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WELLPOSEDNESS OF THE TORNADO-HURRICANE EQUATIONS

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Abstract. We prove local-in-time existence of a unique mild solution for the tornado-hurricane equations in a Hilbert space setting. The well-posedness is shown simultaneously in a halfspace, a layer, and a cylinder and for various types of boundary conditions which admit discontinuities at the edges of the cylinder. By an approach based on symmetric forms we first prove maximal regularity for a linearized system. An application of the contraction mapping principle then yields the existence of local-in-time mild solution.

1. Introduction

The aim of this note is to present an analytic approach to the system

$$\begin{cases} \partial_t u - \mathrm{div} D(u) + (u\cdot\nabla)u + \frac{\nabla q}{\rho} + \Omega e_3 \times u - e_3 g \frac{\vartheta - \overline{\vartheta}}{\overline{\vartheta}} = 0 & \text{in } J \times G, \\ \mathrm{div}\, \rho u = 0 & \text{in } J \times G, \\ \partial_t \vartheta - \nu\Delta\vartheta + (u\cdot\nabla)\vartheta = 0 & \text{in } J \times G, \\ (\alpha^v D(u)n + \beta^v u)_\tau = 0 & \text{on } J \times \Gamma, \\ \alpha^\vartheta \partial_n \vartheta + \beta^\vartheta \vartheta = 0 & \text{on } J \times \Gamma, \\ n \cdot u = 0 & \text{on } J \times \Gamma, \\ u|_{t=0} = u_0 & \text{in } G, \\ \vartheta|_{t=0} = \vartheta_0 & \text{in } G, \end{cases} \tag{1.1}$$

which is known as the *tornado-hurricane equations* (see [12]). As the domain $G \subseteq \mathbb{R}^3$ we consider simultaneously a half-space, a layer with finite height d, or a cylinder with a fixed finite height d and radius R. We use the notation $\Gamma = \partial G$ for the boundary of G, whereas $J = (0,T)$ denotes a time interval. The first line in (1.1) represents the anelastic equations of momentum with the stress tensor $D(u) = \nu(\nabla u + (\nabla u)^T)$ and where u describes the velocity of a particle and q the corresponding pressure. In contrast to the compressible Navier-Stokes equations the density ρ here is assumed to be a given time independent positive function. The symbol ν denotes the eddy viscosity, g stands for gravity, and Ω is twice the angular velocity of earth's rotation, where we assume rotation around $e_3 = (0,0,1)^T$ for simplicity. The term $\Omega e_3 \times u$ therefore represents the Coriolis force and $-e_3 g \frac{\vartheta - \overline{\vartheta}}{\overline{\vartheta}}$ bouyancy acting only in vertical direction. Furthermore, ϑ denotes the temperature varying

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around a given mean value $\overline{\vartheta} = \overline{\vartheta}(x)$. The second line in (1.1) is the anelastic incompressibility condition arising from conservation of mass, whereas the third line reflects conservation of energy. These first three lines with the unknowns velocity u, temperature ϑ, and pressure q are known as the *weakly compressible Navier-Stokes equations.*

In the boundary conditions (line 4 to 6) n stands for the outer normal at the boundary Γ and the subscript τ denotes the tangential part of a vector field (see the next section for a precise definition). The boundary conditions are chosen such that there is no transition of the fluid through the boundary in normal direction (line 6) and such that there can be no slip, no stress, or partial slip in tangential direction on each component of Γ depending on the values of α^v, α^ϑ, β^v, and β^ϑ. More precisely, the boundary coefficients are assumed to be piecewise constant on the components of $\Gamma = \partial G$. For instance, if Γ_j, $j = 1, 2, 3$, denote bottom, barrel, and top of the cylinder we have

$$\alpha^v : \Gamma \to [0,1], \quad x \mapsto \alpha^v(x) = \begin{cases} \alpha_1^v, & x \in \Gamma_1, \\ \alpha_2^v, & x \in \Gamma_2, \\ \alpha_3^v, & x \in \Gamma_3, \end{cases} \tag{1.2}$$

with $\alpha_j^v \in [0,1]$, $j = 1, 2, 3$. The parameters β^v, α^ϑ, and β^ϑ are defined analogously. Note that on a layer the coefficients attain only two values at the top and the bottom whereas on a half-space we have just one value at the bottom. For this reason we write e.g.

$$\alpha_j^v, \quad j \in \{1, \ldots, m_G\}, \tag{1.3}$$

where $m_G = 3$ if G is a cylinder, $m_G = 2$ if G is a layer, and $m_G = 1$ if G is a half-space. Furthermore, we assume that $\alpha_j^v, \beta_j^v, \alpha_j^\vartheta, \beta_j^\vartheta \geq 0$, and, since we deal with homogeneous traces, for simplicity we also impose that $\alpha_j^v + \beta_j^v = 1$, $\alpha_j^\vartheta + \beta_j^\vartheta = 1$ for $j \in \{1, \ldots, m_G\}$.

In applications ρ and $\overline{\vartheta}$ usually only vary in vertical and radial direction, i.e., $\rho = \rho(|x|, x_3)$ and $\overline{\vartheta} = \overline{\vartheta}(|x|, x_3)$. However, we will see that in our approach this assumption is not necessary, i.e., we suppose $\rho = \rho(x)$ and $\overline{\vartheta} = \overline{\vartheta}(x)$. Furthermore, many different types of boundary conditions are used to describe various phenomena. Varying boundary conditions might even essentially influence stability of solutions and therefore the trajectory of a big cyclone. This is the reason why we consider the general form of the boundary conditions in (1.1) that admit a different type of condition on each part of the boundary. It seems that up to now there is no literature available treating the tornado-hurricane equations rigorously from a mathematical point of view. The intention of this note therefore is to give a first analytical approach to problem (1.1) and to present a starting point for further discussions also concerning significant stability questions. Forthcoming works of the author and some of his collaborators in this direction are in preparation. For a treatment of the incompressible Navier-Stokes equations in a half-space with partial slip type boundary conditions we refer to [14] and [15], for a bounded domain see also [17]. A basic mathematical approach to the Ekman boundary layer problem, which is geophysically related to the evolution of hurricanes (see e.g. [11]) and which represents the incompressible Navier-Stokes equations with rotation effect, can be found in [8]. However, the standard methods that work for the incompressible Navier-Stokes equations do not directly apply to system (1.1). This relies to the weak comressibility, that is, to the fact that the density ρ depends on space and to the coupling with a nonlinear heat equation. For geophysical literature we refer to [12], [11], [4], [6] and the literature cited therein. For an introduction to rotating fluids we refer to the monographs [9] and [13].

We proceed with the rigorous statement of our main result. For this purpose we impose conditions on the density and on existence and properties of a tornado-hurricane vortex which should be a stationary solution of system (1.1). For the density ρ the essential assumption is that there is nowhere a vacuum in $\overline{G}$. More precisely, we require the given function $\rho : \overline{G} \to [0, \infty)$ to satisfy the following conditions:

$$\rho \in W^{2,\infty}(G), \tag{1.4}$$
$$\exists\, c_0, C_0 > 0\ \forall\, x \in \overline{G} \ :\ c_0 \leq \rho(x) \leq C_0. \tag{1.5}$$

Note that the Sobolev embedding and condition (1.4) imply ρ to be a once continuously diffentiable function on $\overline{G}$. Next we assume that there exists a stationary solution $\overline{U} = (\overline{u}, \overline{\vartheta})^T$, the tornado-hurricane vortex, with a corresponding pressure $\overline{q}$ of system (1.1) such that

$$\overline{U} \in W^{1,\infty}(G, \mathbb{R}^4), \tag{1.6}$$
$$\exists\, c_1 > 0\ \forall\, x \in \overline{G} \ :\ \overline{\vartheta}(x) \geq c_1. \tag{1.7}$$

For the construction of such tornado-hurricane vortices we refer to [12] and [16]. The ground space for the construction of solutions is $H^r_{\mathbb{P}_\rho} := L^2_{\mathbb{P}_\rho} \cap H^r(G, \mathbb{R}^4)$, where

$$L^2_{\mathbb{P}_\rho} = \left\{ U = (u, \vartheta) \in L^2(G, \mathbb{R}^4) :\ \operatorname{div} \rho u = 0,\ u \cdot n|_\Gamma = 0 \right\}$$

and $H^r(G, \mathbb{R}^4)$ denotes the standard $\mathbb{R}^4$-valued Sobolev space of order $r \geq 0$. For a preciser definition, in particular for the trace $u \cdot n|_\Gamma$, we refer to Section 3. We again emphasize that the present note merely represents a first approach to system (1.1). Our main result therefore might be optimized in one or the other direction. Preciser results seem to be available by developing a systematic theory in L^p-spaces for $1 < p < \infty$. This will be the content of a forthcoming work. Here our main result is

