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Newton's polygon in the theory of singular perturbations of boundary value problems

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Abstract. In this paper we discuss ellipticity conditions for some parameter-dependent boundary value problems which do not satisfy the Agmon–Agranovich–Vishik condition of ellipticity with parameter. The appropriate definition of ellipticity uses the concept of the Newton polygon. For the corresponding boundary value problems with small parameter we construct the formal asymptotic solution, thus explaining the nature of the Shapiro–Lopatinskii condition for these problems.

1. Introduction

Starting with the papers of Agmon [1] and Agranovich–Vishik [2], the theory of ellipticity with parameter was developed which treats boundary value problems depending on a complex parameter λ. Under some ellipticity conditions, uniform (with respect to the parameter) a priori estimates and solvability results were obtained. In the proofs of these results, a key point was the fact that the corresponding λ-dependent symbols were (quasi-)homogeneous with respect to the covariables and the parameter λ.

However, there are some situations where the Agmon–Agranovich–Vishik theory cannot be applied. For instance, let us consider a matrix differential operator

$$A(x,D) = \big(A_{ij}(x,D)\big)_{i,j=1,\dots,N} \tag{1.1}$$

having Douglis–Nirenberg structure; i.e. suppose that there exist nonnegative integers $s_1,\dots,s_N$ and $t_1,\dots,t_N$ with $\operatorname{ord} A_{ij} \le s_i + t_j$ such that the principal symbol (in the sense of Douglis–Nirenberg) $A^0(x,\xi) = \big(A^0_{ij}(x,\xi)\big)_{i,j=1,\dots,N}$ is invertible for $\xi \neq 0$. If we consider $A(x,D) - \lambda I_N$ where I_N stands for the unit matrix, the symbol $A^0(x,\xi) - \lambda I_N$ is, in general, not quasi-homogeneous with respect to ξ and λ. This is due to the fact that the operators on the diagonal may have different orders.

In the case of "constant order", i.e. if $s_1 = \dots = s_N = 0$ and $t_1 = \dots = t_N = 2m$, the Agmon–Agranovich–Vishik (AAV) condition says that there exists a ray $\mathcal{L}$ in the complex plane (starting at the origin) and a constant C, independent of x,ξ and λ, such that

$$\Big|\det\big(A^0(x,\xi) - \lambda I_N\big)\Big| \ge C\,\big(|\xi|^{2m} + |\lambda|\big)^N$$

holds for all x,ξ and $\lambda \in \mathcal{L}$. The AAV theory also includes a parameter-dependent version of the Shapiro–Lopatinskii condition which ensures solvability for boundary value problems connected with the operator $A(x,D) - \lambda I_N$. What is the analogue of the AAV condition and of the Shapiro–Lopatinskii condition for general operators of the form (1.1)?

The same question can be posed for scalar operator pencils given by

$$P(x,D,\lambda)=\sum_{\alpha,k}a_{\alpha k}(x)\lambda^k D^\alpha \tag{1.2}$$

or for general non-stationary operators

$$P(x,D_x,D_t)=\sum_{\alpha,k}a_{\alpha k}(x,t)D_t^k D_x^\alpha\,. \tag{1.3}$$

In the present paper we concentrate on parameter-dependent operator matrices of the form

$$A(x,D,\lambda)=(A_{11}(x,D)A_{12}(x,D)A_{21}(x,D)A_{22}(x,D)-\lambda)\,. \tag{1.4}$$

Such matrices may be considered as a mixture of a parameter-independent system (possibly of Douglis–Nirenberg structure) and of a parameter-dependent system as discussed above. Operators of the form (1.4) may serve as model problems in the theory of general ellipticity with parameter as it will be clear below (see Remark 2.5). We also want to mention that there are direct applications of operators of the form (1.4) to transmission problems which we want to discuss in a forthcoming paper. Replacing in (1.4) the parameter λ by ε^{-1}, we obtain a problem with small parameter ε; this leads to the theory of singular perturbations of boundary value problems as started by Vishik–Lyusternik [14]. The aim of the present paper is to introduce and study a general notion of ellipticity with parameter for (1.4) and the related problem with small parameter. We will construct (in Section 4) the so-called formal asymptotic solution for this problem; in particular, this construction allows us to understand the appearance of non-standard ellipticity conditions.

