$u_{tt} + \Delta^2 u + \delta\Delta\theta = 0, \tag{2.5}$$

$$\theta_t - \Delta\theta - \delta\Delta u_t = 0, \tag{2.6}$$

$$u(t=0) = u_0, \quad u_t(t=0) = u_1, \quad \theta(t=0) = \theta_0, \tag{2.7}$$

$$u_{|\partial\Omega} = \Delta u_{|\partial\Omega} = 0, \quad \theta_{|\partial\Omega} = 0. \tag{2.8}$$

The local well-posedness of the evolution equations (2.1)–(2.4) is given for data

$$(u_0, u_1, \theta_0) \in D(A^k) \times D(A^{k-1}) \times D(A^{k-1}),$$

where $k \geq 3$. It suffices to assume that the functions M, N and Q are in C^{k-1}. Then there exists a local solution (u, θ) satisfying

$$(u, \theta) \in \bigcap_{j=0}^{k} C^j([0,T], D(A^{k-j})) \times \bigcap_{j=0}^{k-1} C^j([0,T], D(A^{k-j-1}))$$

for some $T > 0$; T depends only on ϱ, where

$$\varrho = (\|u_0\|_{D(A^2)}, \|u_1\|_{D(A)}, \|\theta_0\|_{D(A)})$$

and $T \to \infty$ as $\varrho \to 0$. (Without loss of generality we assumed $l_1 = l_2 = 1$.)

The proof of this well-posedness is somehow standard with a fixed point argument in appropriate spaces. Basic ingredients are so-called energy estimates, which are also crucial for the proofs of the subsequent results, and will be explained exemplarily below.

Looking at the thermoelastic plate, described in (2.5)–(2.8), we remark that solutions v to the plate equation modeled by

$$v_{tt} + \Delta^2 v = 0, \quad v_{|\partial\Omega} = \Delta v_{|\partial\Omega} = 0, \;\; v(t=0) = v_0, \;\; v_t(t=0) = v_1, \tag{2.9}$$

are not as smooth as solutions to the heat equation

$$\chi_t - \Delta\chi = 0, \qquad \chi_{|\partial\Omega} = 0, \quad \chi(t=0) = \chi_0, \tag{2.10}$$

i.e. the system for v does not show the strong smoothing effect as the system (2.10), where weakly smooth data χ_0 lead to arbitrarily smooth $\chi(t)$ if $t > 0$.

We also remark that solutions to the linear second-order thermoelastic equation in one dimension to be discussed in section 3.1,

$$u_{tt} - \tau u_{xx} + \gamma \theta_x = 0, \tag{2.11}$$

$$\theta_t - \kappa \theta_{xx} + \gamma u_{tx} = 0, \tag{2.12}$$

$$u(t=0) = u_0, \quad u_t(t=0) = u_1, \quad \theta(t=0) = \theta_0, \tag{2.13}$$

$$u = \theta_x = 0 \quad \text{on } \partial\Omega, \tag{2.14}$$

where $\tau, \kappa, |\gamma| > 0$, $\Omega \in \{(0,1), (0,\infty), (-\infty,\infty)\}$, are propagating singularities, i.e. the impact of θ is not strong enough to compensate for the non-smoothing of the hyperbolic part in u. So *a priori* it is an open question whether the system (2.1)–(2.4) has a smoothing property or not. The positive answer is given in the next theorem.

Theorem 2.1 *Let* $(u,\theta) \in \bigcap_{j=0}^{2} C^j([0,T], D(A^{2-j})) \times \bigcap_{j=0}^{1} C^j([0,T], D(A^{1-j}))$ *be a solution to* (2.1)–(2.4) *for some* $T > 0$ *with* $(u_0, u_1, \theta_0) \in D(A^2) \times D(A) \times D(A)$. *Then we have for any* $t \in (0,T]$ *and all* $m \in \mathbb{N}$:

$$u(t),\, \theta(t) \in D(A^m).$$

Before presenting the proof in a simple case, we give a general idea why energy estimates are useful to prove local-wellposedness, smoothing effects, the existence of global solutions and the asymptotic behavior as time $t \to \infty$. Let us consider the heat equation in all of $\mathbb{R}^n$,

$$w_t - \Delta w = 0, \quad w(t=0) = w_0. \tag{2.15}$$

Applying the classical Fourier transform

$$\hat{w}(t,\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\xi} w(t,x)\,dx,$$

we obtain

$$\hat{w}_t(t,\xi) + |\xi|^2 \hat{w}(t,\xi) = 0, \;\; \hat{w}(0,\xi) = \hat{w}_0(\xi),$$

hence

$$\hat{w}(t,\xi) = e^{-t|\xi|^2} \hat{w}(0,\xi)$$

or

$$E(t,\xi) := |\hat{w}(t,\xi)|^2 = e^{-2t|\xi|^2} E(0,\xi). \tag{2.16}$$

From (2.16) it follows immediately, applying the inverse Fourier transform, that, e.g.,

$$||w(t)||_{L^\infty} \le c t^{-n/2} ||w_0||_{L^1},$$

and that for $t > 0$, $w(t,\cdot)$ belongs to the Sobolev space $H^m(\mathbb{R}^n)$ for any $m \in \mathbb{N}$, since

$$\xi \mapsto |\xi|^{2m} E(t,\xi) \in L^1(\mathbb{R}^n)$$

if $t > 0$.

With the energy estimate (2.16) an *a priori* estimate is given which can be used for the proof of the local existence as well as — in an associated nonlinear system — for the proof of global existence for small (or large) data. Replacing $\mathbb{R}^n$ by a bounded domain, the Fourier series expansion plays the rôle of the Fourier transform and leads to exponential decay of the solution immediately.

Back to the abstract system (2.1)–(2.4) we have to have an equivalent for the Fourier transform. For our purposes the spectral theorem for self-adjoint operators in the following form can be used:

There exists a Hilbert space $\mathcal{H} = \int_{\oplus} \mathcal{H}(\lambda) d\mu(\lambda)$, a direct integral of Hilbert spaces $\mathcal{H}(\lambda), \lambda \in \mathbb{R}$, with respect to a measure μ, and a unitary operator $\mathcal{A} : H \to \mathcal{H}$ such that for $m \in \mathbb{N}_0$

$$\begin{aligned} D(A^m) &= \{v \in H |\, \lambda \mapsto \lambda^m (\mathcal{A}v)(\lambda) \in \mathcal{H}\} \\ &= \{v \in H |\, \int_{-\infty}^{\infty} \lambda^{2m} |\mathcal{A}v(\lambda)|^2 d\mu(\lambda) < \infty\}, \end{aligned}$$

and

$$\mathcal{A}(A^m v)(\lambda) = \lambda^m (\mathcal{A}v)(\lambda).$$

Now we present the Proof of Theorem 2.1 in the simpler linear case

$$M = Q = 1, \quad N = -\delta > 0.$$

Let

$$v(t,\lambda) := \mathcal{A}u(t,\cdot)(\lambda), \quad \psi(t,\lambda) := \mathcal{A}\theta(t,\cdot)(\lambda).$$

Then we get from (2.1), (2.2)

$$v_{tt}(t,\lambda) + \lambda^2 v(t,\lambda) - \delta\lambda\psi(t,\lambda) = 0, \tag{2.17}$$

$$\psi_t(t,\lambda) + \lambda\psi(t,\lambda) + \delta\lambda v_t(t,\lambda) = 0. \tag{2.18}$$

For fixed $\lambda \in \mathbb{R}$ this is a system of ordinary differential equations for $v(\cdot,\lambda)$, $\psi(\cdot,\lambda)$.

