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Wave equations with non-dissipative damping*

Jaime E. Muñoz Rivera and Reinhard Racke

Abstract: We consider the nonlinear wave equation $u_{tt} - \sigma(u_x)_x + a(x)u_t = 0$ in a bounded interval $(0, L) \subset \mathbb{R}^1$. The function a is allowed to change sign, but has to satisfy $\int_0^L a(x)dx > 0$. For this non-dissipative situation we prove the exponential stability of the corresponding linearized system for small a, as well as the global existence of smooth, small solutions to the nonlinear system if in particular the negative part of a is small enough.

1 Introduction

We consider the following nonlinear wave equation

$$u_{tt} - \sigma(u_x)_x + a(x)u_t = 0 \tag{1.1}$$

for a function $u = u(t, x)$, $t \geq 0$, $x \in (0, L) \subset \mathbb{R}^1$, $L > 0$ fixed, with initial conditions

$$u(t = 0) = u_0, \quad u_t(t = 0) = u_1 \tag{1.2}$$

and Dirichlet type boundary conditions

$$u(\cdot, 0) = u(\cdot, L) = 0. \tag{1.3}$$

The function a is assumed to satisfy $a \in L^\infty((0, L))$ for the part on the exponential stability of the associated semigroup, and $a \in C^3([0, L])$ for the discussion of the nonlinear system, as well as

$$\int_0^L a(x)dx > 0, \tag{1.4}$$

in particular a may change sign in $[0, L]$ or be zero in open subsets.
The nonlinear function σ is assumed to satisfy

$$\sigma \in C^3(\mathbb{R}), \qquad d_0 := \sigma'(0) > 0, \text{ and } \sigma''(0) = 0. \tag{1.5}$$

Remark: This is, for instance, satisfied for σ corresponding to a vibrating string,

$$\sigma(y) = \frac{y}{\sqrt{1 + y^2}}.$$

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Rewriting (1.1) as

$$u_{tt} - d_0 u_{xx} + a u_t = b(u_x) u_{xx} \tag{1.6}$$

with

$$b(u_x) := \sigma'(u_x) - d_0 = \sigma'(u_x) - \sigma'(0) \tag{1.7}$$

the associated linearized system is

$$u_{tt} - d_0 u_{xx} + a u_t = 0 \tag{1.8}$$

together with the initial conditions (1.2) and the boundary conditions (1.3).
Since a may change sign we have a non-dissipative system still regarding au_t to be a non-local damping. There are many papers on solutions to (1.1) or on decay rates for (1.1) or (1.8) if $a \geq 0$ i.e. if a does not change sign, see for example the papers of Cox and Overton [3], Cox and Zuazua [4], Kawashima, Nakao and Ono [7], Nakao [9, 10], da Silva Ferreira [14] or Zuazua [16] and the references therein. If $a(x) \geq a_0 > 0$ is strictly positive, the exponential decay of solutions to (1.8) and also to (1.1), for small data, easily follows.
Here we consider (perhaps for the first time) a non-dissipative case, and we show that solutions to the linearized system (1.8), (1.2), (1.3) are exponentially stable when a is a small L^∞-function and satisfies (1.4), i.e. when the mean value of a is positive. This is not "a small data result perturbing an exponentially stable system" since for $a \equiv 0$ the system is conservative with no damping at all. For the nonlinear system(1.1)–(1.3), again assuming (1.4), it will be proved that a global solution exists for small data provided the negative part is not too large compared to the mean value of a, more precisely: If α_0 denotes the decay rate for the linear system,

$$\int_0^L (u_t^2 + u_x^2)(t,x) \leq c_1 e^{-2\alpha_0 t}, \quad c_1 > 0,$$

see section 2, and if

$$a = a^+ - a^-, \qquad a^+ = \max(a,0), \quad a^- = \max(-a,0)$$

then a^- has to satisfy in particular

$$||a^-||_{L^\infty} < \alpha_0, \tag{1.9}$$

see section 3.
In section 2 we shall prove the exponential stability for the linearized system. This is the crucial part, and the method will be the spectral one characterizing exponentially stable semigroups in terms of the spectrum of the associated generator of the semigroup. It is possible to give an explicit lower bound on the decay rate which, in turn, is necessary to make (1.9) a reasonable condition in the nonlinear case.
In section 3 the global existence of small solutions to the nonlinear system is investigated. Using the result from section 2 and pertubation arguments the condition (1.9) is shown to be sufficient to guarantee the global existence and also the exponential stability of the nonlinear system.

Summarizing the contributions of our paper we present a result on exponential stability for the wave equation when the function a may change sign. We present an explicit description of the decay rate and of the type of the associated semigroup, and also a discussion of a corresponding nonlinear problem with global existence and stability. Finally, our approach can be applied to other one-dimensional models.

We use standard notations. e.g. for Sobolev spaces.

2 Exponential stability for the linearized system

We first consider the linearized system

$$u_{tt} - d_0 u_{xx} + a(x)u_t = 0 \quad \text{in } (0,\infty)\times(0,L) \tag{2.1}$$

$$u(0,\cdot) = u_0, \quad u_t(0,\cdot) = u_1 \quad \text{in } (0,L) \tag{2.2}$$

$$u(\cdot,0) = u(\cdot,L) = 0 \quad \text{in } (0,\infty). \tag{2.3}$$

We assume that $a \in L^\infty((0,L))$ and satisfies (1.4). Without loss of generality we shall assume

$$d_0 = 1, \tag{2.4}$$

see the remarks at the end of this section. The aim is to prove that the energy

$$E_0(t) := \frac{1}{2}\int_0^L (u_t^2 + u_x^2)(t,x)dx$$

decays to zero exponentially as time t tends to infinity. To do this we introduce the variables

$$p := u_t - u_x, \qquad q := u_t + u_x$$

such that

$$p_t + p_x = -a(x)u_t, \qquad q_t - q_x = -a(x)u_t. \tag{2.5}$$

Since

$$u_t = \frac{p+q}{2}$$

we can rewrite (2.1) as

$$U_t + KU_x + \frac{a}{2}BU = 0 \tag{2.6}$$

where

$$U := \begin{pmatrix} p \\ q \end{pmatrix}, \; K := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; B := \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

The boundary conditions for U are, for $t \geq 0$,

$$p(t,0) + q(t,0) = p(t,L) + q(t,L) = 0. \tag{2.7}$$

Moreover, we have, for $t \geq 0$,

$$\int_0^L p(t,s) - q(t,s)ds = 0 \tag{2.8}$$

and the initial condition

$$U(0,\cdot) \equiv U_0 := \begin{pmatrix} p(0,\cdot) \\ q(0,\cdot) \end{pmatrix}. \tag{2.9}$$

