Universität Konstanz

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Konstanzer Schriften in Mathematik

Nr. 266, Mai 2010

ISSN 1430-3558

ON THE MAXIMAL L_p-REGULARITY OF PARABOLIC MIXED ORDER SYSTEMS

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ABSTRACT. We study maximal L_p-regularity for a class of pseudodifferential mixed order systems on a space-time cylinder $\mathbb{R}^n \times \mathbb{R}$ or $X \times \mathbb{R}$, where X is a closed smooth manifold. To this end we construct a calculus of Volterra pseudodifferential operators and characterize the parabolicity of a system by the invertibility of certain associated symbols. A parabolic system is shown to induce isomorphisms between suitable L_p-Sobolev spaces of Bessel potential or Besov type. If the cross section of the space-time cylinder is compact, the inverse of a parabolic system belongs to the calculus again. As applications we discuss time-dependent Douglis-Nirenberg systems and a linear system arising in the study of the Stefan problem with Gibbs-Thomson correction.

1. Introduction

Motivated by many applications arising in mathematics, mathematical physics and applied sciences, parabolic initial boundary value problems have been studied systematically from a general point of view at least since the 1960's. Classical works in this direction are Agranovich-Vishik [1], Solonnikov [21] and Eidel'man [6]. Generalizations covering wider classes of systems and more general boundary conditions have been obtained subsequently, for example by Kozhevnikov [10], Gindikin, Volevich [9], Volevich [24], and Denk, Mennicken, Volevich [2] .

A standard approach in the analysis of boundary value problems are reduction to the boundary techniques. In this way the investigation of an initial boundary value problem is reduced to studying a system on the domain's boundary corresponding to the so-called Lopatinskij matrix. This system has two characteristic properties: it has a pseudodifferential structure (even if the underlying boundary value problem is purely differential) and it is, in general, a mixed order system. In this paper we develope a calculus of Volterra pseudodifferential operators that allows to obtain solvability results for a wide class of such kind of systems.

Our approach combines pseudodifferential techniques with ideas of Volevich [24] and Denk, Volevich [5], where boundary value problems with dynamical boundary conditions are investigated. Though quite general in nature the analysis in [5] is limited to an L_2-setting for model problems on a half-space and all involved operators have constant coefficients, both in time and space (i.e., the framework is that of Fourier multipliers rather than general pseudodifferential operators). We avoid these restrictions and treat more general problems on smoothly bounded domains and work within scales of L_p-Sobolev spaces both of Bessel potential and of Besov type. Our results also extend those of Denk, Saal, Seiler [4] where the authors use a

2010 *Mathematics Subject Classification.* 35G40, 35S05, 35R35.

Key words and phrases. Pseudodifferential operators, mixed-order systems, maximal regularity.

Kalton-Weis approach to investigate in an L_p-setting mixed order systems on the half-space, where the reduced system on the boundary has constant coefficient symbols depending on τ and $|\xi|$ only (with τ and ξ being the time and space co-variable, respectively).

The use of pseudodifferential analysis for studying partial differential equations is by now a rather classic method. The principal idea is to embed problems of a certain class of interest in an algebra of operators with a 'symbol structure' and to obtain qualitative statements on solvability by characterizing invertibility (possibly modulo good remainders) of operators within the algebra in terms of conditions on the associated symbols. By knowing the precise symbolic structure of the inverses, at the same time one also obtains information on the shape of the solutions. It was Piriou [17], [18] who introduced the concept of Volterra pseudodifferential operators to tackle parabolic problems. Roughly speaking, the novel feature of a Volterra calculus is that the pseudodifferential symbols extend in the time co-variable holomorphically to the complex lower half-plane, and satisfy there certain symbol estimates. This Volterra property has two striking consequences: it allows to modify parametrices (in the sense of elliptic theory) to become exact inverses and it ensures that the operators preserve 'vanishing in the past', i.e., if a function vanishes before a time t_0, so does its image. The latter condition is important in handling initial values, while the first implies unique solvability. The concept of the Volterra property has been employed in Krainer, Schulze [13] and Krainer [12] in the study of long-time asymptotics for parabolic equations. Though the modification of holomorphy in the covariable looks rather innocent, it brings along certain technical difficulties; for instance the standard procedure of asymptotic summation cannot be performed in the usual way, since excisions in the covariable destroy the holomorphy. We develope a calculus of Volterra pseudodifferential operators that is adapted to the kind of mixed order systems as described above, and find explicit 'parabolicity' conditions on a system that imply the existence of an inverse within the calculus, leading to unique solvability in exponentially weighted spaces.

As a straight-forward application of our calculus, we establish maximal regularity (in the sense of isomorphisms between suitable Sobolev spaces) of time-dependent Douglis-Nirenberg systems. This generalizes results of [3] where we have discussed the existence of a bounded H^∞-calculus for a (stationary) system of Douglis-Nirenberg type. Moreover, we show that the linearized Stefan problem with Gibbs-Thomson correction fits into our framework. This free boundary value problem with dynamic boundary conditions leads to an inhomogeneous structure of the Lopatinskij matrix and cannot be dealt with classical parabolic theory. We discuss when our results can be applied to problems with dynamical boundary conditions of more general form.

2. Volterra pseudodifferential operators on $\mathbb{R}^n \times \mathbb{R}$

In this section we develop a pseudodifferential calculus for symbols having the *Volterra property*, i.e., the time co-variable extends holomorphically to the lower complex half-plane. We discuss all standard properties of a pseudodifferential calculus, like composition, asymptotic summation and continuity in associated Sobolev spaces.

2.1. **Weight functions and symbol spaces.** In the sequel we let E denote a Fréchet space and set

$$\mathbb{H} := \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau \le 0\}.$$

or y belonging to $\mathbb{R}^n$ or $\mathbb{C}^n$ with some $n \in \mathbb{N}$, we write

$$\langle y\rangle := (1 + |y|^2)^{1/2}.$$

We shall use the standard multi-index notation for partial derivatives $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, where $D = -i\partial$.

Definition 2.1. *A* ***weight function*** *is a function $\omega : \mathbb{R}^n \times \mathbb{H} \to \mathbb{C}$ having the following properties:*

i) *ω depends smoothly on ξ and holomorphically on τ, i.e.*

$$\omega \in \mathscr{C}^\infty\big(\mathbb{H}, \mathscr{C}^\infty(\mathbb{R}^n)\big) \cap \mathscr{A}\big(\operatorname{int}\mathbb{H}, \mathscr{C}^\infty(\mathbb{R}^n)\big),$$

ii) *for all multi-indices $\alpha \in \mathbb{N}_0^n$ and $\beta \in \mathbb{N}_0^2$ there exist constants $C_{\alpha\beta}$ such that*

$$|D_\xi^\alpha D_\tau^\beta \omega(\xi, \tau)| \le C_{\alpha\beta}\, |\omega(\xi, \tau)| \langle\xi\rangle^{-|\alpha|} \langle\tau\rangle^{-|\beta|}$$

(identifying $\mathbb{C}$ with $\mathbb{R}^2$),

iii) *there exist constants $C, M \ge 0$ such that*

$$|\omega(\xi + \xi', \tau + \tau')| \le C\, |\omega(\xi, \tau)| \langle\xi'\rangle^M \langle\tau'\rangle^M.$$

The estimates in ii) and iii) are uniform in $\xi, \xi' \in \mathbb{R}^n$ and $\tau, \tau' \in \mathbb{H}$.

The third property implies that, for suitable positive constants $c, C \ge 0$,

$$c\langle\xi\rangle^{-M}\langle\tau\rangle^{-M} \le |\omega(\xi, \tau)| \le C\langle\xi\rangle^M\langle\tau\rangle^M. \tag{2.1}$$

From the fact that $D_\xi^\alpha D_\tau^\beta \frac{1}{\omega}$ is a finite linear combination of terms of the form

$$(D_\xi^{\alpha_1} D_\tau^{\beta_1}\omega) \cdot \ldots \cdot (D_\xi^{\alpha_k} D_\tau^{\beta_k}\omega) \frac{1}{\omega^{k+1}}$$

with $1 \le k \le |\alpha| + |\beta|$ and $\alpha_1 + \ldots + \alpha_k = \alpha$, $\beta_1 + \ldots + \beta_k = \beta$, it is obvious that with ω also $1/\omega$ is a weight function. By product rule the product of two weight functions is a weight function, again.

Definition 2.2. *For $\mu, \nu \in \mathbb{R}$ let $S^{\mu,\nu;\omega}(\mathbb{R}^n \times \mathbb{H}; E)$ consist of all smooth functions $a : \mathbb{R}^n \times \mathbb{H} \to E$ which satisfy estimates*

$$|\!|\!| D_\xi^\alpha D_\tau^\beta a(\xi, \tau) |\!|\!| \le C\, |\omega(\xi, \tau)| \langle\xi\rangle^{\mu - |\alpha|} \langle\tau\rangle^{\nu - |\beta|} \qquad \forall\ (\xi, \tau) \in \mathbb{R}^n \times \mathbb{H}$$

for all multi-indices $\alpha \in \mathbb{N}_0^n$, $\beta \in \mathbb{N}_0^2$, and each continuous semi-norm $|\!|\!| \cdot |\!|\!|$ of E (with C depending only on α, β, and the semi-norm). Moreover, let

$$S_V^{\mu,\nu;\omega}(\mathbb{R}^n \times \mathbb{H}; E) := S^{\mu,\nu;\omega}(\mathbb{R}^n \times \mathbb{H}; E) \cap \mathscr{A}\big(\operatorname{int}\mathbb{H}, \mathscr{C}^\infty(\mathbb{R}_\xi^n, E)\big)$$

be the space of all symbols from $S^{\mu,\nu;\omega}(\mathbb{R}^n \times \mathbb{H}; E)$ that additionally depend holomorphically on $\tau \in \operatorname{int}\mathbb{H}$. For notational convenience we shall often use the short-hand notation $S^{\mu,\nu;\omega}$ and $S_V^{\mu,\nu;\omega}$, respectively. If $\omega = 1$ we suppress it from the notation.

The subscript V stands for *Volterra property*; see below for further explanation. In the standard way, we may also define spaces $S^{\mu,-\infty;\omega}$, $S^{-\infty,\nu;\omega}$, $S^{-\infty,-\infty;\omega}$, and those with subscript V. In fact, the last space does not depend on ω and therefore we shall write $S^{-\infty,-\infty}_{(V)}$. All these symbol spaces are Fréchet spaces in a canonical way. Note that a weight function ω belongs to $S^{0,0;\omega}_V$ and that

$$S^{\mu,\nu;\omega}_{(V)}(\mathbb{R}^n\times\mathbb{H};E)=S^{0,0;\omega_{\mu,\nu}}_{(V)}(\mathbb{R}^n\times\mathbb{H};E),$$
$$\omega_{\mu,\nu}(\xi,\tau)=\omega(\xi,\tau)\langle\xi\rangle^{\mu}(1+i\tau)^{\nu}.$$

Hence often we can assume without loss of generality that $\mu=\nu=0$.

Due to Definition 2.1.(iii) and estimate (2.1) we obtain the embeddings

$$S^{\mu-M,\nu-M}_{(V)}\hookrightarrow S^{\mu,\nu;\omega}_{(V)}\hookrightarrow S^{\mu+M,\nu+M}_{(V)}$$

and

$$S^{\mu-M,\nu;\omega^0}_{(V)}\hookrightarrow S^{\mu,\nu;\omega}_{(V)}\hookrightarrow S^{\mu+M,\nu;\omega^0}_{(V)},\qquad \omega^0(\tau):=\omega(0,\tau),$$

as well as

$$S^{\mu,\nu-M;\omega_0}_{(V)}\hookrightarrow S^{\mu,\nu;\omega}_{(V)}\hookrightarrow S^{\mu,\nu+M;\omega_0}_{(V)},\qquad \omega_0(\xi):=\omega(\xi,0).$$

Note that Cauchy's integral formula implies that

$$S^{\mu,\nu;\omega}_{V}(\mathbb{R}^n\times\mathbb{H};E)\subset\mathscr{A}\big(\mathrm{int}\,\mathbb{H},S^{\mu;\omega_0}(\mathbb{R}^n_\xi;E)\big)$$

where $S^{\mu;\omega_0}(\mathbb{R}^n_\xi;E)$ is defined in the obvious way.

Definition 2.3. *By choosing the Frèchet space E as*

$$E_m:=\mathscr{C}^\infty_b\big(\mathbb{R}^n_x,S^m(\mathbb{R}_t)\big),\qquad m\in\mathbb{R},^{1}$$

we[2] define

$$S^{\mu,(\nu,m);\omega}_{(V)}(\mathbb{R}^n\times\mathbb{R};\mathbb{R}^n\times\mathbb{H}):=S^{\mu,\nu;\omega}_{(V)}(\mathbb{R}^n\times\mathbb{H};E_m).\tag{2.2}$$

This is then a space of symbols $a=a(x,t,\xi,\tau)$ with variables x, t and corresponding covariables ξ and τ, respectively. In case $m=0$, we write $S^{\mu,\nu;\omega}_{(V)}$ instead of $S^{\mu,(\nu,0);\omega}_{(V)}$.

The type of t-dependence of the symbols from $S^{\mu,(\nu,m);\omega}_{(V)}$ is known from the calculus of SG-pseudodifferential operators, cf. [16]. By product rule it is obvious that pointwise multiplication yields a continuous map

$$S^{\mu,(\nu,m);\omega}_{(V)}\times S^{\mu',(\nu',m');\omega'}_{(V)}\longrightarrow S^{\mu+\mu',(\nu+\nu',m+m');\omega\omega'}_{(V)}.\tag{2.3}$$

For a symbol $a\in S^{\mu,(\nu,m);\omega}$ the associated pseudodifferential operator

$$a(x,t,D_x,D_t):\mathscr{S}(\mathbb{R}^n\times\mathbb{R})\longrightarrow\mathscr{S}(\mathbb{R}^n\times\mathbb{R})$$

is defined in the usual way by

$$[a(x,t,D_x,D_t)u](x,t)=\int e^{i(x,t)(\xi,\tau)}a(x,t,\xi,\tau)\widehat{u}(\xi,\tau)\,đ(\xi,\tau),$$

[2]i.e. the space of all smooth functions $p:\mathbb{R}^n\times\mathbb{R}\to\mathbb{C}$ satisfying estimates $|D^\alpha_xD^l_tp(x,t)|\le C_{\alpha l}\langle t\rangle^{m-l}$ uniformly in $(x,t)\in\mathbb{R}^n\times\mathbb{R}$, for all α and l.

where $đ(\xi,\tau)=d(\xi,\tau)/(2\pi)^{n+1}$ and $\widehat{u}$ denotes the Fourier transform of u.

