Coulomb effects in magnetotransport of a nondegenerate two-dimensional electron system

Dissertation zur Erlangung des akademischen Grades eines Doktors der Naturwissenschaften an der Universität Konstanz Fachbereich Physik

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Tag der mündlichen Prüfung: 16. Februar 2004

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Two-dimensional electron systems (2DES) have turned out to be crucial to both technology and fundamental research, as is demonstrated for instance by the MOSFET and the discovery of the integer and fractional quantum Hall effects, to name only some of the most prominent examples. Among the great (and ever increasing) variety of 2DES, one system has always elicited a special interest due to a combination of rather unusual properties, namely surface electrons bound to the free surface of liquid helium (SE), which were first theoretically investigated and experimentally realized around 1970 (for a recent overview of 2D electrons on cryogenic substrates see Ref. [1]). The quantizing potential well arises here from the fact that an electron in the vapor phase above liquid $^4$He is attracted to the liquid surface by weak long-range polarization forces, but prevented from entering the liquid itself by a strong short-range repulsion resulting from the noble gas properties of $^4$He.

SE are often considered as a system for complementary study. One of the most notable differences to the usual Fermi-degenerate 2DES in MOSFETs or semiconductor heterostructures is that SE are a nondegenerate 2DES. Here, a high effective mass (equal to the free electron mass $m_e$, see table 1) and low particle density result in a Fermi temperature of only some ten mK or less, which is usually negligible to the temperatures of about 0.1–1 K at which experiments are commonly conducted. Another way of stating this is that the mean distance between electrons is much larger than the thermal wavelength of the single electron. (“Surface electrons” means here and in the following always electrons on bulk liquid $^4$He, similar systems can also exist on other cryogenic substrates, for instance on helium films [1], and have then somewhat different properties, usually more resembling those of 2DES in semiconductors.)

Moreover, SE exist in an environment that is extremely clean, simple and well defined. All impurities are frozen out, a surrounding atomic lattice is absent, and the 2D motion of a single electron along the $^4$He surface is close to that of a free electron moving in vacuum; the only relevant interactions consist in scattering at quantized surface oscillations of the

<table>
<thead>
<tr>
<th></th>
<th>electrons on $^4$He</th>
<th>electrons in GaAs/GaAlAs</th>
</tr>
</thead>
<tbody>
<tr>
<td>effective mass</td>
<td>$m_e \lesssim 2 \times 10^9$</td>
<td>$\sim 0.07m_e$</td>
</tr>
<tr>
<td>particle density [cm$^{-2}$]</td>
<td>$\lesssim 0.05$</td>
<td>$\gg 5 \times 10^{10}$</td>
</tr>
<tr>
<td>Fermi temperature [K]</td>
<td>$\lesssim 6$</td>
<td>$\gg 20$</td>
</tr>
<tr>
<td>scattering time [ns]</td>
<td>$\lesssim 1.0572$</td>
<td>$\gg 0.32$</td>
</tr>
<tr>
<td>mean Coulomb energy/k$_B$ [K]</td>
<td>$\lesssim 130$</td>
<td>$\gg 50$</td>
</tr>
</tbody>
</table>

**Table 1:** Some typical properties of SE in contrast to those of a 2DES in a GaAs/GaAlAs heterostructure (data are mostly taken from Ref. [2])
Introduction and outline

liquid, which is analogous to phonon scattering in solids, and scattering at $^4$He atoms in the vapor phase, which constitute almost ideal weak random pointlike scattering centers. The latter is the dominant scattering mechanism at temperatures above 1 K, where the experiments for this thesis were conducted. SE appear thus as a sort of model 2DES; they lack some of the interesting complexities of usual 2DES, but are in many aspects more accessible to accurate and rigorous theoretical descriptions.

Another, and for this thesis the most relevant property of SE is the high importance of their mutual Coulomb interaction. In 2DES in semiconductor structures, the dielectric properties of the surrounding substrate strongly shield the Coulomb interaction, moreover, its influence as a potential energy is counteracted by the high Fermi energy, which is a kinetic energy. In contrast, in SE the Coulomb interaction is almost unshielded and usually much stronger than the kinetic energy, which is here thermal energy only. The ratio of average potential to average kinetic energy can reach very high values in SE, more than hundred at temperatures around 1 K, and strongly affect the system behavior.

Probably the most famous effect resulting from strong Coulomb interaction is the solidification of a 2DES into a 2D crystal with hexagonal structure, a possibility that was first discussed in 1934 by Wigner for 3D electron systems. SE were the first system where the crystal transition was observed [3], and it has been an area of intense research ever since. Although not directly relevant for this thesis (the experiments were conducted at too high temperature to enter the solid phase), the phenomenon serves well to illustrate some important general aspects about the role of interaction in this system: In SE, the crystal phase is entered with increasing particle density (at fixed temperature), as this increases the Coulomb interaction which must here essentially overcome thermal fluctuations only. The phase transition is in this case a classical one. Quantum effects are negligible due to the mean electron separation being much larger than the electron wavelength. In contrast, in the high-density, degenerate 2DES in semiconductor structures the crystal rather forms towards low density, and the phase transition is a quantum one. Here, the interaction must compete with quantum fluctuations, i.e., the Fermi energy, the latter depending stronger on density than the Coulomb energy.

The subject of this thesis are the effects of the strong Coulomb interaction on the magnetotransport behavior of SE (also called many-electron effects). The fundamental question with respect to these effects is: how does the interaction affect the electron behavior on the quantum level? Note that this question is unrelated to the solid transition, which is here rather a classical phenomenon and insensitive to quantum effects, as mentioned above. On the other hand, it is clear that transport probes the system on a quantum level, as the scattering behavior is directly determined by the quantum properties of the electrons, i.e., the electron system’s quantum excitation spectrum, even more so in a magnetic field, which enhances the quantum aspects of the system by the formation of discrete Landau levels. The latter is especially true for SE, and quantum magnetotransport has traditionally also been an important area of research in this system. The perturbation by scatterers is here quite weak; this leads to very narrow Landau levels; moreover, for this nondegenerate system the magnetic quantum limit, where the electrons occupy only the lowest Landau level, is reached already at very low magnetic field. Together with the simplicity of the single-electron interactions, these features also single out SE as a system that is especially suited to test quantum transport theories, an example is the discovery of the abnormal Hall effect [4].

The role of Coulomb interaction on the quantum level is rather unique in SE, and its theoretical description requires new and unconventional approaches. A conventional
first approximation to take into account Coulomb interaction in the system’s quantum excitation spectrum is to treat it as a sort of correction to the excitation spectrum of an entirely noninteracting system, like in the random phase approximation (see, for example, Ref. [5]). However, generally these approaches are limited to weakly interacting systems (plasmas), and become unsatisfactory if the interaction increases and electron correlations become more pronounced. They are thus inadequate for SE, which are in a strongly correlated, liquid-like state with a high degree of short-range order already at particle densities far below the Wigner solid transition. Moreover, previous theoretical studies suggested that in a magnetic field the strong interaction may even destroy the single-electron Landau quantization [6]. It is the more surprising that recent theoretical studies, which, among other things, alone can entirely explain the experimental results described in this thesis, found that in a strong magnetic field the quantum excitation spectrum of this strongly interacting system can still be related to that of a noninteracting system by a very simple formula [7].

Coulomb effects were first observed in quantum cyclotron resonance (CR) experiments around 1980 [8, 9]. Subsequent theoretical work [6, 10] proposed a model for the origin of Coulomb effects in terms of the so-called fluctuational electric field (FEF), and could partly explain the experimentally observed behavior. However, this was followed by another experimental CR study [11], conducted explicitly to investigate Coulomb effects, which found its results to be in severe conflict with the theory. Only comparatively recently, extensive experimental studies [12, 13] and theoretical work [14, 15] could give a conclusive description of Coulomb effects for the case of the DC magnetoconductivity. However, the controversial earlier findings in CR remained unresolved.

This situation was the main motivation for the experimental work constituting the most important part of this thesis, namely high-resolution CR experiments at variable high frequency (40–60 GHz), which cover a more extended and previously not investigated parameter range [7, 16, 17]. Also, complementary measurements of the DC magnetoconductivity were performed; this allows direct comparison with the CR results [7, 18]. Moreover, Coulomb effects were studied in the nonlinear conductivity, i.e., when the electron system is heated by a strong microwave excitation under CR conditions.

In relation to the experimental work, a new theory for Coulomb effects was developed by Yu. P. Monarkha [7, 17, 18], basing also on the notion of the FEF but differing from previous approaches by treating each electron in its own local reference frame, which eliminates the FEF locally. This allows to find a simple expression relating the quantum excitation spectrum or dynamic structure factor (DSF) of the strongly interacting electron system to the simple and well-known DSF of a noninteracting system. The same simple expression describes thus the DSF in the whole range from an entirely noninteracting to a strongly interacting system, even into the solid regime. The new theory not only explains the new experimental results described in this thesis quantitatively and without adjustable parameter, but does the same for all known experimental data on Coulomb effects in magnetotransport, including the previously controversial CR results (see Ref. [7] for a recent review). As the discussion of the experimental work in this thesis puts a strong emphasis on a detailed and rigorous comparison with the new theory, which includes also new numerical evaluations of the latter, this thesis contains also a detailed description of the new theory.

To be universal, the treatment of Coulomb effects has to be embedded in a theoretical framework for magnetotransport that covers all relevant cases (like the high-frequency AC as well as the DC limit) in the same formalism. Such a frame is given by the
balance equation formalism, which is a sort of quantum extension of the Drude formalism, and essentially reduces transport treatment to evaluations of the DSF. It will here be employed in a great variety of cases (high and low frequency limit, ripplon and vapor atom scattering, linear and nonlinear transport, and for the latter electrons in 3D as well as 2D states), which are treated in literature in a variety of publications differing partly in their approaches. Moreover, a special case that is important for the quantitative analysis of the nonlinear experiments (Sec. 7.3), namely the energy relaxation of SE due to vapor atom scattering (Sec. 3.3.3), has not yet been treated in literature at all. For these reasons, this thesis contains also a consistent description of the balance equation formalism, which covers all relevant cases. A very elementary approach is used, intended to give a clear physical picture as a basis for the discussion of the experimental results.

Outline of this thesis

Chapter 1 treats basic properties of the interaction with the liquid substrate and the vertical quantization of SE. For the analysis of the nonlinear conductivity (see chapter 7) it is important to quantitatively determine the population of higher surface states, this is here done by the WKB method.

Chapter 2 introduces basic equilibrium and nonequilibrium properties of SE as a 2D system, like the phase diagram in dependence of temperature and particle density, the interaction with the scatterers relevant for the 2D motion of SE, the effect of a vertical magnetic field on the 2D quantum states, and general properties of the 2D quantum excitation spectrum or dynamic structure factor (DSF) of the system. The latter is important both for the general treatment of transport and specifically for the treatment of Coulomb effects.

Chapter 3 summarizes the most important general theoretical treatments of magnetotransport, with the focus being on the balance equation method.

Chapter 4 describes the new theory for Coulomb effects developed by Monarkha, which is compared to the experimental results in Chapter 7.

Chapter 5 gives a short overview of the previous experimental and theoretical work on Coulomb effects.

Chapter 6 describes the experimental setup allowing to simultaneously measure both the AC conductivity (cyclotron resonance) and the DC conductivity, and the methods of signal evaluation.