Theorem 1.1. *Let $r \in (3/4, 1)$ and let $G \subseteq \mathbb{R}^3$ be a half-space, a layer, or a cylinder with boundary coefficients α^v, β^v, α^ϑ, and β^ϑ as prescribed above. Let the density ρ satisfy conditions (1.4), (1.5) and let the stationary tornado-hurricane vortex $\overline{U} = (\overline{u}, \overline{\vartheta})^T$ be given as in (1.6) and (1.7). Then* *for each* $U_0 = (u_0, \vartheta_0) \in H^r_{\mathbb{P}_\rho} + \overline{U}$ *there exists a* $T_0 > 0$ *and a unique mild solution* $U = (u, \theta)^T$ *of the tornado-hurricane equations (1.1) such that*

$$\begin{aligned} U - \overline{U} &\in \mathrm{BC}([0,T_0), H^r_{\mathbb{P}_\rho}) := C([0,T_0), H^r_{\mathbb{P}_\rho}) \cap L^\infty([0,T_0), H^r_{\mathbb{P}_\rho}), \\ & \quad \|U(t) - U_0\|_{H^r_{\mathbb{P}_\rho}} \to 0 \quad (t \to 0). \end{aligned}$$

Remark 1.2. Due to possible discontinuities of the boundary coefficients α^v, α^ϑ, β^v, and β^ϑ at the edges of a cylinder it is not clear, whether the mild solution of Theorem 1.1 is a strong and/or a classical one. However, if G is a layer or a half-space, no discontinuities appear. In these cases the solution is expected to be strong (under the correct assumptions on the data) and classical. This will also be included in a forthcoming paper.

The paper is organized as follows. In Section 2 we transform (1.1) to a perturbed system by subtracting the stationary tornado-hurricane vortex from the solution of (1.1). The linearized version of the perturbed system is treated in Section 3. By means of coercive bilinear forms there we derive maximal regularity for the linearized tornado-hurricane operator (see Theorem 3.8). A main difference to the standard incompressible Navier-Stokes equations lies in the fact that the density ρ is not constant. To circumvent this difficulty we regard ρ as a weight and work in weighted L^2-spaces, where the standard solenoidality condition $\operatorname{div} u = 0$ is replaced by the anelastic condition $\operatorname{div} \rho u = 0$. Based on suitable estimates for fractional powers of the tornado-hurricane operator and the generated analytic semigroup we will finally prove the existence of a local-in-time mild solution in Section 4 by applying the contraction mapping principle.

2. Transformation to an equivalent system

First we derive a representation for the boundary conditions which is more appropriate for our purposes. If n denotes the outer normal vector at Γ, normal and tangential part of a vector $F \in \mathbb{R}^3$ are given by

$$F_n = (n \cdot F)n \quad \text{and} \quad F_\tau = F - F_n.$$

Obviously, then we have

$$F = F_n + F_\tau \quad \text{and} \quad F_n \cdot H_\tau = 0 \quad (F, H \in \mathbb{R}^3).$$

Lemma 2.1. *Let* n *be the outer normal and* ℓ *be a tangential vector at* $\Gamma = \partial G$, *i.e.,* $|\ell| = 1$ *and* $\ell \cdot n = 0$. *Let* $u : \Gamma \to \mathbb{R}^3$ *be a* C^1 *vector field such that* $n \cdot u = 0$. *Then on* Γ *we have*

(i) $\ell^T D(u) n = \ell \cdot \partial_n u + n \cdot \partial_\ell u$,

(ii) $\ell \cdot \partial_n u = \partial_n(\ell \cdot u)$ *and* $n \cdot \partial_\ell u = \partial_\ell (n \cdot u)$,

(iii) $(\alpha^v D(u) n + \beta^v u)_\tau = \alpha^v \partial_n u_\tau + \beta^v u_\tau$.

Proof. Relation (i) follows immediately from the structure of $D(u)$. (ii) is a well-known differential geometric fact (Note that for the $G \subseteq \mathbb{R}^3$ considered here ℓ does obviously not depend on the normal direction n). (iii) is an immediate consequence of (i), (ii), and $n \cdot u = 0$. $\square$

Now let $(\overline{u}, \overline{\vartheta}, \overline{q})$ be a tornado-hurricane vortex, i.e., a stationary solution of (1.1). We set

$$\begin{aligned} v &= u - \overline{u}, \\ \theta &= \vartheta - \overline{\vartheta}, \\ p &= q - \overline{q}. \end{aligned}$$

By virtue of Lemma 2.1 then (u, ϑ, q) solves (1.1) if and only if the triple (v, θ, p) solves the perturbed system

$$\left\{ \begin{aligned} \partial_t v - \operatorname{div} D(v) + (v \cdot \nabla)v + \Omega e_3 \times v \quad & & & \\ + (\overline{u} \cdot \nabla)v + (v \cdot \nabla)\overline{u} + \tfrac{1}{\rho}\nabla p - e_3 g \tfrac{\theta}{\overline{\vartheta}} &= 0 & & \text{in } J \times G, \\ \operatorname{div} \rho v &= 0 & & \text{in } J \times G, \\ \partial_t \theta - \nu\Delta\theta + (v \cdot \nabla)\theta + (\overline{u} \cdot \nabla)\theta + (v \cdot \nabla)\overline{\vartheta} &= 0 & & \text{in } J \times G, \\ \alpha^v \partial_n v_\tau + \beta^v v_\tau &= 0 & & \text{on } J \times \Gamma, \\ \alpha^\vartheta \partial_n \theta + \beta^\vartheta \theta &= 0 & & \text{on } J \times \Gamma, \\ n \cdot v &= 0 & & \text{on } J \times \Gamma, \\ v|_{t=0} &= v_0 & & \text{in } G, \\ \theta|_{t=0} &= \theta_0 & & \text{in } G, \end{aligned} \right. \tag{2.1}$$

where $v_0 = u_0 - \overline{u}$, $\theta_0 = \vartheta_0 - \overline{\vartheta}$, and $J = (0, T)$. The results in the next two sections we will provide the existence of a local-in-time mild solution for system (2.1), which then in turn implies Theorem 1.1.

3. Linear theory

As in the first two sections throughout the rest of the paper $G \subseteq \mathbb{R}^3$ is assumed to be be a half-space, a layer, or a cylinder with boundary coefficients α^v, β^v, α^ϑ, and β^ϑ as prescribed in the introduction. The aim of this section is to develop a systematic approach to the linearized system

$$\left\{ \begin{aligned} \partial_t v - \operatorname{div} D(v) + (\overline{u} \cdot \nabla)v + (v \cdot \nabla)\overline{u} + \Omega e_3 \times v + \tfrac{\nabla p}{\rho} - e_3 g \tfrac{\theta}{\overline{\vartheta}} &= f_v & & \text{in } J \times G, \\ \operatorname{div} \rho v &= 0 & & \text{in } J \times G, \\ \partial_t \theta - \nu\Delta\theta + (\overline{u} \cdot \nabla)\theta + (v \cdot \nabla)\overline{\vartheta} &= f_\vartheta & & \text{in } J \times G, \\ \alpha^v \partial_n v_\tau + \beta^v v_\tau &= 0 & & \text{on } J \times \Gamma, \\ \alpha^\vartheta \partial_n \theta + \beta^\vartheta \theta &= 0 & & \text{on } J \times \Gamma, \\ n \cdot v &= 0 & & \text{on } J \times \Gamma, \\ v|_{t=0} &= v_0 & & \text{in } G, \\ \theta|_{t=0} &= \theta_0 & & \text{in } G. \end{aligned} \right. \tag{3.1}$$

For this purpose let us introduce some notation. We will use standard terminology throughout the paper. For instance, $C^k(G, X)$ denotes the space of continuously differentiable X-valued functions of order $k \in \mathbb{N}_0 \cup \{\infty\}$, whereas $C_c^k(G,X)$ denotes its subspace of compactly supported functions. We also use the notation $\mathrm{BC}(G,X)$ for the space of bounded and continuous functions. As usual $L^2(G,X)$ is the X-valued Lebesgue space of square integrable functions, whereas $H^k(G,X)$ denotes the corresponding Sobolev space of order $k \in \mathbb{N}_0$. By $\widehat{H}^k(G,X)$ we denote its homogeneous version. If no confusion seems likely, we will omit G and X in the notation and just write L^2 and H^k. If $X = \mathbb{R}^m$, on H^k we have the scalar product

$$(u,v)_{H^k} = \sum_{|\alpha| \leq k} (\partial^\alpha u, \partial^\alpha v),$$

where $(u,v) = \int_G u(x) \cdot v(x) \mathrm{d}x$ and where $u \cdot v$ denotes the standard scalar product in $\mathbb{R}^m$. Furthermore, if A is a closed operator in the Banach space X, we denote by $\mathscr{D}(A)$, $\sigma(A)$, and $\rho(A)$ its domain, spectrum, and resolvent set, respectively.