To verify the equivalence of the systems (2.1)–(2.3) and (2.6)–(2.9), let U solve (2.6)–(2.9) for appropriate initial conditions. Let

$$w := \frac{p+q}{2}, \qquad v := \frac{q-p}{2}.$$

Then

$$w_x - v_t = 0$$

hence there exists a function u with

$$\begin{pmatrix} u_x \\ u_t \end{pmatrix} = \begin{pmatrix} v \\ w \end{pmatrix}.$$

We can choose u such that $u(0,0) = 0$. It is not difficult to see that u satisfies (2.1). Moreover,

$$0 = w(t,0) = u_t(t,0), \qquad 0 = w(t,L) = u_t(t,L).$$

Using $u(0,0) = 0$ we obtain

$$u(t,0) = 0, \qquad t \geq 0.$$

With (2.8) we conclude

$$0 = \int_0^L u_x(t,s)ds = u(t,L) - u(t,0)$$

hence

$$u(t,L) = 0, \qquad t \geq 0,$$

and we conclude that the systems (2.1)–(2.3) and (2.6)–(2.9) are equivalent.
Let $\mathcal{A}$ denote the operator given by

$$\mathcal{A}U := -KU_x - \frac{a}{2}BU$$

with domain

$$D(\mathcal{A}) := \left\{ \begin{pmatrix} p \\ q \end{pmatrix} \in H^1((0,L)) \times H^1((0,L)) \mid p(0) + q(0) = p(L) + q(L) = 0, \int_0^L p(s) - q(s)ds = 0 \right\}$$

in the Hilbert space

$$\mathcal{H} := \left\{ \begin{pmatrix} p \\ q \end{pmatrix} \in L^2((0,L)) \times L^2((0,L)) \mid \int_0^L p(s) - q(s)ds = 0 \right\}$$

with the $L^2((0,L))$ inner product.
$D(\mathcal{A})$ is dense in $\mathcal{H}$ and $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $\{e^{\mathcal{A}t}\}_{t\geq 0}$. The latter will be clear after having proved the characterization of the spectrum of $\mathcal{A}$ in the next two lemmata.

Lemma 2.1 *$\mathcal{A}^{-1}$ is compact.*

PROOF:

(i) $\mathcal{A}^{-1}$ exists:
Let $U \in D(\mathcal{A})$ with $\mathcal{A}U = 0$. Then

$$p_x + \frac{a}{2}(p+q) = 0, \qquad -q_x + \frac{a}{2}(p+q) = 0,$$

$$p(x) = -q(x) + c, \qquad p(x) = p(0) - \frac{c}{2}\int_0^x a(s)ds,$$

where c is a constant. From $\int_0^L p(s) - q(s)ds = 0$ we conclude

$$p(0) = q(0) = \frac{c}{2}.$$

Since $0 = p(0) + q(0)$ we get $c = 0$ and finally

$$p = q = 0.$$

(ii) $\mathcal{A}U = -F$ is solvable for $F \in \mathcal{H}$, $\mathcal{A}^{-1}$ is bounded:
With

$$M_0(x) := \frac{a(x)}{2}KB = \frac{a(x)}{2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

$\mathcal{A}U = -F$ is equivalent to

$$U_x + M_0(x)U = KF$$

or

$$U(x) = e^{-\int_0^x M_0(s)ds}U_0 + \int_0^x e^{-\int_s^x M_0(t)dt}KF(s)ds$$

where U_0 has to be determined appropriately.
Since

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

the corresponding series $e^{-\int_0^x M_0(s)ds}$ has only two terms and therefore we have for $U = (p,q)'$, $U_0 = (p_0, q_0)'$, $F = (f,g)'$, $Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, that

$$\begin{aligned}\begin{pmatrix} p \\ q \end{pmatrix}(x) &= \left(Id - \frac{1}{2}\int_0^x a(s)ds \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\right)\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \\ &+ \int_0^x \left(Id - \frac{1}{2}\int_s^x a(t)dt \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\right)\begin{pmatrix} f(s) \\ -g(s) \end{pmatrix} ds.\end{aligned}$$

The boundary conditions require $p_0 + q_0 = 0$, hence

$$\begin{pmatrix} p \\ q \end{pmatrix}(x) = \begin{pmatrix} p_0 \\ -p_0 \end{pmatrix} + \int\limits_0^x \begin{pmatrix} f(s) \\ -g(s) \end{pmatrix} ds - \int\limits_0^x \begin{pmatrix} f(s) - g(s) \\ -f(s) + g(s) \end{pmatrix} \frac{1}{2} \int\limits_s^x a(t)dt\, ds.$$

Then $p(L) + q(L) = 0$ is satisfied since $F \in \mathcal{H}$. Finally the condition $\int\limits_0^L p(s) - q(s)ds = 0$ determines p_0 uniquely by

$$p_0 = \frac{1}{2L} \left\{ \int\limits_0^L \int\limits_0^x (f(s) - g(s)) \int\limits_s^x a(t)dt\, ds\, dx + \int\limits_0^L \int\limits_0^x f(s) + g(s) ds\, dx \right\}.$$

Hence $\mathcal{A}U = -F$ is uniquely solvable and

$$\|U\| \le c\|F\|,$$

where $\|\cdot\|$ denotes the $L^2((0,L))$-norm. Thus $0 \in \varrho(\mathcal{A})$ (resolvent set).

(iii) $\mathcal{A}^{-1}$ is compact:
Let $(F_n)_n \subset \mathcal{H}$ be bounded, let $U_n := \mathcal{A}^{-1}F_n$. Then $(U_n)_n$ is bounded according to (ii). This implies that $(p_n, q_n)_n$ is bounded in $H^1((0,L))$ and hence has a convergent subsequence in $L^2((0,L))$.

Q.E.D.

Lemma 2.1 implies that the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ consists of eigenvalues $(\lambda_n)_n$ only, without any finite accumulation point.

