# (REFLECTED) BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS AND CONTINGENT CLAIMS

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**key words:** reflected backward stochastic differential equations, american claims, Black-Scholes, singular control, stopping

ABSTRACT. We review the relations between adjoints of stochastic control problems with the derivative of the value function, and the latter with the value function of a stopping problem. These results are applied to the pricing of contingent claims.

## 1. INTRODUCTION

In [koh] we examined the properties of adjoint processes in stochastic control on the basis of the theory of stochastic flows. We extended earlier results by Baras-Elliott-Kohlmann [bar] and made extensive use of the important contributions to the theory by Pardoux-Peng [par0][par1][pen][pen2], Zhou [zho3][zho4], and Renner [ren][koh1]and of the results on stochastic partial differential equations by Antonelli [ant], Pardoux-Peng [par0][pen1][pen3], Zhou [zho1], which extend the pioneering works by Bensoussan [ben], Bismut [bis1][bis2][bis3], Kunita [kun][kun1] and many others.

Instead of going into the theoretical details of this work we will here apply some of the results to the theory of pricing contingent claims. The valuation process will take the form of a possibly reflected BSDE. These BSDEs and RBSDEs were considered by Duffie, ElKaroui, Pardoux, Peng, Quenez [duf][duf1][duf2][duf3][elk1][par0], where we find the basic results on the possibilities of representing the arbitrage free price of a contingent claim in perfect and imperfect markets, and the relation to the hedging point of view. The main advantage of the pricing approach using BSDEs is the fact that there is no need to refer to the risk-neutral measure, as it is required in the hedging theory.

In this article the valuation process will be interpreted as the adjoint of a trivial dummy control problem, which takes the form of a possibly reflected BSDE. This will lead to a stochastic Black-Scholes formula (see [ma1]), which we will examine from the point of view of establishing a price system in different situations of abstraction.

When we consider American claims we have to go into the theory of RSBDEs which will also be interpreted as adjoints for a dummy control problem. As this adjoint is the derivative of a singular control problem, and as this derivative may be seen as the value function of a stopping problem, the relation between the RBSDE and the price of the American claims is obvious.

In [boe] we proved the equivalence between a singular control problem and the limit of a sequence of stopping problems. With this result we will then examine the relation between the superhedging strategy and a problem of sequential hedging, where at the random exercise time a new hedging problem starts until we reach the finite horizon. Such sequential hedging problems or rolling hedges are used to compare long-time hedges to sequences of short-time hedges.

In this article we will not give the detailed proofs. Also only the main assumptions will be given, for details the reader is referred to [boe1][koh1][ren]. In this way we try to make this application of results from control theory more readable.

The paper is organized as follows: First from a trivial control problem we derive in a most general setting a BSDE for the price of a European claim. Then a pricing system is introduced and two special cases are considered on the basis of a very general stochastic pricing system. In section 4 we relate the American price of a contingent claim to an RBSDE and apply results on the duality between singular control and stopping to describe a problem of sequential hedging. The techniques we use are adopted from control theory. Part of the results were presented at the Hanzhou Conference on Distributed Parameter Systems and Stochastic Control in June 1998.

## 2. A trivial control problem

A bond/stock price-asset is given by

$$dP_t^0 = r(P_t^0, \omega)dt \text{ (bond)}$$
$$dP_t^i = b_i(t, P_t, \omega)dt + \sigma_{ij}(t, P_t, \omega)dw_t^j \text{ (stock)}$$

on the time interval $[0, T]$ . Here $r$ is the interest rate, $b$ the appreciation rate, and $\sigma$ the volatility.All processes live on a space $(\Omega, F, F_t, P)$ which satisfies the usual conditions and which carries a d-dimensional Brownian motion. Let $r, b, \sigma$ be bounded, progressively measurable in $\omega$ and let $(P_t^{sx})$ be the strong solution of the SDE with initials$(s, x) \in [0, T] \times \Re^n$. This solution is assumed to exist in order to avoid writing down appropriate conditions. Finally, let $\sigma$ satisfy conditions such that a unique risk premium process

$$\theta(t,P_t,\omega)=\sigma^{-1}\left(t,P_t,\omega\right)\left[b(t,P_t,\omega)-r\left(t,P_t,\omega\right)\cdot 1_n\right]$$

exists. With this we are in an arbitrage free world (for details see e.g. [kar]).We will assume that $n=d=1$ . In the first part of this article this is no real restriction. Most results can be generalized to higher dimensions. In the last section, however, this assumption becomes crucial, as there we make use of comparison theorems where processes are used which have similar properties as local times. As in [boe] this makes an extension to higher dimensions impossible, at least at the moment. Also we should note that some of the results on spdes used below only hold when $(P_t)$ is replaced by $(\log P_t)$.

