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TOWARDS AN L^1-THEORY FOR VECTOR-VALUED ELLIPTIC BOUNDARY VALUE PROBLEMS

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1. Introduction

Vector-valued elliptic and parabolic boundary value problems subject to general boundary conditions have been investigated recently in [DHP01] in the L_p-context for $1 < p < \infty$. One of the main goals of this paper was to deduce a maximal L_p-regularity result for the solution of the parabolic initial boundary value problem. A classical reference in the elliptic context are the celebrated papers of Agmon, Douglis and Nirenberg [ADN59]. For further references and information on the scalar and vector-valued case we refer to the [Ama01] and the list of references given in [DHP01].

Vector-valued elliptic and parabolic problems on all of $\mathbb{R}^n$ we considered first by Amann [Ama01] on a large scale of function spaces, including $L_1(\mathbb{R}^n; E)$. Here E denotes an arbitrary Banach space. He proved in particular that the L_1-realization of such problems generates an analytic C_0-semigroup provided the top-order coefficients of the underlying operators are uniformly bounded and Hölder continuous.

In this note, we consider vector-valued boundary value problems with constant coefficients in the L_1-setting for a half space. Following the approach described in [DHP01], we assume the Lopatinskii-Shapiro to be true; we then obtain a representation of the solution u of the elliptic problem by integral operators which allows to deduce a-priori estimates for u in the $L_1(\mathbb{R}^{n+1}_+; E)$-norm. Here E denotes again an arbitrary Banach space. These estimates imply in particular that the $L_1(\mathbb{R}^{n+1}_+; E)$-realization of an elliptic boundary value problem with constant coefficients in the half space $\mathbb{R}^{n+1}_+$ generates an analytic C_0-semigroups on $L_1(\mathbb{R}^{n+1}_+; E)$. For different approaches and results with variable coefficients in the scalar-valued case we refer to Amann [Ama83], Di Blasio [DiB91], Guidetti [Gui93] and Tanabe [Tan97], Section 5.4.

2. Elliptic Problems on $L^1(\mathbb{R}^n; E)$

Throughout this section, let E be a Banach space. Following the notion of [Ama01] or Section 5 of [DHP01], we call a homogeneous $\mathcal{B}(E)$-valued polynomial $\mathcal{A}(\cdot)$ of degree $m \in \mathbb{N}$ *parameter-elliptic* if there is an angle $\phi \in [0, \pi)$ such that the spectrum $\sigma(\mathcal{A}(\xi))$ satisfies

$$\sigma(\mathcal{A}(\xi)) \subset \Sigma_\phi \quad \text{for all } \xi \in \mathbb{R}^n,\ |\xi| = 1. \tag{1}$$

We then call

$$\phi_{\mathcal{A}} := \inf\{\phi : \text{ (1) holds}\} = \sup_{|\xi|=1} |\arg \sigma(\mathcal{A}(\xi))|$$

the *angle of ellipticity* of $\mathcal{A}$. For $D = -i(\partial_1, \dots, \partial_n)$ we call $\mathcal{A}(D) = \sum_{|\alpha|=m} a_\alpha D^\alpha$ parameter elliptic, if its symbol $\mathcal{A}(\xi)$ is parameter-elliptic.

Assume now that $\mathcal{A}(D)$ is a parameter-elliptic operator with angle of ellipticity $\phi_{\mathcal{A}}$. It was proved in Theorem 5.2 and Corollary 5.3 of [DHP01] that for $\phi > \phi_{\mathcal{A}}$ and $k \in \mathbb{N}$ there are constants $c_{\phi,k}$, $C_{\phi,k}$ such that the solution γ_λ of

$$\lambda u + \mathcal{A}(D)u = \delta_0$$

satisfies the estimate

$$|D^\beta \gamma_\lambda(x)| \le C_{\phi,k}|\lambda|^{\frac{n+k}{m}-1} p^n_{m,k}(c_{\phi,k}|x|), \quad x \in \mathbb{R}^n, |\arg\lambda| \le \pi - \phi, |\beta| = k, \tag{2}$$

where $p^n_{m,k}$ is given by

$$p^n_{m,k}(r) = \int_0^\infty \frac{s^{n-2}}{(1+s)^{m-k-1}} e^{-r(1+s)} ds.$$

We define the $L_1(\mathbb{R}^n; E)$-realization A of $\mathcal{A}(D)$ by means of $A = \overline{A}_0$, where

$$[A_0 u](x) = \mathcal{A}(D)u(x), \quad x \in \mathbb{R}^n,\ u \in D(A_0) = W^m_1(\mathbb{R}^n; E).$$