In oder to solve system (3.1) we make use of the Leray projector $P : L^2 \to L^2_\sigma$ associated to the Helmholtz-Weyl decomposition

$$L^2 = L^2_\sigma \oplus G_2,$$

where $L^2_\sigma = \overline{C^\infty_{c,\sigma}(G,\mathbb{R}^3)}^{\|\cdot\|_2}$, $C^\infty_{c,\sigma}(G,\mathbb{R}^3) = \{u \in C^\infty_c(G,\mathbb{R}^3) : \ \operatorname{div} u = 0\}$, and $G_2 = \{\nabla p : \ p \in \widehat{H}^1\}$. Note that the space of solenoidal fields L^2_σ can be represented as

$$L^2_\sigma = \{u \in L^2 : \ \operatorname{div} u = 0, \ n \cdot u|_\Gamma = 0\},$$

where the trace is to understand in the usual sense given for L^2-fields satisfying $\operatorname{div} u \in L^2$. We refer to standard textbooks as [7], [18] for the existence of the Helmholtz-Weyl decomposition and basic facts on the Stokes and Navier-Stokes equations. We also remark that the method we present here in order to solve system (3.1) is closely related to the approach to the Stokes equations used in [18]. We set

$$V = \begin{pmatrix} v \\ \theta \end{pmatrix}, \quad \mathbb{P} = \begin{pmatrix} P \\ 1 \end{pmatrix}.$$

Note that for vector fields V we have the decomposition

$$L^2 = L^2_{\mathbb{P}} \oplus G_{\mathbb{P}}$$

with $L^2_{\mathbb{P}} := \mathbb{P}L^2 = L^2_\sigma \times L^2(G,\mathbb{R})$ and $G_{\mathbb{P}} := (I - \mathbb{P})L^2 = G_2 \times \{0\}$.

Next we establish a suitable definition of the operator '$-\nu\mathbb{P}\Delta$' in $L^2_{\mathbb{P}}$ subject to the boundary conditions $T_r V = 0$ for trace operators of the form

$$T_r V := \begin{pmatrix} \alpha^v \partial_n v_\tau + (\alpha^v r^v + \beta^v) v_\tau \\ \alpha^\vartheta \partial_n \theta + (\alpha^\vartheta r^\vartheta + \beta^\vartheta)\theta \end{pmatrix}\Bigg|_\Gamma + \begin{pmatrix} v_\nu \\ 0 \end{pmatrix}\Bigg|_\Gamma, \tag{3.2}$$

with functions $r^v, r^\vartheta \in L^\infty(\Gamma)$ and α^v, β^v, α^ϑ, and β^ϑ as defined in (1.2). To this end, we introduce the space

$$\begin{aligned}\mathbb{H} \;:=\; & \big\{U = (u,\vartheta)^T \in H^1(G,\mathbb{R}^4): \\ & u_\tau = 0 \text{ on } \Gamma_j \text{ for all } j \in \{k \in \{1,\dots,m_G\}:\ \alpha_k^v = 0\}, \\ & \vartheta = 0 \text{ on } \Gamma_j \text{ for all } j \in \{k \in \{1,\dots,m_G\}:\ \alpha_k^\vartheta = 0\}\big\},\end{aligned}$$

which takes care of the boundary components on which the Dirichlet condition is imposed. Obviously, equipped with the $H^1(G,\mathbb{R}^4)$-scalar product the space $\mathbb{H}$ is a Hilbert space.

Now we set $\mathbb{H}_\mathbb{P} := \mathbb{H} \cap L^2_\mathbb{P}$. On $\mathbb{H}_\mathbb{P}$ we define for $\lambda > 0$ the bilinear form

$$\begin{aligned} & a_\lambda : \mathbb{H}_\mathbb{P} \times \mathbb{H}_\mathbb{P} \to \mathbb{R}, \\ a_\lambda(U,V) \;=\; & \lambda(U,V) + \nu(\nabla U, \nabla V) + \nu\langle U,V\rangle_{\Gamma,r},\end{aligned}$$

where

$$(\nabla U, \nabla V) := \sum_{k=1}^4 (\nabla U^k, \nabla V^k)$$

and

$$\begin{aligned}\langle U,V\rangle_{\Gamma,r} \;:=\; & \sum_{j\in\{k:\ \alpha_k^v \neq 0\}} \int_{\Gamma_j} \left(\frac{\beta_j^v}{\alpha_j^v} + r^v(x)\right) u_\tau(x) v_\tau(x)\,\mathrm{d}\sigma(x) \\ & + \sum_{j\in\{k:\ \alpha_k^\vartheta \neq 0\}} \int_{\Gamma_j} \left(\frac{\beta_j^\vartheta}{\alpha_j^\vartheta} + r^\vartheta(x)\right) \vartheta(x)\theta(x)\,\mathrm{d}\sigma(x).\end{aligned}$$

Induced by a_0 we define the operator $\mathcal{A}$ in $L^2_\mathbb{P}$ with domain $\mathscr{D}(\mathcal{A})$ by

$$\begin{aligned}\mathscr{D}(\mathcal{A}) \;:=\; & \big\{U \in \mathbb{H}_\mathbb{P};\ \exists\, f \in L^2_\mathbb{P}\ \forall\, V \in \mathbb{H}_\mathbb{P}:\ a_0(U,V) = (f,V)\big\}, \\ \mathcal{A}U \;:=\; & f.\end{aligned}$$

Note that it is not so clear (at least to the author), whether $\mathscr{D}(\mathcal{A}) \subset H^2(G,\mathbb{R}^4)$ holds for all types of domains G (i.p. for a cylinder) and all variants of boundary conditions that we consider. Therefore we prefer the approach via forms as given above. It remains to show that the definition of $\mathcal{A}$ is meaningful and that it is the 'correct' operator if acting on smooth enough functions. This will be the content of the next lemma and the proposition afterwards.

Lemma 3.1. (a) *For $U \in \mathbb{H} \cap H^2(G,\mathbb{R}^4)$ the following statements are equivalent.*

(i) $T_r U = 0$,

(ii) $(-\Delta U, V) = (\nabla U, \nabla V) + \langle U,V\rangle_{\Gamma,r}$ $(V \in \mathbb{H}_\mathbb{P})$.

(b) *For $U \in \mathscr{D}(\mathcal{A}) \cap H^2(G,\mathbb{R}^4)$ we have*

$$\mathcal{A}U = -\nu\mathbb{P}\Delta U.$$

Proof. (a) The generalized Gauß theorem yields

$$(-\Delta U, V) = (\nabla U, \nabla V) - \int_\Gamma V \cdot \partial_n U \mathrm{d}\sigma. \tag{3.3}$$