2. Newton's polygon and ellipticity with parameter

In this section we will sketch the fundamental concepts and results in the theory of N-ellipticity with parameter which is based on the so-called Newton polygon (see, e.g., [9] and [3]–[5]). We start with operator pencils of the form (1.2) and consider for fixed x the polynomial

$$P(\xi,\lambda)=P(x,\xi,\lambda):=\sum_{\alpha,k}a_{\alpha k}(x)\lambda^k\xi^\alpha\,. \tag{2.1}$$

Define Newton's polygon $N(P)$ of the symbol (2.1) as the convex hull in $\mathbb{R}^2$ of all points $(|\alpha|,k)$ with $a_{\alpha k}(x)\neq 0$, their projections $(|\alpha|,0)$ and $(0,k)$ and the origin $(0,0)$. As a simple example, consider

$$\begin{aligned}P(\xi,\lambda) &= (\lambda+P_{2\mu}(\xi))(\lambda+P_{2m-2\mu}(\xi))\\ &= \lambda^2+(P_{2\mu}(\xi)+P_{2m-2\mu}(\xi))\lambda+P_{2\mu}(\xi)P_{2m-2\mu}(\xi).\end{aligned}$$

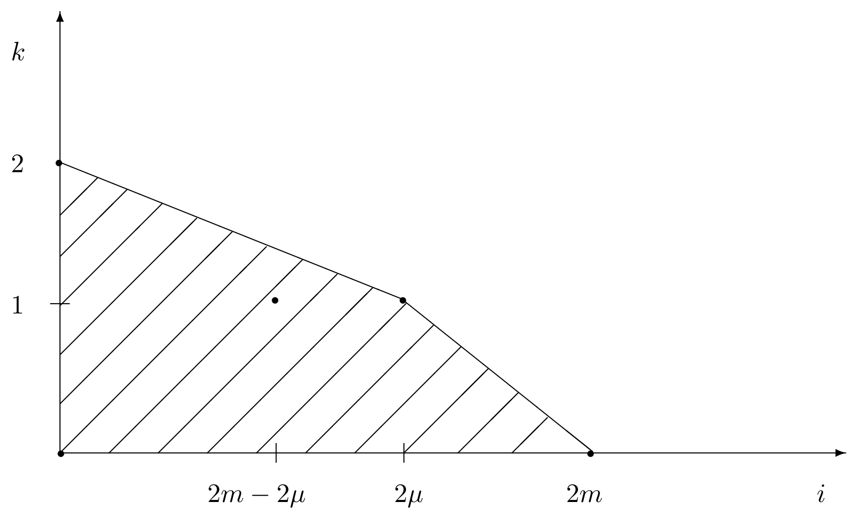

Figure 1: The Newton polygon for the example above.

where $P_{2\mu}$ and $P_{2m-2\mu}$ are polynomials of degree 2μ and $2m-2\mu$, respectively. The Newton polygon of this example can be found in Figure 1.

Definition 2.1. The operator (1.2) is called N-elliptic with parameter in a ray $\mathcal{L}$ in the complex plane if there exists positive constants C and λ_0 such that

$$\big|P(x,\xi,\lambda)\big| \geq C \sum_{(i,k)\in N(P)\cap\mathbb{Z}^2} |\xi|^i|\lambda|^k \tag{2.2}$$

holds for all x and all $\xi \in \mathbb{R}^n$, $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda_0$.

This definition is closely related to the notion of N-parabolicity for operators of the form (1.3).

Definition 2.2. ([8]) The operator (1.3) is called N-parabolic if the following conditions hold:

(i) There exists a real constant λ_0 such that the estimate (2.2) holds for all x, ξ and all λ with $\operatorname{Im}\lambda \leq \lambda_0$.

(ii) The polygon $N(P)$ does not contain edges parallel to the coordinate axes and not belonging to them.

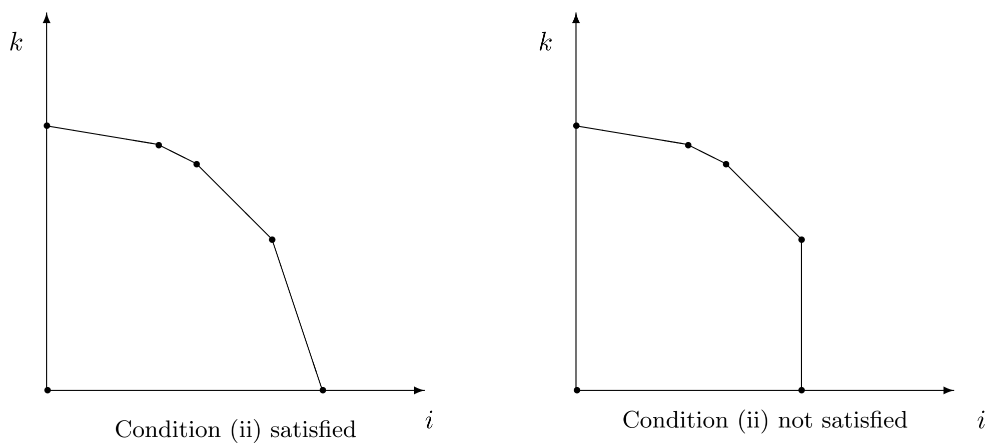

Figure 2: Condition (ii) in Definition 2.2.