Chapter 7 presents the results obtained in the measurements of CR and DC magnetooconductivity of SE and compares them with theory. The most direct way to investigate Coulomb effects is to vary the electron density, which changes directly the strength of the interaction. Also studied was the influence of the interaction with the scatterers and of the magnetic field. Moreover, measurements of the nonlinear conductivity allowed to study the temperature dependence of Coulomb effects, the energy relaxation of SE, and the transition of the electrons to higher surface states.
Chapter 1

Surface electrons on liquid $^4\text{He}$ (SE)

This chapter treats basic properties of surface electrons concerning the interaction with the substrate and the vertical quantization. A characteristic of SE is that they are rather weakly bound, they “hover” over the bulk liquid surface without interacting with it on an atomic scale; this is the basis for their unique, almost free-electron-like two-dimensional (2D) behavior. For later calculations of 2D transport, it is mainly necessary to know the exact properties of the vertical ground state, these are introduced in Secs. 1.1 and 1.1.1. Also, one should consider the conditions for the stability of the ground state, this is done in Sec. 1.1.2. This is necessary to safely manipulate SE experimentally, and especially relevant for the discussion of the nonlinear studies (see Sec. 7.3.3), when the electrons may be excited to higher surface states. The latter requires a quantitative treatment taking into account the microstructure of the vertical electric field using, for example, the WKB method. Finally, Sec. 1.2 deals with the macroscopic instability limiting the attainable surface electron density, and Sec. 1.3 with the possible consequences of surface electrons forming on the helium film covering parts of the experimental setup.

1.1 Vertical quantization

The very basis for the formation of a 2D electron system is clearly the existence of a suitable quantizing potential well. It is a rather unique feature of liquid helium to provide such a potential well outside its free surface, and even a particularly “clean” one.

As the lightest noble gas, helium is the element with the lowest molecular polarizability. Hence, the intermolecular van-der-Waals forces are very weak, which results in the familiar features of lowest known boiling temperature ($4.2\text{ K}$ for $^4\text{He}$), solidification only under pressure, etc. It also makes the liquid a dielectric with a very low dielectric constant, $\epsilon_{\text{He}} \simeq 1.0572$, for $^4\text{He}$ below $1.4\text{ K}$ (Ref. [19] and references therein).

Moreover, the saturated electron shell of the helium atom repels an excess electron on short distance due to the Pauli principle. This repulsion is very strong compared to the also present polarization attraction, and considerably raises the potential energy of an electron injected into the liquid, by $V_0 \sim 1\text{ eV} \sim 10^4\text{ K}$ (for better comparison, energies are usually given in units of Kelvin in the following). The liquid surface constitutes therefore a high potential barrier for electrons [20, 21]. In fact, once injected, an electron relaxes by forming a small “bubble” of about $17\text{ Å}$ radius [22].

A electron can now be trapped just above the liquid surface between this repulsive barrier and the attraction due to the long-range polarization forces. For a flat surface
Surface electrons on liquid $^4$He (SE)

Figure 1.1: Left: Surface electrons on liquid $^4$He are trapped in a potential well primarily formed by the attraction to an image charge inside the liquid and a strong repulsive barrier at the liquid surface. Right: Schematic representation of an experimental setup. Free electrons are produced by thermal emission from an electrically heated filament. To stabilize and control the surface charge density, a vertical electric field is applied by a parallel plate capacitor.

with a steplike vertical density profile, the polarization interaction can be described by an image charge of size $Q = e(\epsilon_{^4\text{He}} - 1)/(\epsilon_{^4\text{He}} + 1) \approx 0.0278$ $e$ for $^4$He (the dielectric constant of $^4$He vapor is negligible below 1.4 K), as indicated in Fig. 1.1, and the total vertical potential for an electron outside the liquid can in a first approximation be written as

$$V_\perp(z) = V_0 \Theta(-z) - \frac{\Lambda}{z} \Theta(z) + eE_\perp z, \quad \Lambda = \frac{e^2 (\epsilon_{^4\text{He}} - 1)}{16\pi \epsilon_0 (\epsilon_{^4\text{He}} + 1)}$$

(\epsilon_0 is the electric constant), which is shown in Fig. 1.2. The first term describes the repulsive barrier at the surface, the second the image attraction. The third term takes into account that to obtain a measurable SE density, it is necessary to additionally apply a vertical electric holding field $E_\perp$ to compensate the field of the SE layer itself, which is commonly done using a capacitor setup as shown in Fig. 1.1. For calculating 2D transport properties, it is usually sufficient to treat the barrier height as infinite and the electric field as a perturbation, as is done in the following.

Figure 1.2: The vertical potential for SE from Eq. (1.1) without holding field (solid curve), together with the energies of ground and first excited level $\epsilon^{(\perp)}_1 \approx -7.6$ K and $\epsilon^{(\perp)}_2 \approx -1.9$ K, given by Eq. (1.2) (horizontal dashed lines), and the ground state probability distribution according to Eq. (1.3) (dotted curve). The dashed curve shows the potential in the presence of a holding electric field.

Setting $V_0 \rightarrow \infty$ and $E_\perp = 0$ in Eq. (1.1), the 1D Schrödinger equation for the vertical part $f(z)$ of the electron wave function can be solved by setting $f(z) = zR(z)$,
the resulting equation for \( R(z) \) is then identical to that for the \( s \)-state radial wave functions of the hydrogen problem [23] and has therefore the well-known hydrogenic spectrum of energy eigenvalues,

\[
\varepsilon^{(l)}_{\perp} = -\frac{m_e \Lambda^2}{2\hbar^2 l^2}, \quad l = 1, 2, \ldots
\]  

(1.2)

(\( m_e \) is the free electron mass). The energies of the ground state and the first excited state are \( \varepsilon^{(1)}_1 \simeq -7.6 \text{ K} \) and \( \varepsilon^{(1)}_2 \simeq -1.9 \text{ K} \), which is sufficiently small compared to the barrier height \( V_0 \) to justify the approximation \( V_0 \to \infty \).

Direct spectroscopic measurements of the vertical energy levels [24, 25] up to the 10th level [26] yield values that are only about 5\% higher than what is expected from Eq. (1.2) together with additional corrections due to a finite electric holding field (the Stark effect) and a finite electron density (see Sec. 1.1.1). These small residual deviations are ascribed to the fact that the potential of Eq. (1.1) is not a good approximation within atomic distances of the liquid surface [27]. However, it is quite sufficient to treat 2D transport.

The ground state wave function for \( E_{\perp} = 0 \) is given by

\[
f_1(z) = 2\sqrt{3} z \exp(-\gamma z), \quad \gamma \big|_{E_{\perp}=0} = \gamma_0 = \frac{m_e \Lambda}{\hbar^2},
\]  

(1.3)

also shown in Fig. 1.2. The characteristic extension, or “Bohr radius” of this wave function is \( \gamma_0^{-1} = 76 \text{ Å} \), and the mean distance of the electron from the liquid surface is then \( \frac{2}{3} \gamma_0^{-1} = 114 \text{ Å} \), which is very large compared to the typical interatomic distance in liquid helium. It is reduced somewhat by a nonzero holding electric field \( E_{\perp} > 0 \), which leads to a Stark shift of the energy levels and a compression of the wave functions. A variational treatment with Eq. (1.3) as trial wave function leads to the correction [19]

\[
\left( \frac{\gamma(E_{\perp})}{\gamma_0} \right)^{-1} = \frac{4}{3\beta} \sinh \left[ \frac{1}{3} \sinh^{-1} \left( \frac{9\beta}{4} \right) \right], \quad \beta = \sqrt{\frac{2m_e E_{\perp}}{\hbar^2 \gamma_0^3}}.
\]  

(1.4)

Typical values of the holding field on bulk helium do not much exceed 100 V/mm, for this value the wave function extension is still \( \gamma^{-1} = 53 \text{ Å} \). Still, the effect must be taken into account in the evaluation of interaction matrix elements in transport theories.

Due to the large wave function extension and the great height of the barrier \( V_0 \), which requires the wave function to be virtually zero at the surface, a SE is quite insensitive to the atomic structure of the liquid. Even when completely quantized vertically, it has therefore basically free-electron-like properties in horizontal direction [28]. In contrast to common 2D carrier systems in semiconductors, there is evidently no band structure, and the effective mass has almost the free electron value [8, 29]. The mobility is only limited by scattering at \(^4\text{He} \) atoms in the vapor phase and at capillary surface waves and can reach high values, more than \( 10^7 \text{cm}^2/\text{Vs} \) (see Fig. 2.4).

Similar electronic surface states are also found on a few other materials with similar properties of highly negative electron affinity in combination with low polarizability, like solid hydrogen [30] and neon [31]. Also, such states are possible on thin superfluid helium films covering a solid substrate (Sec. 1.3). Still, of the possible materials, liquid \(^4\text{He} \) appears as one of the most simple and “clean”, having the least polarizability and best surface quality. The SE are here subject only to rather weak and well understood and controlled interactions.
1.1.1 Electron-electron interaction and correlation

In the preceding section, the vertical quantization was treated for the case of a single electron. One might question the validity of the results if other electrons are present. In this case, the quantizing potential in principle depends itself on the vertical electron wave functions due to Coulomb interaction between the electrons. The problem must then be solved self-consistently, which leads to a Hartree correction [32]. Contrary to usual 2D systems in semiconductors, for SE this effect is negligible. The important point is that SE usually form a 2D system with very low particle density \( n_s \), typically \( n_s \lesssim 2 \times 10^9 \text{ cm}^{-2} \) (see Sec. 1.2), and strong correlation, where electron-electron interaction occurs only on distances that are much larger than the typical wave function extension.

The strength of correlation is characterized by the plasma parameter \( \Gamma_p \), the ratio between the single SE’s mean potential energy \( V_C = e^2 \sqrt{\pi n_s} / 4 \pi \epsilon_0 \) due to Coulomb interaction and its mean kinetic energy, which is essentially thermal energy \( k_B T \) only:

\[
\Gamma_p = \frac{e^2 \sqrt{\pi n_s}}{4 \pi \epsilon_0 k_B T}.
\]

The Coulomb interaction is unshielded for SE and comparatively strong, and relevant values of \( \Gamma_p \) range between ten and hundred (see Sec. 2.1).

\[\text{Figure 1.3: Numerical calculation of the pair correlation function } g(r) \text{ for a plasma parameter } \Gamma_p \text{ of seven ( ) data taken from Ref. [33]), and the analytic interpolation formula Eq. (1.5) for the first peak of } g(r) \text{ at the mean interparticle distance } r \simeq n_s^{-1/2} \text{ (solid line). Short-range order is well developed for } \Gamma_p \gtrsim 10.\]

According to numerical calculations of the pair correlation function \( g(r) \) [33], short-range order with a pronounced first peak of \( g(r) \) at the mean interparticle distance \( r \simeq n_s^{-1/2} \) is well developed for \( \Gamma_p \) above about ten, as can be seen from Fig. 1.3.

From the analytic short-range interpolation formula [33]

\[
g(r) \simeq \exp\left[-\Gamma_p \left(r_s^{-1} + 0.33r_s - 1.18\right)\right], \quad r_s = \sqrt{\pi n_0}r, \tag{1.5}
\]

valid for \( 5 < \Gamma_p < 50 \), and also shown in Fig. 1.3, one can obtain a simple estimate for the half-width \( \delta_p \) of the first peak of \( g(r) \) as

\[
\delta_p \simeq \frac{1}{\sqrt{\Gamma_p}} n_s^{-1/2}, \tag{1.6}
\]

which means that for \( \Gamma_p \gtrsim 10 \) each electron is surrounded by a well-defined area where the probability to find another electron is near zero, a “correlation hole” whose radius

\[\text{The Fermi energy } E_F \text{ is usually negligible, } E_F \lesssim 0.05 \text{ K for SE (see Sec. 2.1).}\]
is of the order of the mean interparticle distance \( n_s^{-1/2} \) itself, and for SE usually larger than 100 nm, which is much larger than the ground wave function extension \( \gamma_0^{-1} = 76 \, \text{Å} \).