Now consider the following trivial control problem

$$dz_{st}=-z_{st}\left[r(t,P_t^{sx},\omega)dt+\theta(t,P_t^{sx},\omega)dw_t\right]$$
$$z_{ss}=1$$

with cost criterion

$$J=E\left[z_{sT}g(P_T^{sx})\right],$$

where $g:\Re\times\Omega\rightarrow\Re^+$ is a nonnegative, bounded, non-anticipative process which is assumed to be once continuously differentiable in the first variable. We interpret this as a control problem with a trivial one-point control space. The formal adjoint process for this control problem is given by the backward equation ( more exactly we had to call it a system of forward-backward sdes)

$$y_t=g(P_T^{sx})-\int_t^T\left[y_ur(u,P_u^{sx})+Z_u\theta(u,P_u^{sx},\omega)\right]du-\int_t^T Z_udw_u$$

for $t\in[s,T]$.

Note that $(z_{st})$ is the deflator process and $(y_t)$ is the price process for the claim $\xi_T:=g(P_T^{sx})$, where the formal duality gives the interpretation

$$y_t=E\left[z_{tT}\xi_T\mid F_t\right],t\in[s,T]$$

Rewrite $(y_t)$ as

$$y_t=\bar{E}\left[\exp(-\int_t^T r(s,P_s)ds)\xi_T\mid F_t\right]$$

where $\bar{E}$ is the expectation with respect to the risk neutral measure associated with the Girsanov functional of $\theta$. In this form we see that $(y_t)$ corresponds to the risk neutral price of the claim in the classical notation. Also note that $z_{st}y_t$ is a P-martingale. So, in order to determine the price of the claim it is necessary to solve the BSDE.

(existence and uniqueness): A solution of the BSDE

$$y_t=g(P_T^{sx})-\int_t^T\left[y_ur(u,P_u^{sx})+Z_u\theta(u,P_u^{sx},\omega)\right]du-\int_t^T Z_udw_u$$

is a pair $(y, Z)$ such that $(y_t)$ is a continuous, adapted process and $(Z_t)$ is a predictable, square integrable process. The solution is unique if both processes are equal $P-a.s.$.

Conditions to guarantee a unique solution in the sense of 2.1 are found in [elk1]. The proofs use comparison theorems or fixed point theorems for functionals on spaces of processes. A very powerful tool to solve more general BSDEs is the four-step-scheme in [ma]

Now we will try to characterize the solution in terms of a pricing system. This is defined to be a mechanism to bring $(y_t)$ and $(P_t)$ into a relation.

## 3. The pricing system

A stochastic pricing system for the claim $\xi_T$ is a function

$$u : [0,T] \times \Re \times \Omega \rightarrow \Re$$

which satisfies

**:** $u$ is progressively measurable
**:** $u(t, P_t^{sx}, \omega) = y_t(\omega)\ P-a.s., t \in [s,T]$.

At this stage we do not impose conditions in the second variable, as such conditions must comply with the real world requirements.

The pricing system will be called *convenient*, if

$$u(t,.,\omega) = u(t,.,\alpha_t(\omega))$$

where $\alpha$ is a given diffusion process. It will be called *differentiable*, if

$$u(t,.,\omega) \in C^{12} \quad P-a.s.$$

and it will be called *deterministic* if

$$u(t,x,\omega) = u(t,x).$$

At first sight one might like to treat the pricing problem as a problem of prediction within filtering theory. Let us have a short look at this approach: The signal is the price of the stock which is observed as

