Universität Konstanz

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Konstanzer Schriften in Mathematik

Nr. 267, Mai 2010

ISSN 1430-3558

WELL-POSEDNESS OF A QUASILINEAR HYPERBOLIC FLUID MODEL

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Abstract. We replace a Fourier type law by a Cattaneo type law in the derivation of the fundamental equations of fluid mechanics. This leads to hyperbolicly perturbed quasilinear Navier-Stokes equations. For this problem the standard approach by means of quasilinear symmetric hyperbolic systems seems to fail by the fact that finite propagation speed might not be expected. Therefore a somewhat different approach via viscosity solutions is developed in order to prove higher regularity energy estimates for the linearized system. Surprisingly, this method yields stronger results than previous methods, by the fact that we can relax the regularity assumptions on the coefficients to a minimum. This leads to a short and elegant proof of a local-in-time existence result for the corresponding first order quasilinear system, hence also for the original hyperbolicly perturbed Navier-Stokes equations.

1. Introduction

Let $n \geq 2$ and $T, \tau > 0$. The intention of this note is to examine the hyperbolicly perturbed Navier-Stokes equations

$$\left\{ \begin{array}{rcll} \tau u_{tt} - \mu\Delta u + \tau(u\cdot\nabla)\partial_t u + ((\tau\partial_t u + u)\cdot\nabla)u + u_t &=& -\nabla\pi & \text{in } (0,T)\times\mathbb{R}^n, \\ \operatorname{div} u &=& 0 & \text{in } (0,T)\times\mathbb{R}^n, \\ u|_{t=0} &=& u_0 & \text{in } \mathbb{R}^n, \\ u_t|_{t=0} &=& u_1 & \text{in } \mathbb{R}^n, \end{array} \right. \tag{1.1}$$

where $u : (0,T)\times\mathbb{R}^n \to \mathbb{R}^n$ denotes the velocity of a fluid and $p : (0,T)\times\mathbb{R}^n \to \mathbb{R}$ the related pressure. System (1.1) is obtained by replacing a Fourier type law by the law of Cattaneo. More precisely, we replace the constitutive law for the deformation tensor

$$S = \frac{\mu}{2}(\nabla u + (\nabla u)') \tag{1.2}$$

with viscosity coefficient $\mu > 0$ by the relation

$$S + \tau S_t = \frac{\mu}{2}(\nabla u + (\nabla u)'), \tag{1.3}$$

which represents the first order Taylor approximation of the delayed deformation condition

$$S(t+\tau) = \frac{\mu}{2}(\nabla u(t) + (\nabla u(t))'), \quad t > 0,$$

for small $\tau > 0$. Relation (1.2) is a Fourier type law. It leads to the well-known paradox of infinite propagation speed for classical parabolic equations. There are applications, however, for that it is more reasonable to work with hyperbolic models, cf. [14] and the references therein. This is also underlined by experiments that document the existence of hyperbolic heat waves.

Recall that the classical Navier-Stokes equations, determined by Fourier's law, are represented by the system

$$\left\{\begin{array}{rcll} u_t + (u\cdot\nabla)u + \nabla p &=& \operatorname{div} 2S & \text{in } (0,T)\times\mathbb{R}^n, \\ \operatorname{div} u &=& 0 & \text{in } (0,T)\times\mathbb{R}^n, \\ u|_{t=0} &=& u_0 & \text{in } \mathbb{R}^n, \end{array}\right. \tag{1.4}$$

where the deformation tensor is given by

$$S(u) = \frac{\mu}{2}(\nabla u + (\nabla u)').$$

In this situation the second line in (1.4) implies that

$$\operatorname{div} 2S(u) = \mu\Delta u.$$

On the other hand, by employing Cattaneo's law (1.3) we have that

$$\operatorname{div} 2(S + \tau S_t) = \mu \operatorname{div}(\nabla u + (\nabla u)') = \mu\Delta u. \tag{1.5}$$

System (1.1) is now obtained as follows. Applying $\tau\partial_t$ to the first line in (1.4) and adding the resulting equation to the original line gives us in view of (1.5) that

$$\begin{aligned} 0 &= \tau u_{tt} + \tau\partial_t(u\cdot\nabla)u + \tau\nabla p_t + (u\cdot\nabla)u + u_t + \nabla p - \operatorname{div} 2(S+\tau S_t) \\ &= \tau u_{tt} + \tau\partial_t(u\cdot\nabla)u + (u\cdot\nabla)u + u_t - \mu\Delta u + \tau\nabla p_t + \nabla p. \end{aligned}$$

Consequently, by introducing the new pressure $\pi = p + \tau p_t$, under the assumption of Cattaneo's law the classical Navier-Stokes equations turn into the hyperbolicly perturbed system (1.1).

The hyperbolic fluid model (1.1) was already derived in [3] and [4]. In these papers on an elementary level the authors discussed consequences and differences of (1.1) compared with the classical model.

In [11] Paicu and Raugel consider the classical Navier-Stokes equations including merely the hyperbolic perturbation τu_{tt} for small $\tau > 0$. The global well-posedness for mild solutions in two dimensions for sufficiently small τ, and the global existence for small data and sufficiently small τ in three dimensions in analogy to the classical case are proved. In [11] also a number of justifications for their model are presented, see the references therein. By just adding the term τu_{tt} to (1.4) the resulting system remains semilinear and therefore methods for the construction of a mild solution can still be applied. This, however, is no longer possible for system (1.1), since due to the third term in the first line of (1.1) this system is a quasilinear one. So, from this point of view system (1.1) rather differs from the the system considered in [11].

We remark that our new Navier-Stokes system is related to the Oldroyd model which considers instead of (1.3) the more general model

$$\tau S_t + S = \mu(\mathcal{E} + \nu\mathcal{E}_t), \tag{1.6}$$

where $\mathcal{E} := \frac{1}{2}(\nabla u + \nabla u^T)$, cf. de Araújo, de Menzenes and Marinho [2] and Joseph [6]; in comparison to our model we have $\nu = 0$ (and $\mu = 1$). If $\nu \neq 0$ then, from the point of derivatives getting involved, S is on a similar level as $\mathcal{E}$, as in the classical case (1.4).

In a first step towards the local-in-time existence result in order, as usually we transform (1.1) into a first order quasilinear system of the form

$$\begin{cases} V_t + \mathcal{A}(V)V + \mathcal{B}(V)V &= 0 \quad \text{in } (0,T)\times\mathbb{R}^n, \\ V|_{t=0} &= V_0 \quad \text{in } \mathbb{R}^n, \end{cases} \tag{1.7}$$

with $V := (u, \partial_1 u, \ldots, \partial_n u, \partial_t u)^T$. A standard approach used for standard quasilinear symmetric hyperbolic systems is to derive a priori estimates in Sobolev spaces of higher order for a linearized version by means of finite propagation speed and then to apply a fixed point iteration to the nonlinear problem. This method, however, seems to fail for the first order system resulting from (1.1). The crucial point here is the finite propagation speed. It seems not to be available (and this can be regarded as a conjecture of the authors) for equations (1.1) neither for the corresponding first order quasilinear system or for the associated linearization. The reason for this conjecture lies in the presence of the pressure gradient in equations (1.1). Of course, as in a standard way for Navier-Stokes equations, ∇p could be removed by applying the Leray-Helmholtz projector onto solenoidal fields to the first line of (1.1) and then dealing with the resulting system. But either way leads to nonlocal terms in the equations which indicates that finite propagation speed might not be expected. (The authors, however, so far have not been able to prove this.) In case of dimension $n = 2$ or $n = 3$ we can obtain finite propagation speed for $\operatorname{curl} u$, for instance. This observation is justified by applying curl to (1.1), since then gradient terms also vanish and (1.1) turns into an equation for the vorticity $\operatorname{curl} u$ (see Section 2). ¿From this point of view, problem (1.1) and the resulting system (1.7) are somewhat different from standard quasilinear symmetric hyperbolic systems.

By the just mentioned fact, in this note we developed a different approach to first order hyperbolic systems, which also covers equations of type (1.1). On a standard way by employing Kato's theory we first prove the existence of strong solutions for a linearized version of (1.7) (see Lemma 4.2). However, the essential step is to derive higher order a priori estimates for the linearized solution, which are required for the application of a fixed point iteration to (1.7). Here we choose an approach via *viscosity solutions*, i.e., we add a small viscous term to (1.7) such that the resulting system becomes parabolic. This method provides a smooth way to justify the formal calculations that lead to higher energy estimates for the solution of the linearized equations. A nice outcome of this method is that we can provide such estimates under minimal regularity assumptions on the coefficients of the linearized operators (see Theorem 4.5). In fact, the regularity assumptions to be made on the coefficients are weaker than the regularity of the obtained solution. Minimal in this context means that we only have to assume the regularity that is required to give sense to the natural energy estimates. Furthermore, these helpful energy estimates for the solution are also provided by the method.

This seems to be different and new in comparison to similar results for standard symmetric hyperbolic systems that are based on finite propagation speed of the displacement. In pertinent textbooks such as [10, Theorem 2.1] or [13, Theorem 5.1], for instance, always the assumed regularity for the coefficients is higher than the regularity obtained for the solutions, and it seems to be difficult or even impossible to improve this to our results by the methods used therein. In [5] an abstract approach to quasilinear evolution equations is developed generalizing results obtained in [7]. But also there the assumed regularity on the coefficients is higher than the obtained for the solution. Only for the approach developed in [8] this is not the case. There the coefficients are assumed to be elements of uniformly local Sobolev spaces. This assumption is enough by the fact that the standard Sobolev embedding and the required algebra properties are still valid. Thus the assumptions in [8] for the coefficients of the linearized system are comparable to ours. On the other hand, it is not so obvious whether the approach to quasilinear hyperbolic systems given in [8] applies to system (1.1) due to the presence of the presssure term $\nabla\pi$ or the Helmholtz projection respectively.

