On the equivalence of pathwise mild and weak solutions for quasilinear SPDEs

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\textbf{ABSTRACT}

The main goal of this work is to relate weak and pathwise mild solutions for parabolic quasilinear stochastic partial differential equations (SPDEs). Extending in a suitable way techniques from the theory of nonautonomous semilinear SPDEs to the quasilinear case, we prove the equivalence of these two solution concepts.

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\section{1. Introduction}

The aim of this article is to relate two solution concepts for quasilinear SPDEs, namely weak and pathwise mild, with a particular emphasis on cross-diffusion systems. Such systems arise in numerous applications, for example, they can be used to describe the dynamics of interacting population species. A well-known model is the deterministic Shigesada–Kawasaki–Teramoto population system, which was introduced in [1] to analyze population segregation between two species by induced cross-diffusion. This system can be formally derived from a random-walk model on lattices for transition rates which depend linearly on the population densities. Generalized population cross-diffusion models are obtained when the dependence of the transition rates on the densities is nonlinear, see for instance [2].

To model population densities for $n \geq 2$ species, we consider cross-diffusion systems of the form

\begin{equation}
\begin{aligned}
\frac{du}{dt} &= \text{div}(B(u) \nabla u) + \sigma(u) \ dW_t, \\
u(0) &= u_0,
\end{aligned}
\end{equation}

on an open, bounded domain $\mathcal{O} \subset \mathbb{R}^d$ ($d \geq 1$), with smooth boundary $\partial \mathcal{O}$. Here, $B = (B_{ij})$ is an $n \times n$ diffusion matrix, $\sigma$ is a nonlinear term and $W = (W_1, \ldots, W_n)$ is a...
Wiener process on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) endowed with a complete right-continuous filtration \((\mathcal{F}_t)_{t \geq 0}\). The precise assumptions on the coefficients and on the noise will be stated in Section 2 and 3. The previous system rewrites componentwise as

\[
\frac{du_i}{dt} - \text{div} \left( \sum_{j=1}^{n} B_{ij}(u) \nabla u_j \right) dt = \sum_{j=1}^{n} \sigma_{ij}(u) dW^j_t, \quad t > 0,
\]

with \(u_i(0) = u^0_i\) in \(\mathcal{O}, i = 1, \ldots, n\), and is augmented by either no-flux boundary conditions

\[
\sum_{j=1}^{n} B_{ij}(u) \nabla u_j \cdot \nu = 0 \quad \text{on } \partial \mathcal{O}, i = 1, \ldots, n, t > 0,
\]

or homogeneous Dirichlet boundary conditions

\[
u_i(t, x) = 0 \quad \text{on } \partial \mathcal{O}, i = 1, \ldots, n, t > 0.
\]

The solution \(u_i : \Omega \times \mathcal{O} \times [0, T] \to \mathbb{R}\) models the density of the \(i\)-th population species at a current location \(x \in \mathcal{O}\) and a certain time \(t > 0\).

To investigate mild solutions for (1.1), we write it as an abstract quasilinear Cauchy problem

\[
\begin{cases}
\frac{du}{dt} = Au \quad \text{dt} + \sigma(u) \quad dW_t \\
u(0) = u^0,
\end{cases}
\]

where the linear operator \(A_u\) is given by \(A_u \nu := \text{div}(B(u) \nabla \nu)\).

Due to their numerous applications, quasilinear SPDEs have attracted considerable interest (e.g., [3–9]) which has also broadened the scope of available solution concepts such as kinetic [8, 10, 11], entropy [12], martingale [8, 13, 14]. Numerous developments for quasilinear SPDEs have been recently made in the context of rough paths theory [15], paracontrolled calculus [16, 17], or regularity structures [18].

Another solution concept, so-called pathwise mild solution, for semilinear parabolic SPDEs with nonautonomous random generators was introduced by Pronk and Veraar in [19], where they bypassed the issue of nonadapted integrand in the definition of the Itô integral by the use of integration by parts. This solution concept was then extended to the case of quasilinear parabolic SPDEs, including stochastic SKT system in [20], where the existence of a unique local-in-time pathwise mild solution for the equations of the form (1.5) was proved. Furthermore, pathwise mild solutions for hyperbolic SPDEs with additive noise were analyzed in [21]. A stochastic process \(u\) is called a pathwise mild solution of (1.5) if

\[
\begin{align*}
u(t) &= U^u(t, 0) u^0 - \int_0^t U^u(t, s) A_u(s) \int_s^t \sigma(u(\tau)) dW_\tau \ ds + U^u(t, 0) \int_0^t \sigma(u(s)) dW_s, \\
\end{align*}
\]

where \(U^u(\cdot, \cdot)\), is the random evolution family generated by \(A_u\), see Section 2 for further details. This formula can be motivated using integration by parts and overcomes the nonadaptedness of the random evolution family \(U^u(\cdot, \cdot)\). More precisely, as observed in [19], the mapping \(\omega \to U^u(t, s, \omega)\) is only \(\mathcal{F}_t\)-measurable. However, to define the stochastic convolution \(\int_0^t U^u(t, s, \omega) \sigma(u(s)) \ dW_s\) as an Itô integral, the \(\mathcal{F}_s\)-measurability of the
mapping \( \omega \rightarrow U^u(t, z, \omega)\sigma(u(z)) \) is required. To solve this issue, one uses the integration by parts formula to obtain (1.6).

The nonadaptedness of the integrand in the definition of a stochastic integral was first discussed by Alós et al. in [22, 23] using the Skorokhod- and the Russo–Vallois [24] forward integral. Similar to [19], in [23] such problems arise for semilinear SPDEs with random, nonautonomous generators. Furthermore, in [22, 23] it was shown that a Skorokhod-mild solution for such SPDEs does not satisfy the weak formulation, whereas the forward mild (based on the Russo–Vallois integral) does. In [19], the authors showed that the pathwise mild solution is equivalent to the forward mild one.

The equivalence results for weak, pathwise- and forward-mild solutions for semilinear SPDEs obtained in [19] and the existence of pathwise mild solutions for quasilinear SPDEs obtained in [20], raise a natural question regarding the equivalence of these solution concepts in the quasilinear case. Such an aspect is also important to study from a numerical point of view, since numerical schemes are aligned to the solution concept at hand. The same holds true for dynamical systems. In fact, many results regarding dynamics and asymptotic behavior of semilinear SPDEs, rely on a semigroup approach. If we take the SKT system as a motivation, then, there are deterministic results regarding the existence of attractors using weak [25] as well as mild [26] solution concepts.

We emphasize that for semilinear PDEs and SPDEs numerous results regarding the equivalence of weak and mild solutions are well-known, see e.g., [27, 28] and [29] for PDEs and, e.g., [30] and [31] for SPDEs. Conversely, for quasilinear PDEs and SPDEs the literature discussing the equivalence of various solution concepts is very scarce. Therefore, we contribute to this aspect and establish the equivalence between pathwise mild and weak solutions for quasilinear SPDEs of the form (1.2), see Theorems 3.10 and 3.11. Results regarding the existence of strong solutions for quasilinear PDEs are available in [32], respectively, for a subclass of quasilinear SPDEs in [33], where restrictive assumptions on the coefficients are imposed. For certain elliptic-parabolic PDEs, using accretive operators and nonlinear semigroups, assertions regarding the equivalence of weak and mild solutions have been derived in [34, 35] and the references specified therein. For nonlinear degenerate problems, the concept of entropy solution was introduced by Carrillo [36]. In [37], the equivalence between weak and entropy solutions was established for an elliptic-parabolic-hyperbolic degenerate PDE. However, to the best of our knowledge there are no other works in the literature discussing the equivalence of various solution concepts for quasilinear SPDEs such as (1.5).

The outline of our article is as follows. In Section 2, we introduce basic notations and collect results from the theory of evolution families generated by nonautonomous, random sectorial operators. These are necessary to introduce the concept of a pathwise mild solution for (1.5). Section 3 contains our main result, which establishes under suitable assumptions on the coefficients, the equivalence of pathwise mild and weak solutions for (1.5). The main idea is to approach the quasilinear SPDE (1.5) as a semilinear SPDE where the nonlinear map \( A_u \) is viewed as a linear nonautonomous map for a fixed \( u \), which is the pathwise mild solution of (1.5). Thereafter we employ similar tools as in [19, 20] to prove the equivalence of weak and pathwise mild solutions for (1.5). Finally, we provide in Section 4 examples of quasilinear SPDEs, to which the theory developed in this article applies. These include the stochastic SKT system.
2. Preliminaries

Let \( T > 0 \) be arbitrary but fixed and \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})\) be a filtered stochastic basis. Let \( X (\cdot \mid \cdot)_X, Y (\cdot \mid \cdot)_Y, \) and \( Z (\cdot \mid \cdot)_Z \) be separable Hilbert spaces such that the embeddings

\[
Z, \hookrightarrow Y, \hookrightarrow X
\]

are continuous. The choice of these Hilbert spaces depends on the corresponding quasi-linear problem, see Section 4 for concrete examples. We identify \( X \) with its topological dual \( X^* \). Let \( H \) denote another separable Hilbert space with orthonormal basis \( \{\eta_n\}_{n \in \mathbb{N}} \) and \((W_t)_{t \in [0, T]}\) is a cylindrical Wiener process over \( X \), taking values in \( H \). The cylindrical Wiener process \((W_t)_{t \geq 0}\) can be written as the series

\[
W_t = \sum_{n=1}^{\infty} e_n \beta_n(t),
\]

where \( \{\beta_n(\cdot)\}_{n \in \mathbb{N}} \) are mutually independent real valued standard Brownian motions and \( \{e_n\}_{n \in \mathbb{N}} \) denotes an orthonormal basis of \( X \) and the sequence \((2.1)\) converges in \( H \) \( \mathbb{P} \)-a.s. The space of Hilbert–Schmidt operators from \( H \) to \( X \) will be denoted by \( \mathcal{L}_2(\mathcal{H}; X) \) and will be endowed with the norm

\[
|L|_{\mathcal{L}_2(\mathcal{H}; X)}^2 := \sum_{k=1}^{\infty} \|L\eta_k\|_X^2.
\]

We recall some auxiliary results related to the regularity of the stochastic integral with respect to a cylindrical Wiener process.

**Proposition 2.1** ([38, Proposition 4.4]). Let \( p \in [2, \infty), 0 < \alpha < 1/2 \) and \( \sigma \) be a strongly measurable adapted process belonging to \( L^0(\Omega; L^p(0, T; \mathcal{L}_2(\mathcal{H}; X))) \). Then, the stochastic integral

\[
\int_0^T \sigma(r) \, dW_r \in L^0(\Omega; W^{\alpha,p}(0, T; X)).
\]

The continuous embedding \( W^{\alpha,p}(0, T; X), \to C^{\alpha-1/p}(0, T; X) \), for \( 1/p < \alpha < 1/2 \), provides Hölder regularity of the stochastic integral, see [19, Proposition 4.1] for the full generality of the statement.