$$dv_t = (r_t v_t + \theta_t)dt + \sigma dw_t$$

We would then like to compute

$$y_t = \bar{E}(g(P_T) \mid F_t^v),$$

where for notational convenience we put $r \doteq 0$. Furthermore let us assume that the system of signal and observation is Markovian. Then we would like to consider the conditional law of $P_T$ given $F_t^v$. Its density satisfies

$$p(t,x) < p(t,x), 1 >^{-1}$$

and the unnormalized density satisfies

$$dp = L^* p dt + B^* p dw_t$$

Here $L^*$ is the adjoint of the differential operator corresponding to $(P_t)$ , and $B^*$ is the adjoint of

$$B = \theta + \sigma \tfrac{\partial}{\partial x}.$$

Obviously we then have

$$y_t =< p(t), g > .$$

The pricing problem reduces in this way to solving the Duncan-Zakai equation for the conditional unnormalized density. For details see e.g. the articles by Pardoux [par], Mitter [mit], Davis [dav], and Kunita [kun1]. Furthermore this approach would give another most desirable property: Following Davis [dav] we apply the Doss-Sussmann technique to find a robust predictor (also see [flo]), that is a function

$$y_t = u(t, v_t).$$

The main drawback of this approach is the fact that in filtering we always assume some sort of independence between the noises in signal and observation to find a satisfactory solution of the Duncan-Zakai equation. This difficulty is made explicit in the article of Zhou [zho1].

The on-going progress in the theory of spdes will certainly contribute to a possible direct application of filtering methods to the pricing problem.

First steps towards this for the Föllmer-Schweizer model are found in [koh].

We will now *derive pricing systems* in different situations and we would like to stress again that we will try to remain as near as possible to methods described in Baras-Elliott-Kohlmann [bar], Elliott-Kohlmann [ell1], Renner [koh1], Zhou [zho4] and Pengh [pen2].

*a) the most general case*
Let

$$E\left[z_{tT} g(P_T^{tx}) \mid F_t\right] = u(t, x, \omega),$$

where $x = P_t^{sx}$. As we may assume that $E\left[z_{tT} g(P_T^{tx}) \mid F_t\right]$ is a special semimartingale (under appropriate conditions) we write $u(t, x, \omega)$ as an integral equation between random fields (see Kunita [kun]for semimartingales with spatial parameters):

$$u(t, x, \omega) = u(0, x, \omega) + \int_0^t p(s, x, \omega) ds + \int_0^t k(s, x, \omega) dw_s.$$

Now apply the Itô-Ventcell formula as generalized in Kunita [kun](also see Bismut [bis2], Elliott-Kohlmann [ell1]) to find

$$\begin{aligned} u(,P_t) &= u(T,P_T) \\ &\quad - \int_t^T \left[p(s,P_s) + 1/2\sigma^2(s,P_s)u_{xx}(s,P_s)\right] ds \\ &\quad - \int_t^T \left[b(s,P_s)u_x(s,P_s + \sigma(s,P_s)k_x(s,P_s)\right] \\ &\quad - \int_t^T \left[k(s,P_s) + \sigma(s,P_s)u_x(s,P_s)\right] dw_s \end{aligned}$$

Compare this to the backward s.d.e for $(y_t)$

$y_t = g(P_T^{sx}) - \int_t^T [y_u r(u, P_u^{sx}) + Z_u \theta(u, P_u^{sx}, \omega)]\, du - Z_u dw_u$

to find

$$\begin{aligned} u(T,P_T) &= g(P_T) = \xi_T \\ p &= -1/2\sigma^2 u_{xx} - bu_x - \sigma k_x + ru + (k + \sigma u_x)\theta \\ &= -1/2\sigma^2 u_{xx} - (b - \sigma\theta)u_x + ru + k\theta - \sigma k_x, \end{aligned}$$

where

$$Z = k + \sigma u_x.$$

This means that the solution of the spde

$du = [1/2\sigma^2 u_{xx} + (b - \sigma\theta)u_x - ru - k\theta + \sigma k_x]\, ds - kdw$

with final condition

$$u(T,x) = g(x)$$

is a stochastic price system in the sense of definition 3.1.

(i) For conditions to ensure existence and uniqueness of this spde the reader is referred to [ma1].