Let the "energy" $\mathcal{E}$ be defined by

$$\mathcal{E}(t,\lambda) := \frac{1}{2}\left\{\|v_t(t,\lambda)\|^2_{\mathcal{H}(\lambda)} + \lambda^2\|v(t,\lambda)\|^2_{\mathcal{H}(\lambda)} + \|\psi(t,\lambda)\|^2_{\mathcal{H}(\lambda)}\right\}.$$

We shall prove that there exist $c, c_1, c_2, c_3 > 0$ such that for all $t \geq 0$ the following estimates hold:

$$\forall\, \lambda \geq c_3 \;:\quad \mathcal{E}(t,\lambda) \leq c\mathcal{E}(0,\lambda) e^{-\frac{c_1}{2}\lambda t}, \tag{2.19}$$

$$\forall\, 0 \leq \lambda \leq c_3 \;:\quad \mathcal{E}(t,\lambda) \leq c\mathcal{E}(0,\lambda) e^{c_2 t}. \tag{2.20}$$

For this purpose we multiply (2.17) by $v_t(t,\lambda)$ in $\mathcal{H}(\lambda)$ and (2.18) with $\psi(t,\lambda)$ in $\mathcal{H}(\lambda)$. Dropping the index $\mathcal{H}(\lambda)$ and the arguments (t,λ) for simplicity, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|v_t\|^2+\lambda^2\|v\|^2)-\delta\lambda\langle\psi,v_t\rangle &= 0,\\ \frac{1}{2}\frac{d}{dt}\|\psi\|^2+\lambda\|\psi\|^2+\delta\lambda\langle v_t,\psi\rangle &= 0.\end{aligned}$$

Taking real parts and summing up, we get

$$\frac{d}{dt}\mathcal{E}=-\lambda\|\psi\|^2. \tag{2.21}$$

Moreover,

$$\begin{aligned}\frac{d}{dt}\mathrm{Re}\,\langle\lambda v_t,v\rangle &= \mathrm{Re}(\lambda\langle v_{tt},v\rangle+\lambda\|v_t\|^2)\\ &= \mathrm{Re}(-\lambda^3\|v\|^2+\delta\lambda^2\langle\psi,v\rangle+\lambda\|v_t\|^2)\\ &\leq \lambda\|v_t\|^2-\lambda^3\|v\|^2+|\delta|\lambda^2\|\psi\|\,\|v\|\\ &\leq \lambda\|v_t\|^2-\frac{\lambda^3}{2}\|v\|^2+\frac{\delta^2}{2}\lambda\|\psi\|^2. \end{aligned}\tag{2.22}$$

$$\begin{aligned}\frac{d}{dt}\mathrm{Re}\,\langle-\psi,v_t\rangle &= \mathrm{Re}\,(\langle-\psi_t,v_t\rangle+\langle-\psi,v_{tt}\rangle)\\ &= \mathrm{Re}\,(\lambda\langle\psi,v_t\rangle+\delta\lambda\|v_t\|^2+\lambda^2\langle\psi,v\rangle-\delta\lambda\|\psi\|^2)\\ &= \lambda\mathrm{Re}\langle\psi,v_t\rangle+\delta\lambda\|v_t\|^2+\lambda^2\mathrm{Re}\,\langle\psi,v\rangle-\delta\lambda\|\psi\|^2\\ &\leq \frac{\delta\lambda}{2}\|v_t\|^2+\frac{\lambda}{-2\delta}\|\psi\|^2-\frac{\delta}{16}\lambda^3\|v\|^2-\frac{4}{\delta}\lambda\|\psi\|^2-\delta\lambda\|\psi\|^2. \end{aligned}\tag{2.23}$$

From (2.22), (2.23) we conclude

$$\frac{d}{dt}\mathrm{Re}\left\{\frac{-\delta}{4}\lambda\langle v_t,v\rangle-\langle\psi,v_t\rangle\right\}\leq\frac{\delta}{4}\lambda\|v_t\|^2+\frac{\delta}{16}\lambda^3\|v\|^2+c\lambda\|\psi\|^2 \tag{2.24}$$

with a positive constant $c=c(\delta)$.
If, for $\varepsilon>0$,

$$\tilde{\mathcal{E}}:=\mathcal{E}+\varepsilon\left(\frac{-\delta}{4}\right)\lambda\mathrm{Re}\,\langle v_t,v\rangle-\varepsilon\mathrm{Re}\,\langle\psi,v_t\rangle,$$

then, for suffiently small ε,

$$\frac{1}{2}\mathcal{E}\leq\tilde{\mathcal{E}}\leq 2\mathcal{E},$$

and (2.21), (2.24) imply

$$\frac{d}{dt}\tilde{\mathcal{E}}\leq -c_1\lambda\tilde{\mathcal{E}}+c_2\tilde{\mathcal{E}}$$

with $c_1,c_2>0$.

1.

$$\lambda \geq c_3 := \frac{2c_2}{c_1} \Rightarrow \frac{d}{dt}\tilde{\mathcal{E}} \leq -\frac{c_1}{2}\lambda\tilde{\mathcal{E}}, \tag{2.25}$$

which gives (2.19).

2.

$$0 \leq \lambda \leq c_3 \Rightarrow \frac{d}{dt}\tilde{\mathcal{E}} \leq c_2\tilde{\mathcal{E}},$$

which gives (2.20).

From (2.19), (2.20) we conclude for $m \in \mathbb{N}_0$, $t > 0$:

$$\int_0^\infty \lambda^{2m}\mathcal{E}(t,\lambda)d\mu(\lambda) \leq c(t,m)\int_0^\infty \mathcal{E}(0,\lambda)d\mu(\lambda).$$

Therefore, $u(t)$, $\theta(t) \in D(A^m)$, which completes the proof.

Q.E.D.

As the proof above shows, it is possible to obtain in this case

$$c_2 \leq 0.$$

Observe that c_2 is of the form

$$c_2 = c\varepsilon - 1/2,$$

which is non-positive, if

$$\varepsilon \leq \frac{1}{2c}.$$

In this case we have also proved the exponential decay, if $A \geq \nu > 0$. In particular, we obtain this for solutions to (2.5)–(2.8).

In some cases it is also possible to obtain information about decay rates even if A is not strictly positive, as the following result shows.

Theorem 2.2 *Let $\Omega = \mathbb{R}^n$ or let $n \geq 3$ and $\Omega = \mathbb{R}^n \setminus B$, where $B \neq \emptyset$ is bounded with smooth boundary and star-shaped. Then the solution (u,θ) to (2.5)–(2.8) satisfies*

$$\|(u_t, \Delta u, \theta)(t)\|_{L^\infty(\Omega)} \leq ct^{-n/2}\|(u_1, \Delta u_0, \theta_0)\|_{L^1(\Omega)}$$

with a positive constant $c > 0$.