**Proposition 2.2.** Let \( p \in [2, \infty), 1/p < \alpha < 1/2 \) and \( \sigma \in L^0(\Omega; L^p(0, T; \mathcal{L}_2(\mathcal{H}; X))) \) be a strongly measurable adapted process. Then, there exists a positive constant \( C_T \) independent of \( \sigma \), converging to 0 for \( T \downarrow 0 \), such that

\[
\left\| \int_0^t \sigma(r) \, dW_r \right\|_{L^p(\Omega; C^{\alpha-1/p}(0, T; X))} \leq C_T \|\sigma\|_{L^p(\Omega; L^p(0, T; \mathcal{L}_2(\mathcal{H}; X)))}.
\]

We let \( \mu > 0 \) and introduce the following function space

\[
\mathcal{Z} := L^0(\Omega; L^\infty([0, T]; Z) \cap C^\mu([0, T]; Y)).
\]

Throughout this section and in Section 3, we choose \( u, v \in \mathcal{Z} \) be \((\mathcal{F}_t)_{t \in [0, T]}\)-adapted stochastic processes.
Recalling (1.5), we write
\[ u \mapsto A_v u = \text{div}(B(v) \nabla u). \] (2.2)
Therefore, \( A_v \) is a linear time-dependent random operator. To highlight this dependence we use the notation
\[ A_v(t, \omega) := A_v(t, \omega). \]
To simplify the notation, we drop the parameter \( \omega \) and simply write \( A_v(t) \).

We now collect essential results regarding evolution families. These are extracted from [19]. For further details regarding evolution systems for nonautonomous operators, we refer to the monographs by Pazy [39] and Yagi [40] as well as to [41]. To deal with the time and \( \omega \)-dependence of \( A_v \) we impose, as in [19], that the Acquistapace-Terreni conditions hold for every \( \omega \in \Omega \). These were introduced in [41] for time-dependent generators and involve a sectoriality condition on \( A_v \) together with a suitable H \( \ddot{o} \)lder-regularity. Since we are dealing with nonlinear generators we additionally impose a certain Lipschitz continuity assumption [20, 40].

**Assumption 1** (Assumptions generators). For \( \vartheta \in (\frac{\pi}{2}, \pi) \), let \( \Sigma_{\vartheta} \) be an open sectorial domain, i.e.,
\[ \Sigma_{\vartheta} := \{ \lambda \in \mathbb{C} : |\arg \lambda| < \vartheta \}. \] (2.3)

(A1) \( A_v \) is a sectorial operator on \( X \), i.e., there exists a \( \vartheta \in (\frac{\pi}{2}, \pi) \), such that for every \( (t, \omega) \in [0, T] \times \Omega \),
\[ \Sigma_{\vartheta} \cup \{0\} \subset \rho(A_v(t, \omega)). \]

(A2) The resolvent operator \( (\lambda \text{Id} - A_v)^{-1} \) satisfies the Hile-Yosida condition, i.e., there exists a constant \( M \geq 1 \) such that for every \( (t, \omega) \in [0, T] \times \Omega \),
\[ \| (\lambda \text{Id} - A_v(t, \omega))^{-1} \|_{L(X)} \leq \frac{M}{|\lambda| + 1}, \text{ for } \lambda \in \rho(A_v(t, \omega)). \]

(A3) There exist two exponents \( \nu, \delta \in (0, 1] \) with \( \nu + \delta > 1 \) such that for every \( \omega \in \Omega \) there exists a constant \( L(\omega) \geq 0 \) such that for all \( s, t \in [0, T] \)
\[ \| (A_v(t, \omega))^{\nu} (A_v(t, \omega)^{-1} - A_v(s, \omega)^{-1}) \|_{L(X)} \leq L(\omega)|t - s|^{\delta}. \]

(A4) Let \( 0 < \nu \leq 1 \) be fixed. Then, for every \( \omega \in \Omega \) there exists a constant \( L(\omega) > 0 \) such that
\[ \| (A_u(t, \omega))^{\nu} (A_u(t, \omega)^{-1} - A_v(t, \omega)^{-1}) \|_{L(X)} \leq L(\omega)\|u(t, \omega) - v(t, \omega)\|_Y. \]

The conditions (A1)–(A3) will be referred to as the (AT) conditions.

Since we aim to relate mild and weak solutions for (1.5), we impose similar assumptions on the adjoint \( A^*_v \) defined on \( X^* \) of the operator \( A_v \) with parameters \( \vartheta^*, M^*, \nu^* \), and \( \delta^* \). Recall that \( X \) is a Hilbert space and we identified it with its dual \( X^* \).

**Assumption 2** (Assumptions adjoint operators).
(A1*) $A_v^*$ is a sectorial operator, i.e., there exists a $\theta^* \in (\frac{\pi}{2}, \pi)$, such that for all $(t, \omega) \in [0, T] \times \Omega$,

$$\Sigma_{\theta^*} \cup \{0\} \subset \rho(A_v^*(t, \omega)),$$

where $\Sigma_{\theta^*}$ is defined as in (2.3).

(A2*) There exists a constant $M^* \geq 1$ such that for every $(t, \omega) \in [0, T] \times \Omega$,

$$\| (\lambda \text{Id} - A_v^*(t, \omega))^{-1} \|_{L(X)} \leq \frac{M^*}{|\lambda| + 1}, \text{ for } \lambda \in \rho(A_v^*(t, \omega)).$$

(A3*) There exist two exponents $\nu^*, \delta^* \in (0, 1]$ with $\nu^* + \delta^* > 1$ such that for every $\omega \in \Omega$ there exists a constant $L^*(\omega) \geq 0$ such that for all $s, t \in [0, T]$ such that for all $s, t \in [0, T]$

$$\| (A_v^*(t, \omega))^{\nu^*} (A_v^*(t, \omega)^{-1} - A_v^*(s, \omega)^{-1}) \|_{L(X)} \leq L^*(\omega) |t - s|^{\delta^*}.$$

(A4*) Let $0 < \nu^* \leq 1$ be fixed. Then, for every $\omega \in \Omega$ there exists a constant $L^*(\omega) > 0$ such that

$$\| (A_v^*(t, \omega))^{\nu^*} (A_v^*(t, \omega)^{-1} - A_v^*(s, \omega)^{-1}) \|_{L(X)} \leq L^*(\omega) \| u(t, \omega) - v(t, \omega) \|_Y.$$

We refer the reader to Appendix for details on the fractional power of the operator $A_v$ and its adjoint.

**Remark 2.3.**

1. We assume that the random variables $L(\omega), L^*(\omega)$ are uniformly bounded with respect to $\omega$. This assumption can be dropped by a suitable localization argument, see [19, Section 5.3].

2. The assumption (A1) implies that $-A_v$ is a sectorial operator. Alternatively, one can assume as in [40], that the spectrum of $A_v(t, \omega)$ is contained in an open sectorial domain with angle $0 < \phi < \frac{\pi}{2}$ for every $t \in [0, T]$ and $\omega \in \Omega$, i.e.,

$$\sigma(A_v(t, \omega)) \subset \Sigma_{\phi} := \{ \lambda \in \mathbb{C} : |\arg\lambda| < \phi\},$$

which would imply that $A_v$ is a sectorial operator.

1. Examples of operators satisfying Assumptions 1 and 2 are given in Section 4.

**Assumption 3** (Constant domains). For simplicity we assume that the domains of $A_v$ and $A_v^*$ are constant, i.e., (A3) and (A3*) are satisfied for $\nu = 1$ and $\nu^* = 1$, respectively. These conditions are called in literature the Kato–Tanabe assumptions [19, 41].

**Definition 2.4.** To emphasize the fact that we are working with constant domains, we introduce the following notations for $t \in [0, T]$ and $\omega \in \Omega$:

$$D_v := D(-A_v(t, \omega)), \quad D_v^* := D((-A_v(t, \omega))^2),$$
$$D_v^\nu := D((-A_v(t, \omega)^*)^\nu), \quad D_v^{2\nu} := D((-A_v(t, \omega))^2)^*.$$
In general, conditions (A1) and (A2) can be difficult to verify for a given system, but in the Hilbert space framework, it suffices to apply the following criterion. According to [40, Chapter 2.1], we can associate to $-A_v$ a bilinear form $a(v, w)$ on a separable Hilbert space $V$ which is densely and continuously embedded in $X$. More precisely, we set

$$a(v; w_1, w_2) := \langle -A_v w_1, w_2 \rangle_X, \quad \text{for all } w_1, w_2 \in V. \quad (2.4)$$

In this case, to verify (A1) and (A2) it suffices to show that [40, Chapter 2.1.1]

$$a(v; w_1, w_2) \leq \kappa \|w_1\|^2_V, \quad \forall w \in V, \quad (2.5)$$

$$|a(v; w_1, w_2)| \leq M \|w_1\|_V \|w_2\|_V, \quad \forall w_1, w_2 \in V, \quad (2.6)$$

for some constants $\kappa > 0$ and $M > 0$.

The assumptions (A1)–(A3) allow us to apply pathwise the deterministic results from [41] for the generation of an evolution family for nonautonomous operators and obtain as in [19, Theorem 2.2] the following statement.

**Theorem 2.5.** Let $\Delta := \{(s, t) \in [0, T]^2 : s \leq t\}$. Assume that (A1)–(A3) hold true for the linear nonautonomous operator $A_v(t, \omega)$. Then, there exists a unique map $U^v : \Delta \times \Omega \to \mathcal{L}(X)$ such that

1. **(T1)** for all $t \in [0, T]$, $U^v(t, t) = \text{Id};$
2. **(T2)** for all $r \leq s \leq t$, $U^v(t, s)U^v(s, r) = U^v(t, r);$
3. **(T3)** for every $\omega \in \Omega$, the map $U(v, \cdot, \omega) : \Delta \to \mathcal{L}(X)$ is strongly continuous;
4. **(T4)** there exists a mapping $C : \Omega \to \mathbb{R}_+$, such that for all $s \leq t$, one has

$$\|U^v(t, s)\|_{\mathcal{L}(X)} \leq C;$$

5. **(T5)** for every $s < t$ it holds pointwise in $\Omega$ that

$$\frac{\partial}{\partial t} U^v(t, s) = A_v(t) U^v(t, s). \quad (2.7)$$

Moreover, there exists a mapping $C : \Omega \to \mathbb{R}_+$ such that

$$\|A_v(t) U^v(t, s)\|_{\mathcal{L}(X)} \leq C(t - s)^{-1}.$$

In [19, Proposition 2.4] the following measurability result for the evolution family $U^v$ was established. This fact prevents us from defining the stochastic convolution as an Itô-integral.

**Proposition 2.6.** The evolution system $U^v : \Delta \times \Omega \to \mathcal{L}(X)$ is strongly measurable in the uniform operator topology. Moreover, for each $t \geq s$, the mapping $\omega \mapsto U^v(t, s, \omega) \in \mathcal{L}(X)$ is strongly $\mathcal{F}_t$-measurable in the uniform operator topology.

In the following, we point out spatial- and time-regularity results of the evolution family $U^v$, cf. [19, Lemma 2.6]. For the convenience of the reader, we provide a brief overview on fractional powers of sectorial operators in Appendix.
Lemma 2.7. Let conditions (A1)–(A3) be satisfied by the linear operator $A_v(t, \omega)$. Then, there exists a mapping $C: \Omega \to \mathbb{R}_+$ such that for all $0 \leq s < t \leq T, \theta \in [0,1], \lambda \in (0,1)$ and $\gamma \in [0, \delta)$, the following estimates are valid

$$\| U^v(t, s)(-A_v(s))\gamma x \|_X \leq C \frac{\| x \|_X}{(t-s)^{\gamma}}, \quad x \in \mathcal{D}_v^\theta,$$  

(2.8)

$$\| (-A_v(t))^\theta U^v(t, s)(-A_v(s))^{-\theta} \|_{\mathcal{L}(X)} \leq C.$$  

(2.9)

Moreover, for $(s, t) \in \Delta$, the map

$$(s, t) \mapsto (-A_v(t))^\theta U^v(t, s)(-A_v(s))^{-\theta}$$

is strongly continuous.