(ii) When we take the hedging point of view as in Karatzas [kar] it is staightforward that the optimal hedging strategy is given by

$$\pi_t = \sigma^{-1} Z_t = u_x(t,P_t) + \sigma^{-1}(t,P_t)k(t,P_t).$$

(iii) Many arguments above become notationally more transparent when we use notations and results from Malliavin's calculus. However, in order to make the results easily comparable to the application below, we refrained from doing so.

(iv) The result

$$k = Z - \sigma u_x$$

is closely related to an equation which appears in the maximum principle of a stochastic control problem, where both drift and diffusion are controlled. There the term corresponds to the second adjoint equation . An important question then arises, namely when equality holds between $Z$ and $\sigma u_x$, i.e.

$$k_t = 0$$

(v) The BSDE as a tool to model evaluations of claims has turned out to be extremely powerful. So it is easy to model the Foellmer-Schweizer hedging within this model: just subtract a martingale orthogonal to the Brownian motion from the original BSDE:

$$y_t = g(P_T^{sx}) - \int_t^T \left[ y_u r(u, P_u^{sx}) + Z_u \theta(u, P_u^{sx}, \omega) \right] du - Z_u dw_u - M_t.$$

Or just as another example for the power of this tool: Recently we could derive the price and portfolio of an informed agent, i.e. an agent with anticipative knowledge about part of the market, by applying the BSDE-techniques to Protter´s [pro] result on the connection between the enlargement of a filtration and Girsanov´s theorem. This simplifies the proof in Elliott et al. [ell3], and extends the result to include the Foellmer-Schweizer model.

The BSDE technique appears to be tailor made for finance purposes.

**To compute the price of the claim and the optimal hedging strategy we have to**

**(i)solve the FBSDE and compute** $\sigma^{-1} Z_t$

**or equivalently**

**(ii) solve the spde** $(u, k)$ **and compute** $u_x(t, P_t) + \sigma^{-1} k(t, P_t)$

**or**

**(iii) find cases where** $k = 0$ **and then compute** $u_x(t, P_t)$ **with different means.**

*b) the convenience rate case*

To solve the problem of 3.3 (iii) we use a result in Renner [ren] where the relation between first and second adjoint was considered in the framework of stochastic control. From the point of view of finance our setting will be more general than necessary for the convenience rate problem, where only the drift coefficient would depend on a further forward SDE. However we hope to treat more general problems as e.g. a problem of pricing an asset which depends on an index of some kind in the coefficients:

Let the index be described by

$$d\alpha_t = a(t, \alpha_t)dt + c(t, \alpha_t)dw_t, a_0 = \alpha$$

and let all randomness in $g, r, b, c$ come from $(P_t, \alpha_t)$, i.e.

$$dP_t = b(t, P_t, \alpha_t)dt + \sigma(t, P_t, \alpha_t)dw_{t.}$$

In this case a quite lengthy technical computation leads to the result

$$k_s = Z_s - \sigma u_x - \sigma u_\alpha = 0,$$

so that here we find a deterministic price system in the form $u(t, x, \alpha)$. The influence of the index on the portfolio is similar to the influence of the volatility of the stock price.

*c) direct computations*

Finally let us apply some results from the theory of stochastic flows to generate an explicit representation of $(y_t)$. We here assume that we are in a Markovian world, that all coefficients are sufficiently differentiable, and that $r = 0$. The last assumption is made at the beginning to make the results from Baras-Elliott-Kohlmann [bar] applicable without change. The general case then is a simple obvious generalization. We are now going to compute the representation of the martingale

$$y_t = \bar{E}\left[g(P_T(x_0)) \mid F_t\right] = \bar{E}(g(P_T(x_0))) + \int_0^t \gamma d\widetilde{w},$$

where $\widetilde{w}$ is a Brownian motion under the risk neutral measure $\widetilde{P}$.
¿From the Markov property we have

$$\begin{aligned} y_t &= \\ &= \bar{E}\left[g(P_T(x_0)) \mid F_t\right] \\ &= E\left[z_{tT}(x) g(P_{tT}(x)) \mid F_t\right] \\ &= E\left[z_{tT}(x) g(P_{tT}(x))\right] \\ &= u(t, x). \end{aligned}$$