For the Proof we only remark that for $\Omega = \mathbb{R}^n$ we use the Fourier transform, and the resulting equations in Fourier space are analogous to (2.17), (2.18), where the Fourier variable ξ plays the rôle of λ. Then the equivalent to (2.25) can be proved for any ξ. In the case of an exterior domain we can use a generalized Fourier transform. There the assumption on star-shapedness is required.

Q.E.D.

Finally, we state a global existence theorem to the nonlinear system (2.1)–(2.4) for small data and strictly positive A.

Theorem 2.3 *Let $A \geq \nu > 0$. Then there is $\mu > 0$ such that if $k \geq 3$ and*

$$(u_0, u_1, \theta_0) \in D(A^k) \times D(A^{k-1}) \times D(A^{k-1})$$

satisfies

$$\|u_0\|_{D(A^2)} + \|u_1\|_{D(A)} + \|\theta_0\|_{D(A)} < \mu,$$

then there exists a unique global solution to (2.1)–(2.4) *satisfying*

$$(u, \theta) \in \bigcap_{j=0}^{k} C^j([0,\infty), D(A^{k-j})) \times \bigcap_{j=0}^{k-1} C^j([0,\infty), D(A^{k-1-j})).$$

Moreover, (u, θ) decays exponentially.

The Proof shows an *a priori* bound for the local solution described above with the help of energy estimates in the spirit of the proof of Theorem 2.1. These bounds allow to continue a local solution up to $t = \infty$. The exponential decay is obtained as a by-product.

Q.E.D.

2.2 Large solutions and relations to viscoelasticity

For the special system

$$u_{tt} + M(\|Au\|^2, \|\theta\|^2)A^2u + N(\|Au\|^2, \|\theta\|^2)A\theta = 0, \tag{2.26}$$

$$R(\|Au\|^2, \|\theta\|^2)\theta_t + Q(\|Au\|^2, \|\theta\|^2)A\theta - N(\|Au\|^2, \|\theta\|^2)Au_t = 0, \tag{2.27}$$

where M, N, R, Q are smooth and M, N^2, R, Q are strictly positive with initial conditions

$$u(t=0) = u_0, \quad u_t(t=0) = u_1, \quad \theta(t=0) = \theta_0, \tag{2.28}$$

as well as the requirement

$$u(t) \in D(A^2), \quad \theta(t) \in D(A), \quad t \geq 0, \tag{2.29}$$

we shall obtain a global existence result for large data. The basic assumption that will assure the global existence is that there exists a function S, for which

$$\frac{\partial S}{\partial \sigma}(\sigma, \tau) = M(\sigma, \tau), \quad \frac{\partial S}{\partial \tau}(\sigma, \tau) = R(\sigma, \tau), \tag{2.30}$$

and

$$\exists\, C > 0 \quad \forall\, \sigma, \tau : \quad S(\sigma, \tau) \geq C(|\sigma| + |\tau|). \tag{2.31}$$

Theorem 2.4 *Let $T > 0$ be arbitrary and assume* (2.30), (2.31). *Let the initial data satisfy*

$$(u_0, u_1, \theta_0) \in D(A^3) \times D(A^2) \times D(A^2).$$

Then there exists a unique solution of the system (2.26)–(2.29) *satisfying*

$$u \in \bigcap_{j=0}^{3} C^j([0,T], D(A^{3-j})), \quad \theta \in \bigcap_{j=0}^{2} C^j([0,T], D(A^{2-j})).$$

For the PROOF we remark that the statements on local existence given at the beginning of Section 2.1 also apply to (2.26)–(2.29). Since the length of the interval of existence depends only on $(\|u_0\|_{D(A^2)}, \ \|u_1\|_{D(A)}, \ \|\theta_0\|_{D(A)})$ it has to be shown that the solution (u, θ) remains bounded in these norms. The additional assumptions (2.30), (2.31) allow to prove the following *a priori* estimate:

$$\sup_t \{\|u_t(t)\|^2 + \|Au(t)\|^2 + \|\theta(t)\|^2 + \int_0^t \|A^{1/2}\theta(\tau)\|^2 d\tau\} \leq \text{const} < \infty,$$

which shows that the so-called first energy is bounded uniformly in time. In turn, this is used to derive bounds for the higher-order energies, finally leading to a global existence theorem by a continuation argument.

The equations (2.26), (2.27) are a special case of the following "α-β-system":

$$u_{tt} + M(\|A^{1/2}u\|^2, \|\theta\|^2)Au + N(\|A^{1/2}u\|, \|\theta\|^2)A^\beta\theta = 0, \quad (2.32)$$

$$R(\|A^{1/2}u\|^2, \|\theta\|^2)\theta_t + Q(\|A^{1/2}u\|^2, \|\theta\|^2)A^\alpha\theta - N(\|A^{1/2}u\|^2, \|\theta\|^2)A^\beta u_t = 0, \quad (2.33)$$

where $\alpha, \beta \geq 0$, namely, if $\alpha = \beta = 1/2$, and if A is replaced by A^2 . If $\alpha = 1$, $\beta = 1/2$ it can be used to describe the one-dimensional thermoelastic system (2.11), (2.12), not literally, but with respect to smoothing and to decay of solutions. If $\alpha = 0$, $\beta = 1/2$, a viscoelastic system is hidden in (2.32), (2.33) which will be discussed later. Hence the α-β-system represents both smoothing ($\alpha = \beta = 1/2$) and non-smoothing ($\alpha = 0$, $\beta = 1/2$ or $\alpha = 1$, $\beta = 1/2$) thermoelastic systems.

With the help of the technique used in the proof of Theorem 2.1, i.e. deriving an estimate like

$$\frac{d}{dt}\mathcal{E}(t,\lambda) \leq \mathcal{E}(0,\lambda)e^{-\lambda^\gamma t} \quad (2.34)$$

for some suitable energy functional $\mathcal{E}$ and some $\gamma = \gamma(\alpha, \beta) > 0$, the smoothing property will follow. The estimate (2.34) can be proved, if α and β satisfy

$$|2\beta - 1| < \alpha < 2\beta. \quad (2.35)$$

For $\beta = 1/2$ these estimates are sharp, i.e. also necessary, $\alpha = 0$ or $\alpha = 1$ is non-smoothing.

For $\beta = 1/2$ we can easily describe the rate of decay of solutions to the following special linear α-β-system which will also give the connection to viscoelasticity below.

$$u_{tt} + Au + b(t)A^{1/2}\theta = 0, \tag{2.36}$$

$$\theta_t + a(t)A^{\alpha}\theta - b(t)A^{1/2}u_t = 0, \tag{2.37}$$

where a, b are continuous, bounded functions satisfying

$$a(t) \geq a_0 > 0, \quad b(t)^2 \geq b_0^2 > 0$$

for some constants a_0, b_0.