The Leibniz product $a\#b$, corresponding to the composition

$$a(x,t,D_x,D_t)b(x,t,D_x,D_t)=(a\#b)(x,t,D_x,D_t),$$

also induces a map as in (2.3), as can be deduced from the formula

$$a\#b(x,t,\xi,\tau)=\iint e^{-i(y,s)(\eta,\sigma)}a(x,t,\xi+\eta,\tau+\sigma)b(x+y,t+s,\xi,\tau)\,d(y,s)đ(\eta,\sigma),$$

where integration is to be understood as an oscillatory integral over an amplitude function on $\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ (cf. [14] for details). Composition of the operator-families $a(x,t,D_x,\tau)$ and $b(x,t,D_x,\tau)$ yields a symbol we denote by $a\#_x b$. Analogously, by passing to the operator-families with respect to t, we get a symbol $a\#_t b$.

Proposition 2.4. *Let $a\in S^{\mu,(\nu,m);\omega}_{(V)}$ and $b\in S^{\mu',(\nu',m');\omega'}_{(V)}$. Then*

$$a\#b\equiv ab+(a\#_x b-ab)+(a\#_t b-ab)\mod S^{\mu+\mu'-1,(\nu+\nu'-1,m+m'-1);\omega\omega'}_{(V)}.$$

Proof. Insert in the above formula for $a\#b$ the expansion

$$\begin{aligned}a(x,t,\xi+\eta,\tau+\rho)=&a(x,t,\xi+\eta,\tau)+\rho\int_0^1(\partial_\tau a)(x,t,\xi+\eta,\tau+\theta_1\rho)\,d\theta_1\\=&a(x,t,\xi+\eta,\tau)+a(x,t,\xi,\tau+\rho)-a(x,t,\xi,\tau)+\\&+\sum_{|\alpha|=1}\rho\eta^\alpha\int_0^1\int_0^1(\partial_\xi^\alpha\partial_\tau a)(x,t,\xi+\theta_2\eta,\tau+\theta_1\rho)\,d\theta_2 d\theta_1.\end{aligned}$$

The first term in this expansion yields $a\#_x b$, the second $a\#_t b$, the third $-ab$, while the last (after integration by parts) yields the remainder term of the stated type. □

The previous proposition shows that the Leibniz-product $a\#b$ is not determined modulo lower order terms by the pointwise product ab. This will have consequences on the parametrix construction, see below.

The terminology Volterra symbols stems from the fact that the (distributional) kernel of the associated pseudodifferential operators vanishes 'above the diagonal'. In fact, if $a\in S^{\mu,(\nu,m);\omega}_V$ then its kernel is

$$\begin{aligned}k(x,t,x',t')=&\iint e^{i(x-x',t-t')(\xi,\tau)}a(x,t,\xi,\tau)\,đ(\xi,\tau)\\=&e^{(t-t')y}\iint e^{i(x-x',t-t')(\xi,\sigma)}a(x,t,\xi,\sigma-iy)\,đ(\xi,\sigma)\end{aligned}$$

for any $y\ge 0$, due to the holomorphy in the covariable τ. Passing to the limit $y\to\infty$, we see that $k(x,x',t,t')=0$ whenever $t-t'<0$. A particular consequence is that pseudodifferential operators with Volterra property preserve the 'time-forward support' of distributions: If we set

$$\mathscr{S}(\mathbb{R}^n\times[t_0,\infty))=\{u\in\mathscr{S}(\mathbb{R}^n\times\mathbb{R})\mid u(x,t)=0\text{ if }t<t_0\}$$

with arbitrary $t_0\in\mathbb{R}$, then

$$a(x,t,D_x,D_t):\mathscr{S}(\mathbb{R}^n\times[t_0,\infty))\longrightarrow\mathscr{S}(\mathbb{R}^n\times[t_0,\infty)).$$

A further fundamental consequence of the Volterra property is the following fact on invertibilty of integral operators:

Theorem 2.5. *Let K be an integral operator with kernel $k \in \mathscr{S}(\mathbb{R}^{n+1}_{(x,t)} \times \mathbb{R}^{n+1}_{(x',t')})$ that vanishes whenever $t < t'$. Then*

$$1 + K : \mathscr{S}(\mathbb{R}^n \times \mathbb{R}) \longrightarrow \mathscr{S}(\mathbb{R}^n \times \mathbb{R})$$

is invertible and $(1+K)^{-1} = 1 + \widetilde{K}$, where $\widetilde{K}$ has the same structure as K.

Proof. First one can show that $1+K : L_2(\mathbb{R}^n \times \mathbb{R}) \to L_2(\mathbb{R}^n \times \mathbb{R})$ is invertible with $(1+K)^{-1} = 1+\widetilde{K}$, where $\widetilde{K}$ has a kernel in $L_2(\mathbb{R}^{n+1}_{(x,t)} \times \mathbb{R}^{n+1}_{(x',t')})$ that vanishes whenever $t < t'$ (see Theorem 4.2.6 in [11]). However, then

$$1 + \widetilde{K} = (1+K)^{-1} = 1 - K + K(1+K)^{-1}K$$

shows that both $\widetilde{K}$ and the L_2-adjoint $\widetilde{K}^*$ map $L_2(\mathbb{R}^n \times \mathbb{R})$ into $\mathscr{S}(\mathbb{R}^n \times \mathbb{R})$. This implies that the kernel of $\widetilde{K}$ is rapidly decreasing in all variables. In fact, first we see that $\widetilde{K}$ has an integral kernel $\widetilde{k}$ in

$$\big(\mathscr{S}(\mathbb{R}^n)\widehat{\otimes}_\pi L_2(\mathbb{R}^n)\big) \cap \big(L_2(\mathbb{R}^n)\widehat{\otimes}_\pi \mathscr{S}(\mathbb{R}^n)\big),$$

where $\widehat{\otimes}_\pi$ denotes the completed projective tensor product. However, this space coincides with $\mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n)$, since the inequality $r^a s^b \le \frac{1}{2}(r^{2a} + s^{2b})$ together with Plancherel's formula allows to estimate the $L_2(\mathbb{R}^{2n})$-norm of $x^{\alpha'} y^{\beta'} D_x^\alpha D_y^\beta \widetilde{k}(x,y)$ by the L_2-norms of $x^{\alpha'_0} D_x^{\alpha_0} \widetilde{k}(x,y)$ and $y^{\beta'_0} D_y^{\beta_0} \widetilde{k}(x,y)$, respectively. □

2.2. **Asymptotic summation of Volterra symbols.** An important feature in any pseudodifferential calculus is the possibility of summing asymptotically a sequence of symbols of decreasing order. The standard technique to achieve this involves excision of symbols in the co-variables. In the present context this technique is not applicable, since excision destroys the holomorphy of Volterra symbols. Hence an alternative approach is used, based on the so-called 'kernel cut-off' procedure, cf. [20]. Again, let E denote a Frèchet space.

Definition 2.6. *For $\mu, \nu \in \mathbb{R}$ let $\Lambda^{\mu,\nu}(\mathbb{R}^n \times \mathbb{H}; E)$ consist of all smooth functions $a : \mathbb{R}^n \times \mathbb{H} \to E$ which satisfy estimates*

$$|||D_\xi^\alpha D_\tau^\beta a(\xi,\tau)||| \le C\, \langle \xi \rangle^{\mu - |\alpha|} \langle \xi, \tau \rangle^{\nu - |\beta|}$$

uniformly in $(\xi,\tau) \in \mathbb{R}^n \times \mathbb{H}$, for all $\alpha \in \mathbb{N}_0^n$, $\beta \in \mathbb{N}_0^2$, and each continuous semi-norm of E. Similarly as above, we denote by $\Lambda_V^{\mu,\nu}(\mathbb{R}^n \times \mathbb{H})$ the subspace of all symbols that, in addition, depend holomorphically on $\tau \in \operatorname{int}\mathbb{H}$.

We may also define spaces $\Lambda^{\mu,-\infty}$, $\Lambda^{-\infty,\nu}$, $\Lambda^{-\infty,-\infty}$, and those with subscript V. Note that $S^{-\infty,-\infty} = \Lambda^{-\infty,-\infty} = \Lambda^{\mu,-\infty}$.

Lemma 2.7. *The map*

$$a(\xi,\tau) \mapsto \widehat{a}(\xi,\tau) := a(\xi, \langle \xi \rangle^{-1}\tau)$$

induces (topological) isomorphisms $S^{\mu,\nu}_{(V)}(\mathbb{R}^n \times \mathbb{H}; E) \to \Lambda^{\mu-\nu,\nu}_{(V)}(\mathbb{R}^n \times \mathbb{H}; E)$ with inverse induced by

$$a(\xi,\tau) \mapsto \widetilde{a}(\xi,\tau) := a(\xi, \langle \xi \rangle \tau).$$

In particular, $a \mapsto \widehat{a} : S^{-\infty,-\infty}_{(V)}(\mathbb{R}^n \times \mathbb{H}; E) \to \Lambda^{-\infty,-\infty}_{(V)}(\mathbb{R}^n \times \mathbb{H}; E)$ *with inverse given by* $a \mapsto \widetilde{a}$.

Proof. Clearly, holomorphy in τ is preserved. Hence it suffices to consider the classes without subscript V. We shall now use the (equivalent) identities

$$\langle \xi, \langle\xi\rangle\tau\rangle = \langle\xi\rangle\langle\tau\rangle, \qquad \langle\langle\xi\rangle^{-1}\tau\rangle = \langle\xi\rangle^{-1}\langle\xi,\tau\rangle.$$

From

$$\begin{aligned} D_{\xi_j}\widehat{a}(\xi,\tau) =&(D_{\xi_j}a)(\xi,\langle\xi\rangle^{-1}\tau) + \tau_1(D_{\xi_j}\langle\xi\rangle^{-1})(D_{\tau_1}a)(\xi,\langle\xi\rangle^{-1}\tau)\\ &+ \tau_2(D_{\xi_j}\langle\xi\rangle^{-1})(D_{\tau_2}a)(\xi,\langle\xi\rangle^{-1}\tau)\\ =&(D_{\xi_j}a)(\xi,\langle\xi\rangle^{-1}\tau) + \langle\xi\rangle(D_{\xi_j}\langle\xi\rangle^{-1})(\tau_1 D_{\tau_1}a)(\xi,\langle\xi\rangle^{-1}\tau)\\ &+ \langle\xi\rangle(D_{\xi_j}\langle\xi\rangle^{-1})(\tau_2 D_{\tau_2}a)(\xi,\langle\xi\rangle^{-1}\tau) \end{aligned}$$

and induction it easily follows that $D^\alpha_\xi\widehat{a}$ is a linear combination of terms $\widehat{b}$ with $b \in S^{\mu-|\alpha|,\nu}$. Consequently, $D^\alpha_\xi D^\beta_\tau\widehat{a}$ is a linear combination of terms $\langle\xi\rangle^{-|\beta|}\widehat{D^\beta_\tau b}$. Therefore,

$$\begin{aligned} |||D^\alpha_\xi D^\beta_\tau\widehat{a}(\xi,\tau)||| &\le C\,\langle\xi\rangle^{-|\beta|}\langle\xi\rangle^{\mu-|\alpha|}\langle\langle\xi\rangle^{-1}\tau\rangle^{\nu-|\beta|}\\ &= C\,\langle\xi\rangle^{-|\beta|}\langle\xi\rangle^{\mu-|\alpha|}\langle\xi,\tau\rangle^{\nu-|\beta|}\langle\xi\rangle^{|\beta|-\nu}\\ &= C\,\langle\xi\rangle^{\mu-\nu-|\alpha|}\langle\xi,\tau\rangle^{\nu-|\beta|}, \end{aligned}$$

provided $a \in S^{\mu,\nu}$. Vice versa, if $a \in \Lambda^{\mu,\nu}$, then $D^\alpha_\xi\widetilde{a}$ is a linear combination of terms $\widetilde{b}$ with $b \in \Lambda^{\mu-|\alpha|,\nu}$. Hence $D^\alpha_\xi D^\beta_\tau\widetilde{a}$ is a linear combination of terms $\langle\xi\rangle^{|\beta|}(D^\beta_\tau b)^\sim$. It follows that

$$\begin{aligned} |||D^\alpha_\xi D^\beta_\tau\widetilde{a}(\xi,\tau)||| &\le C\,\langle\xi\rangle^{|\beta|}\langle\xi\rangle^{\mu-|\alpha|}\langle\xi,\langle\xi\rangle\tau\rangle^{\nu-|\beta|}\\ &= C\,\langle\xi\rangle^{|\beta|}\langle\xi\rangle^{\mu-|\alpha|}\langle\xi\rangle^{\nu-|\beta|}\langle\tau\rangle^{\nu-|\beta|}\\ &= C\,\langle\xi\rangle^{\mu+\nu-|\alpha|}\langle\tau\rangle^{\nu-|\beta|}. \end{aligned}$$

This shows that $\widetilde{a}$ belongs to $S^{\mu+\nu,\nu}$. □

The main reason for introducing the symbol spaces $\Lambda^{\mu,\nu}$ is that differentiation with respect to τ improves the behaviour of the symbols both in τ and ξ. This property shall be used frequently below.

Proposition 2.8 (Kernel cut-off)**.** *Let* $\varphi \in \mathscr{C}^\infty_{\mathrm{comp}}(\mathbb{R})$ *with* $\varphi \equiv 1$ *in some neighborhood of* 0. *The map*

$$a(\xi,\tau) \mapsto [h(\varphi)a](\xi,\tau) := \iint e^{-is\sigma}\varphi(s)a(\xi,\tau-\sigma)\,ds\,đ\sigma \tag{2.4}$$

(*oscillatory integral*) *has the following properties*:

a) $h(\varphi) : \Lambda^{\mu,\nu}_V(\mathbb{R}^n \times \mathbb{H}; E) \longrightarrow \Lambda^{\mu,\nu}_V(\mathbb{R}^n \times \mathbb{H}; E)$,
b) $1 - h(\varphi) : \Lambda^{\mu,\nu}_V(\mathbb{R}^n \times \mathbb{H}; E) \longrightarrow \Lambda^{-\infty,-\infty}_V(\mathbb{R}^n \times \mathbb{H}; E)$.