Note that for pure Dirichlet boundary conditions, i.e., $\alpha^v = \alpha^\vartheta = 0$ the assertion already follows. Thus we may assume that there is at least one $j \in \{1, \dots, m_G\}$ such that $\alpha_j^v \neq 0$ or $\alpha_j^\vartheta \neq 0$. For the last term in (3.3) we calculate

$$\begin{aligned}\int_\Gamma V \cdot \partial_n U \mathrm{d}\sigma &= \sum_{j=1}^{m_G} \int_{\Gamma_j} v_\tau \partial_n u_\tau \mathrm{d}\sigma \\ &\quad + \sum_{j=1}^{m_G} \int_{\Gamma_j} \theta \partial_n \vartheta \mathrm{d}\sigma + \int_\Gamma v_n \partial_n u_n \mathrm{d}\sigma,\end{aligned}$$

where we used the fact that $k_n \cdot \ell_\tau = 0$ for vectors $k, \ell \in \mathbb{R}^3$ and Lemma 2.1(ii). Observing that

$$v_\tau \partial_n u_\tau = \frac{1}{\alpha_j^v} v_\tau \left(\alpha_j^v \partial_n u_\tau + \left(\alpha_j^v r^v + \beta_j^v\right) u_\tau\right) - \left(r^v + \frac{\beta_j^v}{\alpha_j^v}\right) v_\tau u_\tau$$

on Γ_j for $j \in \{k : \ \alpha_k^v \neq 0\}$ and that

$$\vartheta \partial_n \theta = \frac{1}{\alpha_j^\vartheta} \theta \left(\alpha_j^\vartheta \partial_n \vartheta + \left(\alpha_j^\vartheta r^\vartheta + \beta_j^\vartheta\right) \vartheta\right) - \left(r^\vartheta + \frac{\beta_j^\vartheta}{\alpha_j^\vartheta}\right) \theta \vartheta$$

on Γ_j for $j \in \{k : \ \alpha_k^\vartheta \neq 0\}$, we see that

$$\begin{aligned}(-\Delta U, V) &= (\nabla U, \nabla V) + \langle U, V \rangle_{\Gamma, r} \\ &\quad - \sum_{j \in \{k:\ \alpha_k^v \neq 0\}} \frac{1}{\alpha_j^v} \int_{\Gamma_j} v_\tau \left(\alpha_j^v \partial_n u_\tau + \left(\alpha_j^v r^v + \beta_j^v\right) u_\tau\right) \mathrm{d}\sigma \\ &\quad - \sum_{j \in \{k:\ \alpha_k^\vartheta \neq 0\}} \frac{1}{\alpha_j^\vartheta} \int_{\Gamma_j} \theta \left(\alpha_j^\vartheta \partial_n \vartheta + \left(\alpha_j^\vartheta r^\vartheta + \beta_j^\vartheta\right) \vartheta\right) \mathrm{d}\sigma \qquad (3.4) \\ &\quad - \sum_{j \in \{k:\ \alpha_k^v = 0\}} \int_{\Gamma_j} v_\tau \partial_n u_\tau \mathrm{d}\sigma - \sum_{j \in \{k:\ \alpha_k^\vartheta = 0\}} \int_{\Gamma_j} \vartheta \partial_n \theta \mathrm{d}\sigma \\ &\quad - \int_\Gamma v_n \partial_n u_n \mathrm{d}\sigma \quad (U \in \mathbb{H} \cap H^2(G, \mathbb{R}^4),\ V \in \mathbb{H}_\mathbb{P}).\end{aligned}$$

Note that by our regularity assumptions on U and V all appearing traces and integrals are welldefined. Thus $V \in \mathbb{H}_\mathbb{P}$ implies the fourth and the fifth line in (3.4) to vanish. The condition $T_r U = 0$ then immediately implies (ii).

On the other hand, if we assume (ii) to hold, equation (3.4) implies

$$\int_\Gamma V T_r U \mathrm{d}\sigma = 0 \quad (V \in \mathbb{H}_\mathbb{P}).$$

Let $\varphi_j^v \in C_c^\infty(\Gamma_j, \mathbb{R}^3)$ for $j \in \{k:\ \alpha_k^v \neq 0\}$ and $\varphi_j^\vartheta \in C_c^\infty(\Gamma_j, \mathbb{R})$ for $j \in \{k:\ \alpha_k^\vartheta \neq 0\}$. We set $\varphi^v(x) := \varphi_j^v(x)$ if $x \in \Gamma_j$, where $\varphi_j^v := 0$ if $x \in \Gamma_j$ such that $j \in \{k:\ \alpha_k^v = 0\}$. The function φ^ϑ is defined analogously. Then it is well-known that there exists a solution $v \in H^1(G, \mathbb{R}^3) \cap L^2_\sigma$ of the Stokes resolvent problem

$$\begin{cases} (1-\Delta)v + \nabla p &= 0 \quad \text{in } G, \\ \operatorname{div} v &= 0 \quad \text{in } G, \\ v_\tau &= \varphi^v \quad \text{on } \Gamma, \\ n \cdot v &= 0 \quad \text{on } \Gamma, \end{cases}$$

and a solution $\theta \in H^1(G, \mathbb{R})$ of the heat equation

$$\begin{cases} (1-\Delta)\theta &= 0 \quad \text{in } G, \\ \theta &= \varphi^\vartheta \quad \text{on } \Gamma. \end{cases}$$

(These two results can be obtained, e.g., by a standard Hilbert space theoretical argument.) Since φ^v and φ^ϑ vanish on the Dirichlet boundary components we obtain

$$V := \begin{pmatrix} v \\ \theta \end{pmatrix} \in \mathbb{H}_{\mathbb{P}}.$$

This yields

$$\int_\Gamma (\varphi^v, \varphi^\vartheta)^T T_r U \mathrm{d}\sigma = \int_\Gamma V T_r U \mathrm{d}\sigma = 0. \tag{3.5}$$

The fact that $U \in \mathbb{H}$ further implies that the above integral even vanishes for all $(\varphi^v, \varphi^\vartheta)$ not necessarily being zero on the Dirichlet boundary components. More precisely, (3.5) holds for all $(\varphi^v, \varphi^\vartheta)$ such that $\varphi^v = \varphi_j^v$ on Γ_j with arbitrary $\varphi_j^v \in C_c^\infty(\Gamma_j, \mathbb{R}^3)$ and such that $\varphi^\vartheta = \varphi_j^\vartheta$ on Γ_j with arbitrary $\varphi_j^\vartheta \in C_c^\infty(\Gamma_j, \mathbb{R})$ for all $j \in \{1, \dots, m_G\}$. This yields (i) and the assertion is proved.

(b) Thanks to (a) we can calculate for $U \in \mathscr{D}(\mathcal{A}) \cap H^2(G, \mathbb{R}^4)$,

$$\begin{aligned} (\mathcal{A}U, V) &= (f, V) = a_0(U, V) \\ &= \nu \Big[(\nabla U, \nabla V) + \langle U, V \rangle_{\Gamma, r} \Big] \\ &= \nu(-\Delta U, V) \\ &= (-\nu \mathbb{P} \Delta U, V) \quad (V \in \mathbb{H}_{\mathbb{P}}). \end{aligned}$$

□

Proposition 3.2. *The operator $\mathcal{A}$ is selfadjoint on $L^2_{\mathbb{P}}$ and we have $\sigma(\mathcal{A}) \subseteq [\delta_G, \infty)$ for some $\delta_G \in \mathbb{R}$.*

Proof. Clearly, we have $\mathscr{D}(\mathcal{A}) = \mathscr{D}(\lambda + \mathcal{A})$ and $((\lambda + \mathcal{A})U, V) = a_\lambda(U, V)$ for $\lambda > 0$ and all $U \in \mathscr{D}(\mathcal{A})$ and $V \in \mathbb{H}_{\mathbb{P}}$. Obviously the form a_λ, and therefore also the operator $\mathcal{A}$, is symmetric. The fact that the trace operator $\gamma_0 : V \mapsto V|_\Gamma$ is bounded from $H^1(G)$ to $L^2(\Gamma)$ further implies

$$\begin{aligned} |a_\lambda(U,V)| &\leq C(\lambda)\|U\|_{H^1}\|V\|_{H^1} + C(r)\|U\|_{L^2(\Gamma)}\|V\|_{L^2(\Gamma)} \\ &\leq C(\lambda,r)\|U\|_{H^1}\|V\|_{H^1} \quad (U,V \in \mathbb{H}_\mathbb{P}). \end{aligned}$$

Thus a_λ is continuous for each $\lambda > 0$. To see that a_λ is also coercive for $\lambda > \delta_G$ and some $\delta_G \in \mathbb{R}$ we estimate

$$\begin{aligned} a_\lambda(V,V) &\geq (\lambda - \nu)\|V\|_2^2 + \nu\|V\|_{H^1}^2 \\ &\quad - b^v \sum_{j\in\{k:\ \alpha_k^v \neq 0\}} \|v_\tau\|_{L^2(\Gamma_j)}^2 - b^\vartheta \sum_{j\in\{k:\ \alpha_k^\vartheta \neq 0\}} \|\theta\|_{L^2(\Gamma_j)}^2 \end{aligned} \tag{3.6}$$

with

$$\begin{aligned} b^v &= -\inf_{x\in\Gamma,\, j\in\{k:\ \alpha_k^v\neq 0\}} \left(\frac{\beta_j^v}{\alpha_j^v} + r^v(x),\, 0 \right), \\ b^\vartheta &= -\inf_{x\in\Gamma,\, j\in\{k:\ \alpha_k^\vartheta\neq 0\}} \left(\frac{\beta_j^\vartheta}{\alpha_j^\vartheta} + r^\vartheta(x),\, 0 \right). \end{aligned}$$