Based on the linear theory developed here the application of Majda's fixed point iteration, cf [10], in order to construct local-in-time strong solutions to (1.7) becomes rather short and elegant (see Theorem 5.1). This is due to the fact that by the quality of the linear results provided here no smoothing of the data, in particular of the coefficients, for the fixed point iteration is required anymore. By our energy estimates for the linearized solutions, here we also get immediately upper bounds for the approximate solutions of the fixed point iteration. This again is in contrast to [10] (or [13]). There upper bounds have to be derived by estimating the approximate solutions in an elaborate way employing the structure of the underlying quasilinear symmetric hyperbolic system. Also continuity (in time) of the solutions (as given in (5.1)) immediately follows from the linear results. This is also quite different from the approach performed in [10] or [13], where exhausting procedures via the strong convergence in weaker norms and the weak continuity in higher norms have to be applied in order to prove continuity. This seems to be a futher nice advantage of our approach in comparison to previous methods.

We want to emphasize that the approach developed in this note is by no means restricted to first order quasilinear systems arising from equations of type (1.1). In fact, it is quite generally applicable, in particular to standard quasilinear symmetric hyperbolic systems. Thus by our approach on a different (perhabs even more elegant) way we can handle, for example, quasilinear wave equations or systems arising in thermoelasticity such as treated in [10] or [14]. Moreover, the final results for the quasilinear systems are of the same quality as the results obtained by previous methods. On the other hand, obviously the approach presented here is more general, since we can deal as well with problems of type (1.1), which might not produce finite propagation speed. Furthermore, also Oldroyd models such as (1.6) can be covered by our approach which is different from the methods used e.g. in [6].

We proceed with the precise statement of our main results. By virtue of the second line in (1.1) we define the ground space as

$$L^2_\sigma(\mathbb{R}^n) := \{f \in L^2(\mathbb{R}^n) :\ \operatorname{div} f = 0\}.$$

Also note that the symbol $C_b^\infty(\Omega)$ stands for smooth functions whose derivatives of each order $k \in \mathbb{N}_0$ are also bounded on the set Ω.

1.1. **Theorem.** *Let $n \geq 2$ and $m > n/2$. For each*

$$(u_0, u_1) \in \big(H^{m+2}(\mathbb{R}^n) \cap L^2_\sigma(\mathbb{R}^n)\big) \times \big(H^{m+1}(\mathbb{R}^n) \cap L^2_\sigma(\mathbb{R}^n)\big)$$

there exists a time $T_* > 0$ and a unique solution (u, π) of equations (1.1) satisfying

$$\begin{aligned} u &\in C^2([0,T_*], H^m(\mathbb{R}^n)) \cap C^1([0,T_*], H^{m+1}(\mathbb{R}^n)) \\ &\quad \cap C([0,T_*], H^{m+2}(\mathbb{R}^n) \cap L^2_\sigma(\mathbb{R}^n)), \\ \nabla\pi &\in C([0,T_*], H^m(\mathbb{R}^n)). \end{aligned}$$

The existence time T_* can be estimated from below as

$$T_* > \frac{1}{1 + C(\|u_0\|_{H^{m+2}} + \|u_1\|_{H^{m+1}})}$$

with a constant $C > 0$ depending only on m and the dimension n.

As an immediate consequence we also have

1.2. **Corollary.** *In the situation of Theorem 1.1 additionally assume that*

$$u_0, u_1 \in \bigcap_{k=0}^{\infty} H^k(\mathbb{R}^n).$$

Then the solution u, p is classical, i.e. we have

$$u, \nabla\pi \in C_b^\infty([0,T_*] \times \mathbb{R}^n).$$

The paper is organized as follows. We start in Section 2 with a remark on finite propagation speed. In Section 3 we perform the transformation of (1.1) into a first order quasilinear system. Section 4 represents the heart of this work and provides the linear theory. First we prove the existence of strong solutions to a linearized version of (1.7). As mentioned before, the essential point then is to derive higher regularity of this solution. This result is obtained by employing the method of viscosity solutions. In Section 5 we prove the local-in-time existence for the first order quasilinear system, which finally results in our main results Theorem 1.1 and Corollary 1.2 by the equivalence of systems (1.1) and (1.7).

2. Remark on finite propagation speed

For the local solution obtained in the previous section, we can prove the finite propagation speed for the vorticity $v := \operatorname{curl} u = \nabla \times u$. Namely, v satisfies the differential equation

$$\tau v_{tt} - \mu\Delta v + v_t + (\tau u\cdot\nabla)v_t + \Big\{(u\cdot\nabla)v + (\tau u_t\cdot\nabla)v + (2-n)\,(1+\tau\partial_t)J(\nabla u)v)\Big\} = 0, \quad (2.1)$$

where $J(\nabla u)$ denotes the Jacobi matrix of the first derivatives of u. The part in brackets $\{\dots\}$ involves at most first-order derivatives of v. Therefore, the general energy estimates for hyperbolic equations of second order — after transformation to a first-order symmetric-hyperbolic system — apply as described in [13], and give the finite propagation speed. As mentioned before, note that this can still not be expected for u due to the presence of the pressure terms.

3. Transformation into a symmetric system

We start by introducing some notation. Note that we use standard notation throughout this note, for the appearing function spaces see e.g. [1]. Let X be a Banach space and $\Omega \subset \mathbb{R}^n$ be a set. Then $L^p(\Omega, X)$ denotes the standard Lebesgue space of p-integrable X-valued functions for $1 \le p < \infty$. For $p = \infty$, $L^\infty(\Omega, X)$ denotes the space of all (essentially) bounded functions equipped with the standard norm $\operatorname{ess\,sup}_{x\in\Omega} \|\cdot\|_X$. Accordingly, for $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $1 \le p \le \infty$ the symbol $W^{k,p}(\Omega, X)$ denotes Sobolev space of k-th order with norm

$$\|u\|_{k,p} := \|u\|_{W^{k,p}} := \|u\|_{W^{k,p}(\Omega,X)} := \left(\sum_{|\alpha|\le k} \|u\|_X^p \right)^{1/p}.$$

In the case $k = 0$ we also write $\|\cdot\|_p$ for the norm. Moreover, we set $H^k(\Omega, X) := W^{k,2}(\Omega, X)$. In this paper from the just introduced spaces only $L^2(\Omega, X)$, $H^k(\Omega, X)$, $L^\infty(\Omega, X)$ and $W^{k,\infty}(\Omega, X)$ will appear. Also note that if $X = \mathbb{C}^m$ or $X = \mathbb{R}^m$ we write just $L^2(\Omega)$, $H^k(\Omega)$, etc. We will also make use of the homogeneous Sobolev space

$$\widehat{H}^1(\mathbb{R}^n) := \{u \in L^1_{loc} : \ \nabla u \in L^2(\mathbb{R}^n)\}/\mathbb{C},$$

which is equipped with the norm $\|\nabla \cdot\|_2$.

We also use standard notation for spaces of continuous functions. For $k \in \mathbb{N}_0 \cup \{\infty\}$, $C^k(\Omega, X)$ denotes the space of k-times continuously differentiable functions and we write $C(\Omega, X)$ if $k = 0$. If the functions in $C^k(\Omega, X)$ are additionally bounded, we use the symbol $C_b^k(\Omega, X)$ and its subspace of compactly supported functions is denoted by $C_0^k(\Omega, X)$. The (X, X') dual pairing we denote by $\langle \cdot, \cdot \rangle_{X,X'}$. To obtain consistency with the scalar product if X is a Hilbert space, observe that the second argument in $\langle \cdot, \cdot \rangle_{X,X'}$ is defined with complex conjugation, i.e., we have

$$\langle x, x' \rangle_{X,X'} = x'(\overline{x}) \quad (x \in X, \ x' \in X'),$$

if $x'(x)$ denotes the standard dual pairing. If H is a Hilbert space we write $\langle \cdot, \cdot \rangle_H$. From time to time we also omit the subscript and just write $\langle \cdot, \cdot \rangle$, if no confusion seems likely. The space of linear bounded operators from X to a Banach space Y is denoted by $\mathscr{L}(X, Y)$.

Suppose (u, p) with $u : \mathbb{R}_+^{n+1} \to \mathbb{R}^n$ and $p : \mathbb{R}_+^{n+1} \to \mathbb{R}$ is the solution of sytem (1.1). In this section we transform equations (1.1) into a first order quasilinear hyperbolic system for the vector

$$V = (u, \partial_1 u, \dots, \partial_n u, \partial_t u)^T \in (\mathbb{R}^n)^{n+2} = \mathbb{R}^{n(n+2)}.$$

As for the classical Navier-Stokes equations the pressure term ∇p will be eliminated by employing the Leray-Helmholtz projector onto solenoidal fields

$$P : L^2(\mathbb{R}^n) \to L^2_\sigma(\mathbb{R}^n) = \left\{ v \in L^2(\mathbb{R}^n) : \ \operatorname{div} v = 0 \right\}.$$

Observe that $C^\infty_{0,\sigma}(\mathbb{R}^n) := \{u \in C_0^\infty(\mathbb{R}^n) : \ \operatorname{div} u = 0\}$ is dense in $L^2_\sigma(\mathbb{R}^n)$. Also note that P is determined by

$$Pu := u - \nabla \pi,$$

where $\pi \in \widehat{H}^1(\mathbb{R}^n)$ is the unique solution of the weak Neumann poblem

$$\langle \nabla\pi, \nabla\varphi\rangle_{L^2} = \langle u, \nabla\varphi\rangle_{L^2} \qquad (\varphi \in \widehat{H}^1(\mathbb{R}^n)).$$