Proof. The proof is completely analogous to the standard kernel cut-off construction due to [20] (see also [8]). A detailed exposition which can be followed quite closely can be found in [11]. For convenience of the reader we indicate the main steps of the proof: First of all, the definition of $h(\varphi)$ makes sense for any function $\varphi \in \mathscr{C}_b^\infty(\mathbb{R})$, and one shows by explicit regularization of the oscillatory integral that

$$|||[h(\varphi)a](\xi,\tau)||| \le C\,\langle\xi\rangle^\mu\langle\xi,\tau\rangle^\nu.$$

Together with $\partial_\xi^\alpha\partial_\tau^\beta[h(\varphi)a] = h(\varphi)(\partial_\xi^\alpha\partial_\tau^\beta a)$ this yields that $h(\varphi)$ maps into $\Lambda^{\mu,\nu}$. Also holomorphy is preserved, i.e. $h(\varphi)$ maps into $\Lambda_V^{\mu,\nu}$. A Taylor expansion of φ in $s=0$ implies that

$$h(\varphi)a = \sum_{k=0}^{N-1}\Big(\frac{(-1)^k}{k!}D_s^k\varphi(0)\Big)\partial_\tau^k a + h(\psi_{(N)})(\partial_\tau^N a),$$

where $\psi_{(N)} \in \mathscr{C}_b^\infty(\mathbb{R})$. As we have seen above, $h(\psi_{(N)})(\partial_\tau^N a) \in \Lambda_V^{\mu,\nu-N}$. Hence, if $\varphi \in \mathscr{C}_{\mathrm{comp}}^\infty(\mathbb{R})$ and $\varphi\equiv 1$ near $t=0$,

$$a - h(\varphi)a \in \bigcap_{N\in\mathbb{N}}\Lambda_V^{\mu,\nu-N} = \Lambda_V^{-\infty,-\infty}.$$

This proves the claim. □

Theorem 2.9. *Given symbols $a_j \in S_V^{\mu-l_j,\nu-l_j;\omega}(\mathbb{R}^n\times\mathbb{H};E)$, $j\in\mathbb{N}_0$, with a strictly increasing sequence $0\le l_j\to\infty$, there exists a symbol $a\in S_V^{\mu,\nu;\omega}(\mathbb{R}^n\times\mathbb{H};E)$ such that*

$$a - \sum_{j=0}^{N-1} a_j \;\in\; S_V^{\mu-l_N,\nu-l_N;\omega}(\mathbb{R}^n\times\mathbb{H};E) \qquad \forall\, N\in\mathbb{N}_0.$$

We write $a\sim_V \sum_{j=0}^{\infty} a_j$. The symbol a is uniquely determined modulo $S_V^{-\infty,-\infty}$.

Proof. By multiplication with $1/\omega$ we may assume without loss of generality that $\omega=1$. Using Lemma 2.7 we have $\widehat{a}_j\in\Lambda_V^{\mu-\nu,\nu-l_j}$ for each $j\in\mathbb{N}_0$. Following the proof of Theorem 2.1.16 of [11], there exists a sequence of real numbers $(c_j)_j$ with $1\le c_j\to\infty$ such that $b_k := \sum_{j=k}^{\infty} h(\varphi_{c_j})\,\widehat{a}_j$ converges in $\Lambda_V^{\mu-\nu,\nu-l_k}$ for any $k\in\mathbb{N}_0$, where $\varphi_c(t):=\varphi(ct)$ with a fixed $\varphi\in\mathscr{C}_{\mathrm{comp}}^\infty(\mathbb{R})$ being equal to 1 in a neighborhood of $t=0$. Due to Proposition 2.8.b),

$$b_0 - \sum_{j=0}^{N-1}\widehat{a}_j = b_N - \sum_{j=0}^{N-1}(1-h(\varphi_{c_j}))\,\widehat{a}_j \;\in\; \Lambda_V^{\mu-\nu,\nu-l_N}.$$

Then the result follows for $a:=\widetilde{b_0}$, because we have

$$a - \sum_{j=0}^{N-1} a_j = \Big(b_0 - \sum_{j=0}^{N-1}\widehat{a}_j\Big)^{\sim} \in\; S_V^{\mu-l_N,\nu-l_N}$$ □

Let us mention here an alternative way of proving the previous theorem (and, in fact, a slight generalization). To this end assume without loss of generality that $\omega=1$ and let us denote by $H(\varphi)$ the kernel cut-off operator given by (2.4), but now acting on symbols $S_V^{\mu,\nu}(\mathbb{R}^n\times\mathbb{H};E)$. This yields a map $H(\varphi): S_V^{\mu,\nu}(\mathbb{R}^n\times\mathbb{H};E)\to S_V^{\mu,\nu}(\mathbb{R}^n\times\mathbb{H};E)$ with

$$1-H(\varphi): S_V^{\mu,\nu}(\mathbb{R}^n\times\mathbb{H};E)\longrightarrow S_V^{\mu,-\infty}(\mathbb{R}^n\times\mathbb{H};E).$$

If $\psi=\psi(\xi)$ is a zero excision function, multiplication with ψ yields a map $\psi: S_V^{\mu,\nu}(\mathbb{R}^n\times\mathbb{H};E)\to S_V^{\mu,\nu}(\mathbb{R}^n\times\mathbb{H};E)$ with

$$1-\psi: S_V^{\mu,\nu}(\mathbb{R}^n\times\mathbb{H};E)\longrightarrow S_V^{-\infty,\nu}(\mathbb{R}^n\times\mathbb{H};E).$$

Note that ψ and $H(\varphi)$ are commuting. Now let us define

$$\chi(c):=\psi_{1/c}+H(\varphi_c)-\psi_{1/c}H(\varphi_c),\qquad c>0,$$

where $f_c:=f(c\,\cdot)$. Then $\chi(c): S_V^{\mu,\nu}(\mathbb{R}^n\times\mathbb{H};E)\to S_V^{\mu,\nu}(\mathbb{R}^n\times\mathbb{H};E)$ and

$$1-\chi(c): S_V^{\mu,\nu}(\mathbb{R}^n\times\mathbb{H};E)\longrightarrow S_V^{-\infty,-\infty}(\mathbb{R}^n\times\mathbb{H};E),$$

since

$$1-\chi(c)=(1-\psi_{1/c})(1-H(\varphi_c)).$$

Moreover, one can show that

$$\chi(c)a\xrightarrow{c\to\infty}0 \text{ in } S_V^{\mu,\nu}(\mathbb{R}^n\times\mathbb{H};E)$$

whenever $a\in S_V^{\mu-\varepsilon,\nu}(\mathbb{R}^n\times\mathbb{H};E)\cap S_V^{\mu,\nu-\varepsilon}(\mathbb{R}^n\times\mathbb{H};E)$ for some $\varepsilon>0$.
Given a sequence of symbols $a_j\in S_V^{\mu_j,\nu_j}(\mathbb{R}^n\times\mathbb{H};E)$, $j\in\mathbb{N}_0$, with strictly decreasing sequences $\mu_j,\nu_j\xrightarrow{j\to\infty}-\infty$, one can construct a sequence $c_j\xrightarrow{j\to\infty}\infty$ such that $\sum\limits_{j=k}^{\infty}\chi(c_j)a_j$ converges in $S_V^{\mu_k,\nu_k}(\mathbb{R}^n\times\mathbb{H};E)$ for any $k\in\mathbb{N}_0$, see Proposition 1.1.17 of [20]. Thus $a\sim_V\sum\limits_{j=0}^{\infty}a_j$ for $a:=\sum\limits_{j=0}^{\infty}\chi(c_j)a_j$.

2.3. **Sobolev and Besov spaces.** In this section pseudodifferential operators are shown to extend from mappings between the rapidly decreasing functions to continuous maps in suitable distributional spaces.

Definition 2.10. *For $1<p<\infty$ the Sobolev spaces (in the sense of Bessel potential spaces) with respect to a weight function κ are defined as*

$$H_p^\kappa(\mathbb{R}^n\times\mathbb{R}):=\big\{u\in\mathscr{S}'(\mathbb{R}^n\times\mathbb{R})\mid \kappa(D_x,D_t)u\in L_p(\mathbb{R}^n\times\mathbb{R})\big\}\tag{2.5}$$

with the canonical norm

$$\|u\|_{\kappa,p}:=\|\kappa(D_x,D_t)u\|_{L_p(\mathbb{R}^n\times\mathbb{R})}.$$

If $t_0\in\mathbb{R}$ we set

$$H_p^\kappa(\mathbb{R}^n\times[t_0,\infty)):=\big\{u\in H_p^\kappa(\mathbb{R}^n\times\mathbb{R})\mid \operatorname{supp}u\subset\mathbb{R}^n\times[t_0,\infty)\big\}.$$

This is a closed subspace of $H_p^\kappa(\mathbb{R}^n\times\mathbb{R})$. Analogously we introduce the Besov spaces

$$B_{pp}^\kappa(\mathbb{R}^n\times\mathbb{R}),\qquad B_{pp}^\kappa(\mathbb{R}^n\times[t_0,\infty))$$

by replacing $L_p(\mathbb{R}^n\times\mathbb{R})$ in (2.5) by $B_{pp}^0(\mathbb{R}^n\times\mathbb{R})$.

We need the following simple observation concerning the invertibility of maps between interpolation spaces.

Lemma 2.11. *Let X_0, X_1, Y_0, Y_1 be Banach spaces contained in $\mathscr{S}'(\mathbb{R}^l)$ which contain $\mathscr{S}(\mathbb{R}^l)$ as a dense subset. Let*

$$T : \mathscr{S}'(\mathbb{R}^l) \to \mathscr{S}'(\mathbb{R}^l)$$

be a (topological) isomorphism that restricts to isomorphisms $\mathscr{S}(\mathbb{R}^l) \to \mathscr{S}(\mathbb{R}^l)$, $X_0 \to Y_0$, and $X_1 \to Y_1$. Then T restricts to an isomorphism

$$T : (X_0, X_1)_{\theta,p} \longrightarrow (Y_0, Y_1)_{\theta,p},$$

where $(\cdot, \cdot)_{\theta,p}$ denotes the real interpolation method. The inverse is obtained by restricting $T^{-1} : \mathscr{S}'(\mathbb{R}^l) \to \mathscr{S}'(\mathbb{R}^l)$ to $(Y_0, Y_1)_{\theta,p}$.

Proof. Let us write $X = (X_0, X_1)_{\theta,p}$ and $Y = (Y_0, Y_1)_{\theta,p}$. By interpolation, T restricts to a continuous map $T : X \to Y$. Also, T^{-1} restricts to a continuous map $T^{-1} : Y \to X$. However, on $\mathscr{S}(\mathbb{R}^l)$ we have $TT^{-1} = T^{-1}T = 1$. Hence the result follows from density of $\mathscr{S}(\mathbb{R}^l)$ both in X and Y. □

Corollary 2.12. *Let κ be a weight function and*

$$\kappa_s(\xi, \tau) = (\langle\xi\rangle + i\tau)^s \kappa(\xi, \tau).$$

Then, for any real numbers $s_0 \neq s_1$,

$$B^{\kappa_s}_{pp}(\mathbb{R}^n \times \mathbb{R}) = \big(H^{\kappa_{s_0}}_p(\mathbb{R}^n \times \mathbb{R}), H^{\kappa_{s_1}}_p(\mathbb{R}^n \times \mathbb{R})\big)_{\theta,p}, \qquad s = \theta s_0 + (1-\theta)s_1.$$

Proof. For notational convenience, we suppress writing $\mathbb{R}^n \times \mathbb{R}$. The previous lemma applied to $T := \kappa(D_x, D_t)^{-1}$ shows that

$$\kappa(D_x, D_t)^{-1} : (H^{s_0}_p, H^{s_1}_p)_{\theta,p} \longrightarrow (H^{\kappa_{s_0}}_p, H^{\kappa_{s_1}}_p)_{\theta,p}$$

isomorphically. On the other hand, by definition,

$$\kappa(D_x, D_t) : B^{\kappa_s}_{pp} \xrightarrow{\cong} B^s_{pp} = (H^{s_0}, H^{s_1})_{\theta,p}.$$

This already implies the claim. □

Theorem 2.13. *Let κ and ω be weight functions, $a \in S^{0,0;\omega}(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^n \times \mathbb{H})$, and $t_0 \in \mathbb{R}$. Then $a(x, t, D_x, D_t)$ induces continuous operators*

$$H^\kappa_p(\mathbb{R}^n \times \mathbb{R}) \longrightarrow H^{\kappa/\omega}_p(\mathbb{R}^n \times \mathbb{R}), \qquad B^\kappa_{pp}(\mathbb{R}^n \times \mathbb{R}) \longrightarrow B^{\kappa/\omega}_{pp}(\mathbb{R}^n \times \mathbb{R}).$$

In case a has the Volterra property these maps also restrict to

$$H^\kappa_p(\mathbb{R}^n \times [t_0, \infty)) \longrightarrow H^{\kappa/\omega}_p(\mathbb{R}^n \times [t_0, \infty)),$$
$$B^\kappa_{pp}(\mathbb{R}^n \times [t_0, \infty)) \longrightarrow B^{\kappa/\omega}_{pp}(\mathbb{R}^n \times [t_0, \infty)).$$

Proof. By definition of the Sobolev spaces, the first mapping property is equivalent to the continuity of

$$\Big(\frac{\kappa}{\omega}\#a\#\frac{1}{\kappa}\Big)(x, t, D_x, D_t) : L_p(\mathbb{R}^n \times \mathbb{R}) \longrightarrow L_p(\mathbb{R}^n \times \mathbb{R}).$$

This Leibniz-product belongs to the symbol space $S^{0,0;\frac{\kappa}{\omega}\,\omega\,\frac{1}{\kappa}} = S^{0,0;1} = S^{0,0}$. Hence the result follows from Theorem 1 in [25] on the continuity of pseudodifferential operators. The continuity in Sobolev spaces together with interpolation gives the continuity in Besov spaces (e.g., choose above $s_0 = -1$, $s_1 = 1$, and $\theta = 1/2$ to express $B^\kappa_{pp}(\mathbb{R}^n \times \mathbb{R})$ as an interpolation space between two Sobolev spaces). The preservation of the 'time-forward support' is due to the Volterra property, cf. the above discussion at the end of Section 2.1. □

Like Theorem 2.13 many of our results are valid both for Sobolev spaces and for Besov spaces. Whenever this is the case we will indicate this by simply using the short-hand notation

$$\mathscr{H}^\kappa_p(\mathbb{R}^n \times \mathbb{R}), \qquad \mathscr{H}^\kappa_p(\mathbb{R}^n \times [t_0, \infty)). \tag{2.6}$$

3. Parabolicity for symbols with the Volterra property

The main idea of any calculus of Volterra pseudodifferential operators is that under suitable invertibilty conditions on the symbol one can construct an inverse within the calculus. To this end one first constructs a parametrix (similar to standard elliptic theory, but preserving the Volterra property) and then , in a second step, modifies this parametrix to an exact inverse using the Volterra property.

3.1. **Construction of a parametrix.** We shall derive conditions characterizing the existence of a parametrix. These conditions are related to those obtained in [22] in connection with the analysis of 'bisingular' pseudodifferential operators on products of manifolds.