The following result, see [19, Lemma 2.7] for the proof, allows one to improve the regularity of the evolution family $U^v$, provided that the adjoint operator $A_v^*$ satisfies the assumptions $(A1^*)–(A3^*)$.

Lemma 2.8. Let $A_v(t, \omega)$ and $A_v^*(t, \omega)$ satisfy (A1)–(A3) and $(A1^*)–(A3^*)$, respectively. Then, for every $t \in (0, T]$, the map $s \mapsto U^v(t, s)$ belongs to $C^1([0, t), \mathcal{L}(X))$, and for every $x \in \mathcal{D}_v$ and $\omega \in \Omega$, it holds that

$$\frac{\partial}{\partial s} U^v(t, s)x = -U^v(t, s)A_v(s)x.$$  

(2.10)

Moreover, for $\beta \in [0,1], 0 < \gamma < \delta^*, 0 \leq \theta < \delta^*, \mu \in (0,1), \lambda \in (0,1)$, the following inequalities hold:

$$\| U^v(t, s)(-A_v(s))^\beta x \|_X \leq C \frac{\| x \|_X}{(t-s)^{\beta}}, \quad x \in \mathcal{D}_v^\beta,$$  

(2.11)

$$\| U^v(t, s)(-A_v(s))^{1+\theta} x \|_X \leq C \frac{\| x \|_X}{(t-s)^{1+\theta}}, \quad x \in \mathcal{D}_v^{1+\theta},$$  

(2.12)

$$\| (-A_v(t))^{-\gamma} U^v(t, s)(-A_v(s))^{1+\gamma} x \|_X \leq C \frac{\| x \|_X}{(t-s)^{1+\gamma-\lambda}}, \quad x \in \mathcal{D}_v^{1+\gamma}.$$  

(2.13)

Remark 2.9. The estimates (2.12) and (2.13) hold true only if the adjoint operator $(A_v(t, \omega))^*$ satisfies $(A1^*)–(A3^*)$.

Remark 2.10. Lemma 2.7 and Lemma 2.8 remain valid if the roles of $A_v(t)$ and $(A_v(t))^*$ are interchanged. This is due to the fact that we identified $X$ with its dual $X^*$.

For the sake of completeness, we point out the following statement on the adjoint of an evolution family $U^v$. Regarding Assumption 2 we conclude by Theorem 2.5 that for every $(t, \omega) \in (0, T] \times \Omega$ the nonautonomous linear operators $(A_v^*(t - \tau, \omega))_{\tau \in [0, t]}$ generate an evolution family $V^v(t; \tau, s)_{0 \leq \tau \leq t}$. Due to [42, Proposition 2.9]

$$(U^v(t, s))^* = V(t; t - s, 0) \quad \text{for } s, t \in \Delta.$$
Based on Lemma 2.8, we motivate a very useful identity resembling the fundamental theorem of calculus. This will be extensively used in Section 3. For further details we refer to [19].

**Lemma 2.11.** Let \((A1)-(A3)\) and \((A1^*)-(A3^*)\) be satisfied by \(A_v(t,\omega)\) and \(A_v^*(t,\omega)\), respectively. For \(\lambda > \delta^*, x^* \in D_v^{\times},\) the following identity holds true for every \(\omega \in \Omega\) and \(x \in X:\)

\[
\int_0^t \langle U_v(t,s)A_v(s)x,x^* \rangle ds = \langle U_v(t,0)x,x^* \rangle - \langle x,x^* \rangle, \tag{2.14}
\]

where \(\langle \cdot, \cdot \rangle\) denotes the inner product in \(X\).

**Sketch of the proof.** Since \(A_v(t,\omega)\) satisfies \((A1)-(A3),\) by Theorem 2.5 there exists an evolution family \(U_v: \Delta \times \Omega \to L(X)\). In particular, (2.7) holds. Assume \(x \in D_v,\) then, integrating (2.7) with respect to time leads to

\[
U_v(t,t)x - U_v(t,0)x = - \int_0^t U_v(t,s)A_v(s)x \ ds. \tag{2.15}
\]

Testing the above identity by \(x^* \in X^*,\) we obtain

\[
\int_0^t \langle U_v(t,s)A_v(s)x,x^* \rangle ds = \langle U_v(t,0)x,x^* \rangle - \langle x,x^* \rangle. \tag{2.16}
\]

Next, we verify that (2.16) holds for \(x^* \in D_v^{\times}\) and \(x \in X.\) This is justified by (2.13), which implies that for every \(s < t,\) and fixed \(v,\) the operator \(A_v^{-\lambda}(t)U_v(t,s)A_v^{1+\gamma}(s),\) for \(\lambda > \delta^*\) and \(\gamma < \delta^*,\) can be uniquely extended to a bounded linear operator on \(X.\) The claim follows regarding that

\[
\int_0^t \langle U_v(t,s)A_v(s)x,x^* \rangle ds
\]

\[
= \int_0^t \langle (-A_v(t))^{-\lambda}U_v(t,s)(-1)^{1+\gamma}(-A_v(s))^{-\gamma}x,((-A_v(t))^{-\lambda}x^*)ds
\]

\[
\leq \int_0^t \frac{||(A_v(s))^{-\gamma}x||_X||((-A_v(t))^{\gamma}x^*)||_X}{(t-s)^{1+\gamma-\lambda}} ds,
\]

where we used (2.13) in the last step. \(\square\)

**3. The main result**

In this section, we introduce two solution concepts: pathwise mild and weak, for the quasilinear SPDE

\[
\begin{cases}
    du = A_vu \ dt + \sigma(u) \ dW_t \\
    u(0) = u^0.
\end{cases} \tag{3.1}
\]

We are interested in showing the equivalence of these solution concepts. As already mentioned, Pronk and Veraar introduced in [19] the concept of pathwise mild solution for semilinear SPDEs with random nonautonomous generators, which was later
extended in [20] to quasilinear problems. The precise definition of the pathwise mild solution is given below.

**Definition 3.1** (Local pathwise mild solution). A local pathwise mild solution for (3.1) is a pair \((u, \tau)\) where \(\tau\) is a strictly positive stopping time and \(\{u(t) : t \in [0, \tau]\}\) is a \(\mathcal{Z}\)-valued \((\mathcal{F}_t)_{t \in [0, \tau]}\)-adapted stochastic process which satisfies for every \(t \in [0, \tau]\), \(\mathbb{P}\)-a.s.

\[
u(t) = \text{U}u(t, 0)u_0 + \text{U}u(t, 0) \int_0^t \sigma(u(s))dW_s - \int_0^t \text{U}u(t, s)A_u(s) \int_s^t \sigma(u(\tau))dW_\tau \ ds,
\]

where \(\text{U}u\) is the evolution family generated by the operator \(A_u\).

**Definition 3.2** (Maximal solution). We call \(\{u(t) : t \in [0, \tau]\}\) a maximal local pathwise mild solution of (3.1) if for any other local pathwise mild solution \(\{\tilde{u}(t) : t \in [0, \tilde{\tau}]\}\) satisfying \(\tilde{\tau} \geq \tau\) a.s. and \(\tilde{u}|_{[0, \tau]}\) is equivalent to \(u\), one has \(\tilde{\tau} = \tau\) a.s. If \(\{u(t) : t \in [0, \tau]\}\) is a maximal local pathwise mild solution of (3.1), then, the stopping time \(\tau\) is called its lifetime.

In [20], the existence of a maximal local pathwise mild solution was established under additional regularity assumptions on the nonlinear term \(\sigma\). These were necessary for the fixed point argument. For the convenience of the reader, we recall these assumptions.

**Assumption 4** (Existence of pathwise mild solution).

(1) Assumption 1 holds.

(2) The mapping \(\sigma: \Omega \times [0, T] \times X \rightarrow \mathcal{L}_2(H, \mathcal{D}^{2\beta}_u)\) is locally Lipschitz continuous, meaning that there exist constants \(L_\sigma = L_\sigma(u, v) > 0, I_\sigma = I_\sigma(u) > 0\) such that

\[
||\sigma(u) - \sigma(v)||_{\mathcal{L}_2(H, \mathcal{D}^{2\beta}_u)} \leq L_\sigma ||u - v||_X, \quad u, v \in Z,
\]

\[
||\sigma(u)||_{\mathcal{L}_2(H, \mathcal{D}^{2\beta}_u)} \leq I_\sigma (1 + ||u||_X), \quad u \in Z.
\]

Under the above assumptions, the following existence result for (3.1) holds true [20, Theorem 3.11].

**Theorem 3.3.** Let Assumption 4 hold true. Then, there exists a unique maximal local pathwise mild solution \(u\) for (3.1) such that \(u \in L^0(\Omega; \mathcal{B}(0, \tau_\infty); \mathcal{D}^{2\beta}_u)) \cap L^0(\Omega; C^0([0,\tau_\infty); \mathcal{D}^{\alpha}_u)), \) where \(\alpha \in (0,1), \beta \in (1/2,1), \beta > \alpha, \) and \(\delta \in (0, \beta - \alpha)\).

We now give the definition of a weak solution of (3.1).

**Definition 3.4.** We call an \((\mathcal{F}_t)_{t \geq 0}\)-adapted \(\mathcal{Z}\)-valued process \(u\) a weak solution of (3.1), if the following identity holds \(\mathbb{P}\)-a.s.

\[
\langle u(t), \varphi(t) \rangle = \langle u^0, \varphi(0) \rangle - \int_0^t \langle a(u(s); u(s), \varphi(s))ds + \int_0^t \langle u(s), \varphi'(s) \rangle ds
\]

\[
- \int_0^t \left( \int_0^s \sigma(u(\tau))dW_\tau, \varphi'(s) \right) ds + \left( \int_0^t \sigma(u(s))dW_\sigma, \varphi(t) \right),
\]

for every \(\varphi \in L^0(\Omega; C^1([0, T]; \mathcal{D}^{\alpha}_u)), \) where \(a(u; \cdot, \cdot) : V \times V \rightarrow \mathbb{R}\) is the bilinear form corresponding to the operator \(A_u\), as introduced in (2.4).
Remark 3.5. For random nonautonomous or quasilinear problems, it is meaningful to consider time- and path-dependent test functions in the weak formulation, since the domains of the generators might depend on $t$ and $\omega$. Even if we assume constant domains, we prefer to work with time-dependent and measurable test functions for a greater generality of our results. Further, we do not make any assumption regarding the adaptedness of these test functions, but only on their spatial and temporal regularity.