This leads to the well-known orthogonal decomposition

$$L^2(\mathbb{R}^n) = L^2_\sigma(\mathbb{R}^n) \oplus_\perp G_2(\mathbb{R}^n),$$

where $G_2(\mathbb{R}^n) := \{\nabla\pi : \ \pi \in \widehat{H}^1(\mathbb{R}^n)\}$. Applying P to the first line of (1.1), this system is formally reduced to

$$\begin{cases} \tau u_{tt} - \mu\Delta u + \tau P(u\cdot\nabla)\partial_t u + P((\tau\partial_t u + u)\cdot\nabla)u + u_t &= 0 \quad \text{in } (0,T)\times\mathbb{R}^n, \\ u|_{t=0} &= u_0 \quad \text{in } \mathbb{R}^n, \\ u_t|_{t=0} &= u_1 \quad \text{in } \mathbb{R}^n, \end{cases} \tag{3.1}$$

considered in the space $L^2_\sigma(\mathbb{R}^n)$. For the development of the linear theory it will be convenient to get rid of the τ in front of u_{tt} and μ in front of Δu. For this purpose we introduce the dilated function

$$v(t,x) := u(\sqrt{\tau}t, \sqrt{\mu}x).$$

Then u solves (3.1) if and only if v solves

$$\begin{cases} v_{tt} - \Delta v + \sqrt{\tau/\mu}P(v\cdot\nabla)\partial_t v \\ \quad + P((\sqrt{\tau}\partial_t v + v)\cdot\nabla)v/\sqrt{\mu} + v_t/\sqrt{\tau} &= 0 \quad \text{in } (0,T')\times\mathbb{R}^n, \\ v|_{t=0} &= v_0 \quad \text{in } \mathbb{R}^n, \\ v_t|_{t=0} &= v_1 \quad \text{in } \mathbb{R}^n, \end{cases} \tag{3.2}$$

with $T' = T/\sqrt{\tau}$, $v_0 = u_0$, and $v_1 = \sqrt{\tau}u_1$. System (3.2) will be the one which is considered in the sequel and which we transform it into a first order system.

For $j = 1, \dots, n$ we define the symmetric matrices

$$A_j(V) := \begin{pmatrix} 0 & \cdots & & \cdots & & \cdots & 0 & 0 \\ \vdots & \ddots & & & & & \vdots & \vdots \\ & & & & & & & 0 \\ \vdots & & & \ddots & & & \vdots & -I_n \\ & & & & & & & 0 \\ \vdots & & & & & \ddots & \vdots & \vdots \\ 0 & \cdots & & \cdots & & \cdots & 0 & 0 \\ 0 & \cdots & 0 & -I_n & 0 & \cdots & 0 & M_j(V) \end{pmatrix} \in (\mathbb{R}^{n\times n})^{(n+2)\times(n+2)}, \tag{3.3}$$

with I_n the identity in $\mathbb{R}^n$ and where $-I_n$ represents the $(j+1, n+2)$-th and the $(n+2, j+1)$-th entry of $A_j(V)$. The operator M_j is defined as

$$M_j(V) := \sqrt{\tau/\mu}\,(V^1)^j \cdot I_n = \sqrt{\tau/\mu}\,v^j \cdot I_n$$

and correspondes to the quasilinear term in (3.2). We also define the $(n\times n)\cdot((n+2)\times(n+2))$ matrix operators

$$\widetilde{\mathcal{B}}(V) := \begin{pmatrix} 0 & \cdots & \cdots & 0 & -I_n \\ \vdots & \ddots & & \vdots & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 \\ 0 & B_1(V) & \cdots & B_n(V) & I_n/\sqrt{\tau} \end{pmatrix} \tag{3.4}$$

with $B_j(V) := \frac{1}{\sqrt{\mu}}(\sqrt{\tau}(V^{n+2})^j + (V^1)^j)\cdot I_n = \frac{1}{\sqrt{\mu}}(\sqrt{\tau}\partial_t v^j + v^j)\cdot I_n$ and

$$\mathcal{P} := \begin{pmatrix} I_n & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & I_n & 0 \\ 0 & \cdots & 0 & P \end{pmatrix}.$$

Finally, we set

$$\mathcal{A}(V) := \mathcal{P}\sum_{j=1}^n A_j(V)\partial_j \qquad \text{and} \qquad \mathcal{B}(V) := \mathcal{P}\widetilde{\mathcal{B}}(V).$$

Then, it is easily checked that (3.2) is equivalent to the first order quasilinear hyperbolic system

$$\begin{cases} V_t + \mathcal{A}(V)V + \mathcal{B}(V)V &= 0 \quad \text{in } (0,T)\times\mathbb{R}^n, \\ V|_{t=0} &= V_0 \quad \text{in } \mathbb{R}^n, \end{cases} \tag{3.5}$$

with $V := (v, \partial_1 v, \dots, \partial_n v, \partial_t v)^T$ and $V_0 := (v_0, \partial_1 v_0, \dots, \partial_n v_0, v_1)^T$. Observe that the difference to standard quasilinear symmetric hyperbolic systems lies in the presence of the projector $\mathcal{P}$. In the next two sections we will develop the required linear and quasilinear existence theory for systems of the form (3.5).

4. Linear theory

Let $T \in (0,\infty]$. Here we consider a linearized version of system (3.5). To be precise, we assume that A_j and $\mathcal{B}$ are matrices of the form given in (3.3) and (3.4), where $M_j(V)$ and $B_j(V)$ are replace by $a_j I_n$ and $b_j I_n$, respectively, with given functions $a_j, b_j : [0,T)\times\mathbb{R}^n \to \mathbb{R}$. Formally we define for each $t \in [0,T)$ the operator $\mathcal{A}$ in the space

$$\mathcal{H} := L^2(\mathbb{R}^n)^{n(n+1)} \times L^2_\sigma(\mathbb{R}^n)$$

by

$$\mathcal{A}(t) := \sum_{j=1}^n \mathcal{P}A_j(t,\cdot)\partial_j,$$

$$\mathscr{D}(\mathcal{A}) := \mathscr{D}(\mathcal{A}(t)) := \{V \in \mathcal{H} :\ V^{n+2} \in H^1(\mathbb{R}^n),\ P\sum_{j=1}^n \partial_j V^{j+1} \in L^2(\mathbb{R}^n)\}.$$

Observe that it is well-known that in $\mathbb{R}^n$ the Helmholtz projection is bounded on the entire scale of Sobolev spaces, that is, we have $P \in \mathscr{L}(H^m(\mathbb{R}^n))$ for every $m \in \mathbb{Z}$. This, for instance, follows easily by its symbol representation

$$P = \mathcal{F}^{-1}\left[I_n - \frac{\xi\xi^T}{|\xi|^2}\right]\mathcal{F}$$

and Plancherel's theorem, where $\mathcal{F}$ denotes the Fourier transformation. In this spirit the last expression in the definition of $\mathscr{D}(\mathcal{A})$ makes sense, due to $\sum_{j=1}^n \partial_j V^{j+1} \in H^{-1}(\mathbb{R}^n)$. In this section we aim for the well-posedness and higher regularity of the linear nonautonomous first order hyperbolic system

$$\begin{cases} V_t + \mathcal{A}V + \mathcal{B}V &= 0 \quad \text{in } (0,T), \\ V|_{t=0} &= V_0. \end{cases} \tag{4.1}$$

For this purpose we start with the following result for the 'principal' linear part $\mathcal{A}$.

4.1. **Lemma.** *Let $T \in (0,\infty)$ and let $\mathcal{A}$ be as defined above. Assume that*

$$(a_j)_{j=1}^n \subseteq C\left([0,T], L^\infty(\mathbb{R}^n)\right), \quad \operatorname{div}(a^1,\dots,a^n) = 0.$$

Then for every $t \in [0,T]$ the operator $\mathcal{A}(t)$ is skew-selfadjoint, i.e., we have $\mathcal{A}(t)' = -\mathcal{A}(t)$.

Proof. By the definition of A_j we have that

$$A_j\partial_j V = (0,\dots,0,-\partial_j V^{n+2},0,\dots,-\partial_j V^{j+1} + a_j\partial_j V^{n+2})^T.$$

This yields

$$\mathcal{P}\sum_{j=1}^n A_j\partial_j V = \left(0,-\partial_1 V^{n+2},\dots,-\partial_n V^{n+2},-P\sum_{j=1}^n \partial_j V^{j+1} + P\sum_{j=1}^n a_j\partial_j V^{n+2}\right)^T. \tag{4.2}$$

This shows that $\mathcal{A}(t) : \mathscr{D}(\mathcal{A}) \to \mathcal{H}$ is well-defined for each $t \in [0,T]$. Now, let $(V_k)_k \in \mathscr{D}(\mathcal{A})$ such that $V_k \to V$ and $\mathcal{A}(t)V_k \to W$ in $\mathcal{H}$. Then the first $n+1$ components in (4.2) imply that $V^{n+2} \in H^1(\mathbb{R}^n)$ and that $V_k^{n+2} \to V^{n+2}$ in $H^1(\mathbb{R}^n)$. By the last component in (4.2) this, in turn, yields that $P\sum_{j=1}^n \partial_j V_k^{j+1}$ converges in $L^2(\mathbb{R}^n)$. By the fact that $V_k \to V$ in $\mathcal{H}$, we also obtain

$$P\sum_{j=1}^n \partial_j V_k^{j+1} \to P\sum_{j=1}^n \partial_j V^{j+1} \quad \text{in } H^{-1}(\mathbb{R}^n).$$

Since the convergence in L^2 is stronger as the convergence in H^{-1} we conclude that $P\sum_{j=1}^n \partial_j V^{j+1} \in L^2(\mathbb{R}^n)$. Consequently, $V \in \mathscr{D}(\mathcal{A})$ and $\mathcal{A}(t)V = W$ which shows that $\mathcal{A}(t)$ is closed for each $t \in [0,T]$.