Even if an electron moves away from the \(^4\text{He} \) surface, it can be expected to keep its correlation with the remaining electrons in the ground state at vertical distances \( z \) smaller than \( n_s^{-1/2} \). It will therefore feel their presence mainly as an averaged electric field whose vertical component \( E_z \) gradually increases over the same distance, being essentially zero at \( z \ll n^{-1/2} \) and reaching only at \( z \gtrsim n^{-1/2} \) its final long-range value \( E_z = -e n_s/2\varepsilon_0 \). Thus, there is essentially no influence of electron-electron interaction on the vertical ground state, and only a very gradually increasing influence of \( E_z \) on electrons in higher surface states, which slightly reduces the much stronger Stark effect from the usually present holding field \( E_\perp \). Although this screening effect is quite small for the first few higher states, it is still measurable in far-infrared spectroscopy \([26, 34]\), and comparable to theoretical expectations based on the calculated behavior of \( g(r) \).

### 1.1.2 Ground state population and electron lifetime

As the binding energy of the hydrogenic ground state on liquid \(^4\text{He} \) is only \( \varepsilon_{\text{g}}^{(\perp)} = -7.6 \, \text{K} \), the population of the ground state and the lifetime of a SE in it are in general finite. In fact, with the hydrogenic binding energy alone, the thermal population of the ground state should be virtually zero at temperatures above 0.8 K, because of a strong statistical weight of the free continuum states with \( \varepsilon^{(\perp)} > 0 \) in the partition function \([35, 36]\).

It is therefore very important to consider additional interactions, primarily the holding electric field \( E_\perp \). To a certain fixed density \( n_s \), a characteristic value \( E_\perp^{(0)} \) of the holding field is given by the condition of saturation, i.e., when \( E_\perp \) just cancels the long-range vertical field \( -e n_s/2\varepsilon_0 \) of the SE layer itself above the surface:

\[
E_\perp^{(0)} \equiv \frac{e n_s}{2\varepsilon_0}.
\]

Generally, only for \( E_\perp \geq E_\perp^{(0)} \) SE form a stable 2D system with a high ground state population and a sufficient ground state lifetime. For more detail, one has to consider the microstructure of the electric field near the surface. Each SE being surrounded by a correlation hole of radius \( r_0 \sim n_s^{-1/2} \), as discussed in the preceding section 1.1.1, clearly the effective vertical field acting on an electron at saturation will be equal to \( E_\perp^{(0)} \) for \( z \ll r_0 \) and is compensated by the field \( E_z \) of the other electrons only at \( z \gtrsim r_0 \).

One can approximate the resulting effective potential at saturation for \( z > 0 \) by \([37, 38]\)

\[
V_\perp^{(\text{eff})}(z) = \begin{cases} 
-\frac{\Delta}{z} + eE_\perp^{(0)} \left( z - \frac{z^2}{2r_0} \right) & \text{for } z < r_0 \\
\frac{1}{2} \varepsilon_0 E_\perp^{(0)} r_0 \equiv V_\perp^{(\text{max})} & \text{for } z \geq r_0
\end{cases}
\]

(see Fig. 1.4). For quantitative calculations, in the following \( r_0 \) is determined from the condition \( g(r_0) = 1/2 \), with \( g(r) \) given by Eq. (1.5), this takes also into account that at small plasma parameter \( \Gamma_p \), correlation is not yet fully developed \([37]\).

According to Eq. (1.8), the effective single-electron binding energies increase by \( V_\perp^{(\text{max})} \sim \sqrt{\pi \Gamma_p k_B T} \) \((for \ r_0 \sim n_s^{-1/2})\) compared to those of the image potential \([ Eq. (1.2) ]\), which is quite important for \( \Gamma_p \gtrsim 10 \). Also, \( V_\perp^{(\text{eff})}(z) \) is usually steep enough to effectively depopulate higher bound states in favor of a predominant ground state population.
For quantitative considerations, one can approximately determine the energy levels $\varepsilon_l^{(\text{eff})}$ of an electron in the effective potential from the quasiclassical WKB method, i.e., by numerically solving the equation

$$\int_0^{z_l} \sqrt{2m_e \left[ \varepsilon_l^{(\text{eff})} - V_\perp^{(\text{eff})}(z) \right]} \, dz = \pi \hbar, \quad l = 1, 2, \ldots$$

$[z_l]$ is the outer turning point of the classical electron motion, defined by $\varepsilon_l^{(\text{eff})} = V_\perp^{(\text{eff})}(z_l)$. Of course, the WKB method is not necessarily a good approximation for the low energy levels. Still, it turns out that for $E_\perp^{(0)} = 0$ the solutions of the above equation reproduce the hydrogen levels $\varepsilon_l^{(L)}$ from Eq. (1.2) with a relative error of less than $10^{-4}$. One may therefore expect to also obtain results of sufficient accuracy for the general case. Also, the results of the WKB method were compared with those of a variational calculation for the first three levels after Ref. [25], and the deviations found to be minor for the conditions relevant for this work. Most important are here weak to moderately strong $E_\perp^{(0)}$, where the lower levels are still mainly determined by the image potential.

The $\varepsilon_l^{(\text{eff})}$ determined, the relative fraction $\Delta_1$ of electrons populating the ground state $l = 1$ is

$$\Delta_1 = Z^{-1} \exp \left( -\frac{\varepsilon_1^{(\text{eff})}}{k_B T} \right), \quad Z = \sum_l \exp \left( -\frac{\varepsilon_l^{(\text{eff})}}{k_B T} \right)$$

($Z$ is the partition function). We obtain thus, for instance, that for our typical experimental conditions for equilibrium measurements, corresponding to $n_s \gtrsim 10^7 \text{cm}^{-2}$ and $T \lesssim 1.4 \text{K}$, more than 97% of the electrons are in the ground state.

It remains to consider the SE lifetime. If a SE is excited to a free state above $V_\perp^{(\text{max})}$ and escapes from the bulk surface, it will soon reach the $^4\text{He}$-film-covered walls of the experimental cell above the bulk level, and get trapped there effectively forever with a binding energy of at least 100 K (see Sec. 1.3). At $T > 0.5 \text{K}$, a SE is expected to gain the required excitation energy by converting Boltzmann-distributed horizontal momentum to vertical in an elastic collision with a scatterer ($^4\text{He}$ vapor atom or ripplon), and the escape rate per electron, or inverse lifetime, has a thermally activated behavior [39, 40]:

$$\tau_{\text{esc}}^{-1} = (P_a + P_r) \exp \left( -\frac{W}{k_B T} \right),$$

where $P_a$ and $P_r$ are effective collision rates with vapor atoms and ripplons respectively, and $W$ is the electron binding energy. Above 1 K, where the $^4\text{He}$ vapor is dense and $P_a$...
is dominant, the temperature dependence of the escape rate becomes anomalous \cite{37}, which is interpreted as the onset of diffusive behavior with electron backscattering, but the dependence on the binding energy still seems to follow Eq. (1.10).

For example, measurements at $T = 1.23$ K \cite{37} yield $P_a \simeq 3.4 \times 10^7$ s$^{-1}$ \cite{38}. With the hydrogenic ground state binding energy of 7.6 K alone, the lifetime is then very short: $\tau_{\text{esc}} \sim 10$ $\mu$s. Of course, as 1 mV corresponds to 10 K, even a small additional potential difference between the SE layer and the cell walls will prevent escape under all conditions. For $n_s \gtrsim 10^7$ cm$^{-2}$, the additional binding energy of $V_{\perp}^{(\text{max})}$ at saturation alone makes the lifetime virtually infinite.

In fact, at high $n_s$ SE may even be kept oversaturated ($E_{\perp} < E_{\perp}^{(0)}$) for quite some time \cite{41,42}. However, in our usual conditions for experiments at saturation (see Secs. 6.2 and 6.4), fast escape at $E_{\perp} < E_{\perp}^{(0)}$ is always possible via horizontal motion along the liquid surface. To be stable, we expect the SE layer to be slightly undersaturated ($E_{\perp} > E_{\perp}^{(0)}$) with a potential difference of at least a few mV to the cell walls apart from $V_{\perp}^{(\text{max})}$, which agrees with experimental observations discussed in Sec. 6.4.3. In this case, one should in principle add a linear term to the effective potential Eq. (1.8), but the correction is usually negligible if one is not very much below saturation (see Sec. 6.4.3).

### 1.2 Instability of the liquid surface

Liquid $^4$He has a very low density $\rho_{\text{He}} \simeq 145$ kg m$^{-3}$ and surface tension $\alpha_{\text{He}} \simeq 35.7 \times 10^{-6}$ J m$^{-2}$ at $T \simeq 1.3$ K (Ref. \cite{19} and references therein; the $^4$He vapor density is negligible below 1.4 K). It is easily deformed by the electrostatic forces acting on the SE layer. For instance, at saturation, when the long-range total vertical electric field $E_+$ above the charge layer is zero, and thus the long-range field below the layer is $E_- = en_s/\epsilon_0$, the liquid level $h$ of the charged area (see Fig. 1.1) is depressed with respect to the uncharged area by an amount $\Delta h$ determined by

$$\rho_{\text{He}} g \Delta h = \frac{\epsilon_0}{2} E_-^2$$

(1.11)

($g$ is the gravitational constant). In the following, the convention is that $h$ denotes the height of the SE layer above the bottom plate of the capacitor setup shown in Fig. 1.1.

Moreover, the presence of the charge layer “softens” the spectrum of capillary-gravitational surface waves, i.e., it reduces their oscillation frequency \cite{43}: The surface deformation in a wave leads to a redistribution of the charge density, and thus of the electrostatic forces, which just counteracts the restoring forces of gravity and surface tension. Under certain conditions, a flat surface becomes even unstable and develops spontaneous corrugations; this usually leads to a breakthrough of surface charge at saturation \cite{44}, or to a nonhomogeneous charge redistribution due to the formation of stable multi-electron dimples far below saturation \cite{45}.

The instability depends on the geometry and field distribution of the experimental setup \cite{46}. Usually, the capacitor top plate is comparatively far away from the surface. If then the distance $h$ between charge layer and bottom plate is large compared to the capillary length of liquid $^4$He, $a_{\text{He}} = \sqrt{\alpha_{\text{He}}/\rho_{\text{He}} g} \sim 0.5$ mm, the instability develops with a characteristic wave vector $a_{\text{He}}^{-1}$ of the surface corrugations, and when at saturation the density reaches the value

$$n_s^{(\text{max})} = \frac{1}{e} \sqrt{2 \epsilon_0 \rho_{\text{He}} g a_{\text{He}}} \simeq 2.2 \times 10^9$$

cm$^{-2}$. 

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cm$^{-2}$. 