If condition (i) holds then (ii) means hypoellipticity of the corresponding differential operator (see also Figure 2).
The definition of N-ellipticity for Douglis–Nirenberg systems is reduced to that of scalar pencils:

Definition 2.3. ([3]) The matrix differential operator (1.1) is called N-elliptic in $\mathcal{L}$ if the determinant of its symbol

$$P(x,\xi,\lambda) := \det\big(A(x,\xi) - \lambda I_N\big)$$

satisfies the condition of Definition 2.1.

If (1.1) acts on a closed manifold (i.e. on a compact manifold without boundary), the property of N-ellipticity leads to (and is even equivalent to) unique solvability of the system

$$\big(A(x,D) - \lambda I_N\big)u = f$$

for large $|\lambda|$ and uniform estimates in terms of parameter-dependent norms (see [3]).
The class of N-elliptic Douglis–Nirenberg systems coincides with the class of systems previously introduced by Kozhevnikov [11] (see also [12] for boundary value problems).
Set $s_i + t_i = 2r_i$ and assume without loss of generality that $r_1 \geq \cdots \geq r_N \geq 0$. For simplicity, let us consider the case where these inequalities are strict:

$$r_1 > \cdots > r_N > 0\,. \tag{2.3}$$

For $\kappa = 1, \dots, N$ denote by $A(\kappa)(x,\xi)$ the $\kappa \times \kappa$ matrix

$$A(\kappa)(x,\xi) = \begin{pmatrix} A_{11}(x,\xi) & \dots & A_{1\kappa}(x,\xi) \\ \vdots & & \vdots \\ A_{\kappa 1}(x,\xi) & \dots & A_{\kappa\kappa}(x,\xi) \end{pmatrix}$$

and by $E(\kappa)$ the matrix which differs from the $\kappa \times \kappa$ zero matrix only by the element at position (κ, κ) which equals 1. We will write $A^0(\kappa)$ for the principal symbol (in the sense of Douglis–Nirenberg) of $A(\kappa)$.

Theorem 2.4. ([3]) *Assume that* (2.3) *is satisfied. The operator* (1.1) *is N-elliptic in* $\mathcal{L}$ *if and only if the Kozhevnikov conditions are satisfied:*

(i) *For* $\kappa = 1, \dots, N$ *the symbols* $A(\kappa)(x,\xi)$ *are elliptic in the sense of Douglis–Nirenberg.*

(ii) *For* $\kappa = 1, \dots, N$ *we have*

$$\det\big(A^0(\kappa)(x,\xi) - \lambda E(\kappa)\big) \neq 0 \quad \big(\xi \in \mathbb{R}^n \backslash \{0\},\ \lambda \in \mathcal{L}\big)\,.$$

Remark 2.5. Let $A(x,D)$ be a matrix of the form (1.1) and consider the operator $A(x,D) - \lambda I_N$. Using the theory of Newton polygons, it is possible to see that there exists a covering of $\mathbb{R}^n \times \mathcal{L}$ by neighbourhoods U_j and V_j $(j = 1, \dots, J)$ and a subordinated partition of unity

$$\sum_{j=1}^{J} \varphi_j(\xi,\lambda) + \sum_{j=1}^{J} \psi_j(\xi,\lambda) \equiv 1$$

such that the following statements hold:

The operator $\big(A(x,D) - \lambda I_N\big)\varphi_j(D,\lambda)$ is a small regular perturbation of

$$\begin{pmatrix} A(\kappa)(x,D) & 0 \\ 0 & \lambda I_{N-\kappa} \end{pmatrix} \varphi_j(D,\lambda)\,.$$

Here and in the following, we use the pseudodifferential operator notation $\varphi_j(D,\lambda) := F^{-1}\varphi_j(\xi,\lambda)F$ where F stands for the Fourier transform in $\mathbb{R}^n$.

(ii) The operator $\big(A(x,D) - \lambda I_N\big)\psi_j(D,\lambda)$ is a small regular perturbation of

$$\begin{pmatrix} A(\kappa)(x,D) - \lambda E(\kappa) & 0 \\ 0 & \lambda I_{N-\kappa} \end{pmatrix} \psi_j(D,\lambda)\,.$$

(See [3] for details.) This shows that the study of operators of the form (1.4) is of particular interest in the theory of N-ellipticity.