Next, for $V \in \mathscr{D}(\mathcal{A})$ and $U \in \mathcal{H}$ we have

$$\begin{aligned}\langle \mathcal{A}(t)V, U\rangle = &-\sum_{j=1}^n(\partial_j V^{n+2}, U^{j+1}) - (P\sum_{j=1}^n \partial_j V^{j+1}, U^{n+2}) \\ &+\sum_{j=1}^n(a_j\partial_j V^{n+2}, U^{n+2})\end{aligned} \tag{4.3}$$

By the symmetry of P on L^2 and since we use the same symbol for the Helmholtz projection on H^m for different m, we also have $P' = P$ if P is the projection on H^m. For $U \in \mathscr{D}(\mathcal{A})$ we therefore can continue the above calculation as

$$\begin{aligned}\langle \mathcal{A}(t)V, U\rangle &= \sum_{j=1}^{n} \langle PV^{n+2}, \partial_j U^{j+1}\rangle_{H^1,H^{-1}} - \langle \sum_{j=1}^{n} \partial_j V^{j+1}, PU^{n+2}\rangle_{H^{-1},H^1} \\ &\quad + \sum_{j=1}^{n}(a_j\partial_j V^{n+2}, U^{n+2}) \\ &= (V^{n+2}, P\sum_{j=1}^{n} \partial_j U^{j+1}) + \sum_{j=1}^{n}(V^{j+1}, \partial_j U^{n+2}) - \sum_{j=1}^{n}(V^{n+2}, a_j\partial_j U^{n+2}) \\ &= \langle V, -\mathcal{A}(t)U\rangle,\end{aligned}$$

where we used the fact that $\operatorname{div}(a^1,\dots,a^n)^T = 0$ in the second equality. This shows that $\mathcal{A}(t)$ is skew-symmetric and that $\mathscr{D}(\mathcal{A}(t)) \subset \mathscr{D}(\mathcal{A}(t)')$.

For the converse inclusion we pick

$$U \in \mathscr{D}(\mathcal{A}(t)') = \{U \in \mathcal{H};\ \exists\, W \in \mathcal{H}\ \forall\, V \in \mathscr{D}(\mathcal{A}):\ \langle V, W\rangle = \langle \mathcal{A}(t)V, U\rangle\}.$$

First we choose $V \in \mathscr{D}(\mathcal{A})$ such that $V^k = 0$ except for $k = \ell+1$ with fixed $\ell \in \{1,\dots,n\}$ and such that $V^{\ell+1} \in C_0^\infty(\mathbb{R}^n)$. In view of (4.2) we then obtain

$$\begin{aligned}(V^{\ell+1}, W^{\ell+1}) &= \langle V, W\rangle = \langle \mathcal{A}(t)V, U\rangle \\ &= -(\sum_{j=1}^{n} P\partial_j V^{j+1}, U^{n+2}) = \langle V^{\ell+1}, \partial_\ell U^{n+2}\rangle_{H^1,H^{-1}}.\end{aligned}$$

This shows that $\partial_\ell U^{n+2}$ has a representant in $L^2(\mathbb{R}^n)$ for every $\ell \in \{1,\dots,n\}$. Thus $U^{n+2} \in H^1(\mathbb{R}^n)$. Next we choose $V \in \mathscr{D}(\mathcal{A})$ satisfying $V^k = 0$ except for $k = n+2$ and $V^{n+2} \in C_{0,\sigma}^\infty(\mathbb{R}^n) \overset{d}{\hookrightarrow} L^2_\sigma(\mathbb{R}^n)$. By the fact that $U^{n+2} \in H^1(\mathbb{R}^n)$ we can calculate

$$\begin{aligned}(V^{n+2}, W^{n+2}) &= \langle V, W\rangle = \langle \mathcal{A}(t)V, U\rangle \\ &= -\sum_{j=1}^{n}(\partial_j V^{n+2}, U^{j+1}) + \sum_{j=1}^{n}(Pa_j\partial_j V^{n+2}, U^{n+2}) \\ &= \langle V^{n+2}, P\sum_{j=1}^{n}\partial_j U^{j+1}\rangle_{H^1,H^{-1}} - (V^{n+2}, \sum_{j=1}^{n} Pa_j\partial_j U^{n+2}).\end{aligned}$$

Thanks to $W^{n+2}, \sum_{j=1}^n Pa_j\partial_j U^{n+2} \in L^2(\mathbb{R}^n)$, this shows that also $P\sum_{j=1}^n \partial_j U^{j+1}$ belongs to $L^2(\mathbb{R}^n)$. Consequently, $U \in \mathscr{D}(\mathcal{A})$ and we conclude that $\mathscr{D}(\mathcal{A}(t)') \subset \mathscr{D}(\mathcal{A}(t))$. The assertion is therefore proved. □

The full linear operator can now be handled by a perturbation argument.

4.2. **Lemma.** *Let $T \in (0,\infty)$, $\mathcal{A}$ be defined as above, and let $\mathcal{M} = \mathcal{P}M$ with an $(n+2)n \times (n+2)n$ matrix $M \in C_b([0,T]\times\mathbb{R}^n)$. Assume that*

$$(a_j)_{j=1}^n \subset LIP\,([0,T], L^\infty(\mathbb{R}^n)), \quad \operatorname{div}(a_1,\dots,a_n) = 0.$$

Then $\mathcal{A}+\mathcal{M}$ is the propagator of an evolution family

$$(\mathcal{U}(t,s))_{0\leq s\leq t\leq T}\subset \mathscr{L}(\mathcal{H}).$$

Proof. By Lemma 4.1 for every $t\in[0,T]$, $\mathcal{A}(t)$ is skew-selfadjoint on $\mathcal{H}$. Stones's theorem implies that $\mathcal{A}(t)$ is the generator of a unitary C_0-group of contractions on $\mathcal{H}$. Clearly, we also have $\mathscr{D}(\mathcal{A}(t))=\mathscr{D}(\mathcal{A})$ for every $t\in[0,T]$. The Lipschitz continuity assumption on $(a_j)_{j=1}^n$ in t then implies that

$$(t\mapsto \mathcal{A}(t))\in LIP([0,T],\mathscr{L}(\mathscr{D}(\mathcal{A}),\mathcal{H})).$$

Thus, $(\mathcal{A}(t))_{t\in[0,T]}$ is a CD-system. By [9, Section 1.2] (see also [12]) therefore $\mathcal{A}$ is the propagator of an evolution family on $\mathcal{H}$. By the fact that $\mathcal{M}\in C([0,T],\mathscr{L}(\mathcal{H}))$, a standard abstract perturbation argument (cf. [9, Remark 1.1(c)] or [12]) implies that $\mathcal{A}+\mathcal{M}$ is still the propagator of an evolution family on $\mathcal{H}$ as claimed in the lemma. □

Lemma 4.2 and the variation of constant formula imply (for suitable f and V_0) the well-posedness of the problem

$$\begin{cases} \partial_t V+\mathcal{A}V+\mathcal{B}V &= f \quad \text{in } (0,T),\\ V|_{t=0} &= V_0. \end{cases}\tag{4.4}$$

However, in order to prove a local-in-time existence result for the full quasilinear system, higher regularity in Sobolev spaces for the linear problem is required. For this purpose we employ the method of viscosity solutions.

4.3. **Lemma.** *Let $q\in\mathbb{N}_0$, $V_0\in H^{q+2}(\mathbb{R}^n)\cap\mathcal{H}$, and let $a,b\in C_b^\infty([0,T]\times\mathbb{R}^n)$. Then for each $\varepsilon>0$ there exists a unique solution V_ε of*

$$\begin{cases} \partial_t V_\varepsilon-\varepsilon\Delta V_\varepsilon+(\mathcal{A}+\mathcal{B})V_\varepsilon &= 0 \quad \text{in } (0,T),\\ V_\varepsilon(0) &= V_0 \end{cases}\tag{4.5}$$

satisfying

$$V\in C^1([0,T],H^q(\mathbb{R}^n)\cap\mathcal{H})\cap C([0,T],H^{q+2}(\mathbb{R}^n)).\tag{4.6}$$

Proof. It is well-known that $\varepsilon\Delta$ is the generator of an analytic C_0-semigroup on $H^q(\mathbb{R}^n)\cap\mathcal{H}$. Note that by our regularity assumptions on a,b the nonautonomous operator $(\mathcal{A}+\mathcal{B})$ represents a lower order perturbation of $\varepsilon\Delta$ regarded as a propagator on $H^q(\mathbb{R}^n)\cap\mathcal{H}$. By standard abstract perturbation results (cf. [12]) we therefore obtain that $-\varepsilon\Delta+\mathcal{A}+\mathcal{B}$ is the propagator of an evolution family $(\mathcal{U}_\varepsilon(t,s))_{0\leq s\leq t\leq T}$ on $H^q(\mathbb{R}^n)\cap\mathcal{H}$ such that $V(t):=\mathcal{U}_\varepsilon(t,0)V_0$ satisfies (4.5) and (4.6). □

In the proof of the next Theorem we will also frequently make use of the following estimates, which are often quoted as "Moser-type inequalities". For a proof we refer to [13, Lemma 4.9].

4.4. **Lemma.** *Let $m\in\mathbb{N}$. There there is a constant $C=C(m,n)>0$ such that for all f, $g\in W^{m,2}(\mathbb{R}^n)\cap L^\infty(R^n)$ and $\alpha\in N_0^n$, $|\alpha|\leq m$, the following inequalities hold:*

$$\|\nabla^\alpha(fg)\|_2 \leq C(\|f\|_\infty\|\nabla^m g\|_2+\|g\|_\infty\|\nabla^m f\|_2),\tag{4.7}$$

$$\|\nabla^\alpha(fg)-f\cdot\nabla^\alpha g\|_2 \leq C(\|\nabla f\|_\infty\|\nabla^{m-1}g\|_2+\|g\|_\infty\|\nabla^m f\|_2).\tag{4.8}$$

The next result provides higher regularity of the solutions of (4.4) under, and this is essential, in a certain sense minimal regularity assumptions on the data and the coefficients. In particular, in Sobolev spaces of higher order these regularity assumptions are weaker as the obtained regularity for the solutions. This will be very helpful for the construction of time-local strong solutions for the full nonlinear problem in Section 5.