This corresponds also to the depression depth $\Delta h$ becoming equal to the capillary length. If $h$ is comparable to, or smaller than $a_{\text{He}}$ (in our case, typically $h \sim 0.7\,\text{mm}$, see Sec. 6), the increased interaction between the charge layer and the bottom electrode reduces the stability further. For $h < \sqrt{3}a_{\text{He}}$, the characteristic corrugation wave vector goes to zero and the instability develops when the saturation density reaches the $h$-dependent value

$$n_{s}^{(\text{max})} = \frac{1}{e} \sqrt{\epsilon_{0}\rho_{\text{He}}gh}$$

Here, it is important to take $h$ as the height of the charged area [47].

### 1.3 Surface electrons on helium films

Below the Lambda point, $^4\text{He}$ covers any solid surface above the bulk liquid level by a thin saturated superfluid film, stabilized by the van der Waals interaction with the solid substrate. The film has a (uncharged) thickness $d$ of typically $30\,\text{nm}$ at $1\,\text{cm}$ above the bulk level [48], and it can also support surface electrons.

In the following, some main properties of SE on films are shortly summarized. On films, the attractive potential is mainly given by the image interaction with the solid substrate and very strong, leading, for instance, to a binding energy $e^2/16\pi\epsilon_{\text{He}}\epsilon_{0}d \simeq 130\,\text{K}$ on a $30\,\text{nm}$ thick $^4\text{He}$ film above a metallic substrate (the wave function extension is here usually much smaller than $d$ [49]). SE on films have therefore usually a rather low mobility and a very long lifetime, even under strong extraction fields.

The strong van der Waals interaction renders a film very stable with respect to the electrostatic forces from a surface charge layer. Moreover, if increasing electronic pressure reduces the film thickness, this in turn increases strongly the van der Waals interaction, so that a film self-stabilizes and actually never develops a soft mode instability, like the bulk liquid does [48]. Charge is rather lost by tunneling through the film [49], which usually limits the maximum attainable electron density quite severely if the solid substrate is a conductor. But with a dielectric substrate, very high densities were obtained, about $10^{11}\,\text{cm}^{-2}$ [48]. An explanation is that tunneling occurs quite early at local irregularities, but if the solid substrate is insulating, such weak spots are soon blocked by charge accumulating on the substrate surface after tunneling.

Even for the study of SE on bulk helium, the behavior of SE on films is of practical interest. Below the Lambda point, they usually will be present on inner surfaces of the experimental cell above the bulk level, and one should estimate their possible influence.

However, on a metallic surface kept at constant potential, the large capacitance of the thin film should prevent them from seriously affecting the electric fields inside the cell. For example, charging a film of $d = 30\,\text{nm}$ thickness to a density $n_{s} = 10^{10}\,\text{cm}^{-2}$ will increase the static potential at the film surface at most by $en_{s}/\epsilon_{0}d \simeq 50\,\text{mV}$. Also, their low mobility and close vicinity to the metal wall should prevent direct influence on transport measurements of SE on bulk helium. At most, in experiments with SE on bulk helium at strongly undersaturated conditions ($E_{\perp} \gg E_{\perp}^{(0)}$), electrons flowing from film to bulk surface may cause a slow drift of the electron density on bulk.
Chapter 2

SE as a two-dimensional (2D) electron system

This chapter treats basic equilibrium and nonequilibrium properties of surface electrons on liquid $^4$He as a 2D electron system. Sec. 2.1 gives an overview over the basic equilibrium statistics of SE as determined by the relative importance of the different characteristic energies proper to the system. Sec. 2.2 introduces the scatterers relevant in the 2D motion of SE. The interaction with the scatterers is generally rather weak and does negligibly affect equilibrium properties, except in a magnetic field. Sec. 2.3 treats the 2D quantization induced by a normal magnetic field, and Sec. 2.3.1 the effect of the scatterers on it. Sec. 2.4 introduces the quantum excitation spectrum, or dynamic structure factor of the system, which describes its reaction to a small external perturbation. The DSF allows the most general treatment of transport in form of the balance equation/memory function theory in chapter 3, and it is central to the treatment of Coulomb effects in chapter 4.

2.1 The phase diagram

An overview over the basic statistics of the SE system is obtained from a simplified phase diagram of a general 2D electron system in dependence of temperature $T$ and electron density $n_s$, shown in Fig. 2.1. To emphasize their particularities, SE are compared to 2D electron systems in semiconductor structures, like a GaAs/GaAlAs heterostructure.

At zero magnetic field, the state of the system is essentially determined by the interplay of three characteristic energies per particle: the classical thermal energy $E_{\text{th}}$, the (quantum) Fermi energy $E_F$, and the potential energy due to Coulomb interaction $V_C$:

$$E_{\text{th}} = k_B T,$$
$$E_F = \frac{\hbar^2 \pi n_s}{m^*},$$
$$V_C = \frac{e^2}{4\pi \varepsilon \varepsilon_0 \sqrt{n_s}}.$$

Here, $m^*$ is the effective electron mass, and $\varepsilon$ is the dielectric constant of the surrounding medium. Fig. 2.1 shows the phase diagram for the free-electron values $m^* = m_e$ and $\varepsilon = 1$, which are a good approximation for SE, and for comparison also for $m^* = 0.1m_e$ and $\varepsilon = 10$ as typical values for 2D electrons in a semiconductor structure, which results in an increased Fermi energy and reduced Coulomb interaction compared to SE.
A first important distinction is made concerning the relative strength of thermal and Fermi energy, as this determines whether the statistics of the system are classical (Boltzmann) or quantum (Fermi-Dirac). For $E_{\text{th}} \gg E_F$ (high temperature/low density) the system is nondegenerate, and essentially classical in its statistics. An alternative formulation of the above condition is $n_s^{1/2} \gg \lambda_{\text{th}}$, where $\lambda_{\text{th}} \equiv 2\pi\hbar/\sqrt{2m_kT}$ is the 2D de Broglie thermal wavelength, which means that the mean interelectron distance is much larger than the single electron's quantum wavelength. This condition is generally fulfilled for SE, even at the low experimental temperatures of around 0.1–2 K, as the upper limit for the density $n_s \lesssim 2 \times 10^9 \text{ cm}^{-2}$ (see Sec. 1.2) corresponds to a Fermi energy $E_F \lesssim 0.05 \text{ K}$. In contrast, electrons in semiconductors are for the same temperatures a highly degenerate system, as here applies a lower density limit of $n_s \gtrsim 5 \times 10^{10} \text{ cm}^{-2}$, corresponding to $E_F \gtrsim 10 \text{ K}$ (at lower densities, the electrons tend to become localized due to the reduced screening of impurities within the semiconductor [2]).

Secondly, the effect of the Coulomb interaction, being a potential energy, is to favor spacial ordering of the electrons, in contrast to thermal and Fermi energies, which both are kinetic energies and increase disorder. This is best demonstrated by the appearance of a region in the phase diagram where the electron system solidifies to a crystal with hexagonal structure (due to the long-range nature of the Coulomb interaction, there is no clear distinction between gas and liquid phase [50]). The size of the solid region in the $(n_s, T)$-plane depends on the values for $m^*$ and $\epsilon$, and is therefore in Fig. 2.1 smaller for semiconductor electrons than for SE.

**Figure 2.1:** Phase diagram of 2D electron systems at zero magnetic field. The Wigner solid boundary is shown as heavy solid curve for surface electrons on liquid $^4\text{He}$ (effective mass $m^* = m_e$, dielectric constant of surrounding medium $\epsilon = 1$) and as heavy dashed curve for electrons in a semiconductor structure ($m^* = 0.1m_e$, $\epsilon = 10$). The thin solid line indicates the relation $E_{\text{th}} = E_F$ for surface electrons, and the thin dashed line the same relation for electrons in GaAs. The experimentally accessible region for surface electrons on bulk liquid helium is indicated by the dotted lines.

The existence of a solid phase was first suggested by Wigner for a degenerate system with $E_{\text{th}} \ll E_F$. In this case, the relevant ordering parameter is the ratio between Coulomb and Fermi energy, the so-called quantum plasma parameter $r_p = V_C/E_F$. Monte-Carlo simulations [51] yield a theoretical value of $r_p \simeq 37$ for the solid phase boundary. Towards lower $r_p$, i.e., higher density, the wave functions of different electrons start to overlap, and the solid melts due to quantum fluctuations. For the example of semiconductor electrons, $n_s \gtrsim 5 \times 10^{10} \text{ cm}^{-2}$ corresponds to $r_p \lesssim 5$, and the system is at $B = 0$ always a degenerate fluid/gas with comparatively weak Coulomb interaction. But in principle, the solid phase may still be induced by a strong magnetic field, as this reduces the characteristic extension of the electron wave function (see below).

For the case of SE, the situation is quite different. In the nondegenerate limit $E_{\text{th}} \gg$
2.2 Interaction with scatterers

$E_F$, the relevant ordering parameter is the classical plasma parameter

$$\Gamma_p = \frac{V_C}{E_{th}} = \frac{e^2 \sqrt{\pi n_s}}{4 \pi e_0 e_0 k_B T},$$

and from Kosterlitz-Thouless theory, the solid phase boundary is expected at $\Gamma_p \approx 127$. This value is well within the experimentally accessible parameter range, and in fact, SE were the first system where the solid formation was observed [3]. Even in the liquid phase, the plasma parameter is usually larger than ten, and SE are a strongly correlated system with at least a high degree of short-range order, as already discussed in Sec. 1.1.1.

Finally, in a perpendicular magnetic field $B$, the Landau quantization introduces a new energy scale in form of the cyclotron energy $\hbar \omega_c = eB/m^*$. The consequences depend strongly on whether the system is degenerate or not. High in the degenerate regime ($r_p \lesssim 1$) and at sufficiently low temperature, the relevant ordering parameter is the quantum filling factor $\nu = E_F/\hbar \omega_c$, indicating the number of occupied Landau levels, and the system shows the variety of the integral ($\nu > 1$) and fractional ($\nu < 1$) quantum Hall states. At $\nu \ll 1$ the system may undergo a transition to a magnetically induced Wigner crystal, which can be understood as the magnetic field limiting the electron’s wave function extension to the magnetic length $l_B = \sqrt{\hbar/m^* \omega_c}$, and thus reducing wave function overlap between different electrons. $\nu \ll 1$ corresponds to $l_B$ being much smaller than the average distance $n_s^{-1/2}$ between electrons, and the solid state is possible [52].

For nondegenerate SE with $E_F \ll k_B T$, the magnetic field has not this pronounced influence on the state of the whole system, as it acts only on quantum properties, which here are rather unimportant compared to ordinary thermal fluctuations. Even the solid phase boundary depends only on the classical plasma parameter $\Gamma_p$. The relative population of the Landau levels is determined by the ratio $k_B T/\hbar \omega_c$, and as 1 tesla corresponds to about 1.3 kelvin, the quantum limit $\hbar \omega_c \ll k_B T$, where the SE occupy essentially only the lowest Landau level, is easily accessible. This also results in the excitation spectrum of the system having a quite simple structure, as shown in Sec. 2.4.1.