In the case that $b^v = b^\vartheta = 0$ the coerciveness follows. Thus, we may assume that at least one of them is positive. Suppose that $b^v > 0$. In order to estimate the second term in (3.6) we employ the interpolation estimate

$$\begin{aligned} \|u\|_{H^s}^2 &\leq C\|u\|_{H^1}^{2s}\|u\|_2^{2(1-s)} \\ &\leq C\left(\varepsilon\|u\|_{H^1}^2 + C(\varepsilon)\|u\|_2^2\right) \quad (u \in H^1,\ \varepsilon > 0), \end{aligned}$$

which is valid by virtue of $H^s = [L^2, H^1]_s$ and due to Young's inequality. Here $[\cdot,\cdot]_s$ denotes the complex interpolation space (see [19]). We fix an $s > 1/2$. Then the boundedness of the trace operator γ_0 from $H^s(G)$ to $L^2(\Gamma)$ yields

$$\begin{aligned} \|v_\tau\|_{\Gamma_j}^2 &\leq \|v_\tau\|_{L^2(\Gamma)}^2 \leq C\|v_\tau\|_{H^s}^2 \\ &\leq C\left(\varepsilon\|V\|_{H^1}^2 + C(\varepsilon)\|V\|_2^2\right) \quad (\varepsilon > 0). \end{aligned}$$

This gives us

$$-b^v\|v_\tau\|_{L^2(\Gamma_j)}^2 \geq -b^v C\left(\varepsilon\|V\|_{H^1}^2 + C(\varepsilon)\|V\|_2^2\right) \quad (\varepsilon > 0).$$

Completely analogous we can derive an estimate as

$$-b^\vartheta\|\theta\|_{L^2(\Gamma_j)}^2 \geq -b^v C\left(\tilde\varepsilon\|V\|_{H^1}^2 + C(\tilde\varepsilon)\|V\|_2^2\right) \quad (\tilde\varepsilon > 0)$$

in case that $b^\vartheta > 0$. Choosing $\varepsilon := \nu/4b^v C$, $\tilde\varepsilon := \nu/4b^\vartheta C$, and inserting this into (3.6) we arrive at

$$a_\lambda(V,V) \geq \Big(\lambda - C(b^v, b^\vartheta, \nu)\Big)\|V\|_2^2 + \frac{\nu}{2}\|V\|_{H^1}^2 \quad (V \in \mathbb{H}_\mathbb{P}).$$

Consequently, there exists a $\delta_G < C(b^v, b^\vartheta, \nu)$ such that for each $\lambda > \delta_G$ the bilinear form a_λ is coercive. The Lax-Milgram theorem then yields for each $f \in L^2_{\mathbb{P}}$ the existence of a unique $U \in \mathbb{H}_{\mathbb{P}}$ such that

$$a_\lambda(U, V) = (f, V) \quad (V \in \mathbb{H}_{\mathbb{P}}).$$

This implies that $(-\infty, \delta_G) \subseteq \rho(\mathcal{A})$. It is also obvious that $C^\infty_{c,\sigma}(G, \mathbb{R}^3) \times C^\infty_c(G, \mathbb{R}) \subseteq \mathscr{D}(\mathcal{A})$. Thus, $\mathcal{A}$ is densely defined. By general results for symmetric bilinear forms (see [3]) we then obtain that $\mathcal{A}$ is selfadjoint on $L^2_{\mathbb{P}}$. Hence the assertion follows. □

It is a well-known fact that selfadjoint operators admit various further properties as, e.g., the property of having maximal regularity.

Definition 3.3. A closed operator $A : \mathscr{D}(A) \subseteq X \to X$ is said to have *maximal regularity* on the Banach space X, if there exists a $p \in [1, \infty)$ such that for each $T \in (0, \infty)$ and each $(f, u_0) \in L^p(J, X) \times (X, \mathscr{D}(A))_{1-1/p,p}$, where $J = (0, T)$ and the latter space denotes the real interpolation space, there exists a unique solution u of the Cauchy problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u + Au &= f \quad \text{in } (0, T), \\ u(0) &= u_0, \end{cases}$$

satisfying $Au \in L^p(J, X)$. In this case the operator

$$\frac{\mathrm{d}}{\mathrm{d}t} + A : H^1(J, X) \cap L^2(J, \mathscr{D}(A)) \to L^2(J, X) \times (X, \mathscr{D}(A))_{1-1/p,p}$$

is an isomorphism by the open mapping theorem. The class of operators having maximal regularity on X we denote by $\mathrm{MR}(X)$.

It is also well-known that maximal regularity implies A to generate an analytic C_0-semigroup on X (in Hilbert spaces this is even equivalent). For a comprehensive discussion of maximal regularity and related properties we refer to [5]. For instance, it is well-known that the property of maximal regularity is independent of p, i.e., if $A \in \mathrm{MR}(X)$ for one $p \in [1, \infty)$ then it follows $A \in \mathrm{MR}(X)$ for all $p \in (1, \infty)$. This justifies the p-independent notation.

Corollary 3.4. *We have $\mathcal{A} \in \mathrm{MR}(L^2_{\mathbb{P}})$. In particular, $\mathcal{A}$ is the generator of an analytic C_0-semigroup on $L^2_{\mathbb{P}}$.*

The class $\mathrm{MR}(X)$ is known to be stable under relatively bounded perturbations (see [5]). This gives us the following result.

Lemma 3.5. *Let $b_j \in L^\infty(G, \mathbb{R}^{4\times 4})$ for $j = 0, 1, 2, 3$ and set*

$$\mathcal{B}U := \mathbb{P}b_0U + \mathbb{P}\sum_{j=1}^{3} b_j\partial_j U, \quad U \in H^1(G, \mathbb{R}^4).$$

Then we have $\mathcal{A} + \mathcal{B} \in \mathrm{MR}(L^2_{\mathbb{P}})$.

Proof. By the assumptions on b_j, $j = 0, 1, 2, 3$, and the coerciveness of the form a_λ we obtain

$$\begin{aligned} \|\mathcal{B}U\|_2^2 &\le C\|U\|_{H^1}^2 \\ &\le Ca_\lambda(U,U) = C\left((\lambda+\mathcal{A})U,U\right) \\ &\le C\|(\lambda+\mathcal{A})U\|_2\|U\|_2 \\ &\le \varepsilon\|(\lambda+\mathcal{A})U\|_2^2 + C(\varepsilon)\|U\|_2^2 \quad (U \in \mathcal{D}(\mathcal{A}), \varepsilon > 0). \end{aligned}$$

Hence $\mathcal{B}$ is a Kato perturbation which yields the claim. □

Next we set up an L^2-theory for the linearized tornado-hurricane equations (3.1). For this purpose we assume the density ρ to satisfy conditions (1.4) and (1.5). It will be convenient to introduce the scalar product

$$\langle U, V\rangle_\rho := \int_G U(x)\cdot V(x)\rho^2(x)\mathrm{d}x, \quad U, V \in L^2(G,\mathbb{R}^4),$$

on $L^2(G,\mathbb{R}^4)$. By condition (1.5) on ρ the induced norm $\|\cdot\|_{L^2_\rho}$ is equivalent to the standard norm in L^2. If we mean L^2 to be equipped with $\langle\cdot,\cdot\rangle_\rho$ we use the subscript notation L^2_ρ. Then the identity operator $I : L^2 \to L^2_\rho$ or, equivalently, the multiplication operator

$$M_\rho : L^2 \to L^2, \quad M_\rho U := \rho\cdot U$$

is an isomorphism of Hilbert spaces. Obviously, $M_\rho : L^2_\rho \to L^2$ is an isometry and we have $M_\rho^{-1} = M_{1/\rho}$. The decomposition $L^2 = L^2_{\mathbb{P}} \oplus G_{\mathbb{P}}$ therefore induces the decomposition

$$L^2_\rho = L^2_{\mathbb{P}_\rho} \oplus G_{\mathbb{P}_\rho}$$

with $L^2_{\mathbb{P}_\rho} = M_\rho^{-1}L^2_{\mathbb{P}}$, $G_{\mathbb{P}_\rho} = M_\rho^{-1}G_{\mathbb{P}}$, and the projection

$$\mathbb{P}_\rho := M_\rho^{-1}\mathbb{P}M_\rho : L^2_\rho \to L^2_{\mathbb{P}_\rho}.$$

Since $\mathbb{P}$ is orthogonal on L^2, the projection $\mathbb{P}_\rho$ is orthogonal on L^2_ρ. However, observe that $\mathbb{P}_\rho$ regarded as a projection on L^2 is not orthogonal in general.