4.5. **Theorem.** *Let $T \in (0,\infty)$, $m \in \mathbb{N}$, $m > n/2$, $V_0 \in \mathcal{H} \cap H^{m+1}(\mathbb{R}^n)$, and let the coefficients $a = (a_1,\dots,a_n)$ and $b = (b_1,\dots,b_n)$ satisfy the assumptions of Lemma 4.2. Assume additionally that*

$$a, b \in L^1\left((0,T), H^{m+1}(\mathbb{R}^n)\right) \cap C([0,T], H^m(\mathbb{R}^n)). \tag{4.9}$$

Then the unique solution $V = \mathcal{U}(t,0)V_0$ of problem (4.1) satisfies

$$V \in C^1([0,T], H^m(\mathbb{R}^n) \cap \mathcal{H}) \cap C([0,T], H^{m+1}(\mathbb{R}^n)). \tag{4.10}$$

Furthermore, the evolution family $\mathcal{U}$ satisfies the estimates

$$\begin{aligned} \|\mathcal{U}(t,s)V_0\|_{H^{m+1}} &\le C_1\|V_0\|_{H^{m+1}} \exp\left(C_2 \int_s^t \left(\big|(a(r),b(r))\big|_{m+1} + 1\right) \mathrm{d}r\right), && (4.11)\\ \|\partial_t \mathcal{U}(t,0)V_0\|_{H^m} &\le C_1\|V_0\|_{H^{m+1}} \left(\big|(a(t),b(t))\big|_{m+1} + 1\right) \\ &\quad \cdot \exp\left(C_2 \int_0^t \left(\big|(a(r),b(r))\big|_{m+1} + 1\right) \mathrm{d}r\right) && (4.12) \end{aligned}$$

for all $0 \le s \le t \le T$ with constants $C_1, C_2 > 0$ depending only on m and the dimension n, and where we put

$$\big|(a(r),b(r))\big|_{m+1} = \|a(r)\|_{H^{m+1}} + \|b(r)\|_{H^{m+1}}.$$

Proof. The proof is splitted in five steps.

Step 1: construction of suitable approximate solutions $V_{k,\varepsilon}$.

We denote by $J_k^x f$ and $J_k^t f$ the convolution of a function f with the Friedrichs mollifier in the variable x and t, respectively. We set

$$\begin{aligned} V_{0,k} &:= J_k^x V_0 \ \in H^{q+2}(\mathbb{R}^n),\\ a_{j,k} &:= J_k^t E_0 J_k^x a_j|_{[0,T]} \ \in C_b^\infty([0,T]\times\mathbb{R}^n),\\ b_{j,k} &:= J_k^t E_0 J_k^x b_j|_{[0,T]} \ \in C_b^\infty([0,T]\times\mathbb{R}^n) \end{aligned}$$

for $j = 1,\dots,n$ and $k \in \mathbb{N}$, where E_0 denotes the trivial extension by 0 from $[0,T]$ to $\mathbb{R}$. Then we readily obtain

$$\begin{aligned} V_{0,k} &\to V_0 \quad \text{in } H^{m+1}(\mathbb{R}^n) \cap \mathcal{H}, && (4.13)\\ a_k = (a_{1,k},\dots,a_{n,k}) &\to a \quad \text{in } L^1((0,T),H^{m+1}) \cap C([0,T],H^m), && (4.14)\\ \operatorname{div} a_k &= 0 \quad (k \in \mathbb{N}),\\ b_k = (b_{1,k},\dots,b_{n,k}) &\to b \quad \text{in } L^1((0,T),H^{m+1}) \cap C([0,T],H^m). && (4.15) \end{aligned}$$

We fix $q > m+1$ and denote by $\mathcal{A}_k$ and $\mathcal{B}_k$ the operators being defined as $\mathcal{A}$ and $\mathcal{B}$ with coefficients a_k and b_k, respectively. Due to Lemma 4.3 for every $k \in \mathbb{N}$ and $\varepsilon > 0$

there is a viscosity solution, denoted by $V_{k,\varepsilon}$, of the system

$$\left\{\begin{array}{rcl} \partial_t V_{k,\varepsilon} - \varepsilon\Delta V_{k,\varepsilon} + (\mathcal{A}_k + \mathcal{B}_k)V_{k,\varepsilon} & = & 0 \quad \text{in } (0,T), \\ V_{k,\varepsilon}(0) & = & V_{0,k} \end{array}\right. \tag{4.16}$$

satisfying

$$V_{k,\varepsilon} \in C^1([0,T], H^q(\mathbb{R}^n) \cap \mathcal{H}) \cap C([0,T], H^{q+2}(\mathbb{R}^n)). \tag{4.17}$$

Step 2: uniform boundedness of $V_{k,\varepsilon}$.
Let $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \le m+1$. Since $m+1 < q$, we may apply ∂^α to (4.16) to the result

$$\left\{\begin{array}{rcl} \partial_t \partial^\alpha V_{k,\varepsilon} - \varepsilon\Delta\partial^\alpha V_{k,\varepsilon} + \mathcal{A}_k\partial^\alpha V_{k,\varepsilon} & = & F(V_{k,\varepsilon}) \quad \text{in } (0,T), \\ V_{k,\varepsilon}(0) & = & V_{k,0} \end{array}\right. \tag{4.18}$$

with

$$\begin{aligned} F(V_{k,\varepsilon}) = &- e_{n+2}\Big[P\sum_{j=1}^n \Big(\partial^\alpha a_{j,k}\partial_j V_{k,\varepsilon}^{n+2} - a_{j,k}\partial^\alpha\partial_j V_{k,\varepsilon}^{n+2} + \partial^\alpha b_{j,k}V_{k,\varepsilon}^{j+1}\Big) \\ &+ \partial^\alpha V_{k,\varepsilon}^{n+2}/\sqrt{\tau}\Big] + e_1\partial^\alpha V_{k,\varepsilon}^{n+2}. \end{aligned}$$

Inequality (4.8) applied on the terms involving the $a_{j,k}$'s and (4.7) on the terms involving the $b_{j,k}$'s yields

$$\begin{aligned} &\|F(V_{k,\varepsilon})(t)\|_{L^2} \\ &\le\; C(n,m)\bigg(\sum_{j=1}^n\Big[\|a_{j,k}(t)\|_{W^{1,\infty}}\|V_{k,\varepsilon}(t)\|_{H^{m+1}} + \|a_{j,k}(t)\|_{H^{m+1}}\|V_{k,\varepsilon}(t)\|_{W^{1,\infty}}\Big] \\ &\qquad + \sum_{j=1}^n\Big[\|b_{j,k}(t)\|_{L^\infty}\|V_{k,\varepsilon}(t)\|_{H^{m+1}} + \|b_{j,k}(t)\|_{H^{m+1}}\|V_{k,\varepsilon}(t)\|_{L^\infty} + \|V_{k,\varepsilon}(t)\|_{H^{m+1}}\Big]\bigg). \end{aligned}$$

In view of the Sobolev embedding and by our assumption $m > n/2$ we can continue this calculation to the result

$$\begin{aligned} \|F(V_{k,\varepsilon})(t)\|_{L^2} \le\; & C(n,m)\Big(\|a_k(t)\|_{H^{m+1}} + \|b_k(t)\|_{H^{m+1}} + 1\Big)\|V_{k,\varepsilon}(t)\|_{H^{m+1}} \\ \le\; & C(n,m)\Big(\big|(a_k(t), b_k(t))\big|_{m+1} + 1\Big)\|V_{k,\varepsilon}(t)\|_{H^{m+1}} \quad (t \in [0,T]). \end{aligned} \tag{4.19}$$

Forming the dual pairing of (4.18) with $\partial^\alpha V_{k,\varepsilon}$ implies

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial^\alpha V_{k,\varepsilon}(t)\|_{L^2}^2 + \varepsilon\|\partial^\alpha\nabla V_{k,\varepsilon}(t)\|_{L^2}^2 = \langle F(t),\, V_{k,\varepsilon}(t)\rangle \\ &\le C(n,m)\Big(\big|(a_k(t), b_k(t))\big|_{m+1} + 1\Big)\|V_{k,\varepsilon}(t)\|_{H^{m+1}}^2. \end{aligned}$$

Summing up over $|\alpha| \le m+1$ and integrating over t then yields

$$\begin{aligned} &|||V_{k,\varepsilon}(t)|||^2 \\ &\le \|V_{0,k}\|_{H^{m+1}}^2 + C(n,m)\int_0^t \Big(\big|(a_k(r), b_k(r))\big|_{m+1} + 1\Big)|||V_{k,\varepsilon}(r)|||^2\mathrm{d}r, \end{aligned}$$

where

$$|||V_{k,\varepsilon}(t)|||^2 := \|V_{k,\varepsilon}(t)\|^2_{H^{m+1}} + \varepsilon \int_0^t \|\nabla V_{k,\varepsilon}(r)\|^2_{H^{m+1}} \mathrm{d}r, \quad t \in [0,T].$$

Thus, applying Gronwall's lemma and taking into account (4.13)-(4.15), we end up with

$$\begin{aligned} &\|V_{k,\varepsilon}(t)\|^2_{H^{m+1}} + \varepsilon \int_0^t \|\nabla V_{k,\varepsilon}(r)\|^2_{H^{m+1}} \mathrm{d}r \\ &\leq C_1(n,m)\|V_{0,k}\|^2_{H^{m+1}} \exp\left(C_2(n,m) \int_0^t \left(\big|(a_k(r), b_k(r))\big|_{m+1} + 1 \right) \mathrm{d}r \right) \\ &\leq C_1(n,m)\|V_0\|^2_{H^{m+1}} \exp\left(C_2(n,m) \int_0^t \left(\big|(a(r), b(r))\big|_{m+1} + 1 \right) \mathrm{d}r \right) \\ &\leq C_1(n,m,V_0,a,b,T) \quad (t \in [0,T],\ k \in \mathbb{N},\ \varepsilon > 0). \end{aligned} \tag{4.20}$$

This shows that $V_{k,\varepsilon}$ is uniformly bounded in $L^\infty([0,T], H^{m+1}(\mathbb{R}^n))$ and that $\varepsilon \nabla V_{k,\varepsilon}$ is uniformly bounded in $L^2([0,T], H^{m+1}(\mathbb{R}^n))$. Again by an application of (4.7) we therefore obtain that $(\mathcal{A}_k + \mathcal{B}_k)V_{k,\varepsilon}$ is uniformly bounded in $L^\infty([0,T], H^m(\mathbb{R}^n))$. ¿From that, the uniform boundedness of $\varepsilon \Delta V_{k,\varepsilon}$ in $L^2([0,T], H^m(\mathbb{R}^n))$, and the equations (4.16) we infer that also $\partial_t V_{k,\varepsilon}$ is uniformly bounded in $L^2([0,T], H^m(\mathbb{R}^n))$. Thus, we have proved that $V_{k,\varepsilon}$ is uniformly bounded in the class

$$H^1([0,T], H^m(\mathbb{R}^n)) \cap L^\infty([0,T], H^{m+1}(\mathbb{R}^n)). \tag{4.21}$$

Step 3: weak* convergence of $V_{k,\varepsilon}$ to the solution V of (4.1).