We wish to prove that for small $\varepsilon_1 > 0$ and any $\varepsilon_0 > \varepsilon_1$ we can choose a small enough such that for any $\varepsilon \in [\varepsilon_1, \varepsilon_0]$

$$\Gamma_\varepsilon := \{-\frac{1}{2L}\int_0^L a(x)dx + \varepsilon + i\eta \mid \eta \in \mathbb{R}\} \subset \varrho(\mathcal{A}) \tag{2.10}$$

and that

$$\sup_{\varepsilon\in[\varepsilon_1,\varepsilon_0],\ \lambda\in\Gamma_\varepsilon} \|(\lambda - \mathcal{A})^{-1}\| < \infty. \tag{2.11}$$

For this purpose we shall study $\mathcal{A}$ as a perturbation of the following operator $\mathcal{A}_0$, given by

$$\mathcal{A}_0 U := -KU_x - \frac{a}{2}U$$

with domain

$$D(\mathcal{A}_0) := \left\{ \begin{pmatrix} p \\ q \end{pmatrix} \in H^1((0,L)) \times H^1((0,L)) \mid p(0) + q(0) = p(L) + q(L) = 0 \right\}$$

in the Hilbert space

$$\tilde{\mathcal{H}} := L^2((0,L)) \times L^2((0,L))$$

with the usual $L^2((0,L))$ inner product. Then $D(\mathcal{A}) = D(\mathcal{A}_0) \cap \mathcal{H}$ and for $U \in D(\mathcal{A})$ we have

$$\mathcal{A}U = \mathcal{A}_0 U - \frac{a}{2}WU$$

with

$$W := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Lemma 2.2 ${\mathcal{A}_0}^{-1}$ *is compact.*

The proof of this lemma follows the same steps as the proof of Lemma 2.1.
Hence the spectrum $\sigma(\mathcal{A}_0)$ consists of eigenvalues $(\lambda_n)_n$ only, without any finite accumulation point.

Lemma 2.3 $\quad \sigma(\mathcal{A}_0) = \left\{ -\frac{1}{2L} \int\limits_0^L a(y)dy + \frac{k\pi \mathrm{i}}{L} \ \middle| \ k \in \mathbb{Z} \right\}$

PROOF: According to the previous lemma, $\lambda \in \varrho(\mathcal{A}_0)$ is equivalent to the solvability of $(\lambda - \mathcal{A}_0)U = F$ for any $F = (f,g)' \in \tilde{\mathcal{H}}$. With

$$\begin{aligned} E_0(x,s,\lambda) &:= \begin{pmatrix} e^{-\lambda(x-s)-\int_s^x a(y)dy} & 0 \\ 0 & e^{\lambda(x-s)+\int_s^x a(y)dy} \end{pmatrix} \\ &\equiv \begin{pmatrix} e_1(x,s,\lambda) & 0 \\ 0 & e_2(x,s,\lambda) \end{pmatrix} \end{aligned} \tag{2.12}$$

$(\lambda - \mathcal{A}_0)U = F$ equivalent to

$$U(x) = E_0(x,0,\lambda)U_0 + \int_0^x E_0(x,s,\lambda)KF(s)ds \equiv (p(x), q(x))' \tag{2.13}$$

where $U_0 \equiv (p_0, q_0)'$ has to be determined such that the boundary conditions $p(0) + q(0) = p(L) + q(L) = 0$ are satisfied. With $q_0 = -p_0$ these conditions hold if p_0 exists uniquely such that

$$0 = p_0 \left(e_1(L,0,\lambda) - e_2(L,0,\lambda) \right) + \int_0^L e_1(L,s,\lambda)f(s) - e_2(L,s,\lambda)g(s)ds. \tag{2.14}$$

Therefore

$$\begin{aligned} \lambda \in \sigma(\mathcal{A}_0) \ &\Leftrightarrow \ e_1(L,0,\lambda) - e_2(L,0,\lambda) = 0 \ \Leftrightarrow \ e^{-2\lambda L - \int_0^L a(y)dy} = 1 \\ &\Leftrightarrow \ \lambda = \lambda_k = -\frac{1}{2L}\int_0^L a(y)dy + \frac{k\pi \mathrm{i}}{L}, \quad k \in \mathbb{Z}. \end{aligned}$$

Q.E.D.

Now we can prove for sufficiently small a the further characterization of the spectrum of $\mathcal{A}$ given in (2.10) and (2.11). Let $\varepsilon_1 \in (0, \frac{1}{4L}\int_0^L a(y)dy)$ be arbitrary (small), but fixed.

Lemma 2.4 *For any* $\varepsilon_0 > \varepsilon_1$ *there is* $a_0 > 0$ *such that if* $|a|_{L^\infty} < a_0$ *then we have*

(i) $\forall\, \varepsilon \in [\varepsilon_1, \varepsilon_0] : \ \Gamma_\varepsilon = \{ -\frac{1}{2L}\int\limits_0^L a(x)dx + \varepsilon + \mathrm{i}\eta \ | \ \eta \in \mathbb{R} \} \subset \varrho(\mathcal{A}).$

(ii) $\sup\limits_{\varepsilon \in [\varepsilon_1, \varepsilon_0],\ \lambda \in \Gamma_\varepsilon} \|(\lambda - \mathcal{A})^{-1}\| < \infty.$

PROOF: Fix $\varepsilon_0 > 0$. According to Lemma 2.1 we first have to show, for any admitted ε, that for $\lambda \in \Gamma_\varepsilon$ the equation $(\lambda - \mathcal{A})U = F$ is solvable for any $F \in \mathcal{H}$. We shall use a fixed point argument to prove this. Now let $F = (f,g)' \in \mathcal{H}$ be given as well as $\lambda \in \Gamma_\varepsilon$. Let

$$\Phi : \ \tilde{\mathcal{H}} \longrightarrow \tilde{\mathcal{H}}, \quad V \to U = \Phi V$$

be defined as solution $U = (U_1, U_2)'$ to

$$(\lambda - \mathcal{A}_0)U = F - \frac{a}{2}WV$$

which is well defined by Lemma 2.3 since $\lambda \in \varrho(\mathcal{A}_0)$. By (2.13) and (2.14) we have

$$U(x) = (\Phi V)(x) = E_0(x,0,\lambda)U_0 + \int_0^x E_0(x,s,\lambda)\left(KF(s) - \frac{a(s)}{2}KWV(s)\right)ds \tag{2.15}$$

where $U_0 = (p_0, -p_0)$ with

$$p_0 = \frac{\int_0^L e_1(L,s,\lambda)\left(f(s) - \frac{a(s)}{2}V_2(s)\right) - e_2(L,s,\lambda)\left(g(s) - \frac{a(s)}{2}V_1(s)\right)ds}{e_2(L,0,\lambda) - e_1(L,0,\lambda)}. \tag{2.16}$$

Let

$$U^j := \Phi V^j, \quad j = 1, 2.$$

Then

$$U^1(x) - U^2(x) = E_0(x,0,\lambda)(U_0^1 - U_0^2) + \int_0^x E_0(x,s,\lambda)\left(\frac{a(s)}{2}KW(V^2(s) - V^1(s))\right)ds.$$