Remark 3.6. In comparison to the standard weak formulation of (3.1), the weak formulation (3.3) contains two additional terms

\begin{align*}
L_1 &:= \int_0^t \langle u(s), \varphi'(s) \rangle \, ds, \\
L_2 &:= -\int_0^t \left( \int_0^s \sigma(u(\tau)) \, dW_r, \varphi'(s) \right) \, ds.
\end{align*}

The term $L_1$ appears quite naturally when we test (3.1) with a time-dependent function $\varphi$, apply the product rule, i.e.,

\[ \left\langle \frac{d}{ds} u(s), \varphi(s) \right\rangle = \frac{d}{ds} \langle u(s), \varphi(s) \rangle - \langle u(s), \varphi'(s) \rangle \]

and integrate over time. The term $L_2$ can be again justified by the product rule, more precisely

\[ \langle \sigma(u(s)) \, dW_s, \varphi(s) \rangle = \frac{d}{ds} \left\langle \int_0^s \sigma(u(r)) \, dW_r, \varphi(s) \right\rangle - \left\langle \int_0^s \sigma(u(r)) \, dW_r, \varphi'(s) \right\rangle. \]

Integrating this relation with respect to the time variable, and interchanging the stochastic integral and the inner product gives the representation in Definition 3.4.

By Definition 3.4, the weak formulation of Equation (1.1) is given by

\[ \langle u(t), \varphi(t) \rangle = \langle u^0, \varphi(0) \rangle - \int_0^t \langle \nabla u(s), B^T(u) \nabla \varphi(s) \rangle \, ds + \int_0^t \langle u(s), \varphi'(s) \rangle \, ds \]

\[ - \int_0^t \left\langle \int_0^s \sigma(u(\tau)) \, dW_r, \varphi'(s) \right\rangle \, ds + \left\langle \int_0^t \sigma(u(s)) \, dW_s, \varphi(t) \right\rangle. \]

\[ (3.4) \]

The key idea. We describe the intuition behind the approach we use to prove the equivalence of pathwise mild and weak solutions for (3.1). Such an idea is standard in the context of quasilinear problems. Similar to the proof of existence of solutions [20, 40], we firstly work with nonautonomous equations with random coefficients. More precisely, instead of treating (3.1) we consider the linear equation

\[ \begin{cases} 
\frac{du}{dt} = A_t u + \sigma(u) \, dW_t \\
u(0) = u^0,
\end{cases} \]

\[ (3.5) \]

where $A_t = \text{div}(B(v) \nabla u)$ as specified in (2.2). This is a linear nonautonomous Cauchy problem, which due to [19], has a unique pathwise mild solution.
For Definition 3.7, one can show by means of fixed point arguments as in [20], the weak solution (3.7) is equivalent with the pathwise mild solution (3.6). Returning to the (AT) conditions for every test function \( \varphi \in L^0(\Omega; C^1([0, T]; D_u^*)) \). According to [19, Section 4.4], the weak solution (3.7) is equivalent with the pathwise mild solution (3.6). Returning to the quasilinear problem (3.1), one can show by means of fixed point arguments as in [20], that under Assumption 4

\[
\phi(t) = U^u(t, 0)u^0 - \int_0^t U^u(t, s)A_u(s) \int_s^t \sigma(u(\tau))dW_\tau \, ds + U^u(t, 0) \int_0^t \sigma(u(s))dW_s,
\]

is the pathwise mild solution of (3.1). In conclusion, we approach the quasilinear SPDE (3.1) as a semilinear SPDE where the nonlinear map \( A_u \) is viewed as a linear nonautonomous map for a fixed \( u \), which is the pathwise mild solution of the quasilinear SPDE (3.1). In this case, the tools developed in [19, Section 4.4] can be employed to establish the equivalence of weak (3.3) and pathwise mild (3.2) solutions in the quasilinear case.

To prove our main result, we make the following assumptions.

**Assumption 5.**
1. For simplicity, we assume that the initial condition \( u^0 = 0 \).
2. Assumptions 1–3 are satisfied.
3. The diffusion coefficient \( \sigma : \Omega \times [0, T] \times X \rightarrow L_2(H, X) \) is a strongly measurable adapted process. Furthermore \( \sigma(u) \in L^0(\Omega, L^p(0, T; L_2(H, X))) \) for \( p \in (2, \infty) \).

We further introduce an appropriate space that incorporates the time and space regularity of the test functions similar to [19].

**Definition 3.7.** For \( t \in [0, T] \) and \( \beta \geq 0 \), we let \( \Gamma_t^{u, \beta} \) be the subspace of all test functions \( \varphi \in L^0(\Omega; C^1([0, t]; X^\beta)) \), such that

1. for all \( s \in [0, t] \) and \( \omega \in \Omega \), we have \( \varphi(s) \in D^{(\beta+1)s}_u \) and \( \varphi'(s) \in D^\beta_u \).
2. the process \( s \rightarrow A_u^\beta(s) \varphi(s) \) belongs to \( L^0(\Omega; C([0, t]; X^\beta)) \).
3. there is a mapping \( C : \Omega \rightarrow \mathbb{R}_+ \) and \( \varepsilon > 0 \) such that for all \( s \in [0, t] \)

\[
\left\| \left((-A_u(s))^{1+\beta}\right)^{x^\beta} \varphi(s) \right\|_{X^\beta} + \left\| \left((-A_u(s))^\beta\right)^{x^\beta} \varphi'(s) \right\|_{X^\beta} \leq C(t-s)^{-1+\varepsilon}.
\]

In the following, we use test functions of the form \( \varphi(s) = U(t, s)^{x^\beta} \) for \( x^\beta \in D^\beta_u \). Thus, we need to show that such a \( \varphi \in \Gamma_t^{u, \beta} \) for some \( \beta > 0 \). This is established in the next lemma. We recall that the parameters \( \delta, \delta^* \) respectively, \( \delta^* \) stand for the Hölder exponents in the (AT) conditions for \( A_u \) and \( A_u^\beta \) as specified in (A3) and (A3\(^*\)), respectively.
Lemma 3.8. Let \( x^* \in D^*_u \) and \( t \in [0, T] \). For \( \beta \in [0, \delta^* \rangle \), the process \( \varphi : [0, t] \times \Omega \to X^* \) defined as \( \varphi(s) := U^u(t.s)^* x^* \) belongs to \( \Gamma_{t, \beta}^u \).

**Proof.** To show that \( \varphi(s) \in \Gamma_{t, \beta}^u \), we verify the three conditions of Definition 3.7. For (1), using the definition of the norm on \( D_u^{[\beta+1]} \), we have by (2.13) and Remark 2.10

\[
||\varphi(s)||_{D_u^{[\beta+1]}} = ||\left( (-A_u(s))^{1+\beta} \right)^* (U^u(t.s)) x^* ||_{X}.
\]

\[
= ||\left( (-A_u(s))^{\beta+1} \right)^* (U^u(t.s)) ((-A_u(t))^{-1})^* ((-A_u(t))^{1})^* x^* ||_{X}.
\]

\[
\leq ||\left( (-A_u(s))^{\beta+1} \right)^* (U^u(t.s)) ((-A_u(t))^{-1})^* ||_{L(X)} ||((-A_u(t))^{1})^* x^* ||_{X}.
\]

\[
\leq \frac{C}{(t-s)^{1+\beta-\lambda}} ||((-A_u(t))^{1})^* x^* ||_{X} < \infty,
\]

where \( \lambda \in (\beta, \delta^*) \).

Next, we verify condition (2). Using [42, Proposition 2.9], we obtain

\[
\varphi'(s) = \frac{d}{ds} \varphi(s) = (-A_u(s))^* (U^u(t.s))^* x^*.
\]

By Lemma 2.7, we immediately see that \( \varphi \) is continuously differentiable for \( s < t \). For the case \( s = t \), we refer to [41, Theorem 6.5]. Now from the above identity, we have

\[
||\varphi'(s)||_{D_u^{[\beta]}} = ||\left( (-A_u(s))^\beta \right)^* \varphi'(s) ||_{X}.
\]

\[
= ||\left( (-A_u(s))^{1+\beta} \right)^* (U^u(t.s))^* x^* ||_{X} = ||\varphi(s)||_{D_u^{[\beta+1]}} < \infty,
\]

using (1). Condition (3) of Definition 3.7 immediately follows from the previous two estimates, choosing \( \varepsilon := \lambda - \beta \). \( \square \)

Due to Assumption 5 and the regularity of the stochastic integral (recall Propositions 2.1 and 2.2), the terms appearing in (3.3) are well-defined for \( \varphi(s) = U(t.s)^* x^* \in \Gamma_{t, \beta}^u \), where \( x^* \in D^*_u \).

**Lemma 3.9.** Let Assumption 5 hold and let \( \beta \in [0, \delta^* \rangle \). Then, the following mappings

\[
w \to \int_0^t \langle a(u; u, \varphi(s)) \rangle \ ds,
\]

\[
w \to \int_0^t \left( \int_0^s \sigma(u(\tau)) dW_\tau, \varphi'(s) \right) \ ds,
\]

\[
w \to \int_0^t \langle U^u(t.s) A_u(s) \int_0^s \sigma(u(\tau)) dW_\tau, x^* \rangle \ ds
\]

are well-defined from \( Z \) to \( L^0(\Omega; \mathbb{R}) \).

**Proof.** For the first term, we obviously have

\[
|a(u; u, \varphi(t))| = ||(-A_u u, \varphi(t))| = ||u, (-A_u)^* \varphi(t)|| \leq C ||u||_X ||(-A_u)^* \varphi||_X.
\]
For the second integral, we obtain
\[
\int_0^t \left\| \int_0^s \sigma(u(\tau))dW_\tau, \varphi'(s) \right\| ds = \int_0^t \left\| (A_u)^{-\beta} \int_0^s \sigma(u(\tau))dW_\tau, ((A_u)^{-\beta})^* \varphi'(s) \right\| ds
\]
\[
\leq \int_0^t \left\| (A_u)^{-\beta} \int_0^s \sigma(u(\tau))dW_\tau \right\| \|((A_u)^{-\beta})^* \varphi'(s)\| ds
\]
\[
\leq C \sup_{s \in [0,t]} \|((A_u)^{-\beta})^* \varphi'(s)\| ds,
\]
where we use property (3) of Definition 3.7.

For the third term, setting \( \varphi(s) := U^u(t,s)x^* \) for \( x^* \in D_u^x \), we have
\[
\int_0^t \langle U^u(s,r)A_u(r)\int_r^s \sigma(u(\tau))dW_\tau, x^* \rangle \ dr
\]
\[
= \int_0^t \left\| (A_u(s))^{-1} U^u(s,r)(A_u(r))(A_u(r))^{-\beta} \int_0^s \sigma(u(\tau))dW_\tau, (A_u(s))^{-\beta} x^* \right\| ds
\]
\[
= \int_0^t \left\| (A_u(r))^{-\beta} \int_0^r \sigma(u(\tau))dW_\tau, ((A_u(r))^{-\beta})^* U^u(s,r)^* ((A_u(s))^{-\beta}) x^* \right\| ds
\]
\[
\leq \int_0^t \left\| (A_u(r))^{-\beta} \int_0^r \sigma(u(\tau))dW_\tau \right\| \|((A_u(r))^{-\beta})^* U^u(s,r)^* ((A_u(s))^{-\beta}) x^* \| ds
\]
\[
\leq C_u \int_0^t (s-r)^{-1-\beta} \left\| ((A_u(s))^{-\beta}) x^* \right\| ds \leq C_u s^{-\beta} \left\| ((A_u(s))^{-\beta}) x^* \right\| ds,
\]
where we used (2.13) for \( A_u^*(\cdot) \). The last term is bounded for \( \lambda \in (\beta, \delta^*) \).