If $v = \mathcal{A}u$, $\psi = \mathcal{A}\theta$ is the notation as before, the first-order energy E,

$$E(t,\lambda) = \|v_t(t,\lambda)\|^2_{\mathcal{H}(\lambda)} + \lambda\|v(t,\lambda)\|^2_{\mathcal{H}(\lambda)} + \|\psi(t,\lambda)\|^2_{\mathcal{H}(\lambda)},$$

will satisfy an estimate of the type

$$E(t,\lambda) \leq cE(0,\lambda)e^{-\lambda^{r(\alpha)}t},$$

where $r(\alpha) > 0$. If A is strictly positive, this implies the exponential decay of solutions, in the general case $A \geq 0$ the integral

$$\int_0^1 e^{-\lambda^{r(\alpha)}t} d\mu(\lambda)$$

will determine the decay rate, which can be given, if one knows more about the measure μ, e.g. for the linear thermoelastic plate, $\alpha = 1/2$, $a = b = 1$, $A = \Delta^2$ in $\mathbb{R}^n$, μ is represented by the Fourier transform, $A \approx |\xi|^4$, $r(\alpha) = 1/2$ (see below and also Theorem 2.2) and

$$\int_0^1 e^{-\lambda^{r(\alpha)}t} d\mu(\lambda) \approx \int_0^1 e^{-|\xi|^2 t} d\xi \leq ct^{-n/2}$$

Theorem 2.5

(i)

$$A \geq \nu > 0 \Rightarrow \exists\, c > 0 \ \forall\, t \geq 0, \ \ \lambda \geq \nu : E(t,\lambda) \leq cE(0,\lambda)e^{-ct}. \tag{2.38}$$

(ii) $A \geq 0 \Rightarrow$ (2.38) *holds as* $\lambda \to \infty$, *and as* $\lambda \to 0$ *we have:*

for $0 \leq \alpha \leq 1/2, t \geq 0 : \ E(t,\lambda) \leq cE(0,\lambda)e^{-c\lambda^{1-\alpha}t}$,

for $1/2 \leq \alpha \leq 1,\ t \geq 0 : \ E(t,\lambda) \leq cE(0,\lambda)e^{-c\lambda^{\alpha}t}$.

Corollary 2.6 *Let $\Omega = \mathbb{R}^n$ or an exterior domain with star-shaped complement. If $A = (-\Delta)^\gamma$ for some $\gamma > 0$, $-\Delta$ being regarded as the self-adjoint realization in $L^2(\Omega)$ with Dirichlet boundary conditions, then the decay rates for the $L^\infty(\Omega)$-norm of solutions to* (2.36), (2.37) *with L^1-data are given by*

$$\begin{cases} t^{-n/(2\gamma(1-\alpha))} & \text{if} \quad 0 \le \alpha \le 1/2, \\ t^{-n/(2\gamma\alpha)} & \text{if} \quad 1/2 \le \alpha \le 1. \end{cases}$$

Remark: The decay rates are sharp.

We conclude this section with an application to viscoelasticity, i.e. we look at (2.36), (2.37) with $\alpha = 0$. Then we can compute θ from (2.37) leading to a single equation for u. Taking

$$\theta_0 := b(0) A^{1/2} u_0,$$

we arrive at

$$u_{tt} + (1 + b^2(t)) A u - \int_0^t k(t,\tau) A u(\tau) d\tau = 0, \tag{2.39}$$

$$u(t=0) = u_0, \quad u_t(t=0) = u_1, \tag{2.40}$$

where

$$\begin{aligned} k(t,\tau) &= b(t) e^{-\int_\tau^t a(\sigma)d\sigma} \{a(\tau)b(\tau) + b'(\tau)\}, \\ &= p(t) q(\tau) \end{aligned} \tag{2.41}$$

with

$$p(t) = b(t) e^{-\int_0^t a(\sigma)d\sigma}, \quad q(t) = \{a(t)b(t) + b'(t)\} e^{\int_0^t a(\sigma)d\sigma}. \tag{2.42}$$

Therefore, (2.39), (2.40) represent a viscoelastic system with kernel k that is not necessarily of convolution type ($k(t,\tau) \neq \tilde{k}(t-\tau)$ in general).

Observe that also a, b can be determined in terms of p, q. Then, under certain restrictions on p, q arising from conditions on a, b imposed above, Theorem 2.5 and Corollary 2.6 can be applied, leading to decay results also for this class of kernels. Typical examples are:

$$\begin{aligned} q(t) &= \sum_{k=1}^n p_k(t) e^{\mu_k t}, \quad p_k \text{ a positive polynomial}, \\ p(t) &= \frac{p(0)^2}{p(0) + \int_0^t q(\sigma) d\sigma}. \end{aligned}$$

<u>**Notes**</u>: The results in Section 2 are taken from our joint works with Muñoz Rivera [62], [63]. The exponential decay for the linear thermoelastic plate equations (2.5), (2.6) has been shown for one-dimensional models ($n = 1$) and Dirichlet boundary conditions $u = \frac{\partial u}{\partial n} = \theta = 0$ on $\partial\Omega$ (n: exterior normal) by Kim [43], for higher-dimensional models and various boundary

conditions by Liu & Zheng [50] and by Shibata [81]. The analyticity of the semigroup associated to (2.5), (2.6) and the boundary conditions $u = \frac{\partial u}{\partial n} = \theta = 0$ or $u = \Delta u = \theta = 0$ was established by Liu & Renardy [48]. For further results on viscoelastic equations see for example [9], [16], [61].

3 Second-order thermoelasticity

The equations of classical thermoelasticity arise from the equations of balance of linear momentum and the equations of balance of energy. They consist of a coupling of a second-order hyperbolic system — the system of elasticity — and a parabolic equation — the heat equation. In contrast to the situation in Section 2 or in Section 4, even small solutions cannot always be expected to exist for all times, and large data in general lead to a development of singularities in finite time.

The picture for one-dimensional models is more or less complete and will be recalled in Subsection 3.1, while in two or three dimensions only special nonlinear problems could be treated up to now as far as global existence of solutions is concerned, namely the Cauchy problem for initially isotropic and homogenous media and problems with a certain symmetry like radial symmetry. The latter recent result will be discussed in more detail in Subsection 3.2. The basic reason for still many open questions originates in the difficulty of describing the decay behavior of solutions to the *linear* problems; here all possibilities appear e.g. for bounded domains: exponential decay, no decay, arbitrarily slow decay. This depends on the geometry of the domain. The local well-posedness of the nonlinear system is well understood in the general case, of course, also that of the linearized systems. The linearized equations can be described as follows: Let $U = U(t,x)$ be the displacement vector, $\theta = \theta(t,x) := T_a(t,x) - T_0$ be the temperature difference, where T_a is the absolute temperature and T_0 is a constant reference temperature, $T_0 = 1$ without loss of generality. Then the linearized system is

$$\varrho(x)U_{tt} - \mathcal{D}'S(x)\mathcal{D}U + \mathcal{D}'\Gamma(x)\theta = \varrho(x)b(t,x), \tag{3.1}$$

$$c(x)\theta_t - \nabla' L(x)\nabla\theta + \Gamma'(x)\mathcal{D}U_t = r(t,x), \tag{3.2}$$

with initial conditions

$$U(t=0) = U_0, \quad U_t(t=0) = U_1, \quad \theta(t=0) = \theta_0, \tag{3.3}$$

and — for example — Dirichlet type boundary conditions "rigidly clamped, constant temperature"

$$U = 0, \quad \theta = 0 \qquad \text{on } \partial\Omega, \tag{3.4}$$

where $\Omega \subset \mathbb{R}^n$ is the undistorted reference configuration, $n = 1, 2, 3$. Of course, boundary conditions of Neumann type, "traction free, insulated" and combinations can be considered. In (3.1), (3.2) ϱ is a symmetric density matrix, S is a $N \times N$ symmetric, positive definite matrix ($N = 6$ in $\mathbb{R}^3$), Γ is a vector with N coefficients determining the stress-temperature tensor, c is the specific heat, L is the heat conductivity tensor and b and r are the specific extrinsic body force and the extrinsic heat supply, respectively. $\mathcal{D}$ denotes the following generalized gradient,

$$\mathcal{D} = \begin{pmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_2 & 0 \\ 0 & 0 & \partial_3 \\ 0 & \partial_3 & \partial_2 \\ \partial_3 & 0 & \partial_1 \\ \partial_2 & \partial_1 & 0 \end{pmatrix} \text{ in } \mathbb{R}^3, \quad \mathcal{D} = \begin{pmatrix} \partial_1 & 0 \\ 0 & \partial_2 \\ \partial_2 & \partial_1 \end{pmatrix} \text{ in } \mathbb{R}^2, \quad \mathcal{D} = \partial_1 \text{ in } \mathbb{R}^1.$$

Remark: A prime $'$ denotes the transposition of a matrix.