2.2 Interaction with scatterers

The motion of SE along the liquid $^4$He surface is influenced by two kinds of perturbations: scattering at $^4$He atoms in the vapor phase, and scattering at oscillations of the liquid surface, whose normal mode quanta are called ripplons. The relative importance of the two depends essentially on the temperature $T$ and the strength of the holding electric field $E_{\perp}$, as can be seen from Fig. 2.2. Vapor atom scattering, discussed in Sec. 2.2.1, increases strongly with temperature due to the exponential increase of the vapor density, and only weakly with the holding field due to the compression of the vertical wave function entering the coupling. In comparison, ripplon scattering, discussed in Sec. 2.2.2, increases only weakly with temperature, as it involves only low-energy ripplons whose population depends linearly on $T$, but rather strongly with the holding field, which directly enters the coupling of the electron to the liquid surface. Generally, ripplon scattering is dominant at low temperatures $T < 1$K, while for $T > 1$K it is mostly negligible compared to vapor atom scattering, except at high values of the holding field. In the following, the electron is assumed to stay always in the vertical ground state $f_1(z) \equiv \langle z | 1 \rangle$ from Eq. (1.3), so that an effective 2D representation of an interaction Hamiltonian $H_{int}(r, z)$ ($r$ and $z$ are in-plane and vertical electron coordinates) can be
employed, given by the average over the ground state wave function $\langle 1 | H_{\text{int}}(r, z) | 1 \rangle$. For illustration, the main aspects of linear electric transport at zero magnetic field for the two kinds of scatterers are discussed in Sec. 2.2.3 using the Boltzmann equation.

**Figure 2.2:** Mobility at zero magnetic field $\mu_0$ of surface electrons vs. temperature $T$ for two values of the electron density $n_s$ at saturation (holding field $E_\perp = en_s/2\epsilon_0$). $\nu_0$ is the corresponding collision rate [Eq. (2.10)]. Experimental data (○, ●) are from Ref. [53]. Also shown are the results of theoretical calculations (see Sec. 2.2.3): the mobility due to vapor atom scattering alone (dotted curves), the mobility due to ripplon scattering alone (dashed curves), and the total mobility due to both scattering mechanisms combined by Matthiessen’s rule (solid curves). Ripplon scattering has been treated in the complete control approximation [Eq. (2.17)].

### 2.2.1 $^4$He vapor atoms

The 3D particle density $n_a$ of $^4$He vapor atoms is strongly temperature dependent [19]:

$$n_a = \left( \frac{MT}{2\pi\hbar^2} \right)^{3/2} \exp\left( -\frac{Q}{T} \right), \tag{2.1}$$

where $M \approx 6.646 \times 10^{-27}$ kg is the mass of the $^4$He atom, and $Q \approx 7.17$ K is the vaporization energy of liquid $^4$He.

For scattering, the 3D potential of the interaction between a $^4$He vapor atom at position $r_\alpha$, $z_\alpha$ and an electron at $r, z$ can be taken as a simplified contact-type or Fermi pseudopotential, so that the potential energy of an electron in the vapor becomes

$$V(r, z) = U_a \sum_\alpha \delta(r - r_\alpha, z - z_\alpha), \tag{2.2}$$

where $U_a$ is related to the electron-vapor atom scattering cross section $A = m_e^2 U_a^2 / \pi \hbar^2 \approx 4.98 \times 10^{-20}$ m$^2$ [19].

As the mass of the $^4$He atom $M$ is four orders of magnitude larger than the electron mass $m_e$, vapor atoms can often be regarded as entirely static, and scattering at them as entirely elastic, at least for treating the momentum relaxation of SE. Vapor atom scattering can then be treated analogous to scattering at static neutral impurities in usual solids, and it is sufficient to use the simple 2D interaction Hamiltonian

$$H_{e-a} = U_a \sum_\alpha \delta(r - r_\alpha) |f_1(z_\alpha)|^2, \tag{2.3}$$

obtained by averaging Eq. (2.2) over the vertical electron ground state.
However, to treat the energy relaxation of the electrons, one must include also the small inelastic effects, and it is necessary to use a more strict representation of vapor atoms, namely, as free bosonic particles having 3D momentum $\hbar K$, kinetic energy $\varepsilon^{(a)}_K = \hbar^2 K^2 / 2M$, and a Maxwellian momentum distribution

$$f^{(a)}_K = \left( \frac{2\pi \hbar^2}{M k_B T} \right)^{3/2} n_a \exp \left( -\frac{\varepsilon^{(a)}_K}{k_B T} \right). \tag{2.4}$$

The vapor atom density $\rho(r, z)$ is then represented using vapor atom creation and annihilation operators $a^+_K$ and $a_K$ as usual for a system with conserved total number of particles,

$$\rho(r, z) = \sum \delta(r - r_\alpha, z - z_\alpha) = \frac{1}{V} \sum_{K,Q} a^+_K a_K \exp[i (q \cdot r + q_z z)],$$

where $V$ is the volume of the $^4$He vapor space above the liquid surface, and $Q = (q, q_z)$ with $q$ and $q_z$ parallel and normal to the 2D electron plane. The 2D interaction Hamiltonian becomes in this representation [19]

$$H_{e-a} = \frac{U_a}{V} \sum_{K,Q} \eta(q_z) a^+_K a_K \exp(i q \cdot r), \tag{2.5}$$

with $\eta(q_z) \equiv \langle 1 | \exp(i q_z z) | 1 \rangle = (1 - i q_z \gamma/2)^{-3}$. This representation also has the advantage that the interactions with vapor atoms and ripplons (see next section) can be treated in similar representations; moreover, the scatterers’ free-particle properties can conveniently be employed later on in the balance equation treatment of transport (Sec. 3.3).

### 2.2.2 Ripplons

The scattering at surface oscillations can be treated analogous to phonon scattering in usual solids. The normal mode quanta of oscillations of the free helium surface are called ripplons, they retain the classical capillary dispersion relation $\omega_q^{(r)}$ and have a bosonic thermal distribution function $N_q^{(r)}$ ($q$ is the ripplon wave vector) [54]:

$$\omega_q^{(r)} = \sqrt{\frac{\alpha_{He}}{\rho_{He}}} q^2, \quad N_q^{(r)} = \left( e^{\hbar \omega_q^{(r)}/k_B T} - 1 \right)^{-1} \tag{2.6}$$

($\alpha_{He}$ and $\rho_{He}$ are the surface tension and density of liquid $^4$He, see Sec. 1.2). This form of the dispersion assumes the liquid as ideally incompressible, nonviscous, and sufficiently deep. For single-ripplon scattering of SE, the involved ripplons have a typical wave vector of $q \sim 2k_{th}$, where $k_{th} = \sqrt{2m_e k_B T / \hbar} \approx 5 \times 10^7$ m$^{-1}$ at 1 K is the thermal electron wave vector. In this $q$-range, one can neglect in the dispersion the effects of gravitation and of the electrostatic forces from the surface charge layer (they depend linearly and quadratic on $\sqrt{q}$ [30] and are only relevant at lower $q$), and also quantum corrections (these are only relevant at much higher $q$ [55]).

With the concept of ripplons, one can expand a small, arbitrary displacement $\xi(r)$ of the liquid surface in ripplon creation and annihilation operators $b^+_q$ and $b_q$:

$$\xi(r) = \sum_q \xi_q \exp(i q \cdot r), \quad \xi_q = \frac{1}{\sqrt{S}} Q_q (b_q + b^+_q), \quad Q_q = \sqrt{\frac{\hbar q}{2 \rho_{He} \omega_q^{(r)}}}.$$
(S is the liquid surface area).

To obtain the electron-ripplon interaction Hamiltonian, one begins with the potential energy of an electron at position \((r, z)\) above a deformed liquid surface of shape \(\xi(r)\):

\[
V(r, z, \xi) = V_0 \Theta[\xi(r) - z] - \frac{\Lambda}{\pi} \int d^2r' \int_{-\infty}^{\xi(r')} \frac{dz'}{[(r - r')^2 + (z - z')^2]^2} + eE_\perp z. \tag{2.7}
\]

Compared to the unperturbed potential \(V_\perp(z)\) from Eq. (1.1) above a flat surface, here the first term describes the vertical shift of the repulsive surface barrier, and the second is a generalized representation of the polarization interaction. The structure of the integral reflects that the induced dipole interaction between a single electron and a single \(^4\)He atom is proportional to their distance to the fourth power, and the prefactor ensures that for a flat surface \(\xi(r) = 0\) the image form \(-\Lambda/z\) from Eq. (1.1) is recovered.

A peculiar problem of the ripplon interaction is that the straightforward expression of the perturbation as \(\delta V(r, z, \xi) = V(r, z, \xi) - V_\perp(z)\) leads to a wrong result. This completely removes the holding field term, moreover, the vertical shift of the helium surface with respect to a vertically fixed electron wave function leads to a quite strong perturbation, because the polarization term in the interaction potential diverges at the surface, and one also has here the repulsive barrier with \(V_0 \sim 10^4\) K. This is contrary to the experimental evidence that ripplon scattering depends strongly on the holding field, and becomes very weak in the limit \(E_\perp \to 0\).

**Figure 2.3:** Vertical wave function \(f_1\) of a SE above an uneven liquid surface of shape \(\xi(r)\). The dashed curve indicates the SE’s mean distance from the surface. The interaction between a SE and surface oscillations is characterized by the vertical wave function adjusting adiabatically to the surface shape, leading to a dependence of the ripplon scattering rate on the holding electric field.

A simple argument allowing to obtain the proper representation is that the electron, having a small mass, will in zero order adiabatically follow the surface displacement (see Fig. 2.3), i.e., a more adequate representation of the unperturbed ground-state wave function is one that adapts to the local vertical surface shift as \(f_1[z - \xi(r)]\). Reformulated in the new coordinate \(\tilde{z} = z - \xi(r)\), Eq. (2.7) can be presented as

\[
V(r, \tilde{z}, \xi) = V_0 \Theta(-\tilde{z}) - \frac{\Lambda}{\tilde{z}} + eE_\perp \tilde{z}
\]

\[
+ eE_\perp \xi(r) - \frac{\Lambda}{\pi} \int d^2r' \int_{0}^{\xi(r')-\xi(r)} \frac{d\tilde{z}'}{[(r - r')^2 + (\tilde{z} - \tilde{z}')^2]^2}
\]

\[
\equiv V_\perp(\tilde{z}) + \delta V(r, \tilde{z}, \xi),
\]

where the perturbation \(\delta V(r, \tilde{z}, \xi)\) involves now the holding field together with a modified, strongly reduced polarization interaction, while the barrier shift leads not to any perturbation at all. This is only intended as a sketch of the basic idea, as the coordinate transformation will also lead to new terms in the total Hamiltonian that should be shown.
to be negligible \[56\]. In fact, the rigorous derivation of the correct form of the ripplon interaction has been a subject of some controversy, but the above is now well established as a proper approximation, at least for not too large ripplon wave vectors \[57, 58\].

The amplitude of surface oscillations being small compared to the vertical electron wave function extension (thus \(\hat{z}' \ll \hat{z}\)), one can simplify the integral and expand \(\delta V\) in normal modes of surface oscillations. Taking the average over the vertical ground state, one finally obtains the 2D electron-ripplon interaction Hamiltonian in first order as

\[
H_{e-r} = \frac{1}{\sqrt{S}} \sum_q V_q Q_q \left( b_q + b_{-q}^\dagger \right) \exp(i\mathbf{q} \cdot \mathbf{r}),
\]

where \(K_1(x)\) is the modified Bessel function of the second kind or Macdonald function.