The first assertion of the following proposition is a special case of Theorem 5.10 of [Ama01]. The proof given below is based on estimates on the fundamental solution. Via estimate (2) we also obtain some information on the domain of A, which will be important in the following.

Proposition 2.1. *Let $n, m \in \mathbb{N}$, E be a Banach space, $a_\alpha \in \mathcal{B}(E)$ and suppose that $\mathcal{A}(D)$ is parameter elliptic with angle of ellipticity $\phi_{\mathcal{A}} < \frac{\pi}{2}$. Then $-A$ generates an analytic semigroup on $L_1(\mathbb{R}^n; E)$ of angle $\frac{\pi}{2} - \phi_{\mathcal{A}}$ and we have*

$$W^m_1(\mathbb{R}^n; E) \subset D(A) \subset W^{m-1}_1(\mathbb{R}^n; E).$$

Proof. Obviously, A has dense domain. If $f \in L_1(\mathbb{R}^n; E)$, choose a sequence $f_n \in C^\infty_0(\mathbb{R}^n; E)$ such that $f_n \to f$ in $L_1(\mathbb{R}^n; E)$. For $\lambda \in \Sigma_{\pi-\phi}$, $\phi > \phi_{\mathcal{A}}$, we have $u_n := \gamma_\lambda * f_n \in W^m_1(\mathbb{R}^n; E)$ as well as $\lambda u_n + \mathcal{A}(D)u_n = f_n$. Since $u_n \to u = \gamma_\lambda * f$ in $L_1(\mathbb{R}^n; E)$ as $n \to \infty$, we see that $u \in D(A)$ and $\lambda u + Au = f$. This shows that $\lambda + A$ is invertible for each $\lambda \in \Sigma_{\pi-\phi}$ with $(\lambda + A)^{-1} f = \gamma_\lambda * f$. Thus by (2) and Young's inequality we obtain

$$W^{m-1}_1(\mathbb{R}^n; E) \supset D(A) \supset D(A_0) = W^m_1(\mathbb{R}^n; E).$$

Furthermore, estimate (2) yields $-\Sigma_{\pi-\phi} \subset \rho(A)$ and

$$|\lambda(\lambda + A)^{-1}|_{\mathcal{B}(L_1(\mathbb{R}^n;E))} \le M_{\pi-\phi},$$

for each $\phi > \phi_{\mathcal{A}}$. □

3. Elliptic Problems in a Half Space

In this section we consider boundary value problems of the form

$$\begin{aligned} \lambda u + \mathcal{A}(D)u &= f, && \text{in } \mathbb{R}^{n+1}_+ \\ \mathcal{B}_j(D)u &= g_j, && \text{on } \partial\mathbb{R}^{n+1}_+, j = 1, \dots, m. \end{aligned}$$

on a half space. Here $\mathcal{A}(D)$ as well as $\mathcal{B}_j(D)$ for $j \in \{1, \dots, m\}$ are differential operators with operator-valued coefficients. We also assume that $\mathcal{A}(D)$ and $\mathcal{B}_j(D)$ consist only of the principal parts, i.e.