Collecting all the previous deliberations, we now state the main result of this article.

**Theorem 3.10.** Let Assumption 5 be satisfied and let \( \beta \in (0, \delta^*) \). Then, the following assertions are valid.

1. If there exists a pathwise mild solution \( u \) for (3.1) on the interval \([0,T]\). Then, \( u \) satisfies (3.3) for all \( \varphi \in \Gamma_{t,\beta, t \in [0,T]} \), a.s.
2. If there exists a weak solution \( u \) for (3.1) satisfying (3.3) for all \( \varphi \in \Gamma_{t,\beta, t \in [0,T]} \), a.s. Then, \( u \) satisfies (3.2) for \( t \in [0,T] \), a.s.

**Proof.** 1. We start by showing that a pathwise mild solution of (3.1) is also a weak solution. Assume that (3.2) holds and fix \( t \in [0,T] \). Let \( \lambda \in (\beta, \delta^*) \) and \( u \) be the mild solution of (3.1) with zero initial condition, i.e.,

\[
u(t) = - \int_0^t U^u(t,r)A_u(r) \int_r^s \sigma(u(\tau))dW_\tau, dr + U^u(t,0) \int_0^t \sigma(u(\tau))dW_\tau.
\]

Applying \( x^* \in D_u^{\lambda^*} \) to (3.8) and using that \( \int_0^t \sigma(u(\tau))dW_\tau = \int_0^t \sigma(u(\tau))dW_\tau - \int_0^t \sigma(u(\tau))dW_\tau \) to obtain
\[
\int_0^t \left\langle U^u(t, r) A_u(r) \int_0^t \sigma(u(\tau)) dW_\tau, x^s \right\rangle \, dr = \int_0^t \left\langle U^u(t, r) A_u(r) \int_0^t \sigma(u(\tau)) dW_\tau, x^s \right\rangle \, dr
\]

further leads to

\[
\langle u(t), x^s \rangle = -\int_0^t \left\langle U^u(t, r) A_u(r) \int_0^t \sigma(u(\tau)) dW_\tau, x^s \right\rangle \, dr + \left\langle U^u(t, 0) \int_0^t \sigma(u(\tau)) dW_\tau, x^s \right\rangle + \int_0^t \left\langle U^u(t, r) A_u(r) \int_0^t \sigma(u(\tau)) dW_\tau, x^s \right\rangle \, dr.
\]  

(3.9)

We use the identity (2.14) for \( x = \int_0^t \sigma(u(\tau)) dW_\tau \) and rewrite the first term on the right-hand side as

\[
\int_0^t \left\langle U^u(t, r) A_u(r) \int_0^t \sigma(u(\tau)) dW_\tau, x^s \right\rangle \, dr
\]

\[
= \left\langle U^u(t, 0) \int_0^t \sigma(u(\tau)) dW_\tau, x^s \right\rangle - \left\langle \int_0^t \sigma(u(\tau)) dW_\tau, x^s \right\rangle.
\]  

(3.10)

Using (3.10) in (3.9), we obtain

\[
\langle u(t), x^s \rangle = \int_0^t \left\langle U^u(t, r) A_u(r) \int_0^t \sigma(u(\tau)) dW_\tau, x^s \right\rangle \, dr + \left\langle \int_0^t \sigma(u(\tau)) dW_\tau, x^s \right\rangle.
\]  

(3.11)

To show the equivalence with the weak solution, we need to use test functions \( \varphi \in \Gamma_{t, \beta}^u \). To this aim, we choose \( x^s = (-A_u(s))^s \varphi(s) \) in (3.11) and integrate over time to obtain

\[
\int_0^t \langle u(s), (-A_u(s))^s \varphi(s) \rangle \, ds - \int_0^t \left\langle \int_0^t \sigma(u(\tau)) dW_\tau, (-A_u(s))^s \varphi(s) \right\rangle \, ds
\]

\[
= \int_0^t \int_0^t \left\langle U^u(s, r) A_u(r) \int_0^t \sigma(u(\tau)) dW_\tau, (-A_u(s))^s \varphi(s) \right\rangle \, dr \, ds
\]

\[
= \int_0^t \int_0^t \left\langle U^u(s, r) A_u(r) \int_0^t \sigma(u(\tau)) dW_\tau, (-A_u(s))^s \varphi(s) \right\rangle \, ds \, dr,
\]  

(3.12)

where we used Fubini’s theorem in the last step. Using (2.7), we obtain for all \( x \in X \) and \( 0 \leq r \leq t \leq T \),
\[
\int_r^t \left\langle U^u(s, r)A_u(r)x, (-A_u(s))^* \varphi(s) \right\rangle ds \\
\quad = - \int_r^t \left\langle A_u(s)U^u(s, r)A_u(r)x, \varphi(s) \right\rangle ds \\
\quad = - \int_r^t \left\langle \frac{d}{ds} U^u(s, r)A_u(r)x, \varphi(s) \right\rangle ds \\
\quad = - \int_r^t \frac{d}{ds} \left\langle U^u(s, r)A_u(r)x, \varphi(s) \right\rangle ds + \int_r^t \left\langle U^u(s, r)A_u(r)x, \varphi'(s) \right\rangle ds \\
\quad = - \left\langle U^u(t, r)A_u(r)x, \varphi(t) \right\rangle + \left\langle U^u(r, r)A_u(r)x, \varphi(r) \right\rangle + \int_r^t \left\langle U^u(s, r)A_u(r)x, \varphi'(s) \right\rangle ds.
\]

Furthermore
\[
\left\langle U^u(t, r)A_u(r)x, \varphi(t) \right\rangle + \left\langle x, (-A_u(r))^* \varphi(r) \right\rangle \\
\quad = \int_r^t \left\langle U^u(s, r)A_u(r)x; (-A_u(s))^* \varphi(s) \right\rangle ds + \int_r^t \left\langle U^u(s, r)A_u(r)x, \varphi'(s) \right\rangle ds. \tag{3.13}
\]

Note that the expressions above are well-defined. Indeed, for \( \varphi \in \Gamma_{t, \beta}^u \) and \( x \in \mathcal{X} \), choosing \( 1 > \lambda > \theta > 0 \) and applying (2.13), we infer that
\[
\| \left\langle U^u(s, r)A_u(r)x, (-A_u(s))^* \varphi(s) \right\rangle \| \leq \left\langle \left( -A_u(s) \right)^{-\lambda} U^u(s, r)A_u(r)x, \left( -A_u(s) \right)^{1+\lambda} \right\rangle \varphi(s) \| \times \| x \| \| \left( -A_u(s) \right)^{1+\lambda} \varphi(s) \| ,
\]
\[
\leq C(s - r)^{-1+\lambda-\theta}(t - s)^{-1+\theta} \| x \| \| \left( -A_u(s) \right)^{1+\lambda} \varphi(s) \| .
\]

Setting \( x = \int_0^r \sigma(u(\tau))dW_\tau \) in (3.13) further leads to
\[
\left\langle U^u(t, r)A_u(r) \int_0^r \sigma(u(\tau))dW_\tau, \varphi(t) \right\rangle + \left\langle \int_0^r \sigma(u(\tau))dW_\tau, (-A_u(r))^* \varphi(r) \right\rangle \\
\quad = - \int_r^t \left\langle U^u(s, r)A_u(r) \int_0^r \sigma(u(\tau))dW_\tau, (-A_u(s))^* \varphi(s) \right\rangle ds \\
\quad + \int_r^t \left\langle U^u(s, r)A_u(r) \int_0^r \sigma(u(\tau))dW_\tau, \varphi'(s) \right\rangle ds. \tag{3.14}
\]

Next, we use (3.14) to deal with the right-hand side of (3.12). From (3.12), we get
\[
\int_0^t \left\langle u(s), (-A_u(s))^* \varphi(s) \right\rangle ds = \int_0^t \int_r^t \left\langle U^u(s, r)A_u(r) \int_0^r \sigma(u(\tau))dW_\tau, (-A_u(s))^* \varphi(s) \right\rangle ds \ dr \\
\quad + \int_0^t \left\langle \int_0^r \sigma(u(\tau))dW_\tau, (-A_u(s))^* \varphi(s) \right\rangle ds. \tag{3.15}
\]
Thus, using (3.14) in (3.15) results in
\[
\int_0^t \langle u(s), (-A_\tau(s))^\ast \varphi(s) \rangle ds = -\int_0^t \left( \int_0^r U^\ast(t, r)A_\tau(r) \int_0^\tau \sigma(u(\tau))dW_\tau, \varphi(t) \right) dr
- \int_0^t \left( \int_0^r \sigma(u(\tau))dW_\tau, (-A_\tau(r))^\ast \varphi(r) \right) dr
+ \int_0^t \left( \int_0^r U^\ast(s, r)A_\tau(r) \int_0^\tau \sigma(u(\tau))dW_\tau, \varphi'(s) \right) ds dr
+ \int_0^t \left( \int_0^\tau \sigma(u(\tau))dW_\tau, (-A_\tau(s))^\ast \varphi(s) \right) ds.
\]

The above expression on simplification results in
\[
\int_0^t \langle u(s), (-A_\tau(s))^\ast \varphi(s) \rangle ds = -\int_0^t \left( \int_0^r U^\ast(t, r)A_\tau(r) \int_0^\tau \sigma(u(\tau))dW_\tau, \varphi(t) \right) dr
+ \int_0^t \left( \int_0^r U^\ast(s, r)A_\tau(r) \int_0^\tau \sigma(u(\tau))dW_\tau, \varphi'(s) \right) ds dr.
\]  
(3.16)

Moreover, choosing \( x^\ast = \varphi(t) \) in (3.11), we get
\[
\langle u(t), \varphi(t) \rangle = \int_0^t \left( \int_0^r U^\ast(t, r)A_\tau(r) \int_0^\tau \sigma(u(\tau))dW_\tau, \varphi(t) \right) dr
+ \left( \int_0^t \int_0^\tau \sigma(u(\tau))dW_\tau, \varphi(t) \right).
\]  
(3.17)

\[
\int_0^t \langle u(s), (-A_\tau(s))^\ast \varphi(s) \rangle ds
= -\langle u(t), \varphi(t) \rangle + \left( \int_0^t \int_0^\tau \sigma(u(\tau))dW_\tau, \varphi(t) \right) + \left( \int_0^t \int_0^r U^\ast(s, r)A_\tau(r) \int_0^\tau \sigma(u(\tau))dW_\tau, \varphi'(s) \right) ds dr.
\]  
(3.18)

Using (3.17) and (3.16), we obtain

Further, choosing \( x^\ast = \varphi'(t) \) in (3.11), we can express the integrand of the last term in the right-hand side of (3.18) as
\[
\int_0^t \langle U^\ast(t, r)A_\tau(r) \int_0^\tau \sigma(u(\tau))dW_\tau, \varphi'(t) \rangle dr = \langle u(t), \varphi'(t) \rangle - \left( \int_0^t \int_0^\tau \sigma(u(\tau))dW_\tau, \varphi'(t) \right).
\]  
(3.19)

Using Fubini’s theorem and plugging in the relation (3.19) in (3.18), we infer that
\[
\int_0^t \langle u(s), (-A_\tau(s))^\ast \varphi(s) \rangle ds = -\langle u(t), \varphi(t) \rangle + \left( \int_0^t \int_0^\tau \sigma(u(\tau))dW_\tau, \varphi(t) \right)
+ \int_0^t \langle u(s), \varphi'(s) \rangle ds - \int_0^t \left( \int_0^\tau \sigma(u(\tau))dW_\tau, \varphi'(s) \right) ds.
\]