This \(H_{e-r}\) takes into account one-ripplon scattering processes only, which is sufficient here (two-ripplon processes are essentially only relevant for the energy relaxation of SE at low temperature and zero magnetic field \[59, 60\]). Typically, the wavelength of ripplons involved in electron scattering is much larger than the vertical electron wave function extension, i.e., \(q < 2\gamma\), in this case the polarization term has the analytical form \[61\]

\[
eE_q = \Lambda q^2 \frac{2}{w(\frac{q^2}{4\gamma^2})}, \quad w(y) = -\frac{1}{1-y} + \frac{1}{(1-y)^{3/2}} \ln\left(\frac{1 + \sqrt{1-y}}{\sqrt{y}}\right).
\]

### 2.2.3 Transport at zero magnetic field

It is instructive to apply the above results for the interaction to treat DC electrical transport at zero magnetic field using the Boltzmann equation. The electron system is assumed to be isotropic, isothermal, and subject to a weak uniform in-plane DC electric driving field \(E\), resulting in an electric current density per unit length \(j = -e n_u u\), where \(u\) is the average drift velocity of the electrons. Ohm’s law postulates a linear relation between \(j\) and \(E\), which is commonly expressed using the notations

\[
j = \sigma_0 E = en_u \mu_0 E = \frac{e^2 n_k}{m_e} \frac{1}{\nu_0} E,
\]

where \(\sigma_0\) is the conductivity, \(\mu_0\) is the mobility, and \(\nu_0\) is the mean momentum relaxation rate, or momentum collision rate. \(\nu_0\) describes the average effect of electron-scatterer collisions, which randomize the electron momentum and thus limit the drift velocity. It is calculated in the following for scattering at vapor atoms and ripplons.

### Momentum collision rates

At zero field, the unperturbed in-plane electron states are simply plane waves \(|\mathbf{k}\rangle\) with \((\mathbf{r} | \mathbf{k}\rangle = S^{-1/2} \exp(i\mathbf{k} \cdot \mathbf{r})\) (\(S\) is the system area) and the free-electron energy spectrum \(\varepsilon_k = \hbar^2 k^2 / 2m_e\). From Fermi’s “golden rule” and the interaction Hamiltonians introduced in the preceding Secs. 2.2.1 and 2.2.2, one obtains then easily the probability \(P_{k'\mathbf{q}, k}\) per time for one-electron transitions from an occupied state \(|\mathbf{k}\rangle\) to an empty one \(|\mathbf{k} + \mathbf{q}\rangle\) due
to the interaction with vapor atoms and ripplons, respectively:

\[
P_{k+q,k}^{(a)} = \frac{2\pi}{\hbar S} U_a^2 n_\alpha \delta(\varepsilon_{k+q} - \varepsilon_k) \int dz \left| f_1(z) \right|^4,
\]

\[
P_{k+q,k}^{(r)} = \frac{2\pi}{\hbar S} V_q^2 Q_q^2 \left[ N_q^{(r)} \delta(\varepsilon_{k+q} - \varepsilon_k - \hbar \omega_q^{(r)}) + (N_q^{(r)} + 1) \delta(\varepsilon_{k+q} - \varepsilon_k + \hbar \omega_q^{(r)}) \right],
\]

where \( \int_0^\infty dz \left| f_1(z) \right|^4 = 3\gamma/8 \) after Eq. (1.3). For ripplon scattering, \( P_{k+q,k}^{(r)} \) explicitly reflects that the electron either absorbs or emits a ripplon, and in principle scattering is inelastic. However, for typical thermal values of the 2D electron momentum (\( k_{\text{th}} = \sqrt{2m_e k_B T}/\hbar \simeq 5 \times 10^7 \text{ m}^{-1} \) at 1 K), the energy of the involved ripplons is small enough compared to the electron energy (\( \hbar \omega_k^{(r)}/\varepsilon_k < 0.1 \) for \( k > 10^5 \text{ m}^{-1} \)) to treat the scattering as quasi-elastic. For vapor atom scattering, \( P_{k+q,k}^{(a)} \) can be directly derived from Eq. (2.3) already assuming the vapor atoms as ideally static, and scattering as completely elastic.

A more rigorous derivation from Eq. (2.5) of course also shows inelastic effects to be negligible here due to the small mass ratio \( m_e/M \sim 10^{-4} \) [19].

For elastic scattering, the above transition rates lead to simple expressions for the typical momentum relaxation (collision) rate \( \nu(k) \) due to scattering of an electron having initial momentum \( k \). From the usual definition \( \nu(k) = \sum_q \left( 1 - \cos \varphi \right) P_{k+q,k} \) (\( \varphi \) is the angle between \( k \) and \( k + q \), with \( k = |k + q| \) for elastic scattering) [62], one obtains for vapor atom and ripplon scattering, respectively [19, 56],

\[
\nu_a = \frac{m_e U_a^2 n_a 3\gamma}{\hbar^3} \frac{1}{8},
\]

\[
\nu_r(\varepsilon_k) = \frac{m_e}{2\pi \hbar^2} \int_0^{2\pi} d\varphi V_q^2 Q_q^2 \left( 2N_q^{(r)} + 1 \right) (1 - \cos \varphi), \quad q = 2k \sin \varphi/2.
\]

The total collision rate is then given by the sum \( \nu_a + \nu_r(\varepsilon_k) \) (Matthiessen’s rule).

For a typical thermal electron momentum \( k_{\text{th}} \simeq 5 \times 10^7 \text{ m}^{-1} \) at 1 K, the ripplons involved in scattering belong to the low-energy part of the thermal spectrum (\( \hbar \omega_k^{(r)}/k_B T \ll 1 \) for \( k < 10^8 \text{ m}^{-1} \) and \( T \sim 1 \text{ K} \)), so their density \( N_q^{(r)} \) and the typical ripplon collision rate \( \nu_r \) depend linearly on \( T \). Also, \( \nu_r \) depends on the holding field \( E_\perp \) entering directly the coupling \( V_q \) in Eq. (2.8), linearly when it is smaller than the typical thermal value of the polarization term \( E_q \) (both are equal at \( E_\perp \simeq 200 \text{ V/cm} \) at 0.5 K, corresponding to a saturation density \( n_s \simeq 2 \times 10^8 \text{ cm}^{-2} \), and quadratic at stronger fields.

In contrast, \( \nu_a \) depends exponentially on temperature through the \( ^4 \text{He} \) vapor density \( n_a \) [Eq. (2.1)]. The holding field has only a comparatively weak influence through the wave function parameter \( \gamma \) according to Eq. (1.4).

The vapor atom collision rate \( \nu_a \) depends not on \( k \), and can be directly inserted in Eq. (2.10). At weak holding field, \( \nu_a \) is almost \( k \)-independent, too, due to the structure of the polarization term \( E_q \). But \( \nu_r \) rather strongly depends on the kinetic energy, \( \nu_r(\varepsilon_k) \propto 1/\varepsilon_k \), when \( E_\perp \) dominates the coupling \( V_q \). This allows to see a principal difference between transport of noninteracting and strongly interacting electrons [42, 53, 58], because the averaging over the electron energy to obtain the mean collision rate is different for the both cases. This can be seen from the Boltzmann equation.

### Boltzmann equation

For a simple demonstration of the effect of electron interaction on transport, consider the usual linear Boltzmann equation in a DC driving field \( E \) with electron-ripplon and
2.2 Interaction with scatterers

electron-electron scattering, using the relaxation time approximation for the ripplon collision term \( \partial f_k / \partial t \bigg|_{e-r} \) (the temperature is assumed constant):

\[
- \frac{e}{\hbar} \mathbf{E} \cdot \nabla_k f_k^{0} = \frac{\partial f_k}{\partial t} \bigg|_{e-r} + \frac{\partial f_k^0}{\partial t} = -\nu_r(\varepsilon_k) (f_k - f_k^0) .
\] (2.12)

To also include vapor atom scattering, \( \nu_r(\varepsilon_k) \) should be replaced by \( \nu_a + \nu_r(\varepsilon_k) \).

**Noninteracting electrons** For noninteracting electrons, the electron-electron collision term vanishes, \( \partial f_k / \partial t \bigg|_{e-e} = 0 \), and the usual solution for the nonequilibrium distribution function,

\[
f_k = f_k^0 + \frac{eE \cdot \nabla_k f_k^0}{\hbar \nu_r(\varepsilon_k)} ,
\] (2.13)

is immediately obtained, where the equilibrium distribution function is Maxwellian, \( f_k^0 \propto \exp\left(-\varepsilon_k / k_B T\right) \), with the normalization \( \sum_k f_k^0 = N_s \) for nondegenerate SE (\( N_s \) is the total number of electrons). The total current is then

\[
j = -S^{-1} \sum_k e \mathbf{k} f_k / m_e \text{ as usual, and by comparing the result with Eq. (2.10), one finally obtains the mean momentum collision rate \( \nu_{1-e} \) for the noninteracting, or single-electron case to be given by the following average over the electron energy [53]:}

\[
\nu_{1-e} = \left( \int_0^\infty x e^{-x} \frac{1}{\nu_r(xk_BT)} \, dx \right)^{-1} .
\] (2.14)

However, it is clear that SE should rather be considered as a strongly interacting system, and in fact, while the above single-electron result agrees well with experimental data at weak holding field, deviations start to appear as soon as at strong enough \( E \parallel \) the energy dependence of \( \nu_r(\varepsilon_k) \) becomes important, as can be seen in **Fig. 2.4**.

---

**Figure 2.4**: Detailed study of the zero-field mobility \( \mu_0 \) of SE in the ripplon scattering regime at two different densities \( n_s \) at saturation. \( \mu_0 \) is the corresponding collision rate [Eq. (2.10)]. The experimental data (○, ■) are taken from Ref. [63] (the kinks at low \( T \) show the Wigner transition). Also shown are calculations of the total \( \mu_0 \) for vapor atom and ripplon scattering in two different treatments: single-electron treatment Eq. (2.14) (dashed curves), and complete control approximation Eq. (2.17) (solid curves). At low \( n_s \), both treatments identically agree with the experimental data, while at high \( n_s \) (high holding field) the single-electron result deviates more.

To understand the difference, it is instructive to recall that the nonequilibrium distribution function \( f_k \) of Eq. (2.13) can in linear approximation also be written as

\[
f_k = f_k^0(\mathbf{k} - m_e \mathbf{u}_k / \hbar) \quad \text{with} \quad \mathbf{u}_k \equiv -eE / m_e \nu_r(\varepsilon_k) [64].
\] This is as if due to the field \( \mathbf{E} \) each electron in state \( \mathbf{k} \) had acquired a small extra drift velocity \( \mathbf{u}_k \), which shifts the
corresponding part of the equilibrium distribution function \( f_k^0 \) by an amount \( m_e u_k / \hbar \) in \( \mathbf{k} \)-space. If \( u_k \) is energy-dependent, different parts of \( f_k^0 \) will be shifted by different amounts. In the case of SE at strong holding field, when \( |u_k| \propto 1/\nu(\varepsilon_k) \propto \varepsilon_k \), electrons with energy near zero will be almost immobile (no shift of \( f_k^0 \)), while those with high energy acquire the highest drift velocity (strong shift of \( f_k^0 \)), which finally results in \( f_k \) being a distorted Maxwellian, as schematically shown in Fig. 2.5(a). The important point is that this situation is changed by strong electron-electron interaction.

**Figure 2.5:** Maxwellian equilibrium distribution function \( f_k^0 \) (dashed), and nonequilibrium distribution function \( f_k \) (solid) in the presence of a driving field \( \mathbf{E} \) for an energy-dependent momentum collision rate \( \nu(\varepsilon_k) \propto 1/\varepsilon_k \), corresponding to ripplon scattering at very strong holding field: (a) shows the case of noninteracting electrons, where each electron has a different drift velocity \( |u_k| \propto 1/\nu(\varepsilon_k) \propto \varepsilon_k \) (see text). (b) shows the case of strongly interacting electrons, where all electrons have the same drift velocity. The average of the collision rate over energy is different for both cases, which leads to the different theoretical results for the mobility shown in the preceding Fig. 2.4.