$$\begin{aligned} \mathcal{A}(D) &= \sum_{|\alpha|=2m} a_\alpha D^\alpha \\ \mathcal{B}_j(D) &= \sum_{|\beta|=m_j} b_{j\beta} D^\beta, \end{aligned}$$

where $m_j \in \{0, \ldots, 2m-1\}$, $a_\alpha \in \mathcal{B}(E)$ and $b_{j\beta} \in \mathcal{B}(E)$ for $j \in \{1, \ldots, m\}$. In the following we assume that $\mathcal{A}(D)$ is parameter-elliptic with angle of ellipticity $\phi_\mathcal{A} \in [0, \pi)$, i.e. there exists $\phi \in [0, \pi)$ such that

$$\sigma(\mathcal{A}(\xi)) \subset \Sigma_\phi, \quad \xi \in \mathbb{R}^{n+1}, \quad |\xi| = 1 \tag{3}$$

and $\phi_\mathcal{A}$ is defined as the infimum of all ϕ satisfying (3). Here $\mathcal{A}(\cdot)$ is the symbol of $\mathcal{A}(D)$ defined by

$$\mathcal{A}(\xi) = \sum_{|\alpha|=2m} a_\alpha \xi^\alpha, \quad \xi \in \mathbb{R}^{n+1}, a_\alpha \in \mathcal{B}(E). \tag{4}$$

We suppose that the following Lopatinskii-Shapiro Condition holds true:

Lopatinskii-Shapiro Condition:
For each $\xi' \in \mathbb{R}^n$ and $\lambda \in \overline{\Sigma}_{\pi-\phi}$ with $|\xi'| + |\lambda| \neq 0$, the problem

$$\begin{aligned} \lambda v(y) + \mathcal{A}(\xi', D_y)v(y) &= 0, \quad y > 0, \\ \mathcal{B}_j(\xi', D_y)v(0) &= g_j, \quad j = 1, \ldots, m \end{aligned}$$

admits a unique solution $u \in C_0(\mathbb{R}_+; E)$ for each $(g_1, \ldots, g_m)^T \in E^m$.

In the following we are interested in the L_1-theory of the above problem; more specifically, for $\phi > \phi_\mathcal{A}$ we consider the following problem:
Given $\lambda \in \Sigma_{\pi-\phi}$, $f \in L_1(\mathbb{R}^{n+1}_+; E)$ and $g_j \in W_1^{2m-m_j}(\mathbb{R}^{n+1}_+; E)$ for $j \in \{1, \ldots, m\}$, find $u \in W_1^{2m}(\mathbb{R}^{n+1}_+; E)$ which satisfies

$$\lambda u + \mathcal{A}(D)u = f, \quad \text{in } \mathbb{R}^{n+1}_+ \tag{5}$$

$$\mathcal{B}_j(D)u = g_j, \quad \text{on } \partial\mathbb{R}^{n+1}_+, j = 1, \ldots, m. \tag{6}$$

To this end, for $\mathcal{A}(D)$ and $\mathcal{B}_j(D)$ defined as above, we define an operator A in $L_1(\mathbb{R}^{n+1}_+; E)$ associated to the boundary value problem (5) and (6) with $g_j = 0$ for all $j \in \{1, \ldots, m\}$ by means of $A = \overline{A_{min}}$, where $A_{min} : D(A_{min}) \to L_1(\mathbb{R}^{n+1}_+; E) \times \Pi_{j=1}^m W_1^{2m-m_j}(\mathbb{R}^{n+1}_+; E)$ is defined by

$$\begin{aligned} D(A_{min}) &:= W_1^{2m}(\mathbb{R}^{n+1}_+; E) \\ A_{min}u &:= \begin{pmatrix} \mathcal{A}(D) \\ \mathcal{B}_1(D) \\ \vdots \\ \mathcal{B}_m(D) \end{pmatrix} u. \end{aligned}$$

Moreover, we set $g := (g_1, \ldots, g_m)^T$. For $\lambda \in \Sigma_{\pi-\phi}$, $f \in L_1(\mathbb{R}^{n+1}_+; E)$ and $g_j \in W_1^{2m-m_j}(\mathbb{R}^{n+1}_+; E)$ for $j = 1, \ldots, m$, our boundary value problem can rewritten as

$$\lambda J v + A v = \begin{pmatrix} f \\ g \end{pmatrix}, \tag{7}$$

where $Jv = (v, 0, \ldots, 0)^T$. We then have the following theorem.