In the simplest case of a homogenous, isotropic medium in $\mathbb{R}^3$, the equations reduce to

$$U_{tt} - ((2\mu + \lambda)\nabla\nabla' - \mu\nabla \times \nabla\times)U + \gamma\nabla\theta = b, \tag{3.5}$$

$$\delta\theta_t - \beta\Delta\theta + \gamma\nabla' U_t = r, \tag{3.6}$$

where $\mu, \lambda, \delta, \beta$ are constants, μ, λ are the Lamé moduli,

$$\mu > 0, \quad 2\mu + 3\lambda > 0, \quad \delta, \beta > 0, \quad \gamma \neq 0.$$

In one space-dimension one has with $\alpha > 0$ and $x := x_1$

$$U_{tt} - \alpha U_{xx} + \gamma\theta_x = b, \tag{3.7}$$

$$\delta\theta_t - \beta\theta_{xx} + \gamma u_{tx} = r. \tag{3.8}$$

3.1 Results in one space-dimension

For the linear equations (3.7), (3.8) with prescribed initial conditions (3.3) and boundary conditions

$$u = \theta = 0 \text{ on } \partial\Omega, \tag{3.9}$$

or

$$u_x = \theta = 0 \qquad \text{or} \qquad u = \theta_x = 0 \qquad \text{on } \partial\Omega, \tag{3.10}$$

where $u := U$ and $\Omega \subset \mathbb{R}^1$, it is known that for $\Omega = (0,1)$ $u(t)$ and $\theta(t)$ decay to zero exponentially as $t \to \infty$, while for $\Omega = (0, \infty)$ or $\Omega = (-\infty, \infty)$ the L^∞-norm of $u(t)$ and $\theta(t)$ decays to zero like $t^{-1/2}$. Hence, the damping by heat conduction is strong enough to assure that also the hyperbolic part represented by u decays like a solution to a heat equation. Nevertheless, it does not become a kind of parabolic problem, since singularities are propagated, as can be shown e.g. for the boundary conditions (3.10). The proof of this non-smoothing exploits the fact that one can derive a third-order (in t) differential equations for u only, for which the behavior of the associated three characteristic roots can be determined explicitly.

The one-dimensional nonlinear system is given by

$$u_{tt} - \tilde{S}(u_x, \theta)_x = f_1, \tag{3.11}$$

$$(\theta + T_0)N(u_x, \theta)_t - Q(u_x, \theta, \theta_x)_x = f_2. \tag{3.12}$$

We consider initial conditions (3.3) and the boundary conditions (3.9) in $\Omega = (0, 1)$.

$\tilde{S}, N, Q$ are real-valued functions satisfying

$$\frac{\partial \tilde{S}}{\partial u_x}(0,0) = \alpha > 0, \quad \frac{\partial N}{\partial \theta}(0,0) = \delta > 0, \quad -\frac{\partial \tilde{S}}{\partial \theta}(0,0) = \frac{\partial N}{\partial u_x}(0,0) = \gamma \neq 0,$$

$$\frac{1}{T_0}\frac{\partial Q}{\partial \theta_x}(0,0,0) = \beta > 0, \quad \frac{\partial Q}{\partial u_x}(0,0,0) = \frac{\partial Q}{\partial \theta}(0,0,0) = 0.$$

$\tilde{S}$ arises from the stress tensor, N from the entropy and Q from the heat flux. For the next theorem we assume

$$Q = Q(\theta_x)$$

and let $\overline{D}^j$ denote all derivatives with respect to t and to x from order zero up to order $j \in \mathbb{N}_0$. Moreover, let

$$u_j := \frac{\partial^j u}{\partial t^j}(t = 0), \qquad \theta_j := \frac{\partial^j \theta}{\partial t^j}(t = 0).$$

Theorem 3.1 *Assume* $u_i \in H^{3-i}(\Omega) \cap H_0^1(\Omega),\ i = 0, 1, 2,\ \theta_j \in H^{3-j}(\Omega) \cap H_0^1(\Omega),\ j = 0, 1,$ *and* $|\theta_0(x)| < T_0$ *for* $x \in \overline{\Omega}$, $f_1, f_2 \in C^2([0,\infty), L^2(\Omega)) \cap C^1([0,\infty), H^1(\Omega))$. *Let*

$$\lambda(t) := \|\overline{D}^1(f_1, f_2)(t,\cdot)\|^2 + \|\partial_t^2(f_1, f_2)(t,\cdot)\|^2 + \|\partial_t\partial_x(f_1, f_2)(t,\cdot)\|^2.$$

Then there exists $\varepsilon > 0$ *such that if*

$$\|(u_0, u_1, \theta_0, \theta_1)\|_{2,2}^2 + \|u_2\|_{1,2}^2 + \sup_{t \geq 0} \lambda(t) \leq \varepsilon,$$

the initial boundary value problem (3.11), (3.12), (3.3), (3.9) *has a unique global solution*

$$u \in \bigcap_{j=0}^{3} C^j([0,\infty), H^{3-j}(\Omega)), \quad \theta \in \bigcap_{j=0}^{1} C^j([0,\infty), H^{3-j}(\Omega)), \quad \theta \in C^2([0,\infty), L^2(\Omega)).$$

Moreover, there are constants c_1, c_2 *such that for* $t \geq 0$

$$\|\overline{D}^3 u(t,\cdot)\|^2 + \|\overline{D}^2 \theta(t,\cdot)\|^2 + \|\theta_{txx}(t,\cdot)\|^2 + \|\theta_{xxx}(t,\cdot)\|^2$$
$$\leq c_1 e^{-c_2 t}(\|(u_0, u_1, \theta_0, \theta_1)\|_{2,2}^2 + \|u_2\|_{1,2}^2 + \int_0^t e^{c_2 r}\lambda(r)dr).$$

This result shows the global well-posedness for small data and the exponential decay of solutions for exponentially decaying right-hand sides. The PROOF consists of energy estimates for u and θ obtained from the differential equations and their differentiated — with respect to t *and* to x — versions. A tricky treatment of boundary terms, first having been introduced for the linearized equations, is another important ingredient.

Q.E.D.

The Cauchy problem, $(\Omega = \mathbb{R})$ or the half-line $(\Omega = (0, \infty))$ has also been treated. For certain *large* data, a blow-up occurs.