Applying Itô's rule to $u(t, P_{0t}(x))$ under $\tilde{P}$ we find

$$u(t, P_{0t}(x)) = u(0, x_0) + \int_0^t (\tfrac{\partial u}{\partial s} + Lu) ds + \int_0^t \sigma \tfrac{\partial u}{\partial x} d\widetilde{w},$$

where

$$L = (b + \sigma\theta)\tfrac{\partial}{\partial x} + \tfrac{1}{2}\sigma^2 \tfrac{\partial^2}{\partial x^2}.$$

Arguing as in Elliott-Kohlmann [ell1] the integrand $(\gamma_s)$ must be equal to

$$\tfrac{\partial u}{\partial x}(s, P_{0s}(x_0))\sigma(s, P_{0s}(x_0)).$$

Now

$$\tfrac{\partial u}{\partial x} = E\left[\tfrac{\partial z_{tT}(x)}{\partial x} g(P_{0t}(x_0)) + z_{tT}(x)\tfrac{\partial g}{\partial x}(P_{tT}(x))\right],$$

and from Baras-Elliott-Kohlmann [bar]

$$\tfrac{\partial z_{tT}(x)}{\partial x} = z_{tT}(x) \cdot \int_t^T \theta_\xi(P_{tr}(x)) D_{tr}(x) d\widetilde{w}_r$$

where $D_{st}(x) = \frac{\partial}{\partial x} P_{st}(x)$ is the Jacobian of the flow $(P_{st})$. From this we get

$$\frac{\partial u}{\partial x} = E[z_{tT}(x)\{\int_t^T \theta_\xi(P_{tr}(x))P_{tr}(x)d\widetilde{w}_r \bullet g(P_{tT}(x_0)) + g_\xi(P_{tT}(x))D_{tT}(x)\}].$$

It is immediate that then for the general case $r = r(t, P_t) \not\equiv 0$

$$\frac{\partial u}{\partial x} = E[z_{tT}(x)\{\int_t^T (\theta_\xi(r, P_{tr})D_{tr}d\widetilde{w}r + \int_t^T r_\xi(r, P_{tr})D_{tr}dr) \cdot g(P_{0T}(x_0)) + g_\xi(P_{tT}(x))D_{tT}(x)\}]$$

The price of the claim $g(P_T)$ is given by

$$y_t = \bar{E}(g(P_{0T}(x_0)) + \int_0^t \gamma d\widetilde{w}$$

where $\gamma$ is explicitly given by

$$\gamma = \frac{\partial u}{\partial x} \cdot \sigma.$$

As $\frac{\partial u}{\partial s} + Lu = 0$ the following pde holds for $u(t,x)$

$$u_t + \frac{1}{2}\sigma^2 u_{xx} + r u_x - ru = 0$$
$$u(T,x) = g(x),$$

and with this we are back in the classical case.

(i) As in Elliott-Kohlmann [ell1] we can derive a bpde for $\gamma$. This is obvious from the last theorem.

(ii) By repeating the representation in c) over and over we find a chaos decomposition as in [ell2] . From this we can compute the ratios of the claim.

## 4. The American contingent claim

An American option allows to choose the exercise time at any time within the horizon. In order to hedge the risk of early exercise, so-called super-strategies have to be introduced. The price of the option then takes the form of a reflected backward sde as studied in ElKaroui-Pengh-Quenez [elk1].

The reflected backward sde is described by

$$y_t = \xi_T - \int_t^T [ry_s + \theta Z_s]\, ds - \int_t^T Z_s dw_s + \int_t^T dC_t$$

subject to the constraints

(i) $y_t \geq \xi_t$
(ii) $\int_0^T (y_t - \xi_t) dC_t = 0.$

A solution is a triple $(y^{sxC}, Z^{sxC}, C^{sx})$ where $(y, Z)$ has the usual properties and $C$ is an adapted, continuously increasing process with $C_0^{sx} = 0$ , such that (i) and (ii) holds for $\xi_t = g(P_t^{sx})$.

The coefficients here have the same general properties as in the first section.