**Interacting electrons** Strong electron-electron interaction will result in a fast redistribution of momentum between the electrons. The collision rate due to electron-electron interaction can be estimated by the characteristic frequency of short-wavelength oscillations in a Wigner solid, \( \omega_0 \simeq 2.11e n_s^{3/4}/\sqrt{4\pi\varepsilon_0 m_e} \gtrsim 6 \text{ GHz for } n_s \gtrsim 10^7 \text{ cm}^{-2} \) (see Sec. 4.1), to be usually much larger than the typical collision rate due to scattering at ripplons. This will equilibrate the differences in drift velocity \( u_k \) between the electrons found in the noninteracting case, so that finally all will drift with the same velocity \( u \). The nonequilibrium distribution function is then instead of a distorted rather a uniformly shifted or displaced Maxwellian, and one is led to making the following ansatz for solving the Boltzmann equation in this case:

\[
f_k = f^0\left(k - \frac{m_e}{\hbar}u\right),
\]  

which is graphically represented in Fig. 2.5(b). Another way of stating this is that strongly interacting electrons will always attain an equilibrium momentum distribution within their center-of-mass (CM) reference frame. This assumption is also called the complete control approximation [42, 53].

In Eq. (2.15), the common drift velocity \( u \) is initially unknown. To determine \( u \), one can now employ the equation of force balance, which basically states that the total momentum of the whole electron system acquired through the driving field \( \mathbf{E} \) must be equal to the total momentum lost through the scattering at ripplons:

\[
e N_0 \mathbf{E} = \sum_k \hbar k \left. \frac{\partial f_k}{\partial t} \right|_{t=0}. \tag{2.16}
\]

Formally, this relation is derived by multiplying the Boltzmann equation Eq. (2.12) by \( \hbar k \) and summing over all \( k \). The important point is that the electron-electron collision
2.3 Landau quantization in magnetic field

This section outlines the basic properties of SE concerning the horizontal motion in the presence of a perpendicular magnetic field. For this purpose, SE can be regarded as a purely 2D system, where all electrons are in the vertical ground state [Eq. (1.3)].

In the following, first are introduced the properties of an ideal 2D system, i.e., without interaction with scatterers. At zero magnetic field, the horizontal quantum states of the ideal system are plane waves, the energy spectrum is continuous, and the density of states (DOS) is independent of energy (spin interactions are disregarded). In a magnetic field, the quantum states of the ideal system are Landau states, which have a completely discrete energy spectrum, and the DOS consists of a series of singularities (δ functions).

In a real system, the perturbation of these ideal states by the scatterer leads to an energy uncertainty of the Landau levels and a broadening and damping of the singularities in the DOS, this is discussed in Sec. 2.3.1.

The Hamiltonian of a single 2D electron moving in the (x, y)-plane with a magnetic field \( \mathbf{B} = (0, 0, B) \) perpendicular to the 2D plane is

\[
H_0 = \frac{1}{2m_e} (p + eA)^2,
\]  

(2.18)
where $p$ is the momentum operator, and common choices for the 2D vector potential $\mathbf{A}$ are the asymmetric Landau gauge $\mathbf{A} = (0, Bx)$, or the symmetric (Dingle) gauge $\mathbf{A} = (-By/2, Bx/2)$. For constructing free-electron states, the Landau gauge is more convenient. The eigenvalue equation to Eq. (2.18) can then formally be reduced to the one of a one-dimensional harmonic oscillator, oscillating with the cyclotron frequency $\omega_c = eB/m_e$ and having therefore the discrete energy eigenvalues $\varepsilon_N = \hbar \omega_c (N + \frac{1}{2})$, $\omega_c = eB/m_e$, $N = 0, 1, ...$

the so-called Landau levels. The corresponding eigenfunctions $|N, X\rangle$ have the form

$$\langle r | N, X \rangle = \frac{1}{S^{1/4}} \varphi_N(x - X) \exp\left(-iX \frac{r^2}{l_B^2}\right),$$

(2.19)

($S$ is the system area). $\varphi_N(x - X)$ is the $N$-th eigenfunction of a linear harmonic oscillator centered at $x = X$, and $H_N(x)$ is a Hermite polynomial. A characteristic length scale of the oscillator eigenfunctions is the magnetic length $l_B$. The uncertainty in position of $\varphi_N$ is $\delta x \equiv \sqrt{\langle x^2 \rangle - X^2} = l_B \sqrt{N + 1}/2$ and the uncertainty in momentum is $\delta p_x \equiv \sqrt{\langle p_x^2 \rangle} = (\hbar/l_B) \sqrt{N + 1}/2$.

The classical picture has the electron going in a circular cyclotron orbit due to the Lorentz force. For the quantum system, $X$ relates to the position of the orbit center, and $\delta x$ to the cyclotron radius.

As the eigenenergies depend only on the Landau level index $N$, the states $|N, X\rangle$ are degenerate with respect to the center coordinate $X$. The allowed values of $X$ in the 2D plane are spaced by $\Delta X = 2\pi l_B^2/\sqrt{S}$, and sum up to give the number of states per level as $S/2\pi l_B^2$ (as if all states that at zero magnetic field lie within an energy interval $\hbar \omega_c$, and have constant density of states $m_e S/2\pi \hbar^2$, are concentrated into one Landau level). The density of states (DOS) as function of the energy $\varepsilon$ is therefore given by (see Fig. 2.6)

$$D(\varepsilon) = \frac{S}{2\pi l_B^2} \sum_{N=0}^{\infty} \delta(\varepsilon - \varepsilon_N) = \frac{S}{2\pi l_B^2} \sum_{N=0}^{\infty} \frac{1}{\pi} A_N(\varepsilon),$$

(2.20)

where the normalized density of states, or spectral function of the $N$-th Landau level $A_N(\varepsilon) \equiv \pi \delta(\varepsilon - \varepsilon_N)$ for the ideal system has been introduced. For a real system, the perturbations due to the presence of scatterers are taken into account by replacing the ideal spectral functions by broadened and damped ones, as is shown in Fig. 2.6. In the following are discussed the mechanisms of the broadening and the resulting spectral functions, or Landau level shapes.

### 2.3.1 Broadening of Landau levels

In a real 2D system, the ideal electron states are perturbed by the interaction with scatterers $H_{\text{int}}$, so that the total Hamiltonian becomes $H = H_0 + H_{\text{int}}$. For SE, the scatterers are either vapor atoms or ripplons (Sec. 2.2). At zero magnetic field, the influence of a weak interaction on the basic quantum states can usually be neglected.
This is no longer possible in a magnetic field, because the density of states (DOS) of the ideal system is singular [Eq. (2.20)], which makes even very small perturbations important, and leads to unphysical results in theoretical treatments.

For instance, to treat transport, one usually calculates the rate of transitions of a single electron between its basic quantum states due to an interaction (driving field, the scatterers themselves, etc.). With a singular DOS, the rate of ideally elastic transitions diverges, as the density of final states diverges inside a Landau level. On the other hand, any small inelastic effect makes the electron “miss” the infinitely narrow Landau level it started from, so the density of final states and the transition rate become zero.

This dilemma is solved by assuming that the interaction with the scatterers removes the singular DOS and leads to an effective broadening and dampening of the Landau levels. Formally, one replaces the $\delta$ functions in Eq. (2.20) by spectral functions $A_N(E)/\pi$ with the same area, but having a finite width and height (see Fig. 2.6).

One can calculate the level broadening by different theoretical methods. One of the earliest and best established is the self-consistent Born approximation (SCBA) [32, 68]. It takes into account that the level broadening must be calculated self-consistently, as the effect of the scatterers on the DOS depends itself on the DOS. Simple results of a clear physical nature can be obtained for two extreme cases, long- and short-range interaction.

If the range of the electron-scatterer interaction potential is much larger than the electron wavelength, the broadening is of the inhomogeneous type and essentially due to the fact that the potential energy of the electron (averaged over its wave function) depends on its position in the 2D plane, i.e., the level broadening is determined by the variation of $H_{\text{int}}$ in the 2D plane. The broadening is then the same for all levels and independent of the magnetic field.

In SE, however, $^4\text{He}$ vapor atoms rather constitute ideal short-range scatterers, as the interaction potential is given by a $\delta$ function. Moreover, usually many vapor atoms are within the range of the electron wave function, so the spatially averaged potential is essentially constant. The broadening is then rather of the homogeneous type, and can probably be best understood in terms of a finite lifetime of the electron states, limited by collisions of the electron with individual vapor atoms, the so-called collision broadening:

After the uncertainty relation, a finite lifetime $\tau$ is equivalent to an energy uncertainty, or typical level broadening $\Gamma_\alpha \sim \hbar/\tau$. In a first approximation, $1/\tau$ may be considered to be proportional to the zero-field collision rate $\nu_\alpha$ [Eq. (2.11)]. But additionally, one must take into account that the Landau quantization increases the density of states at the (broadened) Landau levels by a typical factor $\hbar\omega_c/\Gamma_\alpha$ compared to the zero-field case. For elastic collisions, this may be considered to effectively increase the density of final states and thus the collision rate by the same factor. One estimates thus $1/\tau \sim \nu_\alpha\hbar\omega_c/\Gamma_\alpha$, which illustrates the need for self-consistency. Together with the energy uncertainty relation, one arrives at $\Gamma_\alpha \sim \sqrt{\hbar\omega_c\nu_\alpha}$. Contrary to the long-range case, here the level broadening increases with magnetic field proportional to $\sqrt{B}$. Full Landau quantization with entirely separated Landau levels is present for $\omega_c > \nu_\alpha$.

**Vapor atom scattering** The following shortly summarizes the results of the SCBA. The spectral function is obtained to have a semielliptic shape,

\[ A_N(\varepsilon) = \frac{2}{\Gamma_N} \sqrt{1 - \left( \frac{\varepsilon - \varepsilon_N}{\Gamma_N} \right)^2}, \quad (2.21) \]
Figure 2.6: Density of states $D(\varepsilon)$ vs. energy $\varepsilon$ of a 2D electron in magnetic field [Eq. (2.20)]. In an ideal system, $D(\varepsilon)$ consists of singularities at the energies $\varepsilon_N = \hbar \omega_c (N + \frac{1}{2})$ (dotted lines). In a real system, the perturbation by scatterers leads to broadening and dampening of the singularities. The resulting $D(\varepsilon)$ is shown for a semielliptic Landau level shape [Eq. (2.21), dashed curve], and a Gaussian shape [Eq. (2.23), solid curve].

where for elastic short-range scattering at vapor atoms the width parameter $\Gamma_N$ is the same for all levels and given by

$$\Gamma_N = \Gamma_a = \hbar \sqrt{\frac{2}{\pi} \omega_c \nu_a}, \quad (2.22)$$

with the zero-field collision rate $\nu_a$ given by Eq. (2.11). $\nu_a$ depends exponentially on temperature and weakly on the holding field (Sec. 2.2), the same applies thus to $\Gamma_a$, besides the $\sqrt{B}$ dependence of the latter.

However, the semielliptic level shape is not very convenient for further theoretical treatment, besides featuring an abrupt cutoff at the sides which is unphysical. For example, another approach, the so-called cumulant expansion method [69], results in a Gaussian Landau level shape,

$$A_N(\varepsilon) = \sqrt{\frac{2\pi}{\Gamma_N}} \exp \left[-2 \left( \frac{\varepsilon - \varepsilon_N}{\Gamma_N} \right)^2 \right], \quad (2.23)$$

with the width parameter $\Gamma_N$ still given by Eq. (2.22).