3.2 Results in three space-dimensions

In three space dimensions, only the Cauchy problem and problems with a certain symmetry (see below) have been dealt with satisfactorily. Satisfactorily means that there are results on the global well-posedness of the nonlinear system. Since the proofs rest on the knowledge of the decay rates to the linearized systems, this will be strongly connected to the geometry of the underlying domain. For the linear equations (3.1), (3.2) in a bounded domain with vanishing b, r it is known that, e.g. for the Dirichlet boundary conditions (3.9), as $t \to \infty$, θ tends to zero and u is asymptotically equal to another function $\tilde{u}$. Whether $\tilde{u}$ is zero depends on the geometry of the domain, e.g. $\tilde{u} = 0$ for a cube but $\tilde{u} \neq 0$ in general for the unit ball. Even if $\tilde{u} = 0$, the rate of decay is not known in general. For the homogeneous, isotropic linear case (3.5), (3.6), i.e.

$$U_{tt} - ((2\mu + \lambda)\nabla\nabla' - \mu\nabla \times \nabla\times)U + \gamma\nabla\theta = 0, \qquad (3.13)$$

$$\delta\theta_t - \beta\Delta\theta + \gamma\nabla' U_t = 0, \qquad (3.14)$$

$$U(t=0) = U_0, \quad U_t(t=0) = U_1, \quad \theta(t=0) = \theta_0, \qquad (3.15)$$

$$U_{|\partial\Omega} = 0, \quad \theta_{|\partial\Omega} = 0, \qquad (3.16)$$

the situation is as follows:

- The solution (U, θ) to (3.13)–(3.16) tends to zero for arbitrary initial data if and only if the vector eigenvalue problem (3.17) with side condition (3.18),

$$-\Delta U + \lambda U = 0, \quad U_{|\partial\Omega} = 0, \qquad (3.17)$$

$$\nabla' U = 0, \qquad (3.18)$$

  does not have a nontrivial solution. Hence the solution tends to zero in a cube, but not necessarily in a ball.

- A sufficient condition for the existence of only the trivial solution $U = 0$ to (3.17), (3.18) is that the *scalar* eigenvalue problem

$$-\Delta v + \lambda v = 0, \quad v_{|\partial\Omega} = 0, \tag{3.19}$$

  has only distinct eigenvalues, i.e. all eigenvalues are simple.

  - Domains $\Omega \subset \mathbb{R}^2$ for which $\partial\Omega$ is smooth and a straight line in a neighborhood of a point, are examples, where (3.19) has only simple eigenvalues.
  - In every C^3-neighborhood of a domain Ω, there is a domain Ω^*, such that (3.19) has only simple eigenvalues. With respect to an appropriate C^3-metric, the set of domains Ω leading to multiple eigenvalues in (3.19) is of first Baire category.

- If (U, θ) decay to zero (essentially: U), the decay may be arbitrarily slow, namely for domains containing a cylinder the ends of which are part of the boundary, e.g. a cube.

- For radially symmetrical domains and data, the solution tends to zero exponentially, more generally, if $\nabla \times U = 0$ (rotation $= 0$) identically.

This shows that all cases of decay or non-decay appear which makes it difficult to prove a general theorem, or rather impossible for the nonlinear situation, since up to now the proofs use the decay behavior strongly.

We deal with the case $\nabla \times U = 0$ in a smoothly bounded domain Ω.

Remark: Defining the rotation of a scalar field f in $\mathbb{R}^2$ to be the vector field

$$\nabla \times f := (\partial_2 f_1, -\partial_1 f)'$$

and the rotation of a vector field F in $\mathbb{R}^2$ to be the scalar

$$\nabla \times F := \partial_1 F_2 - \partial_2 F_1$$

the formula

$$\Delta = \nabla\nabla' - \nabla \times \nabla$$

also holds in $\mathbb{R}^2$ and the following Theorem 3.2 applies to two- and three-dimensional situations.

Theorem 3.2 (i) *Let (u, θ) be the solution to* (3.13)–(3.16) *and assume*

$$\nabla \times U = 0 \quad \text{in } [0, \infty) \times \Omega. \tag{3.20}$$

Then there are constants $\Gamma \geq 1$ and $\kappa > 0$ such that

$$\forall t \geq 0: \ E(t) + \int_0^t e^{\kappa s} \|\nabla\theta_t(s)\|^2 ds \leq \Gamma E(0),$$

where

$$E(t) := e^{\kappa t}\left\{\sum_{k=0}^{2}\|\partial_t^k U(t)\|_{2-k,2}^2 + \|\theta_t(t)\|^2 + \|\theta(t)\|_{2,2}^2\right\}.$$

(ii) (i) *applies to radially symmetrical domains* Ω *and functions* U_0, U_1, θ_0.

Part (ii) easily follows from part (i) since

$$U(t,x) = x\Phi(t,|x|)$$

for some $\Phi : [0,\infty)\times[0,\infty)\to\mathbb{R}$ if radial symmetry is given. We present a sketch of the Proof of part (i): Observe that because of (3.20)

$$\|\nabla U\| = \|\nabla' U\|.$$

Defining $\tau := 2\mu+\lambda$ and the "energies"

$$\begin{aligned}
F_1(t) &:= \frac{1}{2}(\|U_t\|^2 + \tau\|\nabla U\|^2 + \delta\|\theta\|^2)(t),\\
F_2(t) &:= \frac{1}{2}(\|U_{tt}\|^2 + \tau\|\nabla U_t\|^2 + \delta\|\theta_t\|^2)(t),\\
F_3(t) &:= \frac{1}{2}(\|\nabla U_t\|^2 + \tau\|\nabla\nabla' U\|^2 + \delta\|\nabla\theta\|^2)(t),\\
F(t) &:= \sum_{j=1}^{3} F_j(t),
\end{aligned}$$

we have

$$\frac{d}{dt}F_1 = -\beta\|\nabla\theta\|^2, \quad \frac{d}{dt}F_2 = -\beta\|\nabla\theta_t\|^2$$

and

$$\begin{aligned}
\frac{d}{dt}F_3 &= -\beta\|\Delta\theta\|^2 + \gamma\int_{\partial\Omega}\frac{\partial\theta}{\partial n}\nabla' U_t\\
&\leq -\beta\|\Delta\theta\|^2 + \varepsilon\tau\int_{\partial\Omega}|\nabla' U_t|^2 + \frac{C}{\varepsilon}\int_{\partial\Omega}\left|\frac{\partial\theta}{\partial n}\right|^2,
\end{aligned}$$

where n denotes the exterior normal for $\partial\Omega$, $0<\varepsilon<1$ is chosen appropriately later, and $C>0$. Let $\sigma\in(C^1(\overline{\Omega}))^3$ such that $\sigma = n$ on $\partial\Omega$. Then we obtain for

$$H(t) := \{\eta F_1 + \eta F_2 + F_3 - \frac{2\varepsilon\tau}{\mu+\lambda}\sum_{k=1}^{(2)3}\int_\Omega U_{tt}\sigma_k\partial_k U_t + \sqrt{\varepsilon}\int_\Omega U_t U + \sqrt{\varepsilon}\int_\Omega \nabla' U\nabla' U_t\},$$

where η is some positive constant of the order ε^{-2},

$$\forall\, t\geq 0: \quad \frac{d}{dt}H(t) + C\left(\|\nabla\theta_t(t)\|^2 + \int_{\partial\Omega}\left|\frac{\partial U_t(t)}{\partial n}\right|^2 + F(t)\right) \leq 0,$$

whence the assertion of the Theorem follows since $F(t)$, $H(t)$ and $E(t)$ can be estimated each in terms of the others, if ε is chosen small enough.