The price $(y_t)$ is then given by

$$y_t = ess - \sup_{\tau \in [t,T]} E\left[z_{t\tau} \xi_\tau \mid F_t\right], \tau \text{ stopping time. } (*)$$

This is obvious from the following control-theoretical considerations: $(y_t)$ is the first adjoint of a singular control problem. On the other hand from the works of ElKaroui [elk], and Boetius [boe] this first derivative coincides with the value function of a stopping problem, and this is the intuitive meaning of $(*)$.

In [boe] we extended the above mentioned results to general diffusions and established the relation between singular control and optimal stopping for this generalized case. Then it was shown that the singularly influenced process corresponds to a process conctructed from a monotone sequence of stopping times.

Before we go into this problem we state the following variational result for the stochastic pricing system of the American claim.

: Let $u(t, x)$ be a random field which solves the obstacle problem

$$\{(u(t,x) - g(x)) \wedge$$
$$(u(t,x) - g(x) + \int_t^T \tfrac{1}{2}\sigma^2 u_{xx} + r u_x - r u + \sigma k_x - k\theta ds + \int_t^T k dw_s)\}$$
$$= 0$$

with final condition

$$u(T, x) = g(x).$$

This system of variational equalities (in an appropriate space) gives the stochastic price of the American contingent claim

$$u(t, P_t) = y_t$$

From this it is clear that the role of the increasing process $(C_t)$ is to keep $(y_t)$ away from the obstacle (or the forbidden region) $\xi_t$ ($y_t < \xi_t$, respectively). This will allow the seller of the option to fulfill the requirements of the option at any time in $[0, T]$.

Now it is easy to guess, a bit more difficult to prove, but standard, that the optimal stopping time is given by

$$\tau_t = \inf\{T \geqslant s \geq t : y_s = \xi_s\}$$
$$= \inf\{T \geq s \geq t : y_s = g(P_s)\}.$$

Thus $\tau_t$ is the first time of the first move of $(C_t)$, and it is immediate that the American price is the European price with (random) exercise time $\tau_t$.

In [boe] we considered the mathematics behind a problem of installment options and related this to a problem of sequential stopping which turned out to be the limit of a family of impulse control problems. We will apply these results to describe sequential hedging. The idea is easily described: We consider the price of an American claim as decribed above by a RBSDE. At the first exercise time $\tau_t$ the seller offers a new option starting in $(\tau_t, P_{\tau_t})$ and we compute the price of this new claim with these new initials, to get a second stopping time

$$\tau_{\tau_t} = \inf\{T \geq s > \tau_t : y_s^{\tau_t \xi_{\tau_t}} = g(P_s^{\tau_t P_{\tau_t}})\}.$$

The price again coincides with the European price with starting parameters $(\tau_t, P_{\tau_t})$ and exercise time $\tau_{\tau_t}$. In this way we get an increasing sequence of stopping times.

On the other hand let us work backwards to consider an increasing sequence of stopping times $(\tau_j)_{j=1,\dots,n+1}$ and a related obviously increasing family of deterministic states $0 = x_1 \leq x_2 \leq \dots \leq x_n \leq K$ such that

$$\begin{aligned} y_t^{sx(\tau_j x_j)} \quad &= \\ & g(P_T^{sx}) - \int_t^T (r(P_u^{sx})y_u + \theta(P_u^{sx})Z_u)du - \int_t^T Z_s dw_s \\ & +x_n - x_j \end{aligned}$$

for $\tau_j \leq t < \tau_{j+1}$,where the $x_j$ are minimal such that

$$y_t^{sx(\tau_j x_j)} \geq \xi_t \text{ for } \tau_j \leq t < \tau_{j+1}.$$

Define $(\zeta_t^n)$ by

$$\zeta_t^n = x_j \text{ for } \tau_j \leq t < \tau_{j+1},$$

so that $(\zeta_t^n)$ is increasing and right continuous. The resulting impulsed process may thus be identified with a process which at random times $\tau_{j+1}$ jumps to a process with final condition $g(P_t^{sx}) + (x_n - x_j)$ and stays there until $\tau_j$.