Definition 3.1. *A symbol $p \in S_V^{0,0;1/\omega}(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^n \times \mathbb{H})$ is called a parametrix of $a \in S_V^{0,0;\omega}(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^n \times \mathbb{H})$ if both $a\#p - 1$ and $p\#a - 1$ belong to $S_V^{-\infty,-\infty}(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^n \times \mathbb{H})$.*

Lemma 3.2. *For every real $\sigma \geq 0$ the translation operator $T_{i\sigma}$ defined by*

$$(T_{i\sigma} a)(\xi, \tau) := a(\xi, \tau - i\sigma)$$

induces a maps $\Lambda_V^{0,0}(\mathbb{R}^n \times \mathbb{H}; E) \to \Lambda_V^{0,0}(\mathbb{R}^n \times \mathbb{H}; E)$ and $S_V^{0,0;\omega}(\mathbb{R}^n \times \mathbb{H}; E) \to S_V^{0,0;\omega}(\mathbb{R}^n \times \mathbb{H}; E)$ with the property that

$$1 - T_{i\sigma} : \Lambda_V^{0,0}(\mathbb{R}^n \times \mathbb{H}; E) \longrightarrow \Lambda_V^{0,-1}(\mathbb{R}^n \times \mathbb{H}; E),$$
$$1 - T_{i\sigma} : S_V^{0,0;\omega}(\mathbb{R}^n \times \mathbb{H}; E) \longrightarrow S_V^{0,-1;\omega}(\mathbb{R}^n \times \mathbb{H}; E).$$

Proof. Follows directly from

$$a(\xi, \tau - i\sigma) - a(\xi, \tau) = -i\sigma \int_0^1 (\partial_\tau a)(\xi, \tau - i\theta\sigma)\, d\theta.$$ □

Proposition 3.3. *Let $a \in S_V^{0,0}(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^n \times \mathbb{H})$ and set $A(t, \tau) := a(x, t, D_x, \tau)$, considered as an element of $S^0\big(\mathbb{R} \times \mathbb{H}; \mathscr{L}(L_p(\mathbb{R}^n))\big)$. If A is pointwise invertible and*

$$\sup_{(t,\tau) \in \mathbb{R} \times \mathbb{H}} \|A(t, \tau)^{-1}\|_{\mathscr{L}(L_p(\mathbb{R}^n))} < \infty,$$

then there exists a symbol $b \in S_V^{0,0}(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^n \times \mathbb{H})$ such that

$$A(t, \tau)^{-1} = b(x, t, D_x, \tau) \qquad \forall\ (t, \tau) \in \mathbb{R} \times \mathbb{H}.$$

Proof. The proof of this proposition is a parameter-dependent version of the standard proof of the spectral invariance of pseudodifferential operators, see [19] for example. From the spectral invariance of pseudodifferential operators on $\mathbb{R}^n$ it follows that there exists a $b(x,t,\xi,\tau) \in \mathscr{C}^\infty(\mathbb{R}\times\mathbb{H}, S^0(\mathbb{R}^n\times\mathbb{R}^n))$ which also depends holomorphically on $\tau \in \operatorname{int}\mathbb{H}$ such that $A(t,\tau)^{-1} = b(x,t,D_x,\tau)$ for all t and τ. To show that b belongs to $S_V^{0,0} = S_V^0\big(\mathbb{R}\times\mathbb{H}; S^0(\mathbb{R}^n\times\mathbb{R}^n)\big)$ it suffices to show that b is bounded as a function $\mathbb{R}\times\mathbb{H} \to S^0(\mathbb{R}^n\times\mathbb{R}^n)$; in fact, the bounds for derivatives then follow by chain rule. To this end let us use the following notation: If $T : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ is a linear operator we define the commutators

$$\operatorname{ad}(x_j)T := [x_j, T], \qquad \operatorname{ad}(D_{x_j})T := [D_{x_j}, T],$$

where x_j refers to the operator of multiplication with the function $x \mapsto x_j$. For multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ we define the iterated commutators

$$\operatorname{ad}^\beta(x)T = \operatorname{ad}(x_1)^{\beta_1}\cdots\operatorname{ad}(x_n)^{\beta_n}T,$$
$$\operatorname{ad}^\alpha(D)T = \operatorname{ad}(D_1)^{\alpha_1}\cdots\operatorname{ad}(D_n)^{\alpha_n}T.$$

It is well-known that T is a pseudodifferential operator with symbol in $S^0(\mathbb{R}^n\times\mathbb{R}^n)$ if, and only if,

$$\operatorname{ad}^\alpha(D)\operatorname{ad}^\beta(x)T \;\in\; \mathscr{L}(H_p^s(\mathbb{R}^n), H_p^{s+|\beta|}(\mathbb{R}^n))$$

for any multi-indices α, β, and some $s \in \mathbb{R}$ (more precisely, the operator on the left-hand side, which is defined on $\mathscr{S}(\mathbb{R}^n)$, has a continuous extension to an operator belonging to the right-hand side). Moreover, the topology induced by the system of semi-norms

$$p_{\alpha,\beta}(r) = \big\|\operatorname{ad}^\alpha(D)\operatorname{ad}^\beta(x)\, r(x,D)\big\|_{\mathscr{L}(H_p^s(\mathbb{R}^n), H_p^{s+|\beta|}(\mathbb{R}^n))}, \qquad \alpha,\beta \in \mathbb{N}_0,$$

coincides with the standard topology on the symbol space $S^0(\mathbb{R}^n\times\mathbb{R}^n)$. Hence we have to show that $p_{\alpha,\beta}\big(A(t,\tau)^{-1}\big)$ is uniformly bounded in (t,τ) for any multi-indices α,β. To this end observe that in the assumption on A we can replace $L_p(\mathbb{R}^n)$ by any space $H_p^s(\mathbb{R}^n)$ with $s \in \mathbb{R}$. In fact, if $0 \le s \le 1$, then

$$\Lambda^{-s}A(t,\tau)^{-1}\Lambda^s + \Lambda^{-s}A(t,\tau)^{-1}[A(t,\tau),\Lambda^s]A(t,\tau)^{-1}$$

is an inverse of $A(t,\tau) : H_p^s(\mathbb{R}^n) \to H_p^s(\mathbb{R}^n)$, where $\Lambda^s = \langle D_x\rangle^s$ is the standard reduction of orders. Then one iterates the procedure for $j \le s \le j+1$ and $j = 1,2,3,\ldots$. Negative s are treated similarly by using

$$\Lambda^{-s}A(t,\tau)^{-1}\Lambda^s + A(t,\tau)^{-1}[A(t,\tau),\Lambda^{-s}]A(t,\tau)^{-1}\Lambda^s;$$

for details see [19]. Now it remains to observe that $\operatorname{ad}^\alpha(D_x)\operatorname{ad}^\beta(x)\,A(t,\tau)^{-1}$ is a finite linear combination of terms of the form

$$A(t,\tau)^{-1}\big(\operatorname{ad}^{\alpha_1}(D)\operatorname{ad}^{\beta_1}(x)\,A(t,\tau)\big)A(t,\tau)^{-1}\times\ldots$$
$$\ldots\times\big(\operatorname{ad}^{\alpha_k}(D)\operatorname{ad}^{\beta_k}(x)\,A(t,\tau)\big)A(t,\tau)^{-1}$$

with $\alpha_1 + \ldots \alpha_k = \alpha$ and $\beta_1 + \ldots + \beta_k = \beta$. □

Theorem 3.4. *For $a \in S_V^{0,0;\omega}$ the following statements are equivalent:*

a) *There exist symbols* $b_1, b_2 \in S_V^{0,0;1/\omega}$ *such that, for some* $\varepsilon > 0$,

$$a\#_x b_1 - 1,\ b_1\#_x a - 1 \ \in\ S_V^{0,-\varepsilon},$$
$$a\#_t b_2 - 1,\ b_2\#_t a - 1 \ \in\ S_V^{-\varepsilon,0}. \tag{3.1}$$

b) *There exist symbols* $\widetilde{b}_1, \widetilde{b}_2 \in S_V^{0,0;1/\omega}$ *such that*

$$(T_{i\sigma}a)\#_x\widetilde{b}_1 - 1 = \widetilde{b}_1\#_x(T_{i\sigma}a) - 1 = 0,$$
$$\chi_2(a\#_t\widetilde{b}_2 - 1) = \chi_2(\widetilde{b}_2\#_t a - 1) = 0 \tag{3.2}$$

for some $\sigma \geq 0$ *and some zero excision function* $\chi_2(\xi)$.

c) a *has a parametrix.*

Proof. a) $\Rightarrow$ b): Since $r_1 := 1 - a\#_x b_1$ belongs to $S^{0,-\varepsilon}$, we find a $\sigma > 0$ such that

$$\|(T_{i\sigma}r_1)(x,t,D_x,\tau)\|_{\mathscr{L}(L_2(\mathbb{R}^n))} \leq \frac{1}{2} \qquad \forall\ (t,\tau) \in \mathbb{R}\times\mathbb{H}.$$

Proposition 3.3 implies that there exists a symbol $b \in S_V^{0,0}$ such that

$$b(x,t,D_x,\tau) = \big(1 - (T_{i\sigma}r_1)(x,t,D_x,\tau)\big)^{-1} \qquad \forall\ (t,\tau) \in \mathbb{R}\times\mathbb{H}.$$

It follows that $(T_{i\sigma}a)\#_x(T_{i\sigma}b_1)\#_x b = 1$, i.e. we choose $\widetilde{b}_1 = (T_{i\sigma}b_1)\#_x b$. The construction of $\widetilde{b}_2$ is done similarly, using $b(x,t,x,D_t) = \big(1 - \chi(\xi)r_2(x,t,x,D_t)\big)^{-1}$ with $r_2 := 1 - a\#_t b_2$ and a suitable zero excision function χ. The claim then follows for $\widetilde{b}_2 := b_2\#_t b$ and a χ_2 with $\chi_2\chi = \chi_2$.

b) $\Rightarrow$ c): By the first assumption in (3.2)

$$r_1 := 1 - (T_{i\sigma}a)\widetilde{b}_1 = (T_{i\sigma}a)\#_x\widetilde{b}_1 - (T_{i\sigma}a)\widetilde{b}_1 \ \in\ S_V^{-1,0}.$$

Choosing zero excision functions $\widetilde{\chi}(\xi), \widetilde{\chi}_2(\xi)$ with $\widetilde{\chi}_2\widetilde{\chi} = \widetilde{\chi}_2$ and $|\widetilde{\chi}r_1| \leq \frac{1}{2}$, we obtain $\widetilde{\chi}_2(T_{i\sigma}a)^{-1} = \widetilde{\chi}_2\widetilde{b}_1(1-\chi r_1)^{-1} \in S^{0,0;\omega}$. Without loss of generality we may assume $\widetilde{\chi}_2 = \chi_2$.[3] We thus can define

$$b := \widetilde{b}_1 + \chi_2\widetilde{b}_2 - \chi_2(T_{i\sigma}a)^{-1} \ \in\ S^{0,0;\omega}.$$

By direct computation,

$$\widetilde{b}_1 - b = \chi_2(T_{i\sigma}a)^{-1}\big(T_{i\sigma}(1 - a\widetilde{b}_2)\big) - \chi_2(1 - T_{i\sigma})\widetilde{b}_2.$$

Since $\chi_2(1 - a\widetilde{b}_2) = \chi_2(a\#_t\widetilde{b}_2 - a\widetilde{b}_2) \in S_V^{0,-1}$ and $1 - T_{i\sigma} : S_V^{0,0;1/\omega} \to S_V^{0,-1;1/\omega}$ due to Lemma 3.2, we obtain $\widetilde{b}_1 - b \in S_V^{0,-1;1/\omega}$. Similarly $\widetilde{b}_2 - b \in S_V^{-1,0;1/\omega}$, since

$$\widetilde{b}_2 - b = (1-\chi_2)(\widetilde{b}_2 - \widetilde{b}_1) + \chi_2(T_{i\sigma}a)^{-1}(1 - (T_{i\sigma}a)\widetilde{b}_1).$$

This yields

$$a\#_x b \equiv a\#_x\widetilde{b}_1 = (T_{i\sigma}a)\#_x\widetilde{b}_1 + \big((1-T_{i\sigma})a\big)\#_x\widetilde{b}_1 \equiv 1 \mod S_V^{0,-1},$$
$$a\#_t b \equiv a\#_t\widetilde{b}_2 = \chi_2(a\#_t\widetilde{b}_2) + (1-\chi_2)(a\#_t\widetilde{b}_2) \equiv \chi_2 \equiv 1 \mod S_V^{-1,0}.$$

[3] Choose $\widetilde{\chi}_2$ in such a way that $\widetilde{\chi}_2\chi_2 = \widetilde{\chi}_2$. Then the assumption remains true for $\widetilde{\chi}_2$ instead of χ_2.

Therefore,

$$\begin{aligned} a\#b &\equiv ab + (a\#_x b - ab) + (a\#_t b - ab) \\ &= \begin{cases} 1 + (a\#_x b - 1) + (a\#_t b - ab) \equiv 1 & \mod S_V^{0,-1} \\ 1 + (a\#_x b - ab) + (a\#_t b - 1) \equiv 1 & \mod S_V^{-1,0} \end{cases}. \end{aligned}$$

It follows that $r := 1 - a\#b \in S_V^{-1,0} \cap S_V^{0,-1} \subset S_V^{-\frac{1}{2},-\frac{1}{2}}$. By the standard von Neumann series argument, using Theorem 2.9, we can construct a right-parametrix p_R. Analogously there exists a left-parametrix p_L. Then we can choose p both as p_L or p_R.

c) $\Rightarrow$ a): If p is a parametrix of a then

$$1 \equiv a\#p \equiv ap + (a\#_x p - ap) + (a\#_t p - ap) \mod S^{-1,-1}.$$

Hence $a\#_x p - 1 \equiv -(a\#_t p - ap) \equiv 0$ modulo $S_V^{0,-1}$, i.e., $a\#_x p - 1 \in S_V^{0,-1}$. Analogously, $a\#_t p - 1 \in S_V^{-1,0}$. Thus (3.1) holds with $b_1 = b_2 = p$ and $\varepsilon = 1$. □

3.2. **Symbols with coupling property.** It is desirable to derive the existence of a parametrix by more simple conditions than those given in Theorem 3.4. To this end let us introduce the following notion:

Definition 3.5. *Let $J \subset \mathbb{R}$ be a closed interval.*[4] *We call $a \in S_V^{0,0;\omega}(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n \times \mathbb{H})$* **weakly parabolic on J** *if there exists an open interval I containing J such that*

$$|a(x,t,\xi,\tau)| \ge C\,\omega(\xi,\tau) \qquad \forall\,(x,t) \in \mathbb{R}^n \times I \quad \forall\,|(\xi,\tau)| \ge R, \tag{3.3}$$

for some constants $C > 0$ and $R \ge 0$.

In general, weak parabolicity on $\mathbb{R}$ of the symbol $a \in S_V^{0,0;\omega}(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^n \times \mathbb{H})$ does not imply the existence of a parametrix. However, as we shall show below, this is true when additionally imposing that a has the **coupling property**

$$D_{\xi_j} a,\ D_\tau a \ \in\ S_V^{-\varepsilon,-\varepsilon;\omega}(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^n \times \mathbb{H}), \qquad j = 1, \ldots, n, \tag{$\mathbf{C}$}$$

for some $\varepsilon > 0$, i.e., the decay improves simultaneously in both covariables even if derivatives are only taken with respect to one of the covariables. Intuitively this means that there is some coupling between the covariables ξ and τ. For example, symbols from the anisotropic symbol class considered in [11] satisfy ($\mathbf{C}$) (for a suitable choice of ω).