Q.E.D.

The exponential decay described in Theorem 3.2 allows to prove a result for small data for corresponding nonlinear systems. We assume that the medium is homogeneous and initially isotropic. Then the differential equations for U and θ take the form (forces, heat supply = 0)

$$U_{tt} - ((2\mu+\lambda)\nabla\nabla' - \mu\nabla\times\nabla\times)U + \gamma\nabla\theta = F(\nabla U, \nabla^2 U, \theta, \nabla\theta), \tag{3.21}$$

$$\delta\theta_t - \beta\Delta\theta + \gamma\nabla' U_t = g(\nabla U, \nabla^2 U, \nabla U_t, \theta, \nabla\theta, \nabla^2\theta), \tag{3.22}$$

where F, g are smooth and satisfy $F(0) = 0,\ g(0) = 0$.
As mentioned earlier, the local well-posedness is known even in more general cases.

Theorem 3.3 (i) *Let Ω be bounded with smooth boundary, let (U,θ) be a local solution of* (3.21), (3.22), (3.15), (3.16) *in* $[0,T)\times\Omega \equiv \Omega_T$.
If

$$\nabla \times U = 0 \quad in \quad \Omega_T$$

then there exists $\varepsilon > 0$ such that if

$$\sum_{j=0}^{4} \left\| \partial_t^j U(t=0) \right\|^2_{4-j,2} + \sum_{j=0}^{2} \|\partial_t^j \theta(t=0)\|^2_{4-j,2} + \|\partial_t^3\theta(t=0)\|^2 \le \varepsilon^2,$$

then $T = \infty$. Moreover, $\|U(t)\|_{4,2}$ and $\|\theta(t)\|_{4,2}$ decay to zero exponentially.

(ii) (i) *applies to radially symmetrical domains Ω and data U_0, U_1, θ_0.*

The Proof of part (i) derives *a priori* estimates for U and θ in the spirit of the proof of Theorem 3.2. Part (ii) is an easy consequence, then of course, F and g have to satisfy certain restrictions such that radially symmetrical solutions exist.

Q.E.D.

Results were known before for the Cauchy problem $\Omega = \mathbb{R}^3$, where the decay to the linearized problem can be described with the help of the Fourier transform. Global solutions exist for small data if F and g do not contain quadratic terms only involving ∇U, ∇U_t, $\nabla^2 U$ and additionally: either only quadratic terms appear or at least cubic terms and at most one quadratic term of the type $\theta\Delta\theta$. For general nonlinearities a blow-up in finite time has to be expected; in this case the hyperbolic part predominates.

Notes: For a derivation of the equations see Carlson [5], for a comprehensive treatment of linear well-posedness and asymptotic stability in bounded domains see Dafermos [8], for the Cauchy problem or unbounded domains see Leis [45, 46, 47] and the author [66, 67]; compare Kupradze [44] for an approach via integral equations.

The results on the one-dimensional systems, both decay results and global existence theorems for various boundary conditions are contained in the papers by Slemrod [83], Leis [46], Kawashima [40], Zheng [85, 86], Dassios & Grillakis [15], Kawashima & Okada [41], Shen & Zheng [78], Zheng & Shen [88, 89], Hrusa & Tarabek [25], Jiang [29, 33, 34, 35], Kim [43], Muñoz Rivera [58, 60], Shibata [80], S.W. Hansen [20], Henry, Perissinitto & Lopes [23], Shibata, Zheng & the author [74], from where Theorem 3.1 is taken, Liu & Zheng [49], Burns, Liu & Zheng [4], see also [68, 71].

Kawashima & Shibata [42] studied the Neumann problem with time-independent forces. The development of singularities for large data was shown by Dafermos & Hsiao [13], Hrusa & Messaoudi [24] and I. Hansen [21].

Periodic solutions are studied by Feireisl [17] and by Shibata, Zheng & the author [74]. The asymptotic behavior of solutions as $|x| \to \infty$ was investigated by Jiang [30, 32] in both one and three dimensions.

We remark that there exist papers on numerical investigations, for the nonlinear models see for example de Moura [55], de Moura & Suaiden [56] and Jiang [27, 28, 31]. Weak solutions for systems in one space-dimension that do not conduct heat have been studied by Chen & Dafermos [6], and for systems of the type

$$\begin{aligned} u_{tt} - \sigma(u_x) + \theta_x &= 0, \\ \theta_t - \theta_{xx} + u_{tx} &= 0, \end{aligned}$$

weak solutions to various initial boundary value problems have recently been obtained by Rustenbach [77]. For *special semilinear* thermoelastic models global attractors are considered by Hale & Perissinotto [19].

For local existence theorems in one or more space dimensions see Chrzęszczyk [7], Jiang & the author [39], Mukoyama [57], Dan [14] and the references therein. The description of the possible asymptotic behavior of solutions to the linear system given at the beginning of Subsection 3.2 is based on the papers of Dafermos [8, 10], Micheletti [52, 53, 54], Uhlenbeck [84], Henry [22] and Jiang, Muñoz Rivera & the author [38]. For the linear Cauchy problem see Leis [46], Dassios & Grillakis [15], the author [69], Gawinecki [18] and Muñoz Rivera [59], for a cubic medium see Borkenstein [3]. For Theorem 3.2 and Theorem 3.3 see Jiang, Muñoz Rivera & the author [38], also the work of Jiang [37], while for the general Cauchy problem see Ponce & the author [65] and the author [69, 70], the latter for a blow-up result.

4 One-dimensional thermoviscoelasticity

In this final section we shall report on a result on the global existence of solutions and on the asymptotic behavior of solutions to one-dimensional nonlinear thermoviscoelastic systems for large data which are not necessarily smooth. As we pointed out in the previous section, neglecting viscous effects may lead to a blow-up for large data. Now we obtain a global solution $(u, v, \vartheta) =$ (deformation gradient, velocity, temperature) in $L^\infty \times W^{1,\infty} \times H^1$. In particular, the strain has to be in L^∞ only which allows to consider situations where different phases coexist in the material. The constitutive assumptions include models for the study of phase transition problems in shape memory alloys. The referential (Lagrangian) form of the balance laws of mass, momentum and energy for one-dimensional materials with reference densities $\varrho_0 = 1$ can be written as

$$u_t - v_x = 0, \tag{4.1}$$

$$v_t - \sigma_x = 0, \tag{4.2}$$

$$(e + \frac{1}{2}v^2)_t - (\sigma v)_x - q_x = 0, \tag{4.3}$$

and the second law of thermodynamics is expressed by the Clausius-Duhem inequality

$$\eta_t + \left(\frac{q}{\vartheta}\right)_x \geq 0.$$

Here $u, v, \sigma, e, \eta, \vartheta$ and q denote deformation gradient (strain), velocity, stress, internal energy, specific entropy, temperature and heat flux, respectively. The reference configuration Ω is assumed to equal the unit interval (0,1).