The outcome of step 2 implies the existence of a subsequence of $V_{k,\varepsilon}$, for simplicity also denoted by $V_{k,\varepsilon}$, converging weakly* in the class (4.21) for $k \to \infty$ and $\varepsilon \to 0$. Denote by U its limit. Then U also belongs to (4.21). Thanks to the Sobolev embedding we also have

$$U \in H^1([0,T], H^m(\mathbb{R}^n)) \hookrightarrow C([0,T], H^m(\mathbb{R}^n)). \tag{4.22}$$

Next, we show that U solves (4.4). In fact, multiplying

$$\varphi \in C_0^1([0,T), C_c^\infty(\mathbb{R}^n)), \quad \operatorname{div} \varphi^{n+2} = 0$$

to (4.16) and integrating by parts gives us

$$\begin{aligned} 0 &= \int_0^T \langle (\partial_t - \varepsilon\Delta + \mathcal{A}_k(t) + \mathcal{B}_k(t))V_{k,\varepsilon}(t),\, \varphi(t)\rangle \mathrm{d}t \\ &= -\int_0^T \langle V_{k,\varepsilon}(t),\, (\partial_t + \mathcal{A}_k(t) + \mathcal{B}_k(t)')\varphi(t)\rangle \mathrm{d}t - \varepsilon \int_0^T \langle V_{k,\varepsilon}(t),\, \Delta\varphi\rangle \mathrm{d}t + \langle V_{0,k},\, \varphi(0)\rangle. \end{aligned}$$

Due to (4.14), (4.15), and $m > n/2$ we have

$$\begin{aligned} \|(\mathcal{A}_k + \mathcal{B}_k' - \mathcal{A} - \mathcal{B}')\varphi\|_{L^1(\mathcal{H})} &\leq C\left(\|a_k - a\|_{L^\infty} + \|b_k - b\|_{L^\infty}\right)\|\varphi\|_{L^1(H^1)} \\ &\to 0 \quad (k \to \infty). \end{aligned}$$

This shows that

$$(\partial_t + \mathcal{A}_k + \mathcal{B}_k')\varphi \to (\partial_t + \mathcal{A} + \mathcal{B}')\varphi \quad \text{strongly in } L^1([0,T], \mathcal{H}) \quad (k \to \infty).$$

Since $V_{k,\varepsilon} \to U$ weakly* in $L^\infty([0,T],\mathcal{H})$ we obtain

$$\int_0^T \langle V_{k,\varepsilon}(t),\, (\partial_t + \mathcal{A}_k + \mathcal{B}'_k)\varphi(t)\rangle \mathrm{d}t \to \int_0^T \langle U(t),\, (\partial_t + \mathcal{A} + \mathcal{B}')\varphi(t)\rangle \mathrm{d}t \quad (k\to\infty,\ \varepsilon\to 0).$$

The boundedness of $V_{k,\varepsilon}$ in $L^\infty([0,T],\mathcal{H})$ also yields

$$\varepsilon \int_0^T \langle V_{k,\varepsilon},\, \Delta\varphi\rangle \mathrm{d}t \to 0 \quad (k\to\infty,\ \varepsilon\to 0).$$

Thus, letting $k\to\infty$ and $\varepsilon \to 0$ implies

$$\int_0^T \langle U(t),\, (\partial_t + \mathcal{A} + \mathcal{B}')\varphi(t)\rangle \mathrm{d}t = \langle V_0, \varphi(0)\rangle.$$

Thanks to the fact that U belongs to (4.21) and in view of (4.22), we can reverse the integration by parts to the result

$$\int_0^T \langle(\partial_t + \mathcal{A} + \mathcal{B})U(t),\, \varphi(t)\rangle \mathrm{d}t = \langle V_0 - U(0), \varphi(0)\rangle. \quad (\varphi \in C_0^1([0,T), C_0^\infty(\mathbb{R}^n)\cap\mathcal{H})).$$

Choosing $\varphi \in C_0^1((0,T), C_0^\infty(\mathbb{R}^n)\cap\mathcal{H})$ shows that

$$(\partial_t + \mathcal{A} + \mathcal{B})U = 0 \quad \text{a.e.}$$

This, in turn, implies that $U(0) = V_0$, hence that U solves (4.1). By virtue of (4.22) and by the assumptions on a, b, the fact that U solves (4.1) also yields

$$U \in C^1([0,T], H^{m-1}(\mathbb{R}^n)\cap\mathcal{H}). \tag{4.23}$$

Since we assumed that $n \geq 2$, hence that $m > n/2 \geq 2$, we obtain that U is a strong solution of (4.1). Consequently, U is unique and therefore coincides with $V = \mathcal{U}(\cdot,\cdot)V_0$, where $\mathcal{U}$ is the evolution family given by Lemma 4.2.

Step 4: proof of estimates (4.11) and (4.12).

Note that by (4.20) and the fact that $U = V$, we obtain

$$\begin{aligned} \|V(t)\|_{H^{m+1}} &\leq \liminf_{k\to\infty,\ \varepsilon\to 0} \|V_{k,\varepsilon}(t)\|_{H^{m+1}} \\ &\leq C_1(n,m)\|V_0\|_{H^{m+1}} \exp\left(C_2(n,m)\int_0^t \left(\big|(a(r),b(r))\big|_{m+1} + 1\right)\mathrm{d}r\right) \end{aligned}$$

for $t \in [0,T]$. Hence estimate (4.11) is satisfied for V and $s = 0$. In order to get the general case we fix $s \in [0,T]$ and set

$$\begin{aligned} \tilde{\mathcal{U}}(t,0) &:= \mathcal{U}(t+s,s), \\ \tilde{a}(t) &:= a(t+s), \\ \tilde{b}(t) &:= b(t+s) \end{aligned}$$

for $t \in [0,T-s]$. If $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ denote the operators corresponding to the coefficients $\tilde{a}$ and $\tilde{b}$ respectively, we see that $\tilde{V} := \tilde{\mathcal{U}}(t,0)V_0$ solves

$$\left\{\begin{aligned} \partial_t\tilde{V} + (\tilde{\mathcal{A}} + \tilde{\mathcal{B}})\tilde{V} &= 0 \quad \text{in } (0,T-s), \\ \tilde{V}(0) &= V_0 \end{aligned}\right.$$

for $V_0 \in H^{m+1}(\mathbb{R}^n)$. By the just proved facts for the solution of this system we deduce

$$\begin{aligned}
&\|\mathcal{U}(t+s,s)V_0\|_{H^{m+1}} = \|\tilde{\mathcal{U}}(t,0)V_0\|_{H^{m+1}} \\
&\le C_1(n,m)\|V_0\|_{H^{m+1}} \exp\left(C_2(n,m)\int_0^t \left(\left|(\tilde{a}(r),\tilde{b}(r))\right|_{m+1}+1\right)\mathrm{d}r\right) \\
&\le C_1(n,m)\|V_0\|_{H^{m+1}} \exp\left(C_2(n,m)\int_s^{t+s} \left(\left|(a(r),b(r))\right|_{m+1}+1\right)\mathrm{d}r\right),
\end{aligned}$$

hence (4.11). The estimate for the time derivative of $\mathcal{U}$ now easily follows by

$$\begin{aligned}
\|\partial_t\mathcal{U}(t,0)V_0\|_{H^m} &= \|(\mathcal{A}(t)+\mathcal{B}(t))\mathcal{U}(t,0)V_0\|_{H^m} \\
&\le C(n,m)\left(\left|(a(t),b(t))\right|_m+1\right)\|\mathcal{U}(t,0)V_0\|_{H^{m+1}} \quad (t\in[0,T]),
\end{aligned}$$

where we applied once more Lemma 4.4.

Step 5: continuity of V in time.

¿From step 4 and our assumptions on a, b we immediately see that

$$V \in W^{1,\infty}([0,T],H^m(\mathbb{R}^n)) \cap L^\infty([0,T],H^{m+1}(\mathbb{R}^n)). \tag{4.24}$$

It remains to show that in (4.24) $W^{1,\infty}$ and L^∞ can be replaced by C^1 and C, respectively. To this end, we will employ the variation of constant formula.