Remark 2.6. In the same way the study of scalar pencils (1.2) can be reduced to the study of small regular perturbations of homogeneous pencils of the form

$$P(x,D,\lambda) = P_{2m}(x,D) + \lambda P_{2m-1}(x,D) + \dots + \lambda^{2m-2\mu} P_{2\mu}(x,D)$$

where $P_j(s,D)$ are differential operators of order j. Such pencils are discussed in [4], [5] on manifolds with boundary.
Setting $\lambda = \varepsilon^{-1}$ and multiplying the operator by $\varepsilon^{2m-2\mu}$, we obtain a traditional operator with small parameter:

$$P_\varepsilon(x,D) = \varepsilon^{2m-2\mu} P_{2m}(x,D) + \cdots + P_{2\mu}(x,D)$$

as studied, e.g., in [7], [13]. The condition of N-ellipticity means that

$$\left|P_\varepsilon^0(x,\xi)\right| \geq C\, |\xi|^{2\mu} \left(1+\varepsilon|\xi|\right)^{2m-2\mu}.$$

3. N-ellipticity for boundary value problems

Now let us come back to operator matrices of the form (1.4) acting on a smooth compact manifold M with boundary ∂M. For simplicity, we assume that $A_{ij}(x,D)$ is a scalar differential operator of order $2m$; the case of general Douglis–Nirenberg systems is treated in [6]. We set $A(x,D) := (A_{ij}(x,D))_{i,j=1,2}$, thus $A(x,D,\lambda) = A(x,D) - \lambda E(2)$. The operator matrix $A(x,D,\lambda)$ will be supplemented with general boundary conditions

$$B_j(x,D)\,(u_1 u_2) = B_{j1}(x,D)u_1 + B_{j2}(x,D)u_2 = g_j(x) \quad \text{on } \partial M$$

for $j = 1,\dots,2m$. Here $B_j(x,D)$ is a differential operator of order m_j, where we assume that

$$m_1 \leq \cdots \leq m_m < m_{m+1} \leq \cdots \leq m_{2m} < 2m$$

holds. Let $A(x,D,\lambda)$ be N-elliptic with parameter in the ray $\mathcal{L} = [0,\infty)$. What is the proper formulation of the Shapiro–Lopatinskii condition for the boundary value problem $\big(A(x,D) - \lambda E(2), B_1(x,D),\dots,B_{2m}(x,D)\big)$?
As usual, the boundary value problem on a manifold with boundary is reduced, using local coordinates, to a boundary value problem in the half space $\mathbb{R}^n_+ := \{x = (x',x_n) \in \mathbb{R}^n : x_n > 0\}$ with boundary $\partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$. For simplicity of notation, we shall consider only the corresponding model problem, i.e. we assume that the operators A and B_j have constant coefficients and no lower-order terms. So we consider the 2×2 system

$$\begin{array}{rcrcll} A_{11}(D)\,u_1 & + & A_{12}(D)\,u_2 & = & f_1 & \text{in } \mathbb{R}^n_+\,, \\ A_{21}(D)\,u_1 & + & (A_{22}(D)-\lambda)\,u_2 & = & f_2 & \text{in } \mathbb{R}^n_+ \end{array} \tag{3.1}$$

with boundary conditions

$$B_j(D)u(x',0) = g(x') \quad (j=1,\dots,2m) \ \text{ on } \mathbb{R}^{n-1}\,. \tag{3.2}$$

The Newton polygon of $\det\big(A(\xi) - \lambda E(2)\big)$ has the form indicated in Figure 3, and the N-ellipticity condition means that

$$\left|(A_{11}A_{22} - A_{12}A_{21})(\xi) - \lambda A_{11}(\xi)\right| \geq C\,|\xi|^{2m}(\lambda + |\xi|^{2m}) \quad \big((\xi,\lambda) \in \mathbb{R}^n \times \mathcal{L}\big)\,. \tag{3.3}$$

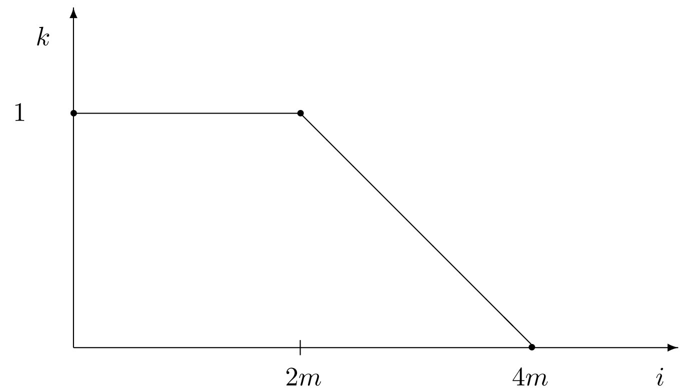

Figure 3: The Newton polygon for the system (3.1).