This implies

$$|U^1(x) - U^2(x)| \leq c_1 e^{(a_\infty + \varepsilon_0)L}|U_0^1 - U_0^2| + c_1 e^{(a_\infty+\varepsilon_0)L} a_\infty \int_0^x |V^1(s) - V^2(s)|ds \tag{2.17}$$

where

$$a_\infty := \|a\|_{L^\infty((0,L))}$$

and c_1 denotes here and in the sequel a positive constant at most depending on ε_1. We conclude from (2.16)

$$|U_0^1 - U_0^2| \leq \frac{c_1 e^{(a_\infty+\varepsilon_0)L} a_\infty}{\sinh(\varepsilon_1 L)} \int_0^L |V^1(s) - V^2(s)|ds. \tag{2.18}$$

The last two inequalities yield

$$\|U^1 - U^2\|^2 \leq c_1 (a_\infty e^{(a_\infty+\varepsilon_0)L})^2 \|V^1 - V^2\|^2.$$

Hence Φ is a contraction mapping for

$$a_\infty < a_0$$

with $a_0 = a_0(\varepsilon_0) > 0$ sufficiently small. Let $U \equiv (p,q)'$ be the unique fixed point. It satisfies

$$\lambda U + KU_x + \frac{a}{2}BU = F. \tag{2.19}$$

By definition we have $U \in D(\mathcal{A}_0) \subset \tilde{\mathcal{H}}$. Since $F \in \mathcal{H}$ we obtain by integration of (2.19)

$$\begin{aligned} 0 = \int_0^L f(s) - g(s)ds &= \lambda \int_0^L p(s) - q(s)ds + \int_0^L (p+q)_x(s)ds \\ &= \lambda \int_0^L p(s) - q(s)ds. \end{aligned}$$

Without loss of generality we can assume that $\lambda \neq 0$. Then we conclude

$$U \in D(\mathcal{A}) \quad \text{ and } (\lambda - \mathcal{A})U = F.$$

This proves $\lambda \in \varrho(\mathcal{A})$, and hence assertion (i). In order to prove assertion (ii) we estimate the inverse $(\lambda - \mathcal{A})^{-1}$ as follows. Let U be still the fixed point, and let

$$\tilde{U} := \Phi(0)$$

or, in other words,

$$(\lambda - \mathcal{A}_0)\tilde{U} = F.$$

Then we get

$$\|U\| - \|\tilde{U}\| \leq \|U - \tilde{U}\| = \|\Phi U - \Phi\tilde{U}\| \leq d\|U\|$$

where $d < 1$ describes the contraction mapping property and depends on a_∞. It follows

$$\|U\| \leq \frac{1}{1-d}\|\tilde{U}\|.$$

On the other hand we obtain from (2.15), (2.16) (cp. (2.17), (2.18))

$$\|\tilde{U}\| \leq c_1 a_\infty e^{(a_\infty+\varepsilon_0)L}\frac{1}{1-d}\|F\|$$

where $c_1 = c_1(\varepsilon_1) \to \infty$ as $\varepsilon_1 \to 0$. Hence we have proved

$$\|(\lambda - \mathcal{A})^{-1}\| \leq c_1 a_\infty e^{(a_\infty+\varepsilon_0)L}\frac{1}{1-d}$$

which proves the assertion (ii).

Q.E.D.

Since $\mathcal{A} - \tau$ is dissipative for τ large enough — e.g. for $\tau = N\|a\|_{L^\infty}$, N large enough, fixed —, we can fix ε_0 large enough — e.g. $\varepsilon_0 > \frac{1}{2L}\int_0^L a(y)dy + 3\tau_0$ —, in order to deal with resolvents $(\lambda - \mathcal{A})^{-1}$ for any λ with $\Re\lambda > \varepsilon_1$. This yields

Theorem 2.5 *For fixed* $\varepsilon_1 \in (0, -\frac{1}{4}\int_0^L a(y)dy)$ *there is* $a_0 > 0$ *such that for* $\|a\|_{L^\infty((0,L))} < a_0$ *we have*

(i) $\omega_\sigma(\mathcal{A}) := \sup\limits_{\Re\lambda;\, \lambda\in\sigma(\mathcal{A})} = -\frac{1}{2}\int_0^L a(y)dy) + \varepsilon_1 < 0.$

(ii) $\sup_{\Re\lambda \geq -\frac{1}{2}\int_0^L a(y)dy)+\varepsilon_1} \|(\lambda - \mathcal{A})^{-1}\| < \infty.$

Hence we conclude by well-known characterizations of C_0-semigroups (see e.g. [8, Thm 1.3.1]) the exponential stability.

Let

$$\mathcal{E}(t) := \|U(t,\cdot)\|^2 = \|e^{t\mathcal{A}}U_0\|^2$$

be the associated energy to (2.6)–(2.9). Then

$$\begin{aligned} \mathcal{E}(t) &= \left\| \begin{pmatrix} p \\ q \end{pmatrix}(t,\cdot)\right\|^2 = \left\| \begin{pmatrix} u_t - u_x \\ u_t + u_x \end{pmatrix}(t,\cdot)\right\|^2 \\ &= \int_0^L (u_t^2 + u_x^2)(t,x)dx. \end{aligned}$$

Theorem 2.6 *Under the conditions of Theorem 2.5 we have*

$$\exists\, c_0 \; > 0 \exists\, \alpha_0 > 0 \; \forall t \geq 0 : \; \mathcal{E}(t) \leq c_0 e^{-2\alpha_0 t} \mathcal{E}(0).$$

Using Theorem 2.5, a result of Neves, Ribeiro and Lopes [12] saying that the essential type $\omega_e(\mathcal{A})$ is given by

$$\omega_e(\mathcal{A}) = -\frac{1}{2L} \int_0^L a(y)dy. \tag{2.20}$$

as well as using the general characterization (see [11])

$$\omega_0(\mathcal{A}) = \max\{\omega_e(\mathcal{A}), \omega_\sigma(\mathcal{A})\},$$

where $\omega_0(\mathcal{A})$ denotes the type of the semigroup,

$$\omega_0(\mathcal{A}) = \lim_{t\to\infty} \frac{\ln\|e^{\mathcal{A}t}\|}{t}.$$

Using this we can establish

Theorem 2.7 *Under the conditions of Theorem 2.5 we have*

$$\exists\, c_0 > 0 \;\exists\, \alpha_0 = \alpha_0 \left(\frac{1}{L} \int\limits_0^L a(x)dx \right) > 0 \;\forall t \geq 0 : \; \mathcal{E}(t) \leq c_0 e^{-2\alpha_0 t} \mathcal{E}(0).$$

α_0 can be chosen as any number $-\overline{\alpha}$ with

$$0 > \overline{\alpha} > -\frac{1}{2L} \int_0^L a(y)dy + \varepsilon_1, \qquad e.g. \qquad \alpha_0 := \frac{1}{4L} \int_0^L a(y)dy$$

We finish this section giving some higher norm estimates. Differentiating (2.1) with respect to t,

$$(\partial_t^j u)_{tt} - (\partial_t^j u)_{xx} + a(x)(\partial_t^j u)_t = 0, \quad j \in \mathbb{N},$$

and using the fact that derivatives with respect to x can be computed from the differential equation successively, we get as a consequence of $a \in C^0([0, L], \mathbb{R})$, and $a \in C^{s-2}([0, L], \mathbb{R})$ if $s \geq 2$ the following theorem.