Theorem 3.1. *Let $\mathcal{A}(D)$ be a parameter-elliptic operator of order $2m$ and angle of ellipticity $\phi_{\mathcal{A}}$. Let $\phi > \phi_{\mathcal{A}}$. For $j \in \{1, \dots, m\}$ let $\mathcal{B}_j(D)$ be boundary operators of order $m_j < 2m$. Assume that the Lopatinskii-Shapiro condition holds. Let E be a Banach space, $f \in L_1(\mathbb{R}^{n+1}_+; E)$ and $g_j \in W_1^{2m-m_j}(\mathbb{R}^{n+1}_+; E)$ for $j = 1, \dots, m$. Let $\lambda \in \Sigma_{\pi-\phi}$ and let A be defined as above. Then there exists a unique function $u \in W_1^{2m-1}(\mathbb{R}^{n+1}_+; E) \cap D(A)$ satisfying*

$$\lambda J u + A u = \begin{pmatrix} f \\ g \end{pmatrix}. \tag{8}$$

Moreover, u is given by

$$u = P(\lambda + A_{\mathbb{R}^{n+1}})^{-1} E_0 f + \sum_{j=1}^{m} R_\lambda^j f + \sum_{j=1}^{m} S_\lambda^j g_j, \tag{9}$$

where R_λ^j and S_λ^j are kernel operators as defined in Propositions 6.8 and 6.9 of [DHP01]. *Furthermore, there exists a constant $C > 0$ such that for $0 \le |\alpha| \le 2m-1$ and $\lambda \in \Sigma_{\pi-\phi}$ we have*

$$\begin{aligned} |\lambda^{1-\frac{|\alpha|}{2m}} D^\alpha u|_{L_1(\mathbb{R}^{n+1}_+;E)} \quad \le \quad & C[|f|_{L_1(\mathbb{R}^{n+1}_+;E)} + \sum_{j=1}^{m} |(-\Delta + |\lambda|^{\frac{1}{m}})^{\frac{2m-m_j}{2}} g_j|_{L_1(\mathbb{R}^{n+1}_+;E)} \\ & + \sum_{j=1}^{m} |(-\Delta + |\lambda|^{\frac{1}{m}})^{\frac{2m-m_j-1}{2}} D_y g_j|_{L_1(\mathbb{R}^{n+1}_+;E)}]. \end{aligned}$$

Proof. The following proof is a modification of the arguments given in the proof of Theorem 6.10 of [DHP01] to the L_1-situation. Indeed, note that under the given assumptions the problem (8) has a unique solution $u \in W_1^{2m-1}(\mathbb{R}^{n+1}_+; E) \cap D(A)$ if and only if u defined as in (9) belongs to $W_1^{2m-1}(\mathbb{R}^{n+1}_+; E) \cap D(A)$.

Given $f \in L_1(\mathbb{R}^{n+1}_+; E)$, it follows from Proposition 2.1 (and its proof in case $\phi > \pi/2$) that the first term on the right hand side of (9) belongs to the required regularity class.

In order to treat the second term, notice that by Proposition 6.6 of [DHP01] the kernel $k_\lambda^{R,j}$ of R_λ^j satisfies for $0 \le |\alpha| \le 2m-1$ and $\lambda \in \Sigma_{\pi-\phi}$ an estimate of the form

$$|D^\alpha K_\lambda^{R,j}(\cdot, y, y')|_{L^1(\mathbb{R}^n;\mathcal{B}(E))} \le C|\lambda|^{\frac{-2m+|\alpha|+1}{2m}} p^{n+1}_{2m+n,|\alpha|}(c|\lambda|^{\frac{1}{2m}}(y+y')), \quad y, y' > 0. \tag{10}$$

The above kernel estimates allow us to derive L^1-estimates for the second term on the right hand side of (9) via the following simple lemma on L^1-continuity of integral operators acting in half spaces.