Inequality (3.3) implies (cf. [4]) that $A_{11}(D)$ and $A(D)$ are elliptic. To formulate the Shapiro–Lopatinskii condition for (3.1)–(3.2), we consider, as usual, a problem on the half-line $t \geq 0$:

$$\begin{aligned} \big(A(\xi', D_t) - \lambda E(2)\big)v(t) &= 0 \quad (t > 0), \\ B_j(\xi', D_t)v(0) &= g_j \quad (j = 1, \dots, 2m), \\ v(t) &\to 0 \quad (t \to \infty). \end{aligned} \tag{3.4}$$

Definition 3.1. The boundary value problem

$$\big(A(x, D) - \lambda E(2), B_1(x, D), \dots, B_{2m}(x, D)\big)$$

in $\mathbb{R}^n_+$ is called N-elliptic with parameter in $\mathcal{L}$ if $A(D) - \lambda E(2)$ is N-elliptic (i.e. if (3.3) holds) and if the following conditions are satisfied:

(i) For every $\xi' \in \mathbb{R}^{n-1} \backslash \{0\}$, every $\lambda \in \mathcal{L}$ and every $(g_1, \dots, g_{2m}) \in \mathbb{C}^{2m}$ the problem (3.4) is uniquely solvable.

(ii) For every $\xi' \in \mathbb{R}^{n-1} \backslash \{0\}$ and every $(g_1, \dots, g_m) \in \mathbb{C}^m$ the problem

$$\begin{aligned} A_{11}(\xi', D_t)v_1(t) &= 0 \quad (t > 0), \\ B_{j1}(\xi', D_t)v_1(0) &= g_j \quad (j = 1, \dots, m), \\ v_1(t) &\to 0 \quad (t \to \infty) \end{aligned} \tag{3.5}$$

is uniquely solvable (i.e. $(A_{11}(D), B_{11}(D), \dots, B_{m1}(D))$ satisfies the standard Shapiro–Lopatinskii condition).

(iii) For every $(g_{m+1},\dots,g_{2m})\in\mathbb{C}^m$ the problem

$$\begin{aligned}\big(A(0,D_t)-E(2)\big)w(t) &= 0 \quad (t>0),\\ B_j(0,D_t)w(0) &= g_j \quad (j=m+1,\dots,2m),\\ w(t) &\to 0 \quad (t\to\infty)\end{aligned} \tag{3.6}$$

has a unique solution.

In [6] it is shown that under the condition of N-ellipticity uniform a priori estimates in parameter-dependent norms hold. The definition of the norms and the proof of the a priori estimate again use the concept of the Newton polygon.
To explain the appearance of the non-standard conditions (ii) and (iii) in the above definition, we will discuss in the next section the construction of the formal asymptotic solution for the problem with small parameter connected with (3.1)–(3.2).

4. Construction of the formal asymptotic solution

Let us consider the boundary value problem (3.1)–(3.2) with $f_1=f_2=0$. Setting $\lambda=\varepsilon^{-2m}$, we can rewrite (3.1) as

$$\begin{aligned}A_{11}(D)\,u_1+A_{12}(D)\,u_2 &= 0 \quad \text{in } \mathbb{R}^n_+,\\ \varepsilon^{2m}(A_{21}(D)\,u_1+A_{22}(D)\,u_2)-u_2 &= 0 \quad \text{in } \mathbb{R}^n_+.\end{aligned} \tag{4.1}$$

Let us assume that (4.1), (3.2) is N-elliptic with parameter in the ray $[0,\infty)$. Then, due to conditions 3.1 (ii) and (iii), respectively, the boundary value problem $(A_{11},B_{11},\dots,B_{m1})$ is elliptic and the system (3.6) is uniquely solvable. To avoid technical difficulties, let us assume that $(A_{11},B_{11},\dots,B_{m1})$ is uniquely solvable, too. If this is not the case, one has to deal with kernels and co-kernels of the operator related to this boundary value problem.
Similar to the notation in [13], let us call $(A_{11},B_{11},\dots,B_{m1})$ the first limit problem and (3.6) the second limit problem. The aim of this section is to construct a formal asymptotic solution (FAS) of the boundary value problem (4.1), (3.2), i.e. a formal series

$$w(x,\varepsilon)=\sum_{k=0}^{\infty}\varepsilon^k w^{(k)}(x)\,,\quad w^{(k)}(x)=\begin{pmatrix} w_1^{(k)}(x)\\ w_2^{(k)}(x)\end{pmatrix}$$

for which the partial sums $\sum_{k=0}^{N}\varepsilon^k w^{(k)}$ satisfy (4.1), (3.2) up to order $O(\varepsilon^N)$. Following Vishik–Lyusternik [14] (see also [10]), we seek the solution in the form

$$w(x,\varepsilon)=u(x,\varepsilon)+v(x,\varepsilon),$$

where

$$u(x,\varepsilon)=\sum_{k=0}^{\infty}\varepsilon^k u^{(k)}(x),\quad u^{(k)}(x)=\begin{pmatrix} u_1^{(k)}(x)\\ u_2^{(k)}(x)\end{pmatrix} \tag{4.2}$$

is a so-called exterior expansion and

$$v(x,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^{m+k} v^{(k)}\Big(x', \frac{x_n}{\varepsilon}\Big), \quad v^{(k)}(x) = \begin{pmatrix} v_1^{(k)}(x) \\ v_2^{(k)}(x) \end{pmatrix} \tag{4.3}$$

is a so-called interior expansion or boundary layer. Our aim is to find partial differential equations and boundary conditions determining the functions $u^{(k)}$, $v^{(k)}$. We start with the equations in the interior of our domain $\mathbb{R}^n_+$.