Theorem 2.8 *Under the conditions of Theorem 2.5 we have*

$$\forall s \in \mathbb{N} \quad \exists C_s > 0 \quad \forall t \geq 0 : \quad \left\| \begin{pmatrix} u_t \\ u_x \end{pmatrix} (t, \cdot) \right\|_{H^s((0,L))} \leq C_s e^{-\alpha_0 t} \left\| \begin{pmatrix} u_1 \\ u_{0,x} \end{pmatrix} \right\|_{H^s(0,L)}$$

where α_0 is given in Theorem 2.7, and the data are assumed to be sufficiently smooth and to satisfy the usual compatibility conditions.

Remarks:

1. It would be also possible to replace the smallness of $a_\infty = \|a\|_{L^\infty((0,L))}$ by an arbitrary prescribed bound on a_∞ and then choosing L small enough.

2. In Theorem 2.7 we only used $a \in L^\infty((0, L))$.

3. Without loss of generality we studied the equation

$$u_{tt} - d_0 u_{xx} + a(x)u_t = 0, \quad x \in (0, L),$$

for $d_0 = 1$, because if $d_0 > 0$ is arbitrary we may define

$$v(t, y) := u(t, \sqrt{d_0}y), \quad y \in \left(0, \frac{L}{\sqrt{d_0}}\right).$$

Then v satisfies

$$v_{tt} - v_{yy} + \tilde{a}(y)v_t = 0, \quad y \in \left(0, \frac{L}{\sqrt{d_0}}\right),$$

for which Theorem 2.7 can be applied directly replacing a by $\tilde{a}$ and L by $L/\sqrt{d_0}$. The decay rate $\tilde{\alpha}_0 = \tilde{\alpha}_0 \left(\frac{\sqrt{d_0}}{L} \int_0^{L/\sqrt{d_0}} \tilde{a}(y)dy \right)$ turns into $\tilde{\alpha}_0 = \alpha_0 \left(\frac{1}{L} \int_0^L a(y)dy \right)$ again since $\frac{\sqrt{d_0}}{L} \int_0^{L/\sqrt{d_0}} \tilde{a}(y))dy = \frac{1}{L} \int_0^L a(x)ds.$

4. The question of validity of a conjucture of Cox and Overton [3] on the optimality of constant damping is not touched by our results since we only discuss sufficiently small a and stay beyond $\omega_e(\mathcal{A})$. But also the counterexample to this conjecture by Freitas [5] essentially, not exclusively, based on numerical evidence, is not underlined here; see Castro and Cox [2] for a recent discussion.

3 Global existence for the nonlinear system

We now return to the nonlinear system (1.1)–(1.3) assuming again the positivity of the mean value (1.4) and also the condition (1.5) on the nonlinearity, which, for example, is satisfied in the classical model for a nonlinear string, where

$$\sigma(u_x) = \frac{u_x}{\sqrt{1 + u_x^2}}.$$

After recalling the local well-posedness it is the aim to prove a global existence result for data (u_0, u_1) being sufficiently small in $H^4((0, L))$, and, quasi simultaneously, to obtain the exponential stability. The method used imitates that one which is well-known for nonlinear evolution equations and systems, see [13] for a presentation of the general approach for Cauchy problems ($x \in \mathbb{R}^n$, no boundary). Here we shall have to prove so-called high energy estimates and a weighted a priori estimate — describing the expected exponential decay — for a boundary value problem and a non-dissipative problem reflected in the possible negativity of the function a. To deal with the latter the condition (1.9) on the negative part of a, i.e.

$$\|a^-\|_{L^\infty} < \alpha_0,$$

will be used.

Observing that the term $a(x)u_t$ is of lower order, we can recall the following local existence theorem, see for instance [1] or [6, p.97].

Theorem 3.1 *There is $T = T\ (\|(u_0,u_1)\|_{H^4\times H^3}) > 0$ such that (1.1)–(1.3) has a unique local solution*

$$u \in \bigcap_{k=0}^{3} C^k([0,T], H^{4-k}((0,L)) \cap H_0^1((0,L)) \cap C^4([0,T], L^2((0,L)).$$

Remark: Of course u_0, u_1 have to satisfy the usual compatibility conditions.

Now we turn to the high energy estimates. For this purpose it is useful to rewrite (1.1)–(1.3) as a first-order system for

$$V := (u_t, u_x)'.$$

Then V satisfies

$$V_t + \underbrace{\begin{pmatrix} a & -d_0\partial_x \\ -\partial_x & 0 \end{pmatrix}}_{=:-A} V = \begin{pmatrix} b(u_x)\partial_x u_x \\ 0 \end{pmatrix} =: F(V, V_x),$$

$$V(t=0) = (u_1, \partial_x u_0)' =: V_0.$$

The first formally defined operator A generates a C_0-semigroup as usual, for $F = 0$ the solution V is given by

$$V(t) = e^{tA} V_0$$

and the (local) solution to (1.1)–(1.3) satisfies

$$V(t) = e^{tA} V_0 + \int_0^t e^{(t-r)A} F(V, V_x)(r) dr. \tag{3.1}$$

From section 2 we conclude that

$$V := (u_t, u_x)$$

as solution of the linear system (1.1)–(1.3) written in first-order form satisfies

$$V(t) = e^{tA} V(t=0)$$

with a C_0-semigroup $\{e^{tA}\}_{t\geq 0}$ satisfying

$$\|V(t)\|_{H^s} \leq c_s e^{-\alpha_0 t} \|V(t=0)\|_{H^s} \tag{3.2}$$

for $s = 0, 1, 2$ (cp. below). This follows from Theorem 2.1 for $s = 0$ and obtained for $s = 1, 2$ by differentiating the equation (1.8) with respect to t one and then twice, as well as using the differential equation to obtain information for derivatives in x. Let

$$a_\infty^- := \|a^-\|_{L^\infty} \tag{3.3}$$

in the sequel we assume without loss of generality that u_x is small enough a priori, i.e. such that $\sigma'(u_x)$ remains strictly positive (near $u_x = 0$, cp. (1.5)), or in other terms we can assume that there is $\eta > 0$ such that

$$d_0 - b(u_x) \geq \frac{d_0}{2} > 0 \qquad \text{if} \qquad |u_x| < \eta < 1. \tag{3.4}$$