Lemma 3.2. *Let T be an integral operator in $L_1(\mathbb{R}^{n+1}_+; E)$ of the form*

$$(Tf)(x, y) = \int_0^\infty \int_{\mathbb{R}^n} k(x - x', y, y') f(x', y') dx' dy', \quad x \in \mathbb{R}^n, y > 0,$$

where $k : \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathcal{B}(E)$ is a measurable function. If

$$\sup_{y'>0} \int_0^\infty |k(\cdot, y, y')|_1 dy =: M < \infty,$$

then $T \in \mathcal{L}(L_1(\mathbb{R}^{n+1}_+; E))$ and $|T|_{\mathcal{L}(L_1(\mathbb{R}^{n+1}_+;E))} \le M$.

The proof of Lemma 3.2 is consists only of an application of Young's and Hölder's inequality. Combining estimate (10) with Lemma 3.2 it follows that

$$|\lambda^{1-\frac{|\alpha|}{2m}} D^\alpha R^j_\lambda f|_{L_1(\mathbb{R}^{n+1}_+;E)} \leq C|f|_{L_1(\mathbb{R}^{n+1}_+;E)}.$$

Similarly, by Proposition 6.8 of [DHP01] the kernel $k^{S,j}_\lambda$ of S^j_λ satisfies for $0 \leq |\alpha| \leq 2m-1$ and $\lambda \in \Sigma_{\pi-\phi}$ an estimate of the form

$$|D^\alpha K^{S,j}_\lambda(\cdot, y')|_{L^1(\mathbb{R}^n;\mathcal{B}(E))} \leq C|\lambda|^{\frac{-2m+|\alpha|+1}{2m}} p^{n+1}_{2m+n,|\alpha|}(c|\lambda|^{\frac{1}{2m}} y'), \quad y' > 0.$$

Again, together with Lemma 3.2, this estimate implies that

$$\begin{aligned} |\lambda^{1-\frac{|\alpha|}{2m}} D^\alpha S^j_\lambda g_j|_{L_1(\mathbb{R}^{n+1}_+;E)} \leq \quad C \quad & (\sum_{j=1}^m |(-\Delta + |\lambda|^{\frac{1}{m}})^{\frac{2m-m_j}{2}} g_j|_{L_1(\mathbb{R}^{n+1}_+;E)} \\ & + \sum_{j=1}^m |(-\Delta + |\lambda|^{\frac{1}{m}})^{\frac{2m-m_j-1}{2}} D_y g_j|_{L_1(\mathbb{R}^{n+1}_+;E)}). \end{aligned}$$

This proves the assertion. □

For a parameter-elliptic operator $A(D)$ of order $2m$ and angle of ellipticity $\phi_{\mathcal{A}}$, we define the $L_1(\mathbb{R}^{n+1}_+;E)$-realization A^0_B of the boundary value problem (8) with $g=0$ as

(11) $$A^0_B u \quad := \quad \mathcal{A}(D)u$$

(12) $$D(A^0_B) \quad := \quad \{u \in W^{2m}_1(\mathbb{R}^{n+1}_+;E); \mathcal{B}_j(D)u = 0 \quad \text{for all} \quad j = 1, \ldots, m\}$$

and set

$$A_B := \overline{A^0_B}.$$

It follows from Theorem 3.1 that $(\lambda + A_B)$ is invertible for all $\lambda \in \Sigma_{\pi-\phi}$ with $\phi > \phi_{\mathcal{A}}$ and that

$$(\lambda + A_B)^{-1} = P(\lambda + A_{\mathbb{R}^{n+1}})^{-1} E_0 + \sum_{j=1}^m R^j_\lambda.$$

Theorem 3.1 also implies that $-\sum_{\pi-\phi} \subset \rho(A)$ and that $|\lambda(\lambda + A_B)^{-1}| \leq M$ for $\lambda \in \Sigma_{\pi-\phi}$ with $\phi > \phi_{\mathcal{A}}$. We thus have the following corollary.

Corollary 3.3. *Suppose that $\phi_{\mathcal{A}} < \frac{\pi}{2}$. Then $-A_B$ generates an analytic C_0-semigroup on $L_1(\mathbb{R}^{n+1}_+;E)$.*

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