(i) *Differential equations for* $u^{(k)}$. Substituting (4.2) into (4.1), we obtain

$$\begin{aligned}
&\left[\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon^{2m} \end{pmatrix} A(D) - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] \sum_{k=0}^{\infty} \varepsilon^k \begin{pmatrix} u_1^{(k)} \\ u_2^{(k)} \end{pmatrix} \\
&\quad = \sum_{k=0}^{\infty} \varepsilon^k \left[\begin{pmatrix} A_{11}(D) & A_{12}(D) \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u_1^{(k)} \\ u_2^{(k)} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ A_{21}(D) & A_{22}(D) \end{pmatrix} \begin{pmatrix} u_1^{(k-2m)} \\ u_2^{(k-2m)} \end{pmatrix}\right],
\end{aligned}$$

where we have set

$$u^{(k)} := 0 \quad \text{for } k = -2m, -2m+1, \dots, -1. \tag{4.4}$$

Thus we obtain the recurrence relations

$$A_{11}(D) u_1^{(k)} = -A_{12}(D) \Big(A_{21}(D) \quad A_{22}(D)\Big) u^{(k-2m)} \quad (k = 0, 1, 2, \dots), \tag{4.5}$$

$$u_2^{(k)} = \Big(A_{21}(D) \quad A_{22}(D)\Big) u^{(k-2m)} \quad (k = 0, 1, 2, \dots). \tag{4.6}$$

In order to determine $u^{(k)}$ (with starting values (4.4)), we have to impose m boundary conditions on $u_1^{(k)}$, see below.

(ii) *Differential equations for* $v^{(k)}$. To find the corresponding equations for $v^{(k)}$, we note that

$$A(D)\Big[v^{(k)}\Big(x', \frac{x_n}{\varepsilon}\Big)\Big] = \varepsilon^{-2m} \Big[A(\varepsilon D', D_n) v^{(k)}\Big]\Big(x', \frac{x_n}{\varepsilon}\Big),$$

due to homogeneity. Here $D' = (D_1, \dots, D_{n-1})$. Substituting (4.3) into (4.1), we get

$$\begin{aligned}
&\left[\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon^{2m} \end{pmatrix} A(D) - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] \sum_{k=0}^{\infty} \varepsilon^{k+m} v^{(k)}\Big(x', \frac{x_n}{\varepsilon}\Big) \\
&\quad = \begin{pmatrix} \varepsilon^{-m} & 0 \\ 0 & \varepsilon^m \end{pmatrix} \sum_{k=0}^{\infty} \varepsilon^k \left(\left[A(\varepsilon D', D_n) - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] v^{(k)}\right)\Big(x', \frac{x_n}{\varepsilon}\Big).
\end{aligned} \tag{4.7}$$

Now we expand $A(\varepsilon D', D_n)$ in a Taylor series with respect to ε,

$$A(\varepsilon D', D_n) = \sum_{l=0}^{2m} \varepsilon^l A^{(l)}(D', D_n)$$

with $A^{(0)}(D', D_n) = A(0, D_n)$ and $A^{(2m)}(D', D_n)$ being a constant complex 2×2 matrix.
Substituting this expansion into the last sum in (4.7), we see that this sum equals

$$\sum_{k=0}^{\infty} \varepsilon^k \left[\left(A^{(0)}(D) - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) v^{(k)} + \sum_{l=1}^{2m} A^{(l)}(D) v^{(k-l)} \right]$$

where we have set

$$v^{(j)} := 0 \quad (j = -2m, -2m+1, \dots, -1)\,. \tag{4.8}$$

Therefore we obtain the recurrence relations

$$\left[A^{(0)}(D) - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] v^{(k)} = -\sum_{l=1}^{2m} A^{(l)}(D) v^{(k-l)} \quad (k = 0, 1, 2, \dots) \tag{4.9}$$

(iii) *Boundary conditions.* Now we want to find boundary conditions for the functions $u_1^{(k)}$ and $v^{(k)}$ with $k = 0, 1, \dots$. Setting

$$B(D) := \begin{pmatrix} B_1(D) \\ \vdots \\ B_{2m}(D) \end{pmatrix},$$

we trivially have

$$B(D) \sum_{k=0}^{\infty} \varepsilon^k u^{(k)}(x', 0) = \sum_{k=0}^{\infty} B(D) u^{(k)}(x', 0)$$

and, by homogeneity,

$$B_j(D) v^{(k)} \left(x', \frac{x_n}{\varepsilon} \right) \Big|_{x_n = 0} = \varepsilon^{-m_j} B_j(\varepsilon D', D_n) v^{(k)}(x', x_n) \Big|_{x_n = 0}.$$