Lemma 3.2 *There are constants $c_2, c_3 > 0$, not depending on V_0 or T, such that the local solution given in Theorem 3.1 satisfies for $t \in [0,T]$:*

$$\|V(t)\|_{H^3}^2 \le c_2 \|V_0\|_{H^3}^2 e^{a_\infty^- t} e^{c_3 \int_0^t (\|V(r)\|_{H^2} + \|V(r)\|_{H^2}^2 + \|V(r)\|_{H^2}^3) dr}$$

Proof: Multiplying

$$u_{tt} - d_0 u_{xx} + a u_t = b(u_x) u_{xx} \tag{3.5}$$

by u_t in L^2 we obtain $\left(\int \equiv \int\limits_0^L \right)$

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int u_t^2 + d_0 u_x^2 dx &= -\int a u_t^2 dx + \int b(u_x) u_{xx} u_t dx \\ &\le \int a_\infty^- u_t^2 - \int (\partial_x b(u_x)) u_x u_t dx - \int b(u_x) u_x u_{xt} \\ &\equiv I.1 + I.2 + I.3, \end{aligned} \tag{3.6}$$

$$\begin{aligned} |I.2| &\le \frac{1}{2}\|b'(u_x)u_{xx}\|_{L^\infty} \int u_x^2 + u_t^2 dx \\ &\le c\|V\|_{H^2} \int u_x^2 + u_t^2 dx \end{aligned} \tag{3.7}$$

where c will denote a constant not depending on V^0 or on T.

$$\begin{aligned} I.3 &= -\frac{1}{2}\frac{d}{dt}\int b(u_x) u_x^2 dx + \frac{1}{2}\int (\partial_t b(u_x)) u_x^2 \\ &\equiv I.3.1 + I.3.2. \end{aligned} \tag{3.8}$$

The term I.3.2 can be estimated in the same way as I.2 in (3.7):

$$|I.3.2| \le c\|V\|_{H^2} \int u_x^2. \tag{3.9}$$

The term I.3.1 can be incorporated into and be dominated by the left-hand side of inequality (3.6) after an integration with respect to t later on, since

$$\int\limits_0^t I.3.1(r) dr = -\frac{1}{2}\int b(u_x) u_x^2 dx + \frac{1}{2}\int b(u_x(t=0)) u_x^2(t=0) dx. \tag{3.10}$$

Summarizing (3.6)–(3.10) we have proved

$$\|V(t)\|_{L^2}^2 \le c\|V_0\|_{L^2}^2 + \int\limits_0^t (a_\infty^- + c\|V(r)\|_{H^2}) \|V(r)\|_{L^2}^2 dr. \tag{3.11}$$

In order to get estimates for the higher-order derivatives of V (resp. u) we differentiate equation (3.5) with respect to t to get

$$u_{ttt} - d_0 u_{txx} + a u_{tt} = b'(u_x) u_{xt} u_{xx} + b(u_x) u_{txx}. \tag{3.12}$$

Multiplying by u_{tt} in L^2 we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int u_{tt}^2 + d_0 u_{tx}^2 dx &\leq \int a_\infty^- u_{tt}^2 dx + \int b(u_x) u_{txx} u_{tt} dx + \int b'(u_x) u_{xt} u_{xx} u_{tt} dx \\ &\equiv I.4 + I.5 + I.6.\end{aligned} \tag{3.13}$$

The term I.5 can be treated like the term I.2 + I.3 from (3.6), see (3.7)–(3.11).

$$|I.6| \leq c\|V\|_{H^2} \int u_{tt}^2 + u_{tx}^2 dx. \tag{3.14}$$

Observe that the differential equation (3.5) yields the estimate

$$|u_{xx}|^2 \leq c(|u_{tt}|^2 + |u_t|^2). \tag{3.15}$$

Thus we obtain from (3.11), (3.13), (3.14)

$$\|V(t)\|_{H^1}^2 \leq c\|V_0\|_{H^1}^2 + \int_0^t (a_\infty^- + c\|V(r)\|_{H^2})\|V(r)\|_{H^1}^2 dr. \tag{3.16}$$

Differentiating the differential equation (3.12) once more with respect to t we get

$$u_{tttt} - d_0 u_{ttxx} + a u_{ttt} = b''(u_x) u_{xt}^2 u_{xx} + b'(u_x) u_{xtt} u_{xx} + 2b'(u_x) u_{xt} u_{xxt} + b(u_x) u_{ttxx}. \tag{3.17}$$

Multiplying by u_{ttt} in L^2 we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int u_{ttt}^2 + d_0 u_{ttx}^2 dx &\leq \int a_\infty^- u_{ttt}^2 dx + \int b''(u_x) u_{xt}^2 u_{xx} u_{ttt} dx \\ &\quad + \int b'(u_x) u_{xtt} u_{xx} u_{ttt} dx \\ &\quad + 2\int b'(u_x) u_{xt} u_{xxt} u_{ttt} dx + \int b(u_x) u_{ttxx} u_{ttt} dx \\ &\equiv I.7 + I.8 + I.9 + I.10 + I.11.\end{aligned} \tag{3.18}$$

The term I.11 is again dealt with like I.2 + I.3 in (3.7)– (3.11).

$$|I.8| + |I.9| + |I.10| \leq c(\|V\|_{H^2}^2 + \|V\|_{H^2}) \int u_{xt}^2 + u_{ttt}^2 + u_{ttt}^2 + u_{xtt}^2 + u_{xxt}^2 dx. \tag{3.19}$$

Hence we obtain from (3.16), (3.19) using (3.12) to estimate u_{txx},

$$\|V(t)\|_{H^2}^2 \leq c\|V_0\|_{H^2}^2 + \int_0^t a_\infty^- + c(\|V(r)\|_{H^2} + \|V(t)\|_{H^2}^2)\|V(r)\|_{H^2}^2 dr. \tag{3.20}$$

The final estimate is obtained after differentiating the differential equation a last time with respect to t yielding

$$\begin{aligned}u_{ttttt} - d_0 u_{tttxx} + a u_{tttt} &= b'''(u_x) u_{xt}^3 u_{xx} + 3b''(u_x) u_{xt} u_{xtt} u_{xx} \\ &\quad + 3b''(u_x) u_{xt}^2 u_{xxt} + b'(u_x) u_{xttt} u_{xx} + 3b'(u_x) u_{xtt} u_{xxt} \\ &\quad + 3b'(u_x) u_{xt} u_{xxtt} + b(u_x) u_{tttxx} \\ &\equiv \sum_{j=12}^{18} \theta_j.\end{aligned} \tag{3.21}$$