From the previous expression, we conclude that \( u \) satisfies (3.3) and is, therefore, a weak solution of (3.1).
In particular, by Theorem 2.5 there exists an evolution family \( U^u : \Delta \times \Omega \rightarrow \mathcal{L}(X) \) generated by the operator \( A_u \). Using the relation between \( a(u, \cdot, \cdot) \) and \( A_u \), recall (2.4), the previous expression rewrites as

\[
\langle u(t), \varphi(t) \rangle = -\int_0^t \langle u(s), (-A_u(s))^\ast \varphi(s) \rangle ds + \int_0^t \langle u(s), \varphi'(s) \rangle ds
- \int_0^t \left( \int_0^s \sigma(u(\tau)) \, dW_\tau, \varphi'(s) \right) ds + \left( \int_0^t \sigma(u(\tau)) \, dW_\tau, \varphi(t) \right).
\]

From Lemma 3.8, for a fixed \( t \in [0, T] \) and \( x^\ast \in \mathcal{D}_u^\ast \), \( \varphi(s) = U^u(t,s)^\ast x^\ast \) belongs to \( \Gamma_{t, \beta}^u \). Now, with this choice of test functions and using that \( \varphi'(s) = (-A_u(s))^\ast \varphi(s) \), we obtain from (3.20) that

\[
\langle u(t), U^u(t,t)^\ast x^\ast \rangle = -\int_0^t \langle u(s), (-A_u(s))^\ast U^u(t,s)^\ast x^\ast \rangle ds
+ \int_0^t \langle u(s), (-A_u(s))^\ast \varphi(s) \rangle ds - \int_0^t \left( \int_0^s \sigma(u(\tau)) \, dW_\tau, (-A_u(s))^\ast \varphi(s) \right) ds
+ \left( \int_0^t \sigma(u(\tau)) \, dW_\tau, U^u(t,t)^\ast x^\ast \right),
\]

which simplifies further to

\[
\langle u(t), x^\ast \rangle = -\int_0^t \langle u(s), (-A_u(s))^\ast U^u(t,s)^\ast x^\ast \rangle ds
+ \int_0^t \langle u(s), (-A_u(s))^\ast U^u(t,s)^\ast x^\ast \rangle ds + \left( \int_0^t \sigma(u(\tau)) \, dW_\tau, x^\ast \right)
- \int_0^t \left( \int_0^s \sigma(u(\tau)) \, dW_\tau, (-A_u(s))^\ast U^u(t,s)^\ast x^\ast \right) ds
= \left( \int_0^t \sigma(u(\tau)) \, dW_\tau, x^\ast \right) - \int_0^t \left( U^u(t,s)(-A_u(s)) \int_0^s \sigma(u(\tau)) \, dW_\tau, x^\ast \right) ds.
\]

All in all, we obtained that

\[
\langle u(t), x^\ast \rangle = \int_0^t \left( U^u(t,s)A_u(s) \int_0^s \sigma(u(\tau)) \, dW_\tau, x^\ast \right) ds + \left( \int_0^t \sigma(u(\tau)) \, dW_\tau, x^\ast \right).
\]

Splitting the stochastic integral in the first term on the right-hand side, into two stochastic integrals results in
\[ \langle u(t), x^* \rangle = \int_0^t \left\langle U^u(t, s)A_u(s) \left( \int_0^t \sigma(u(\tau))dW_\tau - \int_s^t \sigma(u(\tau))dW_\tau \right), x^* \right\rangle ds \]

\[ + \left\langle \int_0^t \sigma(u(\tau))dW_\tau, x^* \right\rangle \]

\[ = \int_0^t \left\langle U^u(t, s)A_u(s) \int_0^t \sigma(u(\tau))dW_\tau, x^* \right\rangle ds + \left\langle \int_0^t \sigma(u(\tau))dW_\tau, x^* \right\rangle \]

\[ - \int_0^t \left\langle U^u(t, s)A_u(s) \int_s^t \sigma(u(\tau))dW_\tau, x^* \right\rangle ds. \] (3.21)

Using the identity (2.14) with \( x = \int_0^t \sigma(u(\tau))dW_\tau \), the first term on the right-hand side of (3.21) can be written as

\[ \int_0^t \left\langle U^u(t, s)A_u(s) \int_0^t \sigma(u(\tau))dW_\tau, x^* \right\rangle ds \]

\[ = \left\langle U^u(t, 0) \int_0^t \sigma(u(\tau))dW_\tau, x^* \right\rangle - \left\langle \int_0^t \sigma(u(\tau))dW_\tau, x^* \right\rangle. \]

Using the above identity in (3.21) entails

\[ \langle u(t), x^* \rangle = \left\langle U^u(t, 0) \int_0^t \sigma(u(\tau))dW_\tau, x^* \right\rangle - \left\langle \int_0^t \sigma(u(\tau))dW_\tau, x^* \right\rangle ds. \]

In the previous deliberations \( x^* \in \mathcal{D}_u^v \). Since \( \mathcal{D}_u^v \) is dense in \( X \), the result can be extended to every \( x^* \in X \) by the Hahn–Banach theorem. This justifies that the weak solution of (3.1) satisfies the mild formulation (3.2).

For time-independent test functions, we recover from the previous deliberations the equivalence of the pathwise mild solution with the following standard weak formulation

\[ \langle u(t), x^* \rangle = - \int_0^t a(u(s); u(s), x^*) ds + \left\langle \int_0^t \sigma(u(s))dW_\tau, x^* \right\rangle, \] (3.22)

for \( x^* \in \mathcal{D}_u^v \).

**Theorem 3.11.** Let Assumption 5 be satisfied. Then, the following assertions are valid.

1. If there exists a pathwise mild solution \( u \) for (3.1) on the interval \([0, T]\). Then, \( u \) satisfies (3.22) for all \( x^* \in \mathcal{D}_u^v, t \in [0, T], \mathbb{P}\text{-a.s.} \)

2. If there exists a weak solution \( u \) for (3.1) satisfying (3.22) for all \( x^* \in \mathcal{D}_u^v, t \in [0, T], \mathbb{P}\text{-a.s.} \) Then, \( u \) satisfies (3.2) for \( t \in [0, T], \mathbb{P}\text{-a.s.} \)

**Proof.** 1. Let \( t \in [0, T] \). We consider only the case \( \sigma(u(t)) \in \mathcal{D}_u \) such that the process \( t \rightarrow A_u(t)\sigma(u(t)) \) is adapted and belongs to \( L^0(\Omega; L^p(0, T; L_2(H, X))) \).

For \( x^* \in \mathcal{D}_u^v \), we simply set \( \varphi(t) := x^* \). However, such a test function does not belong to \( \Gamma_{u, \beta}^v \) since \( \varphi \notin \mathcal{D}_u^{1+\beta} \). Although, the additional spatial regularity of \( \sigma \) enables us to perform the same proof as before. From (3.12), we obtain for the test function \( x^* \) instead of \( A_u(s)\varphi(s) \)
\[
\int_0^t \langle u(s), x^* \rangle ds - \int_0^t \left( \int_0^s \sigma(u(\tau)) dW_\tau, x^* \right) ds
= \int_0^t \int_r^s \langle U^u(s, r) A_u(r), \int_r^s \sigma(u(\tau)) dW_\tau, x^* \rangle ds dr.
\]

Testing with \((-A_u(s))^* x^*\) instead of \((-A_u(s))^* \varphi(s)\) entails
\[
\int_r^t \langle U^u(s, r) A_u(r) x, A_u(s)^* x^* \rangle ds = \int_r^t \frac{d}{ds} \langle U^u(s, r) A_u(r) x, x^* \rangle ds
= \langle U^u(t, r) A_u(r) x, x^* \rangle - \langle A_u(r) x, x^* \rangle.
\]

Analogously to the proof of Theorem 3.10, the result now follows for \(\sigma(\cdot) \in D_u\). The general case follows from a suitable approximation argument for \(\sigma(\cdot)\) as in [19, Theorem 4.9].

2. We show now that a solution satisfying (3.22), also verifies (3.2). To this aim, we fix \(t \in [0, T]\), and let \(f \in C^1([0, t], X^*) \in D_u^R\). Using the density of \(D_u^R\) in \(X\), it suffices to consider test functions of the form \(\varphi(t) = f(t) \otimes x^*\). Using such test functions in (3.22), we infer that
\[
\langle u(t), \varphi(t) \rangle - \left\langle \int_0^t \sigma(u(s)) dW_s, \varphi(t) \right\rangle
= - \int_0^t \langle u(s), (-A_u(s))^* x^* \rangle ds d\varphi(t).
\]

The integration by parts formula results in
\[
\int_0^t \int_0^s \langle u(r), (-A_u(r))^* x^* \rangle dr d\varphi(t)
= \int_0^t \left( \int_0^s \langle u(s), (-A_u(s))^* x^* \rangle ds - \int_0^t \langle u(s), (-A_u(s))^* \varphi(s) \rangle ds \right) d\varphi(t) - \int_0^t \langle u(s), (-A_u(s))^* \varphi(s) \rangle ds.
\]

The above identity and (3.22) further entail that
\[
\langle u(t), \varphi(t) \rangle - \left\langle \int_0^t \sigma(u(s)) dW_s, \varphi(t) \right\rangle
= - \int_0^t \langle u(s), (-A_u(s))^* \varphi(s) \rangle ds + \int_0^t \langle u(s), x^* \rangle f'(s) ds - \int_0^t \left( \int_0^s \sigma(u(\tau)) dW_\tau, x^* \right) f'(s) ds
= - \int_0^t \langle u(s), (-A_u(s))^* \varphi(s) \rangle ds + \int_0^t \langle u(s), \varphi'(s) \rangle ds - \int_0^t \left( \int_0^s \sigma(u(\tau)) dW_\tau, \varphi'(s) \right) ds.
\]

Recalling \(a(u; u, \varphi) = -\langle A_u u, \varphi \rangle\), we obtain from the previous expression the weak formulation (3.3), for test functions having the structure \(\varphi(t) = f(t) \otimes x^*\). To extend the previous identity to test functions belonging to \(\Gamma^u_R\), we first extend it to simple functions \(\varphi: \Omega \to C^1([0, T]; X) \cap C([0, t]; D_u^R)\). By an approximation argument, this can be further extended to any function \(\varphi \in L^0(\Omega; C^1([0, T]; X) \cap C([0, t]; D_u^R))\). Choosing an arbitrary \(x^* \in D_u^R\) and setting \(\varphi(s) := U^u(t, s)^* x^*\), by Lemma 3.8, we have that \(\varphi(s) \in L^0(\Omega; C^1([0, t]; X) \cap C([0, t]; D_u^R))\). This proves the statement \(\square\)

4. Examples

In this section, we present two examples of parabolic quasilinear SPDEs, to which the theory developed in this article applies. The existence theory for pathwise mild-,
martingale-, and weak solutions for these problems is well-known, see [7, 13, 20]. After introducing these SPDEs and recalling the corresponding existence results, we show that they satisfy our assumptions. Therefore, the pathwise mild and weak solution concepts are equivalent in these cases.