Therefore

$$B(D) \left[\sum_{k=0}^{\infty} \varepsilon^k u^{(k)} + \sum_{k=0}^{\infty} \varepsilon^{m+k} v^{(k)} \left(x', \frac{x_n}{\varepsilon} \right) \right] \Bigg|_{x_n = 0} = \sum_{k=0}^{\infty} \varepsilon^k B(D) u^{(k)}(x', 0)$$
$$+ \sum_{k=0}^{\infty} \begin{pmatrix} \varepsilon^{-m_1} & & \\ & \ddots & \\ & & \varepsilon^{-m_{2m}} \end{pmatrix} \varepsilon^{m+k} B(\varepsilon D', D_n) v^{(k)}(x', 0)\,.$$

Again we use the Taylor expansion with respect to ε,

$$B_j(\varepsilon D', D_n) = \sum_{l=0}^{m_j} \varepsilon^l B_j^{(l)}(D', D_n)\,,$$

where $B_j^{(0)}(D', D_n) = B_j(0, D_n)$. We receive

$$\begin{aligned} B_j(D)(u+v) &= \sum_{k=0}^{\infty} \Big[\varepsilon^k B_j(D) u^{(k)} + \varepsilon^{-m_j+m+k} \sum_{l=0}^{m_j} \varepsilon^l B_j^{(l)}(D) v^{(k)}\Big] \\ &= \sum_{k=\min\{m-m_j,0\}} \varepsilon^k \Big[B_j(D) u^{(k)} + \sum_{l=0}^{m_j} B_j^{(l)}(D) v^{(k+m_j-m-l)}\Big]. \end{aligned} \quad (4.10)$$

Note that here negative powers of ε may appear; for negative values of k in $u^{(k)}$ and $v^{(k)}$ we use (4.4) and (4.8).
From the conditions $B_j(u+v) = g_j$ for $j = 1, \dots, 2m$ we obtain the boundary conditions

$$\begin{aligned} & B_j(D) u^{(k)} + \sum_{l=0}^{m_j} B_j^{(l)}(D) v^{(k+m_j-m-l)} = \delta_{k0}\, g_j \\ & \qquad \text{for } k = \min\{m-m_j, 0\}, \min\{m-m_j, 0\}+1, \dots \\ & \qquad \text{and } j = 1, \dots, 2m. \end{aligned} \quad (k, j)$$

This set of boundary conditions is numbered using the indices k and j. At the first moment it seems to be unclear how conditions (k, j) determine the functions $u^{(k)}$ and $v^{(k)}$. This is the essence of the following theorem.

Theorem 4.1. *Assume that the boundary value problem* (1.1)–(1.2) *is N-elliptic and that the first and the second limit problem are uniquely solvable. Then the recursion formulas* (4.4)–(4.6), (4.8)–(4.9) *with boundary conditions* (k, j) *uniquely determine the functions* $u^{(k)}$ *and* $v^{(k)}$ *for* $k = 0, 1, 2 \dots$. *The boundary conditions for* $u_1^{(k)}$ *and* $v^{(k)}$ *have the form*

$$B_{j1}(D) u_1^{(k)}(x', 0) = g_{jk}(x') \quad (j = 1, \dots, m) \quad (4.11)$$

and

$$B_j(0, D_n) v^{(k)}(x', 0) = g_{jk}(x') \quad (j = m+1, \dots, 2m), \quad (4.12)$$

respectively, where the right-hand sides of (4.11) *and* (4.12) *can be determined recursively and contain only functions* $u^{(l)}$ *and* $v^{(l)}$ *which are already known at step* k.

Proof. Due to the recursion formulas (4.5)–(4.6) and (4.9) with starting values (4.4) and (4.8), we only have to show that the boundary condition (k, j) determine $u_1^{(k)}$ and $v^{(k)}$. For simplicity, we restrict ourselves to the particular case that

$$\operatorname{ord} B_j = j - 1 \quad (j = 1, \dots, 2m). \quad (4.13)$$

The general case can be treated with the same idea, but is somewhat more delicate. Figure 4 shows for each index pair (k, j) which appears in the above boundary condition the function for which this condition is used. (The index pairs marked with do not appear). The main question is in what order the formula (k, j) has to be applied.