Remark: The derivatives of order five are formally not defined but the estimates aimed at will only involve derivatives of order four. A usual approximation argument with data $V_0 \in H^4((0,2))$ and the lower semicontinuity of the norms justifies our calculation finally for $V_0 \in H^3((0,2))$ only.
A multiplication of (3.21) by u_{tttt} in L^2 yields

$$\frac{1}{2}\frac{d}{dt}\int u_{tttt}^2 + u_{tttx}^2 dx \le \int a_\infty^- u_{tttt}^2 + \sum_{j=12}^{18}\int \theta_j u_{tttt} dx. \tag{3.22}$$

The term $\int \theta_{18} u_{tttt} dx$ can be dealt with like I.2. + I.3 in (3.7)–(3.11). The terms $\int \theta_j u_{tttx} dx$ for $j \neq 16$, 18 can be estimated easily as before by

$$\left| \sum_{\substack{j=12\\ j\neq 16}}^{17} \int \theta_j u_{tttt} dx \right| \le c(\|V\|_{H^2} + \|V\|_{H^2}^2 + \|V\|_{H^2}^3)\|V\|_{H^3}^2. \tag{3.23}$$

The only more difficult term is

$$\int \theta_{16} u_{tttt} = 3 \int b'(ux) u_{xtt} u_{xxt} u_{tttt} dx$$

involving three factors of order at least three. This term is estimated as follows using the Gagliardo-Nirenberg inequality.

$$\begin{aligned} \left| \int b'(ux) u_{ttx} u_{txx} u_{tttt} dx \right| &\le c\|b'(u_x) u_{ttx}\|_{L^2} \|u_{tttt}\|_{L^2} \\ &\le c\|b'(u_x) u_{ttx} u_{txx}\|_{L^2} \|V\|_{H^3}. \end{aligned} \tag{3.24}$$

$$\begin{aligned} \|b'(u_x) u_{ttx} u_{txx}\|_{L^2} &\le \|\partial_x(b'(u_x)u_{tt})\partial_x(u_{tx})\|_{L^2} + \|b''(u_x) u_{xx} u_{tt} u_{txx}\|_{L^2} \\ &\le \|\partial_x(b'(u_x)u_{tt})\partial_x(u_{tx})\|_{L^2} + c\|V\|_{H^2}^2 \|V\|_{H^3}. \end{aligned} \tag{3.25}$$

Using Young's inequality and the Gagliardo-Nirenberg inequality, and observing $L^\infty \hookrightarrow H^1$, we conclude

$$\begin{aligned} \|\partial_x(b'(u_x)u_{tt})\partial_x(u_{tx})\|_{L^2} &\le \|\partial_x(b'(u_x)u_{tt})\|_{L^4} \|\partial_x(u_{tx})\|_{L^4} \\ &\le c\|b'(u_x)u_{tt}\|_{H^2}^{1/2} \|b'(u_x)u_{tt}\|_{L^\infty}^{1/2} \|u_{tx}\|_{H^2}^{1/2} \|u_{tx}\|_{L^\infty}^{1/2} \\ &= c\left(\|b'(u_x)u_{tt}\|_{L^\infty} \|u_{tx}\|_{H^2}\right)^{1/2} \left(\|b'(u_x)u_{tt}\|_{H^2} \|u_{tx}\|_\infty\right)^{1/2} \\ &\le c\left(\|b'(u_x)u_{tt}\|_{L^\infty} \|u_{tx}\|_{H^2} + \|u_{tx}\|_{L^\infty} \|b'(u_x)u_{tt}\|_{H^2}\right) \\ &\le c\|V\|_{H^2} \|V\|_{H^3}. \end{aligned} \tag{3.26}$$

From (3.20) and (3.22)–(3.26) we conclude

$$\|V(t)\|_{H^3}^2 \le c\|V_0\|_{H^3}^2 + \int_0^t a_\infty^- + c(\|V\|_{H^2} + \|V\|_{H^2}^2 + \|V\|_{H^2}^3)(r)\|V(r)\|_{H^3}^2 dr$$

which yields the assertion of Lemma 3.2 using Gronwall's inequality.

Q.E.D.

Next we want to prove a weighted a priori estimate for $\|V(t)\|_{H^2}$.

Remark: Observe that we did not yet use the assumption (1.5) requiring $b'(0) = \sigma''(0) = 0$. Indeed, with this assumption it would be possible to remove the linear term $\|V(t)\|^1_{H^2}$ in the exponential in the estimate for $\|V(t)\|_{H^3}$ in Lemma 4.2, i.e. the estimate would read

$$\|V(t)\|^2_{H^3} \leq C\|V_0\|^2_{H^3} e^{a_\infty^- t} e^{c\int_0^t (\|V(r)\|^2_{H^2} + \|V(r)\|^3_{H^2})dr} \tag{3.27}$$

and without loss of generality the a priori boundedness — being proved a posteriori — of $\|V(t)\|_{H^2}$ would be used to achieve

$$\|V(t)\|^2_{H^3} \leq C\|V_0\|^2_{H^3} e^{a_\infty^- t} e^{c\int_0^t (\|V(r)\|^2_{H^2} dr}. \tag{3.28}$$

Since we aim at exponential decay it will not matter if we use (3.28), (3.27) or the statement of Lemma 3.2.
Using the representation (3.1) and Theorem 2.8 — observing that the nonlinearity satisfies the compatibility conditions to estimate the H^2-norm — we can estimate

$$\begin{aligned} \|V(t)\|_{H^2} &\leq \|e^{At}V_0\|_{H^2} + \int_0^t \|e^{(t-r)A}\mathcal{F}(V, V_x)(r)\|_{H^2} dr \\ &\leq c_1 e^{-\alpha_0 t}\|V_0\|_{H^2} + c_1 \int_0^t e^{-\alpha_0(t-r)}\|F(V, V_x)\|_{H^2} dr \end{aligned} \tag{3.29}$$

and it will be in the following estimate for $\|F(V, V_x)\|_{H^2}$, where we really use assumption (1.5) to get an estimate we need later on in the weighted a priori estimate.

Lemma 3.3 $\exists\, c > 0\; \forall\, W \in H^3 \,:\, \|F(W, W_x)\|_{H^2} \leq c\|W\|^2_{H^2}\|W\|_{H^3}.$

PROOF: (cp. [13] in $\mathbb{R}^n$) Let $u := W^1$.