We assume that the material has viscoelastic damping of rate type, i.e.

$$\sigma = \frac{\partial \psi}{\partial u}(u, \vartheta) + \mu v_x$$

with constant $\mu > 0$, where

$$\psi = e - \vartheta\eta$$

is the Helmholtz free energy satisfying

$$\eta = -\psi_\vartheta.$$

Moreover, we assume that for some constant $k > 0$

$$q = k\vartheta_x$$

and

$$\psi(u, \vartheta) = -c_v \vartheta \ln \vartheta + c_1 \vartheta + F_1(u)\vartheta + F_2(u),$$

with constants $c_v, c_1 > 0$ and smooth F_1, F_2. Then (4.1)–(4.3) turn into

$$u_t - v_x = 0, \tag{4.4}$$

$$v_t - (f_1(u)\vartheta + f_2(u) + \mu v_x)_x = 0, \tag{4.5}$$

$$c_v\vartheta_t - \vartheta f_1(u)v_x - \mu v_x^2 - k\vartheta_{xx} = 0, \tag{4.6}$$

where

$$f_1 = F_1', \quad f_2 = F_2', \text{ and } \sigma = f_1(u)\vartheta + f_2(u) + \mu v_x.$$

Initial conditions are prescribed by

$$u(t=0) = u_0, \quad u_t(t=0) = u_1, \quad \vartheta(t=0) = \vartheta_0, \tag{4.7}$$

and the boundary conditions are given by

$$\vartheta_x = 0 \qquad \text{for } x = 0, 1 \tag{4.8}$$

and

$$\sigma = 0 \qquad \text{for } x = 0, 1 \tag{4.9}$$

or

$$v = 0 \qquad \text{for } x = 0 \qquad \text{and} \qquad \sigma = 0 \qquad \text{for } x = 1. \tag{4.10}$$

Concerning f_1, f_2 we assume

$$\exists\, u_- < u_+ : f_1(u), f_2(u) \left\{ \begin{array}{c} < \\ > \end{array} \right\} 0 \text{ as } u \left\{ \begin{array}{cc} \leq & u_- \\ \geq & u_+ \end{array} \right\}. \tag{4.11}$$

Then the global existence for any data

$$u_0 \in L^\infty(\Omega), \;\; v_0 \in W^{1,\infty}(\Omega), \;\; \vartheta_0 \in H^1(\Omega), \text{ with } \vartheta_0 > 0 \text{ for } x \in [0,1] \tag{4.12}$$

is assured by the following theorem.

Theorem 4.1 *Under the assumptions above, in particular* (4.11), (4.12), *there exists a unique weak solution* (u, v, ϑ) *to the problem* (4.4)–(4.9) *resp.* (4.10) *such that*

$$(u,\vartheta) \in C^0([0,\infty), L^\infty(\Omega) \times H^1(\Omega)), v \in C^0((0,\infty), W^{1,\infty}(\Omega)) \cap L^\infty([0,\infty), W^{1,\infty}(\Omega))$$

$$u_t \in L^\infty([0,\infty), L^\infty(\Omega)), \qquad \int_1^x v\, dx \in W^{1,\infty}([0,\infty), L^\infty(\Omega)),$$

$$\vartheta_x \in L^2([0,\infty), H^1(\Omega)), \quad \vartheta_t \in L^2([0,\infty), L^2(\Omega))$$

with $\vartheta > 0$ *in* $[0,\infty) \times \Omega$. *Moreover, as* $t \to \infty$,

(i) $\|v(t)\|_{1,2} \to 0, \quad \|\vartheta_x(t)\| \to 0,$

(ii) $\|\vartheta(t) - \overline{\vartheta}(t)\|_{L^\infty} \to 0$, *where* $\overline{\vartheta}(t) := \int_0^1 \vartheta(t,x)dx$, $\|\sigma(t)\| \to 0.$

If additionally $F_2 = 0$ *then also*

(iii) $\|\vartheta(t) - \frac{E}{c_v}\|_{L^\infty} \to 0$, *where* $E := \int_0^1 (c_v\vartheta_0 + \frac{1}{2}v_0^2)(x)dx$,

(iv) $\exists\, u_\infty \in L^\infty(\Omega) : u(\cdot,t) \to u_\infty$ *almost everywhere, and* $f_1(u_\infty) = 0$, *i.e.* u_∞ *is a stationary state.*

A weak solution — for the boundary condition (4.10) in the sequel — is a solution with the regularity described in Theorem 4.1 that satisfies the differential equations (4.4)–(4.6) in the distributional sense, and for which for any $T > 0$ and $\varphi \in H^1(\Omega)$, $\varphi_{|x=0} = 0$, $w \in L^2((0,T), H^1(\Omega))$ the following holds:

$$-\int_0^1 \left(\partial_t \int_1^x v\,dy\right) \varphi_x dx + \int_0^1 \sigma\varphi_x dx = 0,$$

$$\int_0^T \int_0^1 (c_v\vartheta_t - f_1(u)\vartheta v_x - \mu v_x^2)w + k\vartheta_x w_x \,dx\,dt = 0,$$

and for which the boundary condition $v_{|x=0} = 0$ and the initial conditions (4.7) are satisfied.

Sketch of the Proof of Theorem 4.1:

First a local existence theorem is proved by using the transformation

$$p(t,x) := \int_1^x v(t,y)dy, \quad q := \mu u - p \tag{4.13}$$

and Green's function for the heat equation which transform the differential equations into a system of integral equations. This can be solved in appropriate spaces using energy estimates and delicate estimates in $W^{2,\infty}(\Omega)$. Observe that q satisfies an ordinary differential equation in t (for fixed x):

$$q_t = -f_1\left(\frac{p+q}{\mu}\right)\vartheta - f_2\left(\frac{p+q}{\mu}\right).$$

Then one has to prove the uniform boundedness of

$$\|u(t)\|_{L^\infty}, \quad \|v(t)\|_{W^{1,\infty}}, \quad \|\vartheta(t)\|_{1,2}.$$

The positivity of ϑ is necessary for this and can be proved with a maximum principle for weak solutions to the heat equation. The proof of the boundedness of $\|u(t)\|_{L^\infty}$ uses the assumption (4.11). The *a priori* estimates proving the uniform boundedness of $\|v(t)\|_{W^{1,\infty}}$ and $\|\vartheta(t)\|_{1,2}$

as well as asymptotic behavior claimed in Theorem 4.1 are elaborate combining different techniques and will not be reproduced here.

Notes: The results in this section are based on a paper of Zheng & the author [75]. The case where $v = 0$ in $x = 0$ and $x = 1$ was recently dealt with by Shen, Zheng & Zhu [79]. The transformation (4.13) goes back to Pego [64] and Andrews [2]. The existence of classical solutions has been investigated by Dafermos [11], Dafermos & Hsiao [12] and Jiang [36]; Luo [51] resp. Hsiao & Luo [26] gave a description of the asymptotic behavior for $F_2 = 0$. For related papers in phase transition in shape memory alloys see [75] and also Zheng [87].
For smooth solutions for *small* data also in more than one space dimension we refer to Shen & Zheng [78], Shibata [82] and Rohn [76].

The following list of references is not assumed to be complete. Any comments and suggestions for future notes are welcome.

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