Theorem 3.6. *Let $a \in S_V^{0,0;\omega}(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n \times \mathbb{H})$ satisfy at least one of the following assumptions*:

(i) *a has constant coefficients,*
(ii) *a has the coupling property* ($\mathbf{C}$).

Then a has a parametrix if and only if it is weakly parabolic on $\mathbb{R}$.

Proof. First assume that a has constant coefficients. Clearly estimate (3.3) follows from the existence of a parametrix. For the reverse implication we may assume without loss of generality that $\omega = 1$. We shall construct a parametrix. By the von Neumann series argument it suffices

[4] This includes $J = \mathbb{R}$ and $J = [t_0, \infty)$.

to find a parametrix modulo $S_V^{-1,-1}$. We make use of the mappings $\hat{\ }$ and $\tilde{\ }$ from Lemma 2.7. Observe that $|(\xi,\langle\xi\rangle^{-1}\tau)|\ge R$ whenever $|\xi|\ge R$. If $|\xi|\le R$ then $|(\xi,\langle\xi\rangle^{-1}\tau)|\ge\langle\xi\rangle^{-1}|\tau|\ge R$ whenever $|\tau|\ge R\langle R\rangle$. It follows that

$$|\widehat{a}(\xi,\tau)^{-1}|\le C \qquad \forall\,|(\xi,\tau)|\ge S:=R\sqrt{1+\langle R\rangle^2}.$$

Hence, for $\sigma\ge S$,

$$|(T_{i\sigma}\widehat{a})(\xi,\tau)^{-1}|\le C \qquad \forall\,(\xi,\tau)\in\mathbb{R}^n\times\mathbb{H},$$

i.e. $(T_{i\sigma}\widehat{a})^{-1}\in\Lambda_V^{0,0}$. Now let $b:=[(T_{i\sigma}\widehat{a})^{-1}]^{\sim}$. Then $b\in S_V^{0,0}$ and

$$\widehat{ab}=\widehat{a}(T_{i\sigma}\widehat{a})^{-1}=(\widehat{a}-T_{i\sigma}\widehat{a})(T_{i\sigma}\widehat{a})^{-1}+1\equiv 1 \mod \Lambda_V^{0,-1}$$

according to Lemma 3.2. Applying the map $\tilde{\ }$ we obtain $ab-1\in S_V^{-1,-1}$. The argument for $ba-1$ is the same.

Now assume that a has the coupling property and that a^{-1} satisfies estimate (3.3). Then, for sufficiently large σ,

$$b:=(T_{i\sigma}a)^{-1}\;\in\;S_V^{0,0;1/\omega}.$$

By chain-rule also b satisfies (**C**). By (**C**), $a-T_{i\sigma}a\;\in\;S_V^{-\varepsilon,-\varepsilon;\omega}$ (see the formula in the proof of Lemma 3.2). Hence

$$ab=(a-T_{i\sigma}a)b+(T_{i\sigma}a)b\equiv 1 \mod S_V^{-\varepsilon,-\varepsilon}.$$

Moreover,

$$a\#_x b-ab\equiv(T_{i\sigma}a)\#_x b-(T_{i\sigma}a)b \mod S_V^{-\varepsilon,-\varepsilon}.$$

However, the symbol on the right-hand side equals

$$\sum_{|\alpha|=1}\int_0^1\Big\{\iint e^{-iy\eta}(\partial_\xi^\alpha a)(x,t,\xi+\theta\eta,\tau-i\sigma)(D_x^\alpha b)(x+y,t,\xi,\tau)\,dy\,\bar{d}\eta\Big\}\,d\theta,$$

(oscillatory integral) which is easily seen to belong to $S_V^{-\varepsilon,-\varepsilon}$ due to (**C**). Thus $a\#_x b-ab\in S_V^{-\varepsilon,-\varepsilon}$. Analogously one shows that $a\#_t b-ab\in S_V^{-\varepsilon,-\varepsilon}$. Altogether we have obtained that

$$a\#b-1\equiv(ab-1)+(a\#_x b-ab)+(a\#_t b-ab)\equiv 0 \mod S_V^{-\varepsilon,-\varepsilon}.$$

Arguing in the same way, also $b\#a-1\in S_V^{-\varepsilon,-\varepsilon}$. With the standard von Neumann series argument we then can construct a parametrix to a.

If a has the coupling property and possesses a parametrix p, we write

$$ap-1\equiv(a\#p-1)-(a\#_x p-ap)-(a\#_t p-ap) \mod S_V^{-1,-1}.$$

Due to (**C**), the second and third summand on the right-hand side belong to $S_V^{-\varepsilon,-\varepsilon}$ (replace above $T_{i\sigma}a$ by a and b by p). Hence $ap-1\in S_V^{-\varepsilon,-\varepsilon}$ and the desired estimate follows. □

4. Equations on a space-time cylinder with closed cross-section

While in the previous section we considered operators on $\mathbb{R}^n\times\mathbb{R}$ we shall now focus on operators on $X\times\mathbb{R}$ for a smooth closed manifold X.

4.1. **Invariance under coordinate changes.** Consider a weight function ω which has the following property: For each constant $M \geq 1$ there exists a constant $c \geq 1$ such that

$$\frac{1}{c}\,|\omega(\xi,\tau)| \leq |\omega(\eta,\tau)| \leq c\,|\omega(\xi,\tau)| \qquad \forall\,\tau \quad \forall\,M\,\frac{1}{|\xi|} \leq |\eta| \leq M\,|\xi|. \tag{4.1}$$

Moreover, let $\kappa : \mathbb{R}^n \to \mathbb{R}^n$ a smooth diffeomorphism satisfying

$$D_j\kappa_{kl} \in \mathscr{C}_b^\infty(\mathbb{R}^n) \qquad \forall\, j,k,l = 1,\ldots,n,$$

and

$$\frac{1}{c} \leq |\det \kappa'(x)| \leq c \qquad \forall\, x \in \mathbb{R}^n$$

for some constant $c \geq 1$. Note that then the inverse diffeomorphism κ^{-1} has the analogous properties.

Now let $a \in S_V^{0,0;\omega}(\mathbb{R}^n \times \mathbb{R};\mathbb{R}^n \times \mathbb{H})$. We want to show that the push-forward of the associated operator under the coordinate change $\kappa \times 1$, i.e., $(y,t) = (\kappa(x),t)$, belongs to the same class. Recall (see the embeddings stated before Definition 2.3) that for $\omega^0(\tau) := \omega(0,\tau)$ we have

$$S_V^{0,0;\omega}(\mathbb{R}^n \times \mathbb{R};\mathbb{R}^n \times \mathbb{H}) \subset S_V^{0;\omega^0}\big(\mathbb{R};\mathbb{H};S^M(\mathbb{R}^n;\mathbb{R}^n)\big)$$

for some $M \geq 0$. As we only change the x-coordinate we can deduce from the invariance of operators on $\mathbb{R}^n$ with symbols belonging to $S^m(\mathbb{R}^n \times \mathbb{R}^n)$ that

$$(\kappa \times 1)_* a(x,t,D_x,D_t) = b(x,t,D_x,D_t)$$

with a symbol $b \in S_V^{0;\omega^0}\big(\mathbb{R};\mathbb{H};S^M(\mathbb{R}^n;\mathbb{R}^n)\big)$. The standard formula for the asymptotic expansion of b reads then as

$$b(y,t,\eta,\tau) \equiv \sum_{|\alpha|=0}^{N-1} (D_\xi^\alpha a)\big(\kappa^{-1}(y),t,\varphi(y)\eta,\tau\big)\cdot\Phi_\alpha(y,\eta) \;+\; r_N(y,t,\eta,\tau)$$

with a remainder

$$r_N \in S_V^{0;\omega^0}\big(\mathbb{R};\mathbb{H};S^{M-N}(\mathbb{R}^n;\mathbb{R}^n)\big);$$

here, we have set $\varphi(y) := \kappa'(\kappa^{-1}(y))^t$ and the Φ_α are universal functions belonging to $S^{|\alpha|/2}(\mathbb{R}^n;\mathbb{R}^n)$ (in fact, polynomials in η of degree at most $|\alpha|/2$). In particular, $\Phi_0 \equiv 1$. Choosing $N > 2M+1$, we have

$$r_N \in S_V^{0,-1;\omega}\big(\mathbb{R}^n \times \mathbb{R};\mathbb{R}^n \times \mathbb{H}).$$

By assumption on κ we have that both φ and $\varphi(\cdot)^{-1} = (\kappa^{-1})'(\cdot)^t$ belong to $\mathscr{C}_b^\infty$. In particular, we find a constant $M \geq 1$ such that

$$\frac{1}{M}|\eta| \leq |\varphi(y)\eta| \leq M|\eta| \qquad \forall\, y,\eta \in \mathbb{R}^n.$$

Then, using (4.1), it follows easily that in the above expansion the summand for the index α belongs to $S_V^{0,-|\alpha|;\omega}\big(\mathbb{R}^n \times \mathbb{R};\mathbb{R}^n \times \mathbb{H}).$

The previous discussion extends also to symbols having the coupling property. More precisely we obtain:

Proposition 4.1. *Let ω be a weight function with property (4.1) and $\kappa:\mathbb{R}^n\to\mathbb{R}^n$ a diffeomorphism as above. If $a\in S_V^{0,0;\omega}(\mathbb{R}^n\times\mathbb{R};\mathbb{R}^n\times\mathbb{H})$ then*

$$(\kappa\times 1)_*a(x,t,D_x,D_t)=b(y,t,D_y,D_t)$$

with a symbol $b\in S_V^{0,0;\omega}(\mathbb{R}^n\times\mathbb{R};\mathbb{R}^n\times\mathbb{H})$. If a has the coupling property then so has b and

$$b(y,t,\eta,\tau)-a\big(\kappa^{-1}(y),t,\kappa'(\kappa^{-1}(y))^t\eta,\tau\big)\ \in\ S_V^{-\varepsilon,-\varepsilon;\omega}(\mathbb{R}^n\times\mathbb{R};\mathbb{R}^n\times\mathbb{H}).$$

4.2. **Operators on manifolds.** The result of the previous section allows us to define operators on $X\times\mathbb{R}$, where X is assumed to be a closed smooth manifold, provided the weight function has property (4.1). This we shall assume from now on for any weight function whenever we consider a manifold X.

We denote by $L^\mu(X)$ the standard Frechét space of pseudo-differential operators of order μ on X, which is based on the local symbol class $S^\mu(\mathbb{R}^n;\mathbb{R}^n)$.

We fix the following data: $X=\Omega_1\cup\ldots\cup\Omega_L$ with coordinate neighborhoods $\chi_j:\Omega_j\to\chi_j(\Omega_j)\subset\mathbb{R}^n$. We assume without loss of generality that each χ_j extends to a diffeomorphism between an open neighborhood U_j of $\overline{\Omega_j}$ and an open neighborhood V_j of $\overline{\chi_j(\Omega_j)}$.

Definition 4.2. *Let ω be a weight function and set $\omega^0(\tau)=\omega(0,\tau)$. We say that $A(t,\tau)$ belongs to the space $L_V^{\mu,(\nu,m);\omega}(X\times\mathbb{R};\mathbb{H})$ if* [5]

a) *$A(t,\tau)\in\mathscr{C}^\infty(\mathbb{R}\times\mathbb{H};L^M(X))$ for some $M\in\mathbb{R}$,*
b) *whenever $\varphi,\psi\in\mathscr{C}^\infty(X)$ have disjoint support then, for some $\varepsilon>0$,*

$$\varphi\,A(t,\tau)\,\psi\in S_V^{(\nu-\varepsilon,m);\omega^0}(\mathbb{R};\mathbb{H};L^{-\infty}(X)),$$

c) *whenever $\varphi,\psi\in\mathscr{C}_0^\infty(\Omega_j)$ and $\psi\equiv 1$ near the support of φ, then*

$$\varphi\,A(t,\tau)\,\psi=\varphi\,\chi_j^*a_j(x,t,D_x,D_t)\,\psi$$

with an $a_j\in S_V^{\mu,(\nu,m);\omega}(\mathbb{R}^n\times\mathbb{R};\mathbb{R}^n\times\mathbb{H})$ having the coupling property.

The L-tuple $(a_1,\ldots,a_L)$ of local symbols we call a complete symbol of A. If the order in t is $m=0$, we suppress m from the notation, as before.

Note that in the previous definition we included the coupling property. Hence from now on it will not be pointed out explicitly any more.

Via local coordinates we can define Sobolev spaces $H_p^\kappa(X\times\mathbb{R})$ (and subspaces $H_p^\kappa(X\times[t_0,\infty))$ of those elements supported in $X\times[t_0,\infty)$). The space

$$\mathscr{S}(X\times\mathbb{R}):=\mathscr{S}(\mathbb{R},\mathscr{C}^\infty(X))$$

is a dense subspace in the Sobolev spaces. With $A(t,\tau)\in L_V^{0,(0,m);\omega}(X\times\mathbb{R};\mathbb{H})$ we associate an operator

$$A(t,D_t):\mathscr{S}(X\times\mathbb{R})\longrightarrow\mathscr{S}(X\times\mathbb{R})$$

by considering $A(t,\tau)$ as an operator-valued symbol. This operator extends continuously to

$$A(t,D_t):H_p^\kappa(X\times\mathbb{R})\longrightarrow H_p^{\kappa/\omega}(X\times\mathbb{R});$$

[5] recall that m indicates the order of $A(t,\tau)$ with respect to t

the subspaces of elements with support in $X \times [t_0, \infty)$ are preserved due to the Volterra property. Analogous definitions and statements hold true for Besov spaces $B^\kappa_p(X \times \mathbb{R})$.

Definition 4.3. *Let $J \subset \mathbb{R}$ be a closed interval. Then we call $A(t,\tau) \in L^{0,0;\omega}_V(X \times \mathbb{R}; \mathbb{H})$* **parabolic on J** *if one (and then any) complete symbol consists of local symbols which are weakly parabolic on J.*[6]

Definition 4.4. *Let $J = [t_0, \infty)$ or $J = \mathbb{R}$. Then $A(t,\tau) \in L^{0,0;\omega}_V(X \times \mathbb{R}; \mathbb{H})$ is called* **globally parabolic on J** *if it is parabolic in the sense of Definition 4.3 and, additionally, there exist constants $C, M, T \geq 0$ and a weight function κ such that $A(t,\tau) : H^{\kappa_0}_p(X) \to H^{\kappa_0/\omega_0}_p(X)$ is an isomorphism for all $t \in J$ with $|t| \geq T$ and all $\tau \in \mathbb{H}$, satisfying*

$$\|A(t,\tau)^{-1}\|_{\mathscr{L}(H^{\kappa_0}_p(X), H^{\kappa_0/\omega_0}_p(X))} \leq C\langle\tau\rangle^M \qquad \forall\, |t| \geq T \quad \forall\, \tau \in \mathbb{H}. \tag{4.2}$$

The above definitions can be extended to $A(t,\tau) \in L^{0,(0,m);\omega}_V(X \times \mathbb{R}; \mathbb{H})$ by requiring that $\langle t\rangle^{-m} A(t,\tau)$ is (globally) parabolic in the above sense.