Lemma 3.6. *The spaces $L^2_{\mathbb{P}_\rho}$ and $G_{\mathbb{P}_\rho}$ are Banach spaces and we have the characterizations*

$$\begin{aligned} L^2_{\mathbb{P}_\rho} &= \left\{V = (v,\theta)^T \in L^2 : \operatorname{div}\rho v = 0,\ v\cdot n|_\Gamma = 0\right\}, \\ G_{\mathbb{P}_\rho} &= \left\{\frac{1}{\rho}\nabla p :\ p \in \widehat{H}^1\right\} \times \{0\}. \end{aligned}$$

Proof. First observe that by condition (1.5) we have

$$n\cdot\rho v|_\Gamma = 0 \quad \Leftrightarrow \quad n\cdot v|_\Gamma = 0, \tag{3.7}$$

if one of the two traces is defined. However, the trace $n\cdot\rho v|_\Gamma$ can be defined as in the usual sense for solenoidal L^2 vector fields. Now, the fact that $V \in L^2_{\mathbb{P}_\rho}$ is equivalent to say that $M_\rho V \in L^2_{\mathbb{P}}$ which means $\operatorname{div}\rho V = 0$ and

$n\cdot\rho v|_\Gamma = 0$. In view of (3.7) this shows (i). Relation (ii) immediately follows from the characterization of $G_{\mathbb{P}}$. □

By the assumptions on ρ it is also not difficult to see that

$$\langle U,V\rangle_{H^k_\rho} := \sum_{|\alpha|\le k} \langle \partial^\alpha U, \partial^\alpha V\rangle_\rho$$

is an equivalent scalar product on H^k and that $M_\rho : H^k \to H^k$ and $M_\rho : H^k_\rho \to H^k$ are isomorphisms for $k = 0,1,2$. The Sobolev spaces of fractional order we define as usual by complex interpolation, i.e, $H^r := [L^2, H^2]_{r/2}$ for $r\in[0,2]$. For the definition of the weighted versions we set $H^r_\rho := [L^2_\rho, H^2_\rho]_{r/2}$ for $r\in[0,2]$. Then we obviously have $H^r_\rho = M_\rho^{-1}H^r$.

For the transformation of the operator $-\mathbb{P}_\rho(\operatorname{div} D, \nu\Delta) = -\nu\mathbb{P}_\rho(\operatorname{div}\nabla + \nabla\operatorname{div}, \Delta)^T$ we calculate formally

$$\begin{aligned}
& M_\rho(-\nu\mathbb{P}_\rho(\operatorname{div}\nabla + \nabla\operatorname{div}, \Delta)^T)M_\rho^{-1}U \\
= \;& -\nu\mathbb{P}M_\rho\left(\Delta + \begin{pmatrix}\nabla\operatorname{div}\\ 0\end{pmatrix}\right)M_\rho^{-1}U \\
= \;& -\nu\mathbb{P}\Big[\Delta U + \sum_{j=1}^3(2\rho\partial_j\rho^{-1})\partial_j U + U\rho\Delta\rho^{-1} \\
& +\rho\begin{pmatrix}\nabla(\sum_{j=1}^3 u^j\partial_j\rho^{-1} + \rho^{-1}\operatorname{div} u)\\ 0\end{pmatrix}\Big] \\
= \;& -\nu\mathbb{P}\Delta + \nu\mathbb{P}\left(2\sum_{j=1}^3\frac{\partial_j\rho}{\rho}\partial_j U + \left(\frac{\Delta\rho}{\rho} - 2\frac{|\nabla\rho|^2}{\rho^2}\right)U\right. \\
& \left. + \sum_{j=1}^3\frac{\partial_j\rho}{\rho}\begin{pmatrix}\nabla u^j\\ 0\end{pmatrix} + \sum_{j=1}^3\left[\frac{1}{\rho}\partial_j\begin{pmatrix}\nabla\rho\\ 0\end{pmatrix} - 2\begin{pmatrix}\nabla\rho\\ 0\end{pmatrix}\frac{\partial_j\rho}{\rho^2}\right]u^j\right),
\end{aligned} \tag{3.8}$$

where we assumed that $\operatorname{div} u = 0$. If we set

$$\begin{aligned}
B_1 V &:= \mathbb{P}_\rho(\overline{u}\cdot\nabla)V, \\
B_2 V &:= \mathbb{P}_\rho\left[(v\cdot\nabla)\overline{U} + \begin{pmatrix}\Omega e_3\times v\\ 0\end{pmatrix} - \begin{pmatrix}\frac{e_3 g}{\overline{\vartheta}}\theta\\ 0\end{pmatrix}\right],
\end{aligned}$$

where $V = (v,\theta)^T$, $\overline{U} = (\overline{u},\overline{\vartheta})^T$, and $e_3 = (0,0,1)^T$, we obtain

$$M_\rho B_1 M_\rho^{-1}U := \mathbb{P}\left(\sum_{j=1}^3 \overline{u}^j\partial_j U - (\overline{u}\cdot\frac{\nabla\rho}{\rho})U\right), \tag{3.9}$$

$$M_\rho B_2 M_\rho^{-1}U := \mathbb{P}\left(\sum_{j=1}^3(\partial_j\overline{U})u^j + \begin{pmatrix}\Omega e_3\times u\\ 0\end{pmatrix} - \begin{pmatrix}\frac{e_3 g}{\overline{\vartheta}}\vartheta\\ 0\end{pmatrix}\right) \tag{3.10}$$

for $U = (u, \vartheta)^T$. The transform of the boundary operator T_0 in (3.1) reads as

$$M_\rho T_0 M_\rho^{-1} U = \begin{pmatrix} \alpha^v \partial_n u_\tau + (\beta^v - \alpha^v \frac{\partial_n \rho}{\rho}) u_\tau \\ \alpha^\vartheta \partial_n \vartheta + (\beta^\vartheta - \alpha^\vartheta \frac{\partial_n \rho}{\rho}) \vartheta \end{pmatrix}\Bigg|_\Gamma + \begin{pmatrix} u_\nu \\ 0 \end{pmatrix}\Bigg|_\Gamma = T_r U, \tag{3.11}$$

with $r^v = r^\vartheta := -\partial_n \rho / \rho$. So, if we set

$$\mathcal{A}_{TH} := -\nu \mathbb{P}_\rho (\mathrm{div}\, \nabla + \nabla \mathrm{div}\,, \Delta)^T + B_1 + B_2 \tag{3.12}$$

by (3.8), (3.9), and (3.10) it is not difficult to see that the formal transform of this operator is represented as

$$M_\rho \mathcal{A}_{TH} M_\rho^{-1} = -\nu \mathbb{P} \Delta + \mathcal{B} \tag{3.13}$$

with a lower order term $\mathcal{B} = \mathbb{P}(b_0 + \sum_{j=1}^3 b_j \partial_j)$ and certain 4×4 matrices b_j, $j = 0, 1, 2, 3$. By Lemma 3.1(b) this motivates the rigorous definition of $\mathcal{A}_{TH} : \mathscr{D}(\mathcal{A}_{TH}) \to L^2_{\mathbb{P}_\rho}$ by

$$\begin{aligned} \mathcal{A}_{TH} V &:= M_\rho^{-1} (\mathcal{A} + \mathcal{B}) M_\rho V, \\ V &\in \mathscr{D}(\mathcal{A}_{TH}) := M_\rho^{-1} \mathscr{D}(\mathcal{A}). \end{aligned}$$

We call $\mathcal{A}_{TH}$ the *tornado-hurricane operator.*

Lemma 3.7. *We have*

(i) $V \in \mathscr{D}(\mathcal{A}_{TH}) \cap H^2_\rho(G, \mathbb{R}^4) \quad \Rightarrow \quad T_0 V = 0,$

(ii) $\mathscr{D}(\mathcal{A}_{TH}) \hookrightarrow \mathbb{H}_{\mathbb{P}_\rho} := \mathbb{H}_\rho \cap L^2_{\mathbb{P}_\rho} = M_\rho^{-1} \mathbb{H}_{\mathbb{P}},$