Thanks to (4.22) and (4.23) we have

$$V = \mathcal{U}(t,0)V_0 \in C^1([0,T],H^{m-1}(\mathbb{R}^n)) \cap C([0,T],H^m(\mathbb{R}^n)) \tag{4.25}$$

for arbitrary $V_0 \in H^{m+1}(\mathbb{R}^n)$. In view of $m \ge 2$, we may apply ∂^α for $|\alpha| \le 1$ to (4.1). This leads to

$$\left\{\begin{array}{rcl} \partial_t\partial^\alpha V + (\mathcal{A}+\mathcal{B})\partial^\alpha V &=& F(V) \quad \text{in } (0,T), \\ \partial^\alpha V|_{t=0} &=& \partial^\alpha V_0. \end{array}\right. \tag{4.26}$$

with

$$F(V) = -e_{n+2}P\sum_{j=1}^n\Big[(\partial^\alpha a_j)\partial_j V^{n+2} + (\partial^\alpha b_j)V^{j+1}\Big].$$

Very similar to the calculations that lead to (4.19) we can derive

$$\|F(V)(t)\|_{H^m} \le C(n,m)\left|(a(t),b(t))\right|_{m+1}\|V(t)\|_{H^{m+1}} \quad (t\in[0,T]).$$

By virtue of our assumptions on a, b and since

$$V \in L^\infty([0,T],H^{m+1}(\mathbb{R}^n))$$

we observe that

$$F(V) \in L^1((0,T),H^m(\mathbb{R}^n)).$$

On the other hand, by applying the Hölder inequality we can also estimate as

$$\|F(V)(t)\|_2 \le C\bigg(\|\nabla a(t)\|_4 + \|b(t)\|_4\bigg)\|\nabla V(t)\|_4 \quad (t\in[0,T]).$$

Since $m-1 \ge m/2 > n/4$ for $m \ge 2$, the Sobolev embedding implies that $H^{m-1}(\mathbb{R}^n) \hookrightarrow L^4(\mathbb{R}^n)$. Hence the above inequality gives us $F(V) \in L^\infty((0,T),\mathcal{H})$. By our asumptions on a and b and in view of (4.25), $F(V)$ is even continuous in time. So, altogether we obtain

$$F(V) \in L^1((0,T), H^m(\mathbb{R}^n)) \cap C([0,T],\mathcal{H}).$$

According to $H^1(\mathbb{R}^n) \cap \mathcal{H} \hookrightarrow \mathscr{D}(\mathcal{A})$, [9, Remark 1.3] therefore implies that $\partial^\alpha V$ is the unique strong solution of (4.26) given by the variation of constant formula

$$\partial^\alpha V(t) = \mathcal{U}(t,0)\partial^\alpha V_0 + \int_0^t \mathcal{U}(t,s)F(V)(s)\mathrm{d}s, \quad t \in [0,T]. \tag{4.27}$$

Here $\mathcal{U}$ still denotes the evolution system generated by the propagator $\mathcal{A}+\mathcal{B}$.

¿From our assumptions (4.9) on a, b and step 4 we know that $\mathcal{U}$ satisfies the estimate

$$\|\mathcal{U}(t,s)\|_{\mathscr{L}(H^{m+1}\cap\mathcal{H})} \le C_1(T) \quad (0 \le s \le t \le T),$$

for some $C_1 > 0$. Since $\mathcal{U}$ is an evolution system on $\mathcal{H}$ we also have

$$\|\mathcal{U}(t,s)\|_{\mathscr{L}(\mathcal{H})} \le C_2(T) \quad (0 \le s \le t \le T),$$

for some $C_2 > 0$. Interpolating these two inequalities yields

$$\|\mathcal{U}(t,s)\|_{\mathscr{L}([\mathcal{H},\, H^{m+1}\cap\mathcal{H}]_\theta)} \le C(T) \quad (0 \le s \le t \le T),$$

with $C = \max(C_1, C_2)$ and where $[\cdot,\cdot]_\theta$ denotes the complex interpolation space for $\theta \in (0,1)$. By the fact that $\mathcal{H}$ is complementary in $L^2(\mathbb{R}^n)$, [16, Theorem 1.17.1.1] implies that

$$[\mathcal{H},\, H^{m+1} \cap \mathcal{H}]_\theta = [L^2(\mathbb{R}^n),\, H^{m+1}]_\theta \cap \mathcal{H} = H^{\theta(m+1)}(\mathbb{R}^n) \cap \mathcal{H}.$$

Consequently, for $\theta = m/(m+1)$ we deduce

$$\|\mathcal{U}(t,s)\|_{\mathscr{L}(H^m\cap\mathcal{H})} \le Ce^{\omega(t-s)} \quad (0 \le s \le t \le T).$$

¿From this we immediately gain the estimate

$$\|\mathcal{U}(t,s)F(V)(s)\|_{H^m} \le C(T)\|F(V)(s)\|_{H^m} \quad (0 \le s \le t \le T).$$

Inserting this into (4.27) while taking the H^m-norm and keeping in mind continuity relation (4.25) and that $F(V) \in L^1((0,T), H^m(\mathbb{R}^n))$ then gives us

$$\begin{aligned}\|\partial^\alpha(V(t) - V_0)\|_{H^m} &\le \|(\mathcal{U}(t,0) - I)\partial^\alpha V_0\|_{H^m} + C(a,b,T)\int_0^t \|F(V)(s)\|_{H^m}\mathrm{d}s \\ &\to 0 \quad (t \to 0,\ |\alpha| \le 1).\end{aligned}$$

This shows that $t \mapsto \mathcal{U}(t,0)$ is strongly continuous in $t = 0$ w.r.t. the H^{m+1}-norm. The fact that $\mathcal{U}$ is an evolution family then implies the continuity on $[0,T]$. So, we have proved

$$V \in C([0,T], H^{m+1}(\mathbb{R}^n)).$$

The assertion that $V \in C^1([0,T], H^m(\mathbb{R}^n))$ then follows again by $\partial_t V = -(\mathcal{A}+\mathcal{B})V$ and by our assumption $a, b \in C([0,T], H^m(\mathbb{R}^n))$ on the coefficients. The result is therefore proved. □

5. Quasilinear local existence

Based on a fixed point iteration here we construct local-in-time solutions to the first order quasilinear system (3.5). The idea of this fixed point iteration goes back to Majda [10]. However, by the strength of our linear result Theorem 4.5 the proof of the quasilinear local-in-time existence performed here becomes much more elegant compared to the methods used in [10] or [13].

5.1. **Theorem.** *Let $m \in \mathbb{N}_0$, $m > n/2$, and let $V_0 \in \mathcal{H} \cap H^{m+1}(\mathbb{R}^n)$. Then, there exists a $T > 0$ and a unique solution*

$$V \in C^1([0,T], H^m(\mathbb{R}^n) \cap \mathcal{H}) \cap C([0,T], H^{m+1}(\mathbb{R}^n)) \tag{5.1}$$

of system (3.5). The existence time T can be estimated from below as

$$T > \frac{1}{1 + C\|V_0\|_{H^{m+1}}} \tag{5.2}$$

with a constant $C > 0$ depending only on m and the dimension n.

Proof. **Step 1:** existence.
Let $V_0 \in H^{m+1}(\mathbb{R}^n) \cap \mathcal{H}$ be an initial value. Set

$$V_0(t,x) := V_0(x) \quad ((t,x) \in [0,T] \times \mathbb{R}^n)$$

and for $k \in \mathbb{N}_0$ let V_{k+1} be inductively defined as the solution of the initial value problem

$$\left\{ \begin{array}{rcl} \partial_t V_{k+1} + (\mathcal{A}(V_k) + \mathcal{B}(V_k))V_{k+1} & = & 0 \quad \text{in } (0,T), \\ V_{k+1}(0) & = & V_0. \end{array} \right. \tag{5.3}$$

By the fact that

$$\begin{aligned} & C^1([0,T], H^m(\mathbb{R}^n)) \cap C([0,T], H^{m+1}(\mathbb{R}^n)) \\ & \hookrightarrow C([0,T], H^m(\mathbb{R}^n)) \cap L^1((0,T), H^{m+1}(\mathbb{R}^n)) \cap LIP([0,T], L^\infty(\mathbb{R}^n)), \end{aligned}$$

we see that Theorem 4.5 (i.p. (4.9) and (4.10)) implies that every solution belongs to the class of the coefficients for the next step. Hence, V_{k+1} is well-defined for every $k \in \mathbb{N}_0$. Next, we will prove the following uniform bounds.

5.2. **Lemma.** *There exist $R, L, T_* > 0$ such that for all $k \in \mathbb{N}_0$ we have*

(i) $\|V_k\|_{L^\infty([0,T_*],H^{m+1})} \leq R$,
(ii) $\|\partial_t V_k\|_{L^\infty([0,T_*],H^m)} \leq L$.

Proof. We use induction over $k \in \mathbb{N}_0$. For $k = 0$ we have

$$\|V_0\|_{L^\infty([0,T],H^{m+1})} = \|V_0\|_{H^{m+1}} \leq R,$$

which is to understand as a first condition on the size of R. In view of $\partial_t V_0 = 0$ we see that L is still arbitrary.

Now, assume that the assertion holds for $k \in \mathbb{N}_0$. Estimate (4.11) in combination with (4.9) and the induction hypothesis imply

$$\begin{aligned} \|V_{k+1}\|_{L^\infty([0,T],H^{m+1})} & \leq C_1\|V_0\|_{H^{m+1}} \exp\left(C_2 \int_0^T (\|V_k(r)\|_{H^{m+1}} + 1) \, \mathrm{d}r \right) \\ & \leq C_1\|V_0\|_{H^{m+1}} \exp\left(C_2(R+1)T\right) \quad (T > 0). \end{aligned}$$

We choose

$$R = R(\|V_0\|_{H^{m+1}}) := C_1\|V_0\|_{H^{m+1}} \exp(C_2) =: C(n,m)\|V_0\|_{H^{m+1}}.$$

Then for

$$T_* \leq \frac{1}{R+1} = \frac{1}{1 + C(n,m)\|V_0\|_{H^{m+1}}}$$

we obtain

$$\|V_{k+1}\|_{L^\infty([0,T_*],H^{m+1})} \leq R.$$

This leads to estimate (5.2) for the size of the existence time.