Using $b(\tau) = \int_0^1 b''(\mu\tau\nu)d\nu\mu d\mu\tau^2$ we obtain

$$\begin{aligned} \|b(u_x)u_{xx}\|_{H^2} &\leq c(\|b(u_x)\|_\infty \|u_{xx}\|_{H^2} + \|b(u_x)\|_{H^2}\|u_{xx}\|_{L^\infty}) \\ &\leq c(\|u_x\|^2_{L^\infty}\|u_x\|_{H^3} + \|u_x\|_{L^\infty}\|u_{xx}\|_{L^\infty}\|u_x\|_{H^3}) \\ &\leq c\|W\|^2_{H^2}\|W\|_{H^3} \end{aligned}$$

Q.E.D.

Using Lemma 3.3 we conclude from (3.29)

$$\|V(t)\|_{H^2} \leq ce^{-\alpha_0 t}\|V_0\|_{H^2} + c\int_0^t e^{-\alpha_0(t-r)}\|V(r)\|^2_{H^2}\|V(r)\|_{H^3} dr \tag{3.30}$$

which is the starting point to prove the following weighted a priori estimate.

Lemma 3.4 *For $0 \leq t \leq T$ let*

$$M_2(t) := \sup_{0 \leq r \leq t} (e^{\tau_0 r} \|V(r)\|_{H^2})$$

where $0 < \tau_0 \leq \alpha_0$. Let (1.9) be satisfied, i.e. $a_\infty^- < d_0$. Then there are $M_0 > 0$ and $\delta > 0$ such that if $\|V_0\|_{H^3} < \delta$ we have for all $0 \leq t \leq T$:

$$M_2(t) \leq M_0 < \infty$$

M_0 is independent of T (and of V_0).

Proof: From (3.30) and the energy estimate in Lemma 3.2 we conclude

$$\begin{aligned} \|V(t)\|_{H^2} \quad &\leq \quad c\|V_0\|_{H^2} e^{-\alpha_0 t} + c\int_0^t e^{-\alpha_0(t-r)} \|V(r)\|_{H^2}^2 \|V_0\|_{H^3} e^{\frac{a_\infty^-}{2} r} \times \\ &\qquad \times e^{c\int_0^r (\|V(\tau)\|_{H^2} + \|V(\tau)\|_{H^2}^2 + \|V(\tau)\|_{H^2}^3) d\tau} dr \end{aligned}$$

If $\|V_0\|_{H^3} \leq \delta$ (δ to be determined) we get

$$\begin{aligned} \|V(t)\|_{H^2} \quad &\leq \quad c\delta e^{-\alpha_0 t} + c\delta e^{c\int_0^t (\|V(\tau)\|_{H^2} + \|V(\tau)\|_{H^2}^2 + \|V(\tau)\|_{H^2}^3) d\tau} \int_0^t e^{-\alpha_0(t-r)} e^{\frac{a_\infty^- r}{2}} \|V(r)\|_{H^2}^2 dr \\ &\leq \quad c\delta e^{-\alpha_0 t} + c\delta e^{c(M_2(t) + M_2^2(t) + M_2^3(t) \int_0^t e^{-\alpha_0 r} + e^{-2\alpha_0 r} + e^{-3\alpha_0 r} dr} \times \\ &\qquad \times M_2^2(t) \int_0^t e^{-\alpha_0(t-r)} e^{\frac{a_\infty^- r}{2}} e^{-2\alpha_0 r} dr \end{aligned}$$

which implies

$$\begin{aligned} M_2(t) \quad &\leq \quad c\delta + c\delta e^{c(M_2(t) + M_2^2(t) + M_2^3(t))} \times \\ &\qquad \times M_2^2(t) \sup_{0 \leq t < \infty} e^{\alpha_0 t} \int_0^t e^{-\alpha_0(t-r)} e^{\frac{a_\infty^-}{2} r} e^{-2\alpha_0 r} dr. \end{aligned} \tag{3.31}$$

Since by assumption (1.9) it easily follows that

$$\sup_{0 \leq t < \infty} e^{\alpha_0 t} \int_0^t e^{-\alpha_0(t-r)} e^{\frac{a_\infty^-}{2} r} e^{-2\alpha_0 r} dr \leq c < \infty$$

we obtain from (3.31) for $0 \leq t \leq T$:

$$M_2(t) \leq c\delta + c\delta M_2^2(t) e^{c(M_2(t) + M_2^2(t) + M_2^3(t))}. \tag{3.32}$$

By standard arguments (cp. e.g. [13]), considering the function

$$f(x) := c\delta(1 + cx^2 e^{c(x + x^2 + x^3)}) - x$$

it follows that $M_2(t)$ is uniformly bounded by the first zero M_0 of f provided δ and $M_2(0)$ are sufficiently small.
This proves Lemma 3.4.

Q.E.D.

Now we can formulate and prove the main theorem on global existence and exponential decay.

Theorem 3.5 *Let the assumptions (1.5) and (1.9) be satisfied. Then there exists* $\delta > 0$ *such that if* $\|V_0\|_{H^3} < \delta$ *there is a unique global solution* u *to (1.1)–(1.3) satisfying*

$$u \in \bigcap_{k=0}^{3} C^k \left([0,\infty),\ H^{4-k}((0,L)) \cap H_0^1((0,L)\right) \cap C^4([0,\infty), L^2(\Omega)).$$

Moreover there are constants $c_0 = c_0(V_0) > 0$ *and* $c_1 > 0$ *such that*

$$\|V(t)\|_{H^2} \leq c_0 e^{-\alpha_0 t}$$

and

$$\|V(t)\|_{H^3} \leq c_1 \|V_0\|_{H^3} e^{a_\infty^- t}, \quad t \geq 0.$$

PROOF: From Lemma 3.2 and Lemma 3.4 we conclude for the local solution

$$\begin{aligned} \|V(t)\|_{H^3} &\leq c\|V_0\|_{H^3} e^{a_\infty^- t} e^{c\int_0^t (\|V(r)\|_{H^2} + \|V(r)\|_{H^2}^2 + \|V(r)\|_{H^2}^3) dr} \\ &\leq c\|V_0\|_{H^3} e^{a_\infty^- t} e^{c(M_0 + M_0^2 + M_0^3)} \\ &\leq c\|V_0\|_{H^3} e^{a_\infty^- t}, \end{aligned}$$

c being independent of t or V_0, from where the global existence follows by the usual continuation argument. The claim on the exponential decay of $\|V(t)\|_{H^2}$ now is a consequence of Lemma 3.4.

Q.E.D.

The assumption (1.9), i.e.

$$a_\infty^- < 2\alpha_0$$

together with the explicit estimates for α_0 from section 2 just requires that the possibly existing negative part of a is not too large in comparison to its positive part.

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