(iii) *that on* $\mathscr{D}(\mathcal{A}_{TH}) \cap H^2_\rho(G, \mathbb{R}^4)$ *representation (3.12) holds.*

Proof. Since $M_\rho : H^2_\rho \to H^2$ is an isomorphism, we have $\mathscr{D}(\mathcal{A}_{TH}) \cap H^2_\rho = M_\rho^{-1}(\mathscr{D}(\mathcal{A}) \cap H^2)$. Hence, $V \in \mathscr{D}(\mathcal{A}_{TH}) \cap H^2$ yields $M_\rho V \in \mathscr{D}(\mathcal{A}) \cap H^2$. Lemma 3.1(a) then implies that $T_r M_\rho V = 0$, which is according to (3.11) equivalent to $T_0 V = 0$ for $r^v = r^\vartheta = -\partial_n \rho / \rho$. This shows (i). By assumption (1.5) on ρ we have

$$(\rho w)|_{\Gamma_j} = 0 \quad \Leftrightarrow \quad w|_{\Gamma_j} = 0 \quad (w \in H^1,\ j = 1, \dots, m_G).$$

Thanks to assumption (1.4) on ρ this shows that $M_\rho : \mathbb{H} \to \mathbb{H}$ is an isomorphism. The fact that $M_\rho : L^2_{\mathbb{P}_\rho} \to L^2_{\mathbb{P}}$ is isomorphic as well proves (ii). Relation (iii) is obtained as a consequence of the definition of $\mathcal{A}_{TH}$, Lemma 3.1(b), and (3.13). □

Lemma 3.7 shows that by construction system (3.1) is (formally) equivalent to the Cauchy problem

$$\left\{ \begin{aligned} V' + \mathcal{A}_{TH} V &= f \quad \text{in } (0, T), \\ V(0) &= V_0, \end{aligned} \right. \tag{3.14}$$

with $f = (f_v, f_\vartheta)$ and $V_0 = (v_0, \theta_0)$. The wellposedness of this problem is given in the next theorem, which is also the main result of the present section.

Theorem 3.8. *Let the density ρ satisfy assumptions (1.4), (1.5) and let the stationary tornado-hurricane vortex $\overline{U} = (\overline{u}, \overline{\vartheta})^T$ be given as in (1.6) and (1.7). Then we have $\mathcal{A}_{TH} \in \mathrm{MR}(L^2_{\mathbb{P}_\rho})$. In particular, the tornado-hurricane operator $\mathcal{A}_{TH}$ is the generator of an analytic C_0-semigroup on $L^2_{\mathbb{P}_\rho}$.*

Proof. We consider the operator $\mathcal{A} + \mathcal{B}$ with $\mathcal{B}$ as defined above. By representations (3.8), (3.9), (3.10), (3.11), and by our assumptions on ρ and $\overline{U}$ it readily follows that $r^v = r^\vartheta = -\partial_n \rho/\rho \in L^\infty(G, \mathbb{R})$ for the boundary coefficient and that $b_j \in L^\infty(G; \mathbb{R}^{4\times 4})$, $j = 0, 1, 2, 3$, for the matrices appearing in the perturbation $\mathcal{B}$. Thus, Corollary 3.4 and Lemma 3.5 immediately imply $\mathcal{A} + \mathcal{B} \in \mathrm{MR}(L^2_{\mathbb{P}})$. Due to the fact that the property of having maximal regularity is invariant under the conjugation with isomorphisms the assertion follows. □

We close this section with some estimates that will turn out to be very helpful in the next section. First observe that for $\omega \geq 0$ large enough the semigroups generated by $\mathcal{A} + w$ and $\mathcal{A} + \mathcal{B} + w$ are exponentially bounded. We fix such an ω. For operators of this type fractional powers A^r, $r \in [-1, 1]$, are well-defined by means of a Dunford integral calculus (see e.g. [2]).

Lemma 3.9. *Let $r \in [0, 1]$. We have*

(i) $(\mathcal{A}_{TH} + \omega)^r = M_\rho^{-1}(\mathcal{A} + \mathcal{B} + \omega)^r M_\rho$,

(ii) *there exists a $C > 0$ such that*

$$\|(\mathcal{A}_{TH} + \omega)^r e^{-t(\mathcal{A}_{TH}+\omega)} V\|_{L^2_\rho} \leq C t^{-r} \|V\|_{L^2_\rho} \quad (t > 0,\ V \in L^2_{\mathbb{P}_\rho}),$$

(iii) *the norms $\|(\mathcal{A}_{TH}+\omega)^{r/2}\cdot\|_{L^2_\rho}$ and $\|\cdot\|_{H^r_\rho}$ are equivalent on the domain $\mathscr{D}((\mathcal{A}_{TH} + \omega)^{r/2})$,*

(iv) *there exists a $C > 0$ such that*

$$\|(\mathcal{A} + \mathcal{B} + \omega)^{-1/2}\mathbb{P}\partial_j V\|_2 \leq C\|V\|_2 \quad (j = 1, 2, 3,\ V \in H^1(G, \mathbb{R}^4)).$$

Proof. The equality

$$(\lambda + (\mathcal{A}_{TH} + \omega))^{-1} = M_\rho^{-1}(\lambda + (\mathcal{A} + \mathcal{B} + \omega))^{-1} M_\rho$$

shows $\rho(\mathcal{A}_{TH} + \omega) = \rho(\mathcal{A} + \mathcal{B} + \omega)$ for the resolvent sets. In particular, by assumption we have that $0 \in \rho(\mathcal{A}_{TH} + \omega)$. By the representations $(\mathcal{A}_{TH} + \omega)^r = [(\mathcal{A}_{TH} + \omega)^{-r}]^{-1}$ and

$$(\mathcal{A}_{TH} + \omega)^{-r} = \frac{1}{2\pi i}\int_\Lambda \lambda^{-r}(\lambda - (\mathcal{A}_{TH} + \omega))^{-1} \mathrm{d}\lambda,$$

where Λ is a suitable path around the spectrum of $\mathcal{A}_{TH} + \omega$, we therefore obtain (i).

Relation (ii) is a well-known fact for analytic semigroups.

In order to see (iii), we note that by Lemma 3.5 and $0 \in \rho((\mathcal{A}+\omega)) \cap \rho(\mathcal{A}+\mathcal{B}+\omega)$ the norms $\|(\mathcal{A}+\omega)\cdot\|_2$ and $\|(\mathcal{A}+\mathcal{B}+\omega)\cdot\|_2$ are equivalent. By the functional calculus for selfadjoint operators it follows that $\mathcal{A}+\omega$ admits a bounded $\mathbb{H}^\infty$-calculus on $L^2_{\mathbb{P}}$ (see [10] or [5] for the definition). It is also well-known that the class of operators admitting an $\mathbb{H}^\infty$-calculus is stable (modulo shifts) under lower order perturbations (see e.g. [5]). Thus also $\mathcal{A}+\mathcal{B}+\omega$ amits a bounded $\mathbb{H}^\infty$-calculus on $L^2_{\mathbb{P}}$. This in turn implies for the domains of fractional powers that

$$\begin{aligned} \mathscr{D}((\mathcal{A}+\omega)^r) &= [L^2_{\mathbb{P}}, \mathscr{D}(\mathcal{A}+\omega)]_r = [L^2_{\mathbb{P}}, \mathscr{D}(\mathcal{A}+\mathcal{B}+\omega)]_r \\ &= \mathscr{D}((\mathcal{A}+\mathcal{B}+\omega)^r) \qquad (r \in [0,1]), \end{aligned} \tag{3.15}$$

where $[\cdot,\cdot]_r$ denotes again the complex interpolation space (see [19]). On the other hand, by the selfadjointness of $\mathcal{A}+\omega$ we can calculate

$$\begin{aligned} \|(\mathcal{A}+\omega)^{1/2}V\|_2^2 &= ((\mathcal{A}+\omega)^{1/2}V, (\mathcal{A}+\omega)^{1/2}V) \\ &= ((\mathcal{A}+\omega)V, V) \\ &= a_\omega(V,V) \qquad (V \in \mathscr{D}(\mathcal{A}+\omega)). \end{aligned} \tag{3.16}$$