**Example 4.1** (The stochastic SKT population model). Let $\mathcal{O} \subset \mathbb{R}^2$ be an open bounded domain with $C^2$ boundary. We fix parameters $\alpha, \beta, \gamma > 0$. We are interested in studying a cross-diffusion SPDE, which was originally introduced by Shigesada et al. [1] in the deterministic setting, to analyze population segregation by induced cross-diffusion in a two-species model. The nonlinear drift term correspond to those arising in the classical Volterra competition model. The stochastic SKT system is given by

$$
\begin{align*}
\frac{du_1}{dt} &= \left(\Delta(z_1u_1 + \gamma_1u_1u_2 + \beta_1u_2^2) + \theta_{11}u_1 - \theta_{12}u_1u_2\right)dt + \sigma_1(u_1, u_2)\,dW_1^1, \\
\frac{du_2}{dt} &= \left(\Delta(z_2u_2 + \gamma_2u_1u_2 + \beta_2u_2^2) + \delta_{11}u_2 - \delta_{12}u_1u_2\right)dt + \sigma_2(u_1, u_2)\,dW_2^2,
\end{align*}
$$

for $t \in [0,T]$ and $x \in \mathcal{O}$ and is supplemented with the following boundary and initial conditions:

$$
\frac{\partial}{\partial n} u_1(t,x) = \frac{\partial}{\partial n} u_2(t,x) = 0, t > 0, x \in \partial \mathcal{O},
$$

$$
u_1(x,0) = u_1^0(x) \geq 0, \quad u_2(x,0) = u_2^0(x) \geq 0, x \in \mathcal{O}.
$$

$W = (W^1, W^2)$ is an $H$-valued cylindrical Wiener process. The solution $u := (u_1, u_2)$, where $u_1 = u_1(x,t)$ and $u_2 = u_2(x,t)$ denote the densities of two competing species $S_1$ and $S_2$ at certain location $x \in \mathcal{O}$, at time $t$. The coefficients $\theta_{11}, \theta_{12}, \theta_{21} > 0$ denote the intraspecies competition rates in $S_1$, respectively, in $S_2$ and $\theta_{12}, \theta_{21} > 0$ stand for the interspecies competition rates between $S_1$ and $S_2$. Furthermore, the terms $\Delta(\beta_1u_1^2)$ and $\Delta(\beta_2u_2^2)$ represent the self-diffusions of $S_1$ and $S_2$ with rates $\beta_1, \beta_2 \geq 0$, and $\Delta(\gamma_1u_1u_2), \Delta(\gamma_2u_1u_2)$ represent the cross-diffusions of $S_1$ and $S_2$ with rates $\gamma_1, \gamma_2 \geq 0$.

The SKT system (4.1) can be rewritten as an abstract quasilinear SPDE:

$$
\begin{align*}
\left\{
\begin{array}{l}
du = [Au + F(u)] \, dt + \sigma(u) \, dW, \quad t \in [0,T] \\
u(0) = u^0,
\end{array}
\right.
\end{align*}
$$

where

$$
A\,u := \text{div}(B(u)\nabla u) - \Gamma u,
$$

with

$$
B(u) = \begin{pmatrix}
\alpha_1 + 2\beta_1u_1 + \gamma_1u_2 \\
\gamma_2u_2 \\
\alpha_2 + 2\beta_2u_2 + \gamma_2u_1
\end{pmatrix},
\quad
\Gamma(u) = \begin{pmatrix}
\theta_{11} & 0 \\
0 & \theta_{21}
\end{pmatrix}.
$$

The nonlinear term $F$ corresponds to the Lotka–Volterra type competition model

$$
F(u) = \begin{pmatrix}
2\theta_{11}u_1 - \theta_{11}u_1^2 - \theta_{12}u_1u_2 \\
2\theta_{21}u_2 - \theta_{21}u_1u_2 - \theta_{22}u_2^2
\end{pmatrix}.
$$

To ensure the positive definiteness of the matrix $B$, the following restriction on the parameters is necessary:
\[ \gamma_1^2 < 8x_1\beta_1 \quad \text{and} \quad \gamma_2^2 < 8x_2\beta_2. \] (4.3)

This assumption is required to show that \( A_u \) generates a parabolic evolution system \( U^u \) for \( u \in Z \), for a natural choice of \( Z \), see below. The (AT) conditions (A1)–(A3) are satisfied for a standard choice of Hilbert spaces \( Z \), \( Y \), where \( X := L^2(O) \), \( Z := H^{1+\varepsilon}(O) \) and \( Y := H^{1+2\varepsilon}(O) \) for \( 0 < \varepsilon < \varepsilon \), see [40, Chapter 15.2.2]. Moreover, according to [40, Chapter 1.8.2], (A1*), (A2*), (A3*) are also satisfied by the adjoint operator \( (A_u)^* \).

**Remark 4.2.** The choice of the space \( Z = H^{1+\varepsilon}(O) \) is natural. Using the Sobolev embedding \( W^{k,p} \), \( \rightarrow C(O) \) for \( k \geq d \), we observe that for fixed \( t \in [0,T] \) and \( d = 2 \), the choice of \( Z = H^{1+\varepsilon}(O) \) ensures that (2.6) is satisfied.

We also emphasize that the domains of the fractional powers of \( A_u \), for \( u \in Z \), can be identified with Sobolev spaces, see [40, Proposition 15.3 and Theorem 16.7]. More precisely, we have

\[
\begin{align*}
\mathcal{D}_u^\theta &= H^{2\theta}(O), \quad \text{for} \quad 0 \leq \theta < \frac{3}{4} \\
\mathcal{D}_u^\theta &= H_N^{2\theta}(O), \quad \text{for} \quad \frac{3}{4} \leq \theta \leq 1,
\end{align*}
\]

where \( H_N^{2\theta}(O) \) incorporates the Neumann boundary conditions. Furthermore, the non-linear drift term is locally Lipschitz continuous on \( X \). Letting \( u^0 \in Z \) a.s. and assuming a local Lipschitz continuity on \( \sigma \) (recall Assumption 4), [20, Theorem 4.3] provides the existence of a local-in-time pathwise mild solution \( u \) of (4.2) such that \( u \in L^0(\Omega; B([0,\tau]; Z)) \cap L^0(\Omega; C^\delta([0,\tau]; Y)) \), where \( \delta \in (0, \frac{1-\varepsilon}{2}) \).

The main result establishes the equivalence of this pathwise mild solution with the weak solution. This statement is a direct consequence of Theorems 3.10 and 3.11. Moreover, the results in Theorems 3.10 and 3.11 hold if an additional drift term \( F \) is incorporated.

**Theorem 4.3.** The local pathwise mild solution \((u, \tau)\) of (4.2) is also a weak solution. More precisely, for \( t \in [0, \tau) \), the following relation holds \( \mathbb{P}\)-a.s.

\[
\begin{align*}
\langle u(t), x^* \rangle &= \langle u_0, x^* \rangle - \int_0^t \langle \nabla u(s), \nabla x^* \rangle ds - \int_0^t \langle \Gamma(u(s)), x^* \rangle ds \\
&\quad + \int_0^t \langle \nabla \phi(s), x^* \rangle ds + \left\langle \int_0^t \sigma(u(s)) dW_s, x^* \right\rangle,
\end{align*}
\]

where \( x^* \in \mathcal{D}_u^\theta \).

**Proof.** By Theorem 3.10 part (1), we infer that the pathwise mild solution \( u = (u_1, u_2) \) satisfies the weak formulation for \( t \in [0, \tau) \)

\[
\begin{align*}
\langle u(t), \varphi(t) \rangle &= \langle u_0, \varphi(0) \rangle - \int_0^t \langle B(u(s)), \nabla \varphi(s) \rangle ds - \int_0^t \langle \Gamma(u(s)), \varphi(s) \rangle ds \\
&\quad + \int_0^t \langle \nabla u(s), \varphi'(s) \rangle ds - \int_0^t \left\langle \int_0^s \sigma(u(\tau)) dW_{\tau}, \varphi'(s) \right\rangle ds \\
&\quad + \int_0^t \langle \nabla \varphi(s), \varphi(s) \rangle ds + \left\langle \int_0^t \sigma(u(s)) dW_s, \varphi(t) \right\rangle,
\end{align*}
\] (4.5)
for every time-dependent test function \( \varphi \in L^0(\Omega; C^1([0, t]; D_u^*)) \). Now, Theorem 3.11 part (1) entails the usual weak formulation (4.4).

Remark 4.4. Theorem 4.3 also provides a regularity result for the weak solution, i.e., \( u \in L^0(\Omega; C^3([0, t]; Y)) \).

Example 4.5. We let \( d \geq 1 \) and consider a quasilinear parabolic stochastic partial differential equation on a \( d \)-dimensional domain \( \mathcal{O} \subset \mathbb{R}^d \) with smooth boundary \( \partial \mathcal{O} \) of the form

\[
\begin{aligned}
  &du = \text{div}(B(u) \nabla u)dt + \sigma(u) dW_t, & \text{in } (0, T) \times \mathcal{O} \\
  &u = 0 & \text{on } [0, T] \times \partial \mathcal{O} \\
  &u(0, x) = u^0(x), & x \in \mathcal{O},
\end{aligned}
\]

(4.6)

where \( (W_t)_{t \in [0, T]} \) is a cylindrical Wiener process taking values in a Hilbert space \( H \supset X := L^2(\mathcal{O}) \).

Such equations have been extensively studied in the literature, see [7–9], under the following assumptions on the coefficients \( B \) and \( \sigma \):

Assumption 6.

1. The coefficients \( B : \mathbb{R} \to \mathbb{R}^{d \times d} \) are nonlinear functions, such that the diffusion matrix \( B = (B_{ij})_{i,j=1}^d \) is of class \( C^1_b \), symmetric, uniformly positive definite and bounded, i.e., there exist constants \( \kappa, C > 0 \) such that

\[
\kappa I \leq B \leq CI.
\]

2. For each \( u \in X \) we consider a mapping \( \sigma(u) : H \to X \) defined by

\[
\sigma(u)e_k = \sigma_k(\cdot, u(\cdot)),
\]

where \( \sigma_k \in C(\mathcal{O} \times \mathbb{R}) \). We further suppose that \( \sigma \) satisfies usual Lipschitz and linear growth conditions, i.e.,

\[
\begin{align*}
\sum_{k \in \mathbb{N}} |\sigma_k(x, \xi_1) - \sigma_k(x, \xi_2)|^2 &\leq C|\xi_1 - \xi_2|^2, & & \forall x \in \mathcal{O}, \xi_1, \xi_2 \in \mathbb{R}, \\
\sum_{k \in \mathbb{N}} |\sigma_k(x, \xi)|^2 &\leq C(1 + |\xi|^2), & & \forall x \in \mathcal{O}, \xi \in \mathbb{R}.
\end{align*}
\]

In particular, these assumptions imply that \( \sigma \) maps \( X \) to \( L^2(H, X) \). Thus, given a predictable process \( u \) that belongs to \( L^2(\Omega, L^2(0, T; X)) \), the stochastic integral is a well-defined \( X \)-valued process.

The previous assumptions on \( \sigma \) can be relaxed and an additional regular drift term can be incorporated [7–9]. An example of such a drift term is given by \( \text{div}(F(u)) \), where

\[
F = (F_1, \ldots, F_d) : \mathbb{R} \to \mathbb{R}^d
\]

is continuously differentiable with bounded derivatives.