Similarly, for the time derivative of V_{k+1} we employ estimate (4.12) in combination with (4.9) to the result

$$\begin{aligned}\|\partial_t V_{k+1}\|_{L^\infty([0,T],H^m)} &\leq C_1\|V_0\|_{H^{m+1}} \left(\|V_k\|_{L^\infty([0,T],H^{m+1})} + 1\right) \\ &\quad \cdot \exp\left(C_2 \int_0^T (\|V_k(r)\|_{H^{m+1}} + 1)\,\mathrm{d}r\right) \\ &\leq C_1\|V_0\|_{H^{m+1}}(R+1)\exp\left(C_2(R+1)T\right) \quad (T > 0).\end{aligned}$$

Thus, again for $T_* \leq 1/(R+1)$ we deduce

$$\|\partial_t V_{k+1}\|_{L^\infty([0,T],H^m)} \leq R(R+1) =: L.$$

This fixes L and the lemma is proved. □

The just proved lemma implies the existence of a subsequence of V_k (for simpicity also denoted by V_k) converging weakly* in $W^{1,\infty}([0,T_*],H^m(\mathbb{R}^n)) \cap L^\infty([0,T_*],H^{m+1}(\mathbb{R}^n))$. Thus there is a limit

$$V \in W^{1,\infty}([0,T_*],H^m(\mathbb{R}^n)) \cap L^\infty([0,T_*],H^{m+1}(\mathbb{R}^n)). \tag{5.4}$$

Due to $m > n/2$ the Sobolev embedding also implies that

$$V \in C([0,T_*],H^m(\mathbb{R}^n)) \hookrightarrow C([0,T_*] \times \mathbb{R}^n). \tag{5.5}$$

Next , let $G \subset \mathbb{R}^n$ be compact. Since the embedding

$$W^{1,\infty}([0,T_*],H^m(G)) \cap L^\infty([0,T_*],H^{m+1}(G)) \hookrightarrow L^2((0,T_*),H^m(G))$$

is compact (cf. [15]), in view of the Sobolev embedding we obtain that

$$V_k \to V \quad \text{strongly in } L^2((0,T_*) \times G). \tag{5.6}$$

Forming the dual pairing of (5.3) with $\varphi \in C_0^1([0,T_*),C_0^\infty(\mathbb{R}^n))$ such that $\operatorname{div}\varphi^{n+2} = 0$ and integrating by parts implies

$$\int_0^{T_*} \langle V_{k+1}(t),\, (\partial_t + \mathcal{A}(V_k(t)) + \mathcal{B}(V_k(t)))\varphi(t)\rangle_{L^2(G)}\mathrm{d}t = \langle V_0,\, \varphi(0)\rangle \tag{5.7}$$

with $G \subset \mathbb{R}^n$ compact so that $\mathrm{supp}\varphi(t) \subset G$ for all $t \in [0,T_*]$. By virtue of (5.6) we observe that

$$(\partial_t + \mathcal{A}(V_k) + \mathcal{B}(V_k))\varphi \to (\partial_t + \mathcal{A}(V) + \mathcal{B}(V))\varphi \quad (k \to \infty)$$

strongly in $L^2((0,T_*) \times G)$. Hence, letting $k \to \infty$ in (5.7) shows that

$$\int_0^{T_*} \langle V(t),\, (\partial_t + \mathcal{A}(V(t)) + \mathcal{B}(V(t)))\varphi(t)\rangle_{L^2(G)}\mathrm{d}t = \langle V_0,\, \varphi(0)\rangle.$$

Thanks to (5.4) and (5.5) we may reverse the integration by parts which yields

$$\int_0^{T_*} \langle (\partial_t + \mathcal{A}(V(t)) + \mathcal{B}(V(t)))V(t),\, \varphi(t)\rangle_{L^2(G)} \mathrm{d}t = \langle V_0 - V(0),\, \varphi(0)\rangle.$$

In the same way as in step 3 of the proof of Theorem 4.5 we therefore obtain that V solves (3.5) for a.e. $(t,x) \in [0,T_*] \times \mathbb{R}^n$.

To see that V satisfies (5.1) we argue as follows. First observe that we have

$$W^{1,\infty}([0,T_*], H^m(\mathbb{R}^n)) \hookrightarrow LIP([0,T_*], H^m(\mathbb{R}^n)).$$

Combining this with (5.4) and (5.5) implies

$$V \in L^\infty([0,T_*], H^{m+1}(\mathbb{R}^n)) \cap C([0,T_*], H^m(\mathbb{R}^n)) \cap LIP([0,T], L^\infty(\mathbb{R}^n)).$$

By this fact we may regard (3.5) as the linear system

$$\left\{ \begin{array}{rcll} \partial_t U + (\mathcal{A} + \mathcal{B})U &=& 0 & \text{in } (0,T_*), \\ U(0) &=& V_0 & \end{array} \right. \tag{5.8}$$

with fixed coefficients

$$a := \sqrt{\tau/\mu}\, V^1, \qquad b := \frac{1}{\sqrt{\mu}}(\sqrt{\tau} V^{n+2} + V^1).$$

Theorem 4.5 implies the existence of a unique solution

$$U \in C^1([0,T_*], H^m(\mathbb{R}^n) \cap \mathcal{H}) \cap C([0,T_*], H^{m+1}(\mathbb{R}^n)).$$

Obviously U is a strong solution of (5.8). On the other hand, in view of (5.5), our assumptions on a, b, and since V solves (3.5) we obtain

$$V \in C^1([0,T_*], \mathcal{H}) \cap C([0,T_*], \mathscr{D}(\mathcal{A})).$$

Thus, V is a strong solution of (5.8) as well. By the uniqueness of strong solutions of the linear system (5.8) we obtain $V = U$, hence (5.1).

Step 2: uniqueness.

Let

$$U, V \in C^1([0,T], \mathcal{H}) \cap C([0,T], H^1(\mathbb{R}^n)) \cap L^\infty([0,T], W^{1,\infty}(\mathbb{R}^n))$$

be solutions of (3.5) to the initial value V_0. Then $W := U - V$ solves

$$\left\{ \begin{array}{rcll} \partial_t W + \mathcal{A}(U)W &=& F & \text{in } (0,T_*), \\ W(0) &=& 0, & \end{array} \right. \tag{5.9}$$

with

$$F = (\mathcal{A}(W) + \mathcal{B}(W))V + \mathcal{B}(U)W,$$

where we used the fact that $V \mapsto \mathcal{A}(V)$ and $V \mapsto \mathcal{B}(V)$ are linear. Our assumptions on U, V yield

$$\|V\|_{L^\infty([0,T], W^{1,\infty}(\mathbb{R}^n))} + \|U\|_{L^\infty([0,T]\times\mathbb{R}^n)} \le C.$$

Thus we can estimate F as

$$\|F(t)\|_{\mathcal{H}} \le C\|W(t)\|_{\mathcal{H}} \left(\|V(t)\|_{W^{1,\infty}} + \|U(t)\|_{L^\infty}\right) \le C\|W(t)\|_{\mathcal{H}} \quad (t \in [0,T]).$$

Forming the dual pairing of (5.9) with W gives us

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|W(t)\|_{L^2}^2 = \langle F(t),\, W(t)\rangle \le C\|W(t)\|_{L^2}^2 \quad (t \in [0,T]).$$

Consequently, $W = 0$ by Gronwall's lemma. This completes the proof of Theorem 5.1. $\square$

We conclude with the proof of our main result Theorem 1.1.

Proof. Let $(u_0, u_1) \in \big(H^{m+2}(\mathbb{R}^n) \cap L^2_\sigma(\mathbb{R}^n)\big) \times \big(H^{m+1}(\mathbb{R}^n) \cap L^2_\sigma(\mathbb{R}^n)\big)$. Then we have $V_0 := (v_0, \partial_1 v_0, \dots, \partial_n v_0, v_1)^T \in H^{m+1}(\mathbb{R}^n) \cap \mathcal{H}$, where $(v_0, v_1) := (u_0, \sqrt{\tau} u_1)$. If V is the solution of system (3.5) in $(0, T)$ we set $v := V^1$. Then by construction of $\mathcal{A} + \mathcal{B}$ we readily see that v satisfies equations (3.2). Regularity relation (5.1) and the fact that $V = (v, \partial_1 v, \dots, \partial_n, \partial_t v)$ imply

$$v \in C^2([0,T], H^m(\mathbb{R}^n)) \cap C^1([0,T], H^{m+1}(\mathbb{R}^n)) \cap C([0,T], H^{m+2}(\mathbb{R}^n) \cap L^2_\sigma(\mathbb{R}^n)).$$

Setting $T_* := \sqrt{\tau} T$ then gives us the claimed regularity for $u(t,x) := v(t/\sqrt{\tau},\, x/\sqrt{\mu})$, the solution of (3.1).

A further application of Lemma 4.4 and the regularity of u show that

$$\tau(u \cdot \nabla)\partial_t u,\ ((\tau\partial_t u + u) \cdot \nabla)u \in C([0, T_*], H^m(\mathbb{R}^n)).$$

(This can also be seen by the construction of V.) Thus, we may recover the pressure term via

$$\begin{aligned} \nabla\pi &:= (I - P)\left(-\tau(u \cdot \nabla)\partial_t u - ((\tau\partial_t u + u) \cdot \nabla)u)\right) \\ &= (1 + \tau\partial_t)(I - P)(u \cdot \nabla)u. \end{aligned} \tag{5.10}$$

This yields that (u, π) is the unique solution of (1.1) with the claimed regularity. $\square$

Corollary 1.2 now is easily obtained as follows

Proof. Assuming $u_0, u_1 \in \bigcap_{k=0}^\infty H^k(\mathbb{R}^n)$ implies that $u \in C^2([0, T_*], H^m(\mathbb{R}^n))$ for every $m \in \mathbb{N}$. By applying ∂_t iteratively to equations (3.1) and taking into account the boundedness of P on every $H^m(\mathbb{R}^n)$, we even obtain that $u \in C^\infty([0, T_*], H^m(\mathbb{R}^n))$ for every $m \in \mathbb{N}$. From representation (5.10) we then deduce the same regularity for $\nabla\pi$. The Sobolev embedding finally yields the assertion. $\square$

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