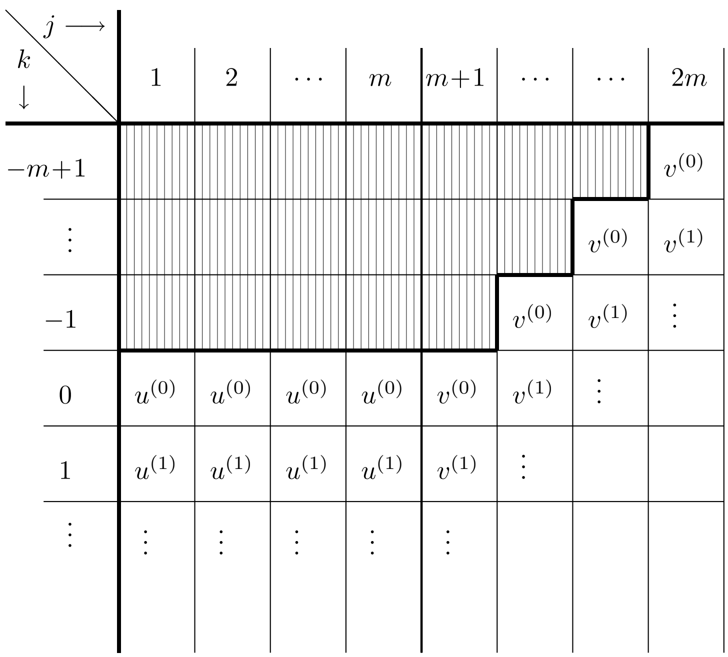

Figure 4: Usage of formula (k, j) in the case (4.13).

We will use the formulas (k, j) with $j \leq m$ as boundary conditions for $u^{(k)}$ and formulas $(k - j + 1 - m, j)$ with $j = m + 1, \dots, 2m$ as boundary conditions for $v^{(k)}$. We still have to show that this can be done in a way such that all functions appearing in formula (k, j) (except $u^{(k)}$ and $v^{(k)}$, respectively) are already known.

Let us assume that in step k we already know the functions $u^{(l)}$ and $v^{(l)}$ with $l < k$. We want to find $u^{(k)}$ and $v^{(k)}$. First let $j \leq m$. Then condition (k, j) contains the functions

$$u^{(k)}, v^{(k+k-m-1)}, v^{(k+j-m-2)}, \dots, v^{(k-m)} .$$

As $k + j - m - 1 < k$ and $u_2^{(k)}$ is defined by (4.6) which only contains $u^{(k-2m)}$, the only unknown function in condition (k, j) is $u_1^{(k)}$. We get (4.11) with

$$g_{jk} := \delta_{k0}\, g_j - B_{j2}(D)u_2^{(k)} - \sum_{l=0}^{j-1} B_j^{(l)}(D)v^{(k+j-1-m-l)}\,.$$

Due to the condition of unique solvability of the first limit theorem, the function $u_1^{(k)}$ is uniquely determined by (4.5) and (4.11).
Now let $j > m$. The boundary condition $(k - j + 1 - m, j)$ contains the functions

$$v^{(k)}, v^{(k-1)}, \dots, v^{(k-j+1)} \text{ and } u^{(k-j+m+1)}.$$

As $k - j + m + 1 \leq k$ and as we already know $u^{(l)}$ for $l \leq k$, this gives m boundary conditions for $v^{(k)}$. Due to unique solvability of the second limit problem, the function $v^{(k)}$ is determined by these boundary conditions, and we can continue with step $k+1$. The boundary conditions for $v^{(k)}$ have the form (4.12) with

$$g_{jk} = \delta_{k-j+m+1,0}\, g_j - B_j(D)u^{(k-j+m+1)} - \sum_{l=1}^{j-1} B_j^{(l)}(D)v^{(k-l)}$$

for $j = m + 1, \dots, 2m$.
Summarizing, we see that we use formula (k, j) as boundary conditions for $u^{(k)}$ and $v^{(k)}$ in the way indicated in Figure 4. Here we compute $u^{(k)}$ and $v^{(k)}$ in the order

$$u^{(0)},\ v^{(0)},\ u^{(1)},\ v^{(1)},\ u^{(2)}, \dots\ .$$

□

Remark 4.2. One can see that the recursion formula for $u_1^{(k)}$ (see equations (4.5) and (4.11)) is given by the first limit problem with appropriate right-hand side. Similarly, the recursion formula for $v^{(k)}$ (cf. equations (4.9) and (4.12)) is exactly the second limit problem with appropriate right-hand side. So we can see that the non-standard ellipticity conditions (ii) and (iii) in Definition 3.1 lead (under the additional assumption that the boundary value problem in 3.1 (ii) is not only elliptic but uniquely solvable) to the existence of a formal asymptotic solution. Here the first limit problem corresponds to the exterior expansion and the second limit problem to the boundary layer. Therefore, from the point of view of singular perturbation theory the non-standard ellipticity conditions are very natural.

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