Theorem 4.5. *Let $J = [t_0, \infty)$ or $J = \mathbb{R}$ and κ some weight function. If $A(t,\tau) \in L^{0,0;\omega}_V(X \times \mathbb{R}; \mathbb{H})$ is globally parabolic on J then*

$$A(t, D_t) : \mathscr{H}^\kappa_p(X \times J) \longrightarrow \mathscr{H}^{\kappa/\omega}_p(X \times J)$$

is an isomorphism. Moreover, there exists a $B(t,\tau) \in L^{0,0;1/\omega}_V(X \times \mathbb{R}; \mathbb{H})$ such that $A(t, D_t)^{-1}$ coincides with

$$B(t, D_t) : \mathscr{H}^{\kappa/\omega}_p(X \times J) \longrightarrow \mathscr{H}^\kappa_p(X \times J).$$

Proof. For simplicity, let us assume that $J = \mathbb{R}$. The case $J = [t_0, \infty)$ is verified similarly, using the fact that

$$(\varphi \# A \# \psi)(t,\tau) \in L^{-\varepsilon,(-\infty,-\infty)}_V(X \times \mathbb{R}; \mathbb{H})$$

whenever $\varphi, \psi \in S^0(\mathbb{R}_t)$ have disjoint support.

Let $(a_1, \ldots, a_L)$ be a complete symbol of $A(t,\tau)$. As in the proof of Theorem 3.6, to each a_j we can construct a b_j such that both $a_j b_j - 1$ and $b_j a_j - 1$ belong to $S^{-\varepsilon,(-\varepsilon,0)}_V(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^n \times \mathbb{H})$. Pulling back $b_j(x, t, D_x, \tau)$ to X and pasting them together with a partition of unity, we obtain a $B(t,\tau) \in L^{0,0;1/\omega}_V(X \times \mathbb{R}; \mathbb{H})$ such that

$$\left.\begin{aligned} R_r(t,\tau) &:= 1 - A(t,\tau)B(t,\tau) \\ R_l(t,\tau) &:= 1 - B(t,\tau)A(t,\tau) \end{aligned}\right\} \in L^{-\varepsilon,(-\varepsilon,0)}_V(X \times \mathbb{R}; \mathbb{H}).$$

By the usual von Neumann argument we may even assume that

$$R_r(t,\tau),\, R_l(t,\tau) \in L^{-\infty,(-\infty,0)}_V(X \times \mathbb{R}; \mathbb{H}).$$

Now let $\chi = \chi(t)$ be a zero excision function vanishing on a neighborhood of $[-T, T]$, where T is as in (4.2). Then

$$\widetilde{P}(t,\tau) := B(t,\tau) + \chi(t)(R_l B + R_l A^{-1} R_r)(t,\tau) \in L^{0,0;\omega}_V(X \times \mathbb{R}; \mathbb{H})$$

[6] To be more precise: In the notation introduced in the beginning of Section 4.2, we require the existence of an open interval I containing J and of constants $C, R > 0$ such that $|a_j(x,t,\xi,\tau)^{-1}| \leq C|\omega(\xi,\tau)^{-1}|$ for all $|(\xi,\tau)| \geq R$, $t \in I$, and $x \in V_j$. For simplicity of presentation, we assume that we can replace V_j by $\mathbb{R}^n$.

(note that the smoothing operators on X are precisely those operators $\mathscr{D}(X) \to \mathscr{D}'(X)$ that induce continuous mappings from $H^s_p(X)$ to $H^t_p(X)$ for any choice of s, t). Then

$$1 - \widetilde{P}(t,\tau)A(t,\tau) = (1-\chi)(t)R_l(t,\tau) \in L_V^{-\infty,(-\infty,-\infty)}(X \times \mathbb{R}; \mathbb{H})$$

Hence, again by the von Neumann argument, we may assume that

$$\widetilde{P}(t,D_t)A(t,D_t) = 1 - R(t,D_t)$$

with $R(t,\tau) \in L_V^{-\infty,(-\infty,-\infty)}(X \times \mathbb{R}; \mathbb{H})$. Then Theorem 2.5 (in the version for operators K with kernels in $\mathscr{S}\big(\mathbb{R}\times\mathbb{R}, \mathscr{C}^\infty(X \times X)\big)$ implies that $1 - R(t,D_t)$ is invertible with

$$(1 - R(t,D_t))^{-1} = 1 - S(t,D_t), \qquad S(t,\tau) \in L_V^{-\infty,(-\infty,-\infty)}(X \times \mathbb{R}; \mathbb{H}).$$

Hence we obtain a left inverse of $A(t,D_t)$,

$$P(t,D_t) = \widetilde{P}(t,D_t) - S(t,D_t)\widetilde{P}(t,D_t).$$

Similarly we get a right-inverse, i.e., $P(t,D_t)$ is the inverse. □

Let us point out the following version of Theorem 4.5, where instead of the closed manifold X we consider $\mathbb{R}^n$.

Theorem 4.6. *Let $J = [t_0,\infty)$ or $J = \mathbb{R}$ and κ some weight function. Assume that $a \in S_V^{0,0;\omega}(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^n \times \mathbb{H})$ has the coupling property and is weakly parabolic on J. Moreover, let a be globally parabolic in a sense analogous to Definition* 4.4[7]. *Then a induces isomorphisms*

$$a(x,t,D_x,D_t) : \mathscr{H}^\kappa_p(\mathbb{R}^n \times J) \longrightarrow \mathscr{H}^{\kappa/\omega}_p(\mathbb{R}^n \times J).$$

In contrast to Theorem 4.5 the inverse of $a(x,t,D_x,D_t)$ does not necessarily belong to the calculus again. This is due to the fact that we employed in the proof of Theorem 4.5 a characterization of smoothing operators on X as those operators acting continuously from $H^s_p(X)$ to $H^t_p(X)$ for any choice of $s, t \in \mathbb{R}$; however, this characterization is true only for closed X, but breaks down for $X = \mathbb{R}^n$.

There is also a version of Theorem 4.5 for operators being only parabolic on a compact interval, which reads as follows:

Remark 4.7. *Let $J = [t_0, t_1]$ and κ some weight function. If $A(t,\tau) \in L_V^{0,0;\omega}(X \times \mathbb{R}; \mathbb{H})$ is parabolic on J then there exists a $P(t,\tau) \in L_V^{0,0;1/\omega}(X \times \mathbb{R}; \mathbb{H})$ such that for any choice of $\varphi, \psi \in \mathscr{C}^\infty_{\text{comp}}((t_0,t_1))$ we have*

$$\varphi\big(A(t,D_t)P(t,D_t) - 1\big)\psi = \varphi\big(P(t,D_t)A(t,D_t) - 1\big)\psi = 0.$$

[7] i.e., (4.2) holds for $A(t,\tau) := a(x,t,D_x,\tau)$ with X replaced by $\mathbb{R}^n$

4.3. **Parabolicity in exponentially weighted Sobolev spaces.** In this section we shall derive a version of Theorem 4.5 only requiring parabolicity, avoiding the assumption (4.2). To this end we consider the operator $A(t,D_t)$ in spaces with exponential weight.
For $A(t,\tau)\in L_V^{\mu,\nu;\omega}(X\times\mathbb{H})$ and $\sigma\ge 0$ define $(T_{i\sigma}A)(t,\tau)\in L_V^{\mu,\nu;\omega}(X\times\mathbb{H})$ by

$$(T_{i\sigma}A)(t,\tau)=A(t,\tau-i\sigma).$$

Then an easy calculation shows that

$$(T_{i\sigma}A)(t,D_t)=e^{-\sigma t}\circ A(t,D_t)\circ e^{\sigma t},$$

where $e^{\pm\sigma t}$ means the operator of multiplication with the function $e^{\pm\sigma\cdot}$.

Definition 4.8. *Let κ be a weight function and $\sigma\ge 0$. We define the weighted spaces*

$$\mathscr{H}_p^{\kappa,\sigma}(X\times\mathbb{R}):=\big\{u\in\mathscr{D}'(X\times\mathbb{R})\mid e^{-\sigma t}u\in\mathscr{H}^{\kappa}(X\times\mathbb{R})\big\}.$$

It is obvious that

$$(T_{i\sigma}A)(t,D_t):\mathscr{H}^{\kappa}(X\times\mathbb{R})\longrightarrow\mathscr{H}^{\kappa/\omega}(X\times\mathbb{R})$$

is an isomorphism if and only if so is

$$A(t,D_t):\mathscr{H}^{\kappa,\sigma}(X\times\mathbb{R})\longrightarrow\mathscr{H}^{\kappa/\omega,\sigma}(X\times\mathbb{R}).$$

Proposition 4.9. *Let $J=[t_0,\infty)$ or $J=\mathbb{R}$ and $A(t,\tau)\in L_V^{0,0;\omega}(X\times\mathbb{H})$. If $A(t,\tau)$ is parabolic on J, there exists a $\sigma_0\ge 0$ such that $(T_{i\sigma}A)(t,\tau)$ is globally parabolic on J for any $\sigma\ge\sigma_0$.*

Proof. Let us assume for simplicity that $J=\mathbb{R}$. As in the proof of Theorem 4.5, we can construct a $P(t,\tau)\in L_V^{0,0;1/\omega}(X\times\mathbb{R};\mathbb{H})$ such that

$$A(t,\tau)P(t,\tau)=1-R_r(t,\tau),\qquad P(t,\tau)A(t,\tau)=1-R_l(t,\tau),$$

where both $R_r(t,\tau)$ and $R_l(t,\tau)$ belong to $L_V^{-\infty,(-\infty,0)}(X\times\mathbb{R};\mathbb{H})$. In particular, they are rapidly decreasing in τ. Hence for $|\tau|$ large enough, say $|\tau|\ge c$, both $1-R_r(t,\tau)$ and $1-R_l(t,\tau)$ are invertible. We obtain that

$$A(t,\tau)^{-1}=P(t,\tau)(1-R_r(t,\tau))^{-1},\qquad |\tau|\ge c.$$

This implies estimate (4.2) for $|\tau|\ge c$, with M chosen in such a way that $|\omega(\xi,\tau)|\le C\langle\tau\rangle^M|\omega(\xi,0)|$. It remains to choose $\sigma_0=c$. □

An immediate consequence is the following theorem:

Theorem 4.10. *Let $J=[t_0,\infty)$ or $J=\mathbb{R}$ and $A(t,\tau)\in L_V^{0,0;\omega}(X\times\mathbb{H})$. If $A(t,\tau)$ is parabolic on J then, for any sufficiently large $\sigma\ge 0$,*

$$A(t,D_t):\mathscr{H}_p^{\kappa,\sigma}(X\times J)\longrightarrow\mathscr{H}_p^{\kappa/\omega,\sigma}(X\times J)$$

is an isomorphism. Moreover, there exists a $B(t,\tau)\in L_V^{0,0;1/\omega}(X\times\mathbb{R};\mathbb{H})$ such that $A(t,D_t)^{-1}$ coincides with

$$e^{\sigma t}\circ B(t,D_t)\circ e^{-\sigma t}:\mathscr{H}_p^{\kappa/\omega,\sigma}(X\times J)\longrightarrow\mathscr{H}_p^{\kappa,\sigma}(X\times J).$$

Analogous to Theorem 4.6 there is a version of the previous theorem for operators on $\mathbb{R}^n$; the invertibilty remains true, while the inverse is not necessarily contained in the calculus again.

5. Mixed order systems

We shall now extend the previously developed calculus to certain systems of mixed order, see below for a precise definition. Since the proofs of the main theorems are very close to the ones given above (essentially they differ only by more involved notation), we restrict ourselves to a formulation of the results in the context of systems without providing proofs.

5.1. **Systems on $\mathbb{R}^n \times \mathbb{R}$.** In the sequel we shall consider a $(q \times q)$-matrix valued symbol

$$A(x,t,\xi,\tau) = \Big(A_{ij}(x,t,\xi,\tau)\Big)_{1\le i,j\le q}.$$

We shall assume that there exist weight functions $m_1,\ldots,m_q,l_1,\ldots,l_q$ such that

$$A_{ij}(x,t,\xi,\tau) \in S_V^{0,0;l_i m_j}(\mathbb{R}^n\times\mathbb{R};\mathbb{R}^n\times\mathbb{H}).$$

In this case we have

$$A(x,t,D_x,D_t): \bigoplus_{j=1}^{q} \mathscr{H}^{\kappa m_j}(\mathbb{R}^n\times\mathbb{R}) \longrightarrow \bigoplus_{i=1}^{q} \mathscr{H}^{\kappa/l_i}(\mathbb{R}^n\times\mathbb{R}), \tag{5.1}$$

where κ is an arbitrary weight function. A matrix symbol

$$P(x,t,\xi,\tau) = \Big(P_{ij}(x,t,\xi,\tau)\Big)_{1\le i,j\le q}, \qquad P_{ij}\in S_V^{0,0;1/(m_i l_j)},$$

is called a parametrix of A if $(A\#P-1)_{ij}$, $(P\#A-1)_{ij}\in S_V^{-\infty,-\infty}$ for all $1\le i,j\le q$.

Theorem 5.1. *For the system A the following conditions are equivalent:*

a) *There exist systems B^1, B^2 with $B^k_{ij}\in S_V^{0,0;1/(m_i l_j)}$ such that, for some $\varepsilon>0$,*

$$\big(A\#_x B^1-1\big)_{ij}\in S_V^{0,-\varepsilon;l_i/l_j}, \quad \big(B^1\#_x A-1\big)_{ij}\in S_V^{0,-\varepsilon;m_j/m_i},$$
$$\big(A\#_t B^2-1\big)_{ij}\in S_V^{-\varepsilon,0;l_i/l_j}, \quad \big(B^2\#_t A-1\big)_{ij}\in S_V^{-\varepsilon,0;m_j/m_i}.$$

b) *There exist systems $\widetilde{B}^1$, $\widetilde{B}^2$ with $\widetilde{B}^k_{ij}\in S_V^{0,0;1/(m_i l_j)}$ such that*

$$(T_{i\sigma}A)\#_x\widetilde{B}^1-1=\widetilde{B}^1\#_x(T_{i\sigma}A)-1=0,$$
$$\chi_2\big(A\#_t\widetilde{B}^2-1\big)=\chi_2\big(\widetilde{B}_2\#_t A-1\big)=0$$

for some $\sigma\ge 0$ and some zero excision function $\chi_2(\xi)$.

c) *A has a parametrix P.*

Definition 5.2. *Let J be a closed interval. We call A* **weakly parabolic on J** *if there exists an open interval I containing J such that*

$$|\det A(x,t,\xi,\tau)| \ge C\prod_{i=1}^{n}\big|l_i(\xi,\tau)\,m_i(\xi,\tau)\big| \qquad \forall\,(x,t)\in\mathbb{R}^n\times I \quad \forall\,|(\xi,\tau)|\ge R$$

for some constants $C>0$ and $R\ge 0$.