Remark 4.6. In contrast to Example 4.1, the diffusion matrix \( B \) additionally satisfies the boundedness assumption (4.7).

Next, we give the definition of a weak solution of (4.6).
Definition 4.7. An \((\mathcal{F}_t)_{t \in [0,T]}\)-adapted, \(X\)-valued continuous process \((u(t))_{t \in [0,T]}\) is called a weak solution for (4.6) if for any \(\varphi \in C^\infty_0(\Omega)\), the following identity holds for \(t \in [0,T], \mathbb{P}\)-a.s.

\[
\langle u(t), \varphi \rangle = \langle u^0, \varphi \rangle - \int_0^t \langle B(u(s)) \nabla u(s), \nabla \varphi \rangle \, ds + \int_0^t \langle \sigma(u(s)) \, dW_s, \varphi \rangle.
\]

Under the previous assumptions on \(B\) and \(\sigma\), together with suitable regularity conditions on the initial data, the existence of a weak solution was established in \([8,9]\).

Moreover, assuming higher spatial regularity on \(\sigma\), the regularity of this weak solution can be improved \([9, \text{Theorem 2.6}]\). For the convenience of the reader we indicate this statement. To this aim we let \(\eta > 0\), set \(D_T := [0,T] \times \Omega\) and consider the Hölder space \(C^{n/2-\eta}(D_T)\) with different time and space regularity, endowed with the norm

\[
\|f\|_{C^{n/2-\eta}(D_T)} = \sup_{(t,x) \in \mathbb{D}_T} |f(t,x)| + \sup_{(t,x) \neq (s,y) \in \mathbb{D}_T} \frac{|f(t,x) - f(s,y)|}{\max\{|t-s|^{\eta/2} + |x-y|^\eta\}}.
\]

Theorem 4.8. Assume that

- \(u^0 \in L^m(\Omega; C^1(\Omega))\) for some \(t > 0\) and all \(m \in [2,\infty)\), and \(u^0 = 0\) on \(\partial \Omega\) a.s.
- \(\|\sigma(u)\|_{L_2(H,H^1_0(\Omega))} \leq C\left(1 + \|u\|_{H^1_0(\Omega)}\right)\).

Then, the weak solution \(u\) of (4.6) belongs to \(L^m(\Omega; C^{n/2-\eta}(\overline{D_T}))\), for all \(m \in [2,\infty)\).

Now, we are ready to state the main result for this example, based on Theorem 4.8.

Theorem 4.9. Let assumptions of Theorem 4.8 and Assumption 6 hold. Then, there exists a pathwise mild solution of (4.6). More precisely, there exists an \((\mathcal{F}_t)_{t \in [0,T]}\)-adapted process \(u\) such that for \(t \in [0,T], \mathbb{P}\)-a.s.

\[
u(t) = U^u(t,0)u^0 + U^u(t,0) \int_0^t \sigma(u(s)) \, dW_s - \int_0^t U^u(t,s) A_u(s) \int_s^t \sigma(u(\tau)) \, dW_\tau \, ds, \tag{4.8}
\]

where \(A_u := \text{div}(B(u) \nabla u)\) and \(U^u(\cdot,\cdot)\) is the evolution family generated by \(A_u\). Moreover, \(u \in L^0(\Omega; B([0,T]; H^1(\Omega))) \cap L^0(\Omega; C^0([0,T]; L^2(\Omega)))\).

Proof. Under Assumption 6, the existence of a weak solution \(u\)

\[
u \in L^2(\Omega; C([0,T]; L^2(\Omega))) \cap L^2(0,T; H^1(\Omega))
\]

was established in \([8,9]\). The bilinear form \(a\) for \(u \in H^1(\Omega)\) given by

\[
a(u;v,w) := \langle B(u) \nabla v, \nabla w \rangle, \quad v,w \in H^1(\Omega)
\]

satisfy conditions (2.5) and (2.6) with \(Z := H^1(\Omega), X = Y := L^2(\Omega)\), thanks to (4.7). Therefore, (A1) and (A2) from Assumption 1 hold. The Lipschitz assumption on \(B\) entails that (A3) holds too. Similarly, it can be shown that the conditions (A1*)–(A3*) of Assumption 2 are satisfied by the adjoint \(A_u^*\), see \([40, \text{Chapter 1.8.2}]\) for details. We are left to verify (A4). To this aim, we require to show that \(u \in C^\alpha([0,T]; L^2(\Omega))\) for
some $\delta > 0$. The weak solution $u \in L^m(\Omega; C^{\eta/2,\eta}(\overline{D_T}))$, for all $m \in [2, \infty)$, by Theorem 4.8. Regarding this along with the following equivalent norm \[43\] on $C^{\eta/2,\eta}(D_T)$

$$
\|u\|_{\eta/2,\eta,D_T} = \sup_{(t,x),(s,x) \in D_T} \frac{|u(t,x) - u(s,x)|}{|t-s|^{\eta/2}} + \sup_{(t,x),(t,y) \in D_T} \frac{|u(t,x) - u(t,y)|}{|x-y|^\eta}
$$

and boundedness of the domain $\mathcal{O}$, we conclude that $u \in C^{\delta}([0,T]; L^2(\mathcal{O}))$ for $\delta = \eta/2$.

Hence, we can infer from Theorem 2.5 that the operator $A_u u = \text{div}(B(u) \nabla u)$ generates an evolution family $U^u$. According to Theorem 3.11 part (2), this evolution family along with the weak solution $u$ satisfies (4.8). Consequently, we obtain that $u$ is a pathwise mild solution of (4.6).

\[\square\]

**Remark 4.10.** The regularity on $\sigma$ assumed in Theorem 4.8 is necessary to obtain the H"older regularity of the solution.

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**References**


Appendix: A fractional powers of sectorial operators

Let $X$ be a Banach space with norm $\| \cdot \|_X$ and $A$ be a linear sectorial operator of $X$ with angle $0 \leq \vartheta_A < \pi$. As before an open sectorial domain $\Sigma_\vartheta$ for $\vartheta_A < \vartheta < \pi$ is given by

$$\Sigma_\vartheta = \{ \lambda \in \mathbb{C}; \ |\arg \lambda| < \vartheta \}, \quad \vartheta_A < \vartheta < \pi.$$  

We define, for each complex number $z$ with $\text{Re} z > 0$, the bounded linear operator

$$A^{-z} = \frac{1}{2\pi i} \int_\Gamma \lambda^{-z}(\lambda - A)^{-1} d\lambda,$$

using the Dunford integral in $L(X)$, where $\Gamma$ is the contour surrounding the spectrum $\sigma(A)$, running counterclockwise in $\mathbb{C} \setminus (\infty, 0] \cap \rho(A)$. If $z = n \in \mathbb{N}$ it can be shown, [40, Chapter 2.7.1], that this definition coincides with the standard definition of $A^{-n} = (A^n)^{-1}$.

$A^{-z}$ is an analytic function for $\text{Re} z > 0$ with values in $L(X)$. The following theorem [40, Theorem 2.21] is concerned with the convergence of $A^{-z}$ as $z \to 0$.

**Theorem A.1.** For any $0 < \phi < \frac{\pi}{2}$, as $z \to 0$ with $z \in \Sigma_\vartheta \setminus \{0\}$, $A^{-z}$ converges to $\text{Id}$ strongly on $X$.

It also holds that $A^{-z}$ satisfies the law of exponent, i.e.,

$$A^{-z}A^{-z'} = A^{-(z+z')}, \quad \text{Re} z > 0, \ \text{Re} z' > 0,$$

which leads to the following theorem, see [40, Theorem 2.22].

**Theorem A.2.** The $L(X)$-valued function $A^{-z}$ is an analytic semigroup defined in the half-plane $\{ z \in \mathbb{C}; \text{Re} z > 0 \}$.

The fractional power $A^z$, for every real number $-\infty < z < \infty$ is defined, see [40, Chapter 2.7.2] for details. The following theorem lists some of the properties of the fractional power $A^z$, cf. [40, Theorem 2.23].

**Theorem A.3.** Let $A$ be a sectorial operator on $X$ with angle $\vartheta_A$. Then,

1. for $-\infty < z < 0$, $A^z$ are bounded operators on $X$, $A^0 = \text{Id}$ on $X$ and $A^z$ are densely defined, closed linear operators of $X$ for $z > 0$. 

2. Let \( 0 \leq \alpha_1 < \alpha_2 < \infty \), then, \( D(A^{\alpha_2}) \subset D(A^{\alpha_1}) \).

3. \( A^\alpha \) satisfies the law of exponent, i.e.,
   \[ A^\alpha A^\beta = A^{\alpha+\beta}, \quad -\infty < \alpha, \beta < \infty. \]

4. For \( 0 < \alpha < 1 \), \( A^\alpha \) is a sectorial operator on \( X \) with angle \( \leq \pi \theta_A \).

Let \( 0 < \alpha < 1 \) and \( A^\alpha \) be the fractional powers of \( A \). Let \( M_\pi \) be the constant appearing in the assumption (A2) with angle \( \theta = \pi \). Let us introduce the spaces

\[ D_\alpha(A) = \left\{ v \in X : \sup_{0 < \rho < \infty} \rho^\alpha ||A(\rho + A)^{-1}v||_X < \infty \right\}, \quad 0 \leq \alpha \leq 1. \]

\( D_\alpha(A) \) are normed spaces, equipped with the norms

\[ ||v||_{D_\alpha(A)} = \sup_{0 < \rho < \infty} \rho^\alpha ||A(\rho + A)^{-1}v||_X. \]

In the following, we compare the domain of fractional powers of the operator \( A \) to that of \( D_\alpha(A) \) [40, Theorem 2.24].

**Theorem A.4.** For any \( 0 < \alpha < 1 \), \( D(A^\alpha) \subset D_\alpha(A) \), and the estimate

\[ ||v||_{D_\alpha(A)} \leq C(1 + M_\pi)^2 ||A^\alpha v||_X, \quad v \in D(A^\alpha) \]

holds true. Conversely, for any \( 0 < \alpha < \alpha' < 1 \), \( D_{\alpha'}(A) \subset D(A^\alpha) \), and the estimate

\[ ||A^\alpha v||_X \leq C_{\alpha, \alpha'} \left[ ||v||_{D_{\alpha'}(A)} + (1 + M_\pi)||v||_X \right], \quad v \in D_{\alpha'}(A), \]

holds true.

Next, we compare domains of fractional powers of two sectorial operators \( A \) and \( B \) of \( X \) for which \( D(A) \subset D(B) \) continuously, i.e., there exists a constant \( C > 0 \) such that

\[ ||Bv||_X \leq C||Av||_X, \quad v \in D(A). \]

**Theorem A.5.** [40, Theorem 2.25] Let \( A \) and \( B \) be two sectorial operators of \( X \) satisfying the above relationship between their domains, as well as assumptions (A1) and (A2). Then, for any \( 0 < \alpha < \alpha' < 1 \), \( D(A^\alpha) \subset D(B^\alpha) \) and the estimate

\[ ||B^\alpha v||_X \leq C_{\alpha, \alpha'} ||A^\alpha v||_X, \quad v \in D(A^\alpha) \]

holds true, where \( C_{\alpha, \alpha'} > 0 \) is determined by \( \alpha, \alpha', M_\pi \) and \( C \).