By choosing more and more support points it was then proved that finally there is a one-to-one correspondence between an exhaustive family of stopping times $\tau^* = (\tau_y)_{y \in J}$ derived from $\tau_j = \tau_{x_j}$ and an increasing continuous process $(\zeta_t)$ defined as the limit of the $(\zeta_t^n)$ constructed from $(\tau_j)$. $(\zeta_t)$ is independent of the approximating sequences so that we identify

$$(y^{sx\zeta}, Z^{sx\zeta}, \zeta) = (y^{sx\tau^*}, Z^{sx\tau^*}).$$

Now let $(y^{sxC}, Z^{sxC}, C^{sx})$ be a solution of the RBSDE above, then

$$(y^{sxC}, Z^{sxC}, C^{sx}) = (y^{sx\tau^*}, Z^{sx\tau^*}),$$

so that we may summarize:

A self-financing superprice of the American claim is a solution of the RBSDE

$$y_t = \xi_T - \int_t^T [ry_s + \theta Z_s]\, ds - \int_t^T Z_s dw_s + \int_t^T dC_t$$

with obstacle

$$y_t \geq \xi_t = g(P_t^{sx})$$

such that $C_0 = 0$ and

$$\int_0^T (y_u - \xi_u) dC_u = 0.$$

The solution is denoted by

$$(y^{sxC}, Z^{sxC}, C^{sx}).$$

Let the correspondence between $C$ and $\tau^*$ be denoted by $\alpha(C) = \tau^*, \pi(\tau^*) = C$,

then

$$(y^{sx\pi(\tau^*)}, Z^{sx\pi(\tau^*)}, \pi(\tau^*)) = (y^{sx\tau^*}, Z^{sx\tau^*})$$

is a self-financing superprice.

For obvious reasons we call this price the price corresponding to the **rolling hedge**

$$(\sigma^{-1} Z^{sx\alpha(C)}).$$

In this way the superprice is characterized by the limit of European prices with random exercise times.

(i) A by-product of the above construction is the possibility of computing and estimating the prices for sequential hedges with finite states.

(ii) In reality the seller of an option might only want to hedge the claim statistically. This means that he might be interested in not keeping $y_t$ above $\xi_t$

for all times. With a given probability he allows $y_t$ to go beyond $\xi_t$ up to a certain level. The problem may be formulated as

$$y_t = g(P_t) - \int_t^T (ry_s + \theta Z_s) ds - \int_t^T Z_s dw_s + \int_t^T dC_t$$

subject to the constraints

$$P(y_t - \xi_t \geq -s(t)) = \alpha,$$

where $s(t)$ is a function on $[0, T]$ will positive values describing the sellers' attitude to security. This problem is still unsolved.

The result of 4 becomes more transparent when we describe it in terms of a boundary in the state space. Let

$$C_0 := \{(t, x, \omega) \mid x > g(P_t)\}$$

be an open set in the state space with regular boundary $\partial C_0$. Assume the sequence of stopping times $\tau^*(y)$ can be written as $v^*(x_0 - y)$ for a pregiven $x_0$. This amounts to considering a bound on the tracking process $(C_t)$ and has its correspondence in singular control in a finite fuel condition. Then

$$v^*(z) = \sup\{t \mid y_t^{sx(tz)} \leq g(P_t)\} = \sup\{t \mid y_t^{sx(tz)} \notin C_0\}.$$

Here, from the above derivation $y_t^{sx(tz)}$ may be seen as the process which on the time interval $[t, T]$ follows the BSDE with terminal condition $g(P_T) + z$.

Then $y_t^{sx\tau^*}$ solves the BSDE

$$\begin{aligned} y_t^{sx\tau^*} \quad = \quad & \\ & g(P_T^{sx}) - \int_t^T \chi_{C_0} \left[ r y_u^{sx\tau^*} + \theta Z_u^{sx\tau^*} \right] du - \int_t^T \chi_{C_0} Z_u^{sx\tau^*} dw_u \\ & + \int_t^T \chi_{\partial C_0} dC_u^* \end{aligned}$$

where $(C_t^*) = \pi(\tau^*)$. Furthermore from the above discussion we have a local time similar property

$$\textstyle\int_s^T \chi_{C_0} dC_u^* = 0,$$

and $y_t^{sx\tau^*}$ stays within $C_0 \bigcup \partial C_0$. Conditions under which this reasoning holds are found in [boe].

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