Also here an important subclass is given by those symbols with a coupling in the covariable, i.e.

$$\textbf{(C)} \qquad D_{\xi_k}A_{ij},\, D_\tau A_{ij} \;\in\; S_V^{-\varepsilon,-\varepsilon;l_i\cdot m_j}(\mathbb{R}^n\times\mathbb{R};\mathbb{R}^n\times\mathbb{H}), \qquad \forall\, i,j,k=1,\ldots,n.$$

Theorem 5.3. *Let A have constant coefficients or satisfy the coupling condition* (**C**)*. Then A possesses a parametrix if and only if A is weakly parabolic on $\mathbb{R}$.*

5.2. **Systems on $X\times\mathbb{R}$.** Let us now consider

$$A(t,\tau)=\Big(A_{ij}(t,\tau)\Big)_{1\le i,j\le q}, \qquad A_{ij}(t,\tau)\in L_V^{0,0;l_im_j}(X\times\mathbb{R};\mathbb{H}).$$

We hereby assume that the weight functions $m_1,\ldots,m_q,l_1,\ldots,l_q$ have the property (4.1). Analogously to Definition 4.2, a complete symbol associated with A consists of matrix symbols as introduced in the previous subsection.

Definition 5.4. *Let J be a closed interval. Then $A(t,\tau)$ is called* **parabolic on J** *if one* (*and then any*) *complete symbol consists of local systems which are weakly parabolic on J in the sense of Definition* 5.2.

Definition 5.5. *Let $J=[t_0,\infty)$ or $J=\mathbb{R}$. Then $A(t,\tau)$ is called* **globally parabolic on** J *if it is parabolic in the sense of Definition* 5.4 *and there exist constants $C,M,T\ge 0$ and a weight function κ such that $A(t,\tau):H_1\to H_2$ is an isomorphism for all $t\in J$ with $|t|\ge T$ and all $\tau\in\mathbb{H}$, satisfying*

$$\|A(t,\tau)^{-1}\|_{\mathscr{L}(H_1,H_2)}\le C\langle\tau\rangle^M \qquad \forall\,|t|\ge T \quad \forall\,\tau\in\mathbb{H}, \tag{5.2}$$

where we used the abbreviations

$$H_1=\bigoplus_{j=1}^q H_p^{\kappa_0 m_j,0}(X\times\mathbb{R}), \qquad H_2=\bigoplus_{i=1}^q H_p^{\kappa_0/l_i,0}(X\times\mathbb{R}).$$

Theorem 5.6. *Let $J=[t_0,\infty)$ or $J=\mathbb{R}$ and κ some weight function. If $A(t,\tau)$ is globally parabolic on J then*

$$A(t,D_t):\bigoplus_{j=1}^q \mathscr{H}^{\kappa m_j}(X\times J)\longrightarrow\bigoplus_{i=1}^q\mathscr{H}^{\kappa/l_i}(X\times J)$$

is invertible with inverse induced by

$$P(t,\tau)=\Big(P_{ij}(t,\tau)\Big)_{1\le i,j\le q}, \qquad P_{ij}(t,\tau)\in L_V^{0,0;1/(l_im_j)}(X\times\mathbb{R};\mathbb{H}).$$

By allowing exponential weights, the assumptions of the previous theorem can be weakened:

Theorem 5.7. *Let $J=[t_0,\infty)$ or $J=\mathbb{R}$. If $A(t,\tau)$ is parabolic on J then, for any sufficiently large $\sigma\ge 0$,*

$$A(t,D_t):\bigoplus_{j=1}^q \mathscr{H}^{\kappa m_j,\sigma}(X\times J)\longrightarrow\bigoplus_{i=1}^q\mathscr{H}^{\kappa/l_i,\sigma}(X\times J)$$

is an isomorphism. Moreover, there exists a system

$$P(t,\tau)=\Big(P_{ij}(t,\tau)\Big)_{1\le i,j\le q}, \qquad P_{ij}(t,\tau)\in L_V^{0,0;1/(l_im_j)}(X\times\mathbb{R};\mathbb{H}),$$

such that the inverse of $A(t,D_t)$ coincides with $e^{\sigma t}\circ P(t,D_t)\circ e^{-\sigma t}$.

As before, there are corresponding versions of the previous two theorems with X replaced by $\mathbb{R}^n$. Then the invertibility statements remain true, but the inverse possibly does not belong to the calculus.

6. Applications

6.1. **Time-dependent Douglis-Nirenberg systems.** Let us consider the symbol $A(t,\tau) = \Big(A_{ij}(t,\tau)\Big)_{1\le i,j\le q}$, where

$$A_{ij}(t,\tau) = i\delta_{ij}\tau + \widetilde{A}_{ij}(t), \qquad \widetilde{A}_{ij}(t) \in S^0\big(\mathbb{R}_t, L^{s_i+t_j}(X)\big)$$

(of course the first i in $i\delta_{ij}\tau$ means the imaginary unit, while δ_{ij} is the usual Kronecker symbol) with real numbers $s_i, t_j \in \mathbb{R}$. In other words, $\widetilde{A}(t)$ is a time-dependent Douglis-Nirenberg system on X. Moreover, setting $r_i = s_i + t_i$, let us assume that $r_1, \ldots, r_q > 0$ and define the weight functions

$$m_j(\xi,\tau) := \big(\langle\xi\rangle^{r_j} + i\tau\big)\langle\xi\rangle^{-s_j}, \qquad l_i(\xi,\tau) := \langle\xi\rangle^{s_i}. \tag{6.1}$$

It is not difficult to verify that $A(t,\tau)$ is a mixed order system as considered in Section 5.2. In fact, if we choose $\varepsilon > 0$ with $\varepsilon < 1/(1+r_i)$ for all $i = 1, \ldots, q$, then

$$\begin{aligned}
|D^\alpha_\xi D^\beta_x D^n_\tau D_{\xi_k} A_{ij}(x,\xi,\tau)| &\le C\,\langle\xi\rangle^{s_i+t_j-1-|\alpha|}\langle\tau\rangle^{-n} \\
&= C\,\big|l_i(\xi,\tau)m_j(\xi,\tau)\big|\frac{\langle\xi\rangle^{\mathrm{r}_j-1-|\alpha|}}{|\langle\xi\rangle^{r_j}+i\tau|}\langle\tau\rangle^{-n} \\
&\le C\,\big|l_i(\xi,\tau)m_j(\xi,\tau)\big|\langle\xi\rangle^{r_j(\varepsilon-1)+r_j-1-|\alpha|}\langle\tau\rangle^{-\varepsilon-n} \\
&\le C\,\big|l_i(\xi,\tau)m_j(\xi,\tau)\big|\langle\xi\rangle^{-\varepsilon-|\alpha|}\langle\tau\rangle^{-\varepsilon-n},
\end{aligned}$$

since $r_j(\varepsilon-1)+r_j-1 = \varepsilon r_j - 1 \le -\varepsilon$. Hence, $D_{\xi_k}A_{ij} \in S^{-\varepsilon,-\varepsilon;l_i\cdot m_j}$. Since $D_\tau A_{ij} = -\delta_{ij}$, it easily follows that also $D_\tau A_{ij} \in S^{-\varepsilon,-\varepsilon;l_i\cdot m_j}$.

If we choose the weight function $\kappa(\xi,\tau) := \langle\xi\rangle^s$, $s \ge 0$, then the mapping property

$$A(t,D_t) = \partial_t + \widetilde{A}(t) : \bigoplus_{j=1}^q H_p^{\kappa m_j}(X\times\mathbb{R}) \longrightarrow \bigoplus_{i=1}^q H_p^{\kappa/l_i}(X\times\mathbb{R}),$$

is equivalent to

$$\partial_t + \widetilde{A}(t) : \bigoplus_{j=1}^q \Big[L_p\big(\mathbb{R}, H_p^{s+t_j}(X)\big) \cap H_p^1\big(\mathbb{R}, H_p^{s-s_j}(X)\big)\Big] \longrightarrow \bigoplus_{i=1}^q L_p\big(\mathbb{R}, H_p^{s-s_i}(X)\big).$$

Parabolicity on J of $A(t,\tau)$ now asks of the local symbols that

$$\big|\det\big(i\tau - \widetilde{A}(t,x,\xi)\big)\big| \ge C(\langle\xi\rangle^{\mathrm{r}_1} + |\tau|)\cdot\ldots\cdot(\langle\xi\rangle^{\mathrm{r}_q} + |\tau|) \tag{6.2}$$

for all $(t,x) \in I \times \mathbb{R}^n$ and all $|(\xi,\tau)| \ge R$. Now it is clear how to formulate Theorems 5.6 and 5.7 in the present context.

Let us remark that in [3] we have shown that (6.2) in case of t-independent $\widetilde{A}$ implies that the operator $\lambda_0 + \widetilde{A}$, considered as an unbounded closed operator in $\bigoplus_{i=1}^q H_p^{s-s_i}(\mathbb{R}^n)$ with domain $\bigoplus_{j=1}^q H_p^{s+t_j}(\mathbb{R}^n)$ admits a bounded H_∞-calculus with H_∞-angle at least $\pi/2$, provided $\lambda_0 \ge 0$ is sufficiently large. From this maximal regularity in the mentioned spaces follows (when $\mathbb{R}$ is replaced by a finite time-interval $[0,T]$). The above example thus allows both to interpret the maximal regularity result of [3] as a consequence of the developed calculus of Volterra pseudodifferential operators as well as to extend the result to time-dependent Douglis-Nirenberg systems.

6.2. **A linearized Stefan problem.** In the investigation of the one-phase Stefan problem with surface tension one is lead (cf. [7])to the study of the system

$$\left\{\begin{aligned} \partial_t u - \Delta u &= f &&\text{in } (0,\infty)\times\Omega,\\ \gamma_0 u + \Delta_{\partial\Omega}\rho &= f_0 &&\text{in } (0,\infty)\times\partial\Omega,\\ \gamma_1 u - \partial_t\rho &= f_1 &&\text{in } (0,\infty)\times\partial\Omega, \end{aligned}\right. \tag{6.3}$$

with initial conditions

$$u(0)=u_0 \text{ in } \Omega, \qquad \rho(0)=\rho_0 \text{ on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^n$ (which we assume here for simplicity to have a smooth boundary), and $u:(0,\infty)\times\Omega\to\mathbb{R}$ and $\rho:(0,\infty)\times\partial\Omega\to\mathbb{R}$ are the unknowns. Moreover, $\gamma_0=\cdot|_{\partial\Omega}$ denotes restriction to the boundary, while $\gamma_1=\gamma_0\circ\frac{\partial}{\partial\nu}$ with the exterior normal ν on $\partial\Omega$. In this formulation, u represents temperature while ρ describes the time evolution of the domain $\Omega=\Omega(0)$, i.e.

$$\partial\Omega(t)=\big\{x+\rho(t)\nu(x)\mid x\in\partial\Omega\big\}, \qquad t>0.$$

For simplicity we shall only consider the case of homogeneous initial conditions, i.e., $u_0=0$ and $\rho_0=0$.

We define operators P and K by asking that $v:=P(f)$ is the solution of the heat equation

$$\partial_t v-\Delta v=f, \qquad \gamma_0 v=0, \qquad v(0)=0,$$

in $(0,\infty)\times\Omega$, while $w:=K(g)$ solves

$$\partial_t w-\Delta w=0, \qquad \gamma_0 w=g, \qquad w(0)=0.$$

Then the 'Lopatinskij-matrix' associated with the above problem is the operator L defined by

$$L\begin{pmatrix} g\\ \rho\end{pmatrix} := \begin{pmatrix}\gamma_0\circ K & \Delta_{\partial\Omega}\\ \gamma_1\circ K & -\partial_t\end{pmatrix}\begin{pmatrix} g\\ \rho\end{pmatrix} = \begin{pmatrix} 1 & \Delta_{\partial\Omega}\\ \gamma_1\circ K & -\partial_t\end{pmatrix}\begin{pmatrix} g\\ \rho\end{pmatrix}. \tag{6.4}$$

If f, f_0, f_1 are the data of the Stefan problem and g,ρ fulfill

$$L(g,\rho)=\big(f_0, f_1-\gamma_1 P(f)\big), \qquad g(0)=0,\ \rho(0)=0,$$

then we get a solution (u,ρ) of the above problem by setting $u:=P(f)+K(g)$.

Let us now show that L is a mixed order system of the sort considered in this paper, and that it satisfies parabolicity conditions that ensure its invertibility in suitable spaces.

The operator $\gamma_1\circ K$ in (6.4) is the parabolic Dirichlet-Neumann operator. From classical results it is known that it is an anisotropic first order pseudodifferential operator with the

Volterra property. To explain this precisely we need to introduce some notation: For $\mu \in \mathbb{R}$ let $S_{\mu;2}(\mathbb{R}^n; \mathbb{R}^n \times \mathbb{H})$ denote the space of symbols $a(x, \tau, \xi)$ satisfying

$$|D_\xi^\alpha D_\tau^\beta D_x^\gamma a(x,\tau,\xi)| \leq C_{\alpha\beta\gamma}\big(1 + |\xi| + |\tau|^{1/2}\big)^{\mu-(|\alpha|+2|\beta|)}$$

uniformly in (x, τ, ξ) for any order of derivatives. The subclass of symbols which are additionally holomorphic in τ is indicated by an additional superscript V.

If now a denotes a local symbol of $\gamma_1 \circ K$ then $a \in S^V_{1;2}(\mathbb{R}^n; \mathbb{R}^n \times \mathbb{H})$. Moreover, if χ is a zero excision function, then

$$a(x,\tau,\xi) = -\chi\big((|\xi|_x^4 + |\tau|^2)^{1/4}\big)(|\xi|_x^2 + i\tau)^{1/2} \mod S_{0;2}(\mathbb{R}^n; \mathbb{R}^n \times \mathbb{H}), \tag{6.5}$$

where $|\xi|_x$ denotes the local expression for the length of the co-vector ξ at x (by smooth extension, we may assume that $|\cdot|_x$ is defined for all $x \in \mathbb{R}$ and coincides with $|\cdot|$ for large x).