However, from the proof of Proposition 3.2 we infer that the norms $\|\cdot\|_{H^1}$ and $\sqrt{a_\omega(V,V)}$ are equivalent on $\mathscr{D}(\mathcal{A}+\omega)$. Moreover, since $\mathscr{D}(\mathcal{A}+\omega)$ lies dense in $L^2_{\mathbb{P}}$, by general results for the complex interpolation functor, $\mathscr{D}(\mathcal{A}+\omega)$ lies also dense in $\mathscr{D}((\mathcal{A}+\omega)^r)$ for every $r \in [0,1]$. Consequently, (3.15) and (3.16) imply that the norms

$$\|(\mathcal{A}+\omega)^{1/2}\cdot\|_2, \quad \|(\mathcal{A}+\mathcal{B}+\omega)^{1/2}\cdot\|_2, \quad \text{and} \quad \|\cdot\|_{H^1} \tag{3.17}$$

are equivalent on $\mathscr{D}((\mathcal{A}+\omega)^{1/2})$. By the fact that $\|\cdot\|_{H^r}$ is equivalent to $\|\cdot\|_{[L^2,H^1]_r}$ we obtain again by virtue of (3.15) and a reiteration argument that the norms

$$\|(\mathcal{A}+\omega)^{r/2}\cdot\|_2, \quad \|(\mathcal{A}+\mathcal{B}+\omega)^{r/2}\cdot\|_2, \quad \text{and} \quad \|\cdot\|_{H^r}$$

are equivalent on $\mathscr{D}((\mathcal{A}+\omega)^{r/2})$. Since $\mathscr{D}(\mathcal{A}_{TH}+\omega) = M_\rho^{-1}\mathscr{D}(\mathcal{A}+\omega)$ and $L^2_{\mathbb{P}_\rho} = M_\rho^{-1}L^2_{\mathbb{P}}$, identity (3.15) obviously yields $\mathscr{D}((\mathcal{A}_{TH}+\omega)^{r/2}) = M_\rho^{-1}\mathscr{D}((\mathcal{A}+\omega)^{r/2})$. By relation (i) and by the fact that $M_\rho : H^r_\rho \to H^r$ is an isomorphism we then arrive at (iii).

Relation (iv) follows from (iii), (3.17) and a simple duality argument. □

4. Proof of the main result

We turn to the proof of Theorem 1.1. Let $T \in (0,\infty)$, $J = [0,T)$, $r \in (3/4, 1)$, and $V_0 \in H^r_{\mathbb{P}_\rho} := H^r_\rho \cap L^2_{\mathbb{P}_\rho}$ be given. We set

$$B_{M,T} := \left\{ V = (v,\theta)^T \in \mathrm{BC}(J, H^r_{\mathbb{P}_\rho}) : \ \|V\|_T := \sup_{t\in J} \|V(t)\|_{H^r_\rho} \le M\|V_0\|_{H^r_\rho} \right\}$$

for $M > 0$ determined later. On $B_{M,T}$ we consider the operator

$$HV(t) = e^{-t\mathcal{A}_{TH}}V_0 + \int_0^t e^{-(t-s)\mathcal{A}_{TH}}\mathbb{P}_\rho(v(s)\cdot\nabla)V(s)\mathrm{d}s, \quad t \in J. \qquad (4.1)$$

Then the fixed point equation $HV = V$ is an equivalent formulation of problem (2.1) by the variation of constant formula. We prove the existence of a fixed point by applying the contraction mapping principle. To see that $H(B_{M,T}) \subseteq B_{M,T}$ for suitably small $T > 0$, by utilizing the results obtained in Lemma 3.9 we estimate for $V \in B_{M,T}$,

$$\begin{aligned}
&\|HV(t)\|_{H^r_\rho} \\
\le\ & \|e^{-t\mathcal{A}_{TH}}V_0\|_{H^r_\rho} \\
&+ C\int_0^t e^{(t-s)\omega}\|(\mathcal{A}_{TH}+\omega)^{r/2}e^{-(t-s)(\mathcal{A}_{TH}+\omega)}\mathbb{P}_\rho(v(s)\cdot\nabla)V(s)\|_{L^2_\rho}\mathrm{d}s \\
\le\ & Ce^{t\omega}\|V_0\|_{H^r_\rho} \\
&+ Ce^{t\omega}\int_0^t \frac{1}{(t-s)^{(1+r)/2}}\|(\mathcal{A}_{TH}+\omega)^{-1/2}\mathbb{P}_\rho(v(s)\cdot\nabla)V(s)\|_{L^2_\rho}\mathrm{d}s. \quad (4.2)
\end{aligned}$$

Observe that for the nonlinear term we have

$$\sum_{j=1}^3(\rho v^j)\partial_j V = \sum_{j=1}^3\partial_j(\rho v^j V)$$

in view of $\operatorname{div}\rho v = 0$. Recalling that $\|M_\rho^{-1}\cdot\|_{L^2_\rho} = \|\cdot\|_2$, $\mathbb{P}_\rho = M_\rho^{-1}\mathbb{P}M_\rho$, and that $\mathcal{A}_{TH}+\omega = M_\rho^{-1}(\mathcal{A}+\mathcal{B}+\omega)M_\rho$, Lemma 3.9(iv) implies that

$$\begin{aligned}
&\|(\mathcal{A}_{TH}+\omega)^{-1/2}\mathbb{P}_\rho(v(s)\cdot\nabla)V(s)\|_{L^2_\rho} \\
=\ & \|(\mathcal{A}+\mathcal{B}+\omega)^{-1/2}\mathbb{P}(\rho v(s)\cdot\nabla)V(s)\|_{L^2} \\
\le\ & \sum_{j=1}^3\|(\mathcal{A}+\mathcal{B}+\omega)^{-1/2}\mathbb{P}\partial_j(\rho v(s)V(s))\|_{L^2} \\
\le\ & C\sum_{j=1}^3\|\rho v^j(s)V(s)\|_{L^2} \le\ C\|V\|^2_{L^4}.
\end{aligned}$$

The Sobolev embedding (see [1]) further implies that $H^r \hookrightarrow L^4$ for $r > 3/4$. Hence we have

$$\|(\mathcal{A}_{TH}+\omega)^{-1/2}\mathbb{P}_\rho(v(s)\cdot\nabla)V(s)\|_{L^2_\rho} \le\ C\|V\|^2_{H^r_\rho}. \qquad (4.3)$$

Inserting (4.3) into (4.2) we arrive at

$$\begin{aligned}
\|HV(t)\|_{H^r_\rho} &\le\ Ce^{\omega t}\left(\|V_0\|_{H^r_\rho} + \int_0^t\frac{1}{(t-s)^{(1+r)/2}}\|V(s)\|^2_{H^r_\rho}\mathrm{d}s\right) \\
&\le\ Ce^{\omega T}\left(\|V_0\|_{H^r_\rho} + M^2T^{(1-r)/2}\|V_0\|^2_{H^r_\rho}\right) \quad (t > 0).
\end{aligned}$$

So, by choosing e.g. $M = 2Ce^{\omega}$ and T sufficiently small we conclude that $\|HV\|_T \le M\|V_0\|_{H^r_\rho}$, hence $H(B_{M,T}) \subseteq B_{M,T}$. Similarly, we see that H is a contraction for $T > 0$ small enough. Indeed, by employing the identity

$$(v \cdot \nabla)V - (u \cdot \nabla)U = (v \cdot \nabla)(V - U) + [(v - u) \cdot \nabla]U$$

we obtain

$$\begin{aligned}
&\|HV(t) - HU(t)\|_{H^r_\rho} \\
&\le Ce^{\omega t} \int_0^t \frac{1}{(t-s)^{(1+r)/2}} \Big(\|v(s)\|_{H^r_\rho} \|V(s) - U(s)\|_{H^r_\rho} \\
&\quad + \|v(s) - u(s)\|_{H^r_\rho} \|U(s)\|_{H^r_\rho} \Big) \, \mathrm{d}s \\
&\le Ce^{\omega T} M T^{(1-r)/2} \|V_0\|_{H^r_\rho} \|V - U\|_T \quad (t > 0,\ V, U \in B_{M,T}).
\end{aligned}$$

Consequently, for $T > 0$ small enough H is a contraction on $B_{M,T}$. The contraction mapping principle then yields a unique mild solution of the transformed system (2.1). Furthermore, from representation (4.1) it readily follows that

$$\|V(t) - V_0\|_{H^r_\rho} \to 0 \qquad (t \to 0).$$

The fact that $U = V + \overline{U}$ for the solution U of (1.1) then implies Theorem 1.1. □

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