Let us now introduce the weight functions

$$\begin{aligned} m_1(\xi,\tau) &= l_1(\xi,\tau) \equiv 1, \\ l_2(\xi,\tau) &= (\langle\xi\rangle^2 + i\tau)^{1/2}, \qquad m_2(\xi,\tau) = (\langle\xi\rangle^4 + i\tau)^{1/2}. \end{aligned} \tag{6.6}$$

It is then obvious that

$$S^V_{1-2s;2}(\mathbb{R}^n; \mathbb{R}^n \times \mathbb{H}) \subset S_V^{-s,-s;l_2}(\mathbb{R}^n; \mathbb{R}^n \times \mathbb{H}) \qquad \forall\, s \geq 0. \tag{6.7}$$

Hence we can conclude the following:

Remark 6.1. *With the choice of the weight functions from* (6.6)*, the Lopatinskij-matrix L from* (6.4) *is a* (2×2)*-system on $\partial\Omega \times \mathbb{R}$ in the sense of Section* 5.2*. Modulo negligible operators of lower order, L is described by the local symbols*

$$A(x,\tau,\xi) = \begin{pmatrix} 1 & -|\xi|_x^2 \\ a(x,\tau,\xi) & -i\tau \end{pmatrix}. \tag{6.8}$$

Using (6.7) *it is easily seen that $A(x, \tau, \xi)$ satisfies the coupling condition.*

Since $|\cdot|_x$ is comparable with $|\cdot|$ and because of (6.5), we can find positive constants C and R such that

$$|\det A(x,\tau,\xi)| \geq C\,\big||\xi|^2(|\xi|^2 + i\tau)^{1/2} + i\tau\big| \geq C\,\big|l_2(\xi,\tau)\, m_2(\xi,\tau)\big|$$

uniformly in x and $|(\tau, \xi)| \geq R$. For the last estimate see [5]. Hence the local symbols are weakly parabolic (recall that $l_1 = m_1 \equiv 1$). We therefore obtain:

Proposition 6.2. *Let $J = [0, \infty)$ and κ a weight function. Then there exists a $\sigma_0 \geq 0$ such that, for any $\sigma \geq \sigma_0$,*

$$L : \begin{array}{c} B_{pp}^{\kappa,\sigma}(\partial\Omega \times J) \\ \oplus \\ B_{pp}^{\kappa m_2,\sigma}(\partial\Omega \times J) \end{array} \longrightarrow \begin{array}{c} B_{pp}^{\kappa,\sigma}(\partial\Omega \times J) \\ \oplus \\ B_{pp}^{\kappa/l_2,\sigma}(\partial\Omega \times J) \end{array}$$

is an isomorphism.

Let us now recall the standard definition of anisotropic Besov spaces.

Remark 6.3. *For $a, s > 0$ and $\sigma \in \mathbb{R}$ set*

$$B_{pp}^{(s,a),\sigma}(\partial\Omega \times J) := B_{pp}^{\kappa_{s,a},\sigma}(\partial\Omega \times J), \qquad \kappa_{s,a}(\tau,\xi) = \big(\langle\xi\rangle^a + i\tau\big)^{s/a}.$$

Thus s measures smoothness while a describes the anisotropy between space and time variables; in fact,

$$B_{pp}^{(s,a)}(\partial\Omega \times J) = L_p\big(J, B_{pp}^s(\partial\Omega)\big) \cap B_{pp}^{s/a}\big(J, L_p(\partial\Omega)\big).$$

Still, σ indicates the exponential weight in time.

If we have $f \in L_p^\sigma(\Omega \times J)$ we get by standard results on the heat equation that $P(f) \in H_p^{(2,2);\sigma}(\Omega \times J)$,[8] hence

$$\gamma_1 P(f) \in B_{pp}^{(1-\frac{1}{p},2),\sigma}(\partial\Omega \times J).$$

Assume further that f_1 has the same regularity as $\gamma_1 P(f)$ and that

$$f_0 \in B_{pp}^{(2-\frac{1}{p},2),\sigma}(\partial\Omega \times J).$$

Then Proposition 6.2 with the choice $\kappa = \kappa_p = \kappa_{2-\frac{1}{p},2}$ implies

$$g \in B_{pp}^{(2-\frac{1}{p},2),\sigma}(\partial\Omega \times J), \qquad \rho \in B_{pp}^{\omega_p;\sigma}(\partial\Omega \times J),$$

with the weight function

$$\omega_p(\xi,\tau) = (\kappa_p \cdot m_2)(\xi,\tau) = (\langle\xi\rangle^2 + i\tau)^{\frac{1}{2}(2-\frac{1}{p})}(\langle\xi\rangle^4 + i\tau)^{1/2}.$$

Again due to standard regularity properties of the heat equation we get

$$u = K(g) + P(f) \in H_p^{(2,2),\sigma}(\Omega \times J).$$

Summarizing, we have shown the following:

Theorem 6.4. *Let $J = [0,\infty)$. Then there exists a $\sigma_0 \geq 0$ such that for any $\sigma \geq \sigma_0$ and given data*

$$f \in L_p^\sigma(\Omega \times J), \quad f_0 \in B_{pp}^{(2-\frac{1}{p},2),\sigma}(\partial\Omega \times J), \quad f_1 \in B_{pp}^{(1-\frac{1}{p},2),\sigma}(\partial\Omega \times J),$$

the above linearized Stefan problem has a unique solution

$$u \in H_p^{(2,2);\sigma}(\Omega \times J), \qquad \rho \in B_{pp}^{\omega_p;\sigma}(\partial\Omega \times J).$$

This result was obtained in Theorem 1.4 of [7] for the half-space case $\Omega = \mathbb{R}^n_+$, using techniques from semi-group theory. Here we cover the case of a smoothly bounded compact domain Ω.

[8] where $H_p^{(s,a)}(\Omega \times J) = L_p\big(J, H_p^s(\Omega)\big) \cap H_p^{s/a}\big(J, L_p(\Omega)\big)$

6.3. **Reduction to the boundary (I).** The approach of the previous section can be seen as a particular example of a more general situation. This we shall describe now in detail. Throughout the section we let Ω denote a smoothly bounded compact domain in $\mathbb{R}^{n+1}$ and set $J = [0,\infty)$. Moreover, $A = a(x,t,D_x,D_t)$ is a second order differential operator. We shall assume

$$\text{(A1)}\quad \begin{pmatrix} A \\ \gamma_0 \end{pmatrix} : H_p^{(2,2)}(\Omega\times J) \longrightarrow \begin{matrix} L_p(\Omega\times J) \\ \oplus \\ B_{pp}^{(2-\frac{1}{p},2)}(\partial\Omega\times J) \end{matrix} \quad \text{is an isomorphism.}$$

For $j=1,2$ let $B_j := \gamma_0 \circ b_j(y,t,D_y,D_t)$ with differential operators $b_j(y,t,D_y,D_t)$ defined in a neighborhood of $\overline{\Omega}$ and let C_j be differential operators on $\partial\Omega\times\mathbb{R}$. We now consider the problem

$$\text{(6.9)}\qquad \begin{cases} Au = f & \text{in } \Omega\times(0,\infty), \\ B_1u + C_1\rho = f_1 & \text{in } \partial\Omega\times(0,\infty), \\ B_2u + C_2\rho = f_2 & \text{in } \partial\Omega\times(0,\infty), \end{cases}$$

with zero initial conditions $u(0)=0$ and $\rho(0)=0$. The unknowns are u and ρ.

If $(P \quad K)$ denotes the inverse of the map in (A1), then the Lopatinskij-matrix associated with (6.9) is

$$L := \begin{pmatrix} B_1K & C_1 \\ B_2K & C_2 \end{pmatrix}.$$

In general, the entries in the first column will be pseudodifferential operators. We shall assume

(A2) There exist weight functions m_1, m_2, l_1, l_2 such that L is a mixed order system in the sense of Section 5.2 which is parabolic on J.

Choosing the weight function κ_p as

$$\text{(6.10)}\qquad \kappa_p(\xi,\tau) = \big(\langle\xi\rangle^2 + i\tau\big)^{\frac{1}{2}(2-\frac{1}{p})}/m_1(\xi,\tau)$$

yields that $B_{pp}^{(2-\frac{1}{p},2),\sigma}(\partial\Omega\times J) = B_{pp}^{\kappa_p m_1,\sigma}(\partial\Omega\times J)$. By assumption,

$$\begin{pmatrix} P & K & 0 \\ 0 & 0 & 1 \end{pmatrix} : \begin{matrix} L_p^\sigma(\Omega\times J) \\ \oplus \\ B_{pp}^{\kappa_p m_1,\sigma}(\partial\Omega\times J) \\ \oplus \\ B_{pp}^{\kappa_p m_2,\sigma}(\partial\Omega\times J) \end{matrix} \longrightarrow \begin{matrix} L_p^\sigma(\Omega\times J) \\ \oplus \\ B_{pp}^{\kappa_p m_2,\sigma}(\partial\Omega\times J) \end{matrix}$$

is an isomophism. Assuming the mapping properties

(A3) $B_j : H_p^{(2,2),\sigma}(\Omega\times J) \to B_{pp}^{\kappa_p/l_j,\sigma}(\partial\Omega\times J)$ for $j=1,2$,

we obtain that the composition

$$\begin{pmatrix} A & 0 \\ B_1 & C_1 \\ B_2 & C_2 \end{pmatrix} \begin{pmatrix} P & K & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ B_1P & B_1K & C_1 \\ B_2P & B_2K & C_2 \end{pmatrix}$$

is an isomorphism as a map (for sufficiently large σ)

$$\begin{array}{ccc} L_p^\sigma(\Omega\times J) & & L_p^\sigma(\Omega\times J) \\ \oplus & & \oplus \\ B_{pp}^{\kappa_p m_1,\sigma}(\partial\Omega\times J) & \longrightarrow & B_{pp}^{\kappa_p/l_1,\sigma}(\partial\Omega\times J). \\ \oplus & & \oplus \\ B_{pp}^{\kappa_p m_2,\sigma}(\partial\Omega\times J) & & B_{pp}^{\kappa_p/l_2,\sigma}(\partial\Omega\times J) \end{array}$$

Here we used the simple fact that a block-matrix $\begin{pmatrix} 1 & 0 \\ T & L \end{pmatrix}$ is invertible whenever L is, with inverse given by $\begin{pmatrix} 1 & 0 \\ -L^{-1}T & L^{-1} \end{pmatrix}$. Altogether we can conclude:

Theorem 6.5. *Under the assumptions* (A1) – (A3) *and with* κ_p *from* (6.10)*,*

$$\begin{pmatrix} A & 0 \\ B_1 & C_1 \\ B_2 & C_2 \end{pmatrix} : \begin{array}{c} H_p^{(2,2),\sigma}(\Omega\times J) \\ \oplus \\ B_{pp}^{\kappa_p m_2,\sigma}(\partial\Omega\times J) \end{array} \longrightarrow \begin{array}{c} L_p^\sigma(\Omega\times J) \\ \oplus \\ B_{pp}^{\kappa_p/l_1,\sigma}(\partial\Omega\times J). \\ \oplus \\ B_{pp}^{\kappa_p/l_2,\sigma}(\partial\Omega\times J) \end{array}$$

is an isomorphisms for any sufficiently large σ*.*

In other words, the system (6.9) possesses for any data

$$f\in L_p^\sigma(\Omega\times J),\qquad f_1\in B_{pp}^{\kappa_p/l_1,\sigma}(\partial\Omega\times J),\qquad f_2\in B_{pp}^{\kappa_p/l_2,\sigma}(\partial\Omega\times J),$$

a unique solution

$$u\in H_p^{(2,2),\sigma}(\Omega\times J),\qquad \rho\in B_{pp}^{\kappa_p m_2,\sigma}(\partial\Omega\times J).$$

6.4. **Reduction to the boundary (II).** In a way very similar to the one of the previous section we can also consider systems for unknowns u and $\rho=(\rho_{\mu+1},\ldots,\rho_{\mu+\nu})$ of the form

$$Au=f \quad \text{in } \Omega\times(0,\infty), \tag{6.11}$$

where $A=a(x,t,D_x,D_t)$ is a differential operator of order 2μ, and with boundary conditions on $\partial\Omega\times(0,\infty)$ of the form

$$B_ju+\sum_{k=\mu+1}^{\mu+\nu}C_{jk}\rho_k=f_j,\qquad j=1,\ldots,\mu+\nu, \tag{6.12}$$

where B_j and C_{jk} are as in the previous section. Again, zero initial conditions are required. Writing $\gamma=(\gamma_0,\ldots,\gamma_{\mu-1})^t$ let us assume that

$$\text{(A1)}\quad \begin{pmatrix} A \\ \gamma \end{pmatrix} : H_p^{(2\mu,2\mu)}(\Omega\times J) \longrightarrow \begin{array}{c} L_p(\Omega\times J) \\ \oplus \\ \bigoplus_{j=1}^{\mu} B_{pp}^{(2\mu-j+1-\frac{1}{p},2\mu)}(\partial\Omega\times J) \end{array} \quad \text{isomorphically.}$$

If $(P \quad K)$ with $K = (K_1\ K_2 \dots K_\mu)$ denotes the inverse, the resulting Lopatinskij-matrix is

$$L = \begin{pmatrix} B_1K_1 & \cdots & B_1K_\mu & C_{1,\mu+1} & \cdots & C_{1,\mu+\nu} \\ \vdots & & \vdots & \vdots & & \vdots \\ B_{\mu+\nu}K_1 & \cdots & B_{\mu+\nu}K_\mu & C_{\mu+\nu,\mu+1} & \cdots & C_{\mu+\nu,\mu+\nu} \end{pmatrix}.$$

We assume the following:

(A2) There exist weight functions $\widetilde{m}_1$, $m_{\mu+1}, \dots, m_{\mu+\nu}$ and $l_1, \dots, l_{\mu+\nu}$ such that with

$$m_j(\xi,\tau) := \widetilde{m}_1(\xi,\tau)(\langle\xi\rangle^{2\mu} + i\tau)^{-\frac{j-1}{2\mu}}, \qquad j = 1, \dots, \mu,$$

the Lopatinskij-matrix L is a parabolic mixed order system with respect to $m_1, \dots, m_{\mu+\nu}$, $l_1, \dots, l_{\mu+\nu}$.

Then defining

$$\kappa_p(\xi,\tau) = (\langle\xi\rangle^{2\mu} + i\tau)^{\frac{1}{2\mu}(2\mu-\frac{1}{p})} \tag{6.13}$$

we have $B_{pp}^{\kappa_p m_j,\sigma}(\partial\Omega \times J) = B_{pp}^{(2\mu-j+1-\frac{1}{p},2\mu),\sigma}(\partial\Omega \times J)$ for $j = 1, \dots, \mu$. If we furthermore assume that

(A3) $B_j : H_p^{(2\mu,2\mu),\sigma}(\Omega \times J) \to B_{pp}^{\kappa_p/l_i,\sigma}(\partial\Omega \times J)$ for $i = 1, \dots, \mu+\nu$,

we can conclude

Theorem 6.6. *Under the assumptions* (A1) – (A3) *and with* κ_p *from* (6.13)*, the problem* (6.11), (6.12) *has for any given data*

$$f \in L_p^\sigma(\Omega \times J), \qquad (f_1, \dots, f_{\mu+\nu}) \in \bigoplus_{i=1}^{\mu+\nu} B_{pp}^{\kappa_p/l_i,\sigma}(\partial\Omega \times J),$$

a unique solution

$$u \in H_p^{(2\mu,2\mu),\sigma}(\Omega \times J), \qquad (\rho_{\mu+1}, \dots, \rho_{\mu+\nu}) \in \bigoplus_{j=\mu+1}^{\mu+\nu} B_{pp}^{\kappa_p m_j,\sigma}(\partial\Omega \times J),$$

provided σ *is sufficiently large.*

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