Both level shapes are shown in Fig. 2.6. The real level shape is probably a kind of mixture of semielliptic and Gaussian [70, 71]. However, the final results of transport theory usually do not much depend on the exact shape of the Landau levels [4, 60]. The reason is that the level shape usually enters calculations in form of a convolution $\int d\varepsilon' A_N(\varepsilon') A_N(\varepsilon' + \varepsilon)$ describing the probability for electron transitions from occupied to empty states. Although the semielliptic and Gaussian level shapes themselves look rather different, the form of convolutions for the two is remarkably similar, see Fig. 3.1 for an example. A noticeable difference in the use of the two shapes may only occur when the thermal occupation of states varies noticeably across the level (like at very high magnetic field), or when the broadening is very different between $A_N$ and $A_{N'}$, which is usually not the case. It is therefore justified and customary to mostly use a Gaussian shape for a more convenient theoretical treatment, as is done in the following.

Ripplon scattering It is considerably more difficult to treat the broadening due to ripplon scattering, and it cannot be simply related to a zero-field collision rate, as in the case of $\Gamma_a$. For quasi-elastic scattering, the most convenient approximate expression for the ripplon broadening parameter $\Gamma_r$ of the ground level is [60]

$$\Gamma_r^2 = \frac{\Lambda^2 k_B T}{4 \alpha_{He} l_B^2} \left\{ \int_0^\infty x w^2 \left( \frac{x}{2 \gamma B^2} \right) + e \frac{E_{\perp} l_B^2}{\Lambda} \left( \frac{x}{2 \gamma B^2} \right) \right\} e^{-x} dx + \left( \frac{e E_{\perp} l_B^2}{\Lambda} \right)^2, \quad (2.24)$$
2.4 Dynamic structure factor (DSF)

where the function $w(y)$ is the same as in Eq. (2.9). At small electron densities, when the holding field terms in Eq. (2.24) are negligible, it can be shown that $\Gamma_r \propto \sqrt{BT}$. In the limit of high densities, $\Gamma_r \propto E_\perp \sqrt{T}$ and independent of $B$ according to Eq. (2.24).

In principle, the ripplon broadening is different for higher levels. But to treat transport later on, mainly ground and first excited level are relevant; moreover, the ripplon broadening will only occur as a minor correction to the vapor atom broadening. In the following, the total broadening due to vapor atom and ripplon scattering together is therefore taken to be approximately the same for all levels, and given by [7, 60]

$$\Gamma_N = \Gamma_{se} = \sqrt{\Gamma_2^2 + \Gamma_r^2}. \quad (2.25)$$

So far, effects due to electron-electron interaction on the level broadening were neglected, i.e., the above results correspond to the so-called single-electron treatment. The effects due to Coulomb interaction will be described later in Sec. 4.3.

2.4 Dynamic structure factor (DSF)

The dynamic structure factor $S(q, \omega)$ is an important means for the characterization of a many-body system. It contains information on its spacial and temporal structure, and gives directly the excitation spectrum of the system in question. Within the scope of this work, the DSF plays a central role, both for the description of electron transport in general, and specifically for the treatment of Coulomb effects.

The meaning of the DSF is best illustrated from its use in scattering theory [72, 73]. Here, it provides a relation between the inelastic scattering cross section as measurable quantity, and the properties of the probed 2D target system, consisting of $N_s$ particles at positions $r_i$ and having unperturbed many-body eigenstates $|j\rangle$ with eigenenergies $\epsilon_j$ before and after scattering. The probe beam is assumed to consist of 2D particles that are in simple plane wave states $|m\rangle$ with momentum $\hbar m$ and energy $\epsilon_m(p)$ before and after scattering, and interact with the target by a position-dependent interaction Hamiltonian

$$H_{\text{int}} = \sum_i V(r_i - r^{(p)}),$$

where $r^{(p)}$ is the probe particle position. In a 3D situation, an effective 2D interaction Hamiltonian may be used (see Sec. 2.2).

The interaction is assumed weak so that first-order perturbation theory can be used. Moreover, it is convenient to introduce the Fourier transforms of the target system particle density, $n_q \equiv \sum_i e^{-iq\cdot r_i}$ [the particle density operator itself is $n(r) \equiv \sum_i \delta(r - r_i)$], and of the interaction potential, $V_q \equiv \int dr V(r)e^{-iq\cdot r}$, to express the interaction as $H_{\text{int}} = S^{-1} \sum_q V_q n_{-q} e^{-iq\cdot r^{(p)}}$ ($S$ is the system area). The matrix element for a specific transition of the whole system, probe particle plus target, in which the probe particle changes from initial state $|m\rangle$ before to final state $|m - q\rangle$ after the scattering event, and simultaneously the target changes from $|j\rangle$ to $|k\rangle$, is then

$$\langle m - q, k | H_{\text{int}} | m, j \rangle = S^{-1} V_q \langle k | n_{-q} | j \rangle$$

The total probability per time $P_{m - q, m}$ for the probe particle to be scattered from state $|m\rangle$ to state $|m - q\rangle$, whereby it transfers momentum $\hbar q$ and energy $\epsilon_m^{(p)} - \epsilon_{m - q}^{(p)} \equiv \hbar \omega$ to the target, is then the sum of the probabilities of all corresponding transitions of
the whole system. The probability for a specific transition is given as usual by Fermi’s “golden rule”, and the sums runs over the initial target states $|j\rangle$ that are occupied with probability $\rho_j$ (in thermal equilibrium the Boltzmann factor divided by the partition function, $\rho_j = e^{-\varepsilon_j/k_B T}/Z$, $Z = \sum_j e^{-\varepsilon_j/k_B T}$), and all target states as final states $|k\rangle$, and is written as

$$P_{m-q,m} = \frac{N_s}{\hbar^2 S^2} |V_q|^2 S(q, \omega),$$

where

$$S(q, \omega) \equiv \frac{2\pi \hbar}{N_s} \sum_{k,j} \rho_j \langle k | n_{-q} | j \rangle^2 \delta(\varepsilon_k - \varepsilon_j - \hbar \omega)$$

(2.26)
is the dynamic structure factor (DSF). The total scattering rate $P_{m-q,m}$ as measurable quantity is thus expressed through the Fourier component of the interaction potential and the DSF, with the latter only depending on internal properties of the target system.

Generally, the DSF characterizes the probability of the target system to simultaneously acquire momentum $\hbar q$ and energy $\hbar \omega$ in a scattering event. It is a measure of the excitation spectrum of the system, fundamentally determined by its wave function structure and density of states (apart from the statistical averaging with $\rho_j$), and represents the maximum information about the system one can obtain in a scattering experiment.

For a spatially isotropic system is $S(q, \omega) = S(q, \omega)$. For a system in thermal equilibrium, positive and negative $\omega$ are related by the condition of detailed balance

$$S(q, -\omega) = e^{-\hbar \omega/k_B T} S(q, \omega),$$

(2.27)

which reflects the fact that $\omega > 0$ corresponds to absorption of energy by the target, while $\omega < 0$ corresponds to energy loss. The relation between the two depends on the initial occupation of target states, determined in thermal equilibrium by the Boltzmann factor.

Another useful relation describes the (nonrelativistic) Doppler effect on the DSF. At a change between two reference frames (1) and (2), with frame (2) moving at a uniform velocity $u$ in frame (1), the DSF of a system transforms as

$$S^{(1)}(q, \omega) = S^{(2)}(q, \omega - q \cdot u),$$

(2.28)
because the momentum transfer $q$ in scattering does not depend on the reference frame, but the energy transfer is affected by the Doppler shift $-\hbar q \cdot u$. This relation allows in quantum transport theory (Sec. 3.3) to easily treat the special nonequilibrium situation of a uniformly moving system that is in internal equilibrium, similarly to the displaced equilibrium distribution function in the Boltzmann treatment in Sec. 2.2.3.

Beyond the special case of a scattering experiment, the DSF has a far more general significance. By the fluctuation-dissipation theorem, it is directly related to the imaginary part of the density response function from linear response theory, whose real part is then also determined by the Kramers-Kronig relations. The density response function describes the system response to any arbitrary form of weak perturbation that couples to the particle density, and for an electron system, it is in turn related to the dielectric function. The DSF describes thus the system behavior in very general terms.

---

1In a scattering experiment, $q$ and $\omega$ are also related by the properties of the specific probe particle, but this is irrelevant to the general derivation and significance of the DSF.
2.4 Dynamic structure factor (DSF)

2.4.1 Single-electron DSF

For simple systems, the DSF is easily obtained from Eq. (2.26). In the following, a single electron is considered; this is equivalent to a noninteracting, nondegenerate electron system (see Sec. 2.4.2). The Fourier transform of the density operator is then simply \( n_q = e^{-iq \cdot r} \) and may also be interpreted as momentum shift operator, which changes the electron momentum by \(-\hbar q\).

Free-electron DSF

For instance, consider a free 2D electron of mass \( m_e \) at zero magnetic field and finite temperature \( T \). The unperturbed wave functions are the usual plane waves \( |k\rangle \) with continuous energy spectrum \( \varepsilon_k = \hbar^2 k^2 / 2m_e \). The matrix element between initial state \( |k\rangle \) and final state \( |k'\rangle \) becomes simply \( \langle k' | e^{i q \cdot r} | k \rangle = \delta_{k', k+q} \), which reflects that plane wave states are momentum eigenstates. Evaluating the sum over final states reduces then the DSF to

\[
S_0(q, \omega) = \frac{2\pi \hbar}{N_s} \sum_k f(\varepsilon_k) \delta(\varepsilon_{k+q} - \varepsilon_k - \hbar \omega),
\]

(2.29)

and as in this case the occupation of initial states is given by a 2D Maxwellian function, \( f(\varepsilon_k) = 2\pi \hbar^2 N_s (Sm_e k_BT)^{-1} \exp(-\varepsilon_k / k_BT) \), this becomes finally [7]

\[
S_0(q, \omega) = \sqrt{\frac{2\pi m_e}{k_BT}} \frac{1}{q} \exp\left(-\frac{(\hbar \omega - \varepsilon_q)^2}{4\varepsilon_q k_BT}\right),
\]

(2.30)

which is graphically represented in Fig. 2.7. In the limit \( T = 0 \), this DSF becomes \( S_0(q, \omega) = 2\pi \hbar \delta(\hbar \omega - \varepsilon_q) \), clearly the excitation spectrum of a particle initially at rest with \( k = 0 \). Equation (2.30) is also the DSF for a three-dimensional free electron [74]; this requires only to treat \( q \) as a 3D wave vector.

**Figure 2.7:** Density plot of the DSF \( S_0(q, \omega) \) of a 2D free electron at zero magnetic field [Eq. (2.30)]. A darker shade of grey means a higher value of \( S_0(q, \omega) \). At \( T = 0 \), the DSF is a \( \delta \) function along the dotted line \( \hbar \omega = \varepsilon_q \). At finite \( T \), the thermal distribution of the initial states leads to broadening of the DSF along the \( \hbar \omega \)-axis with typical scale \( k_BT \). The broadening along the \( q \)-axis corresponds then to the thermal wavevector \( k_{th} = \sqrt{2m_e k_BT / \hbar} \).

DSF of electron in magnetic field

Now consider a single 2D electron in a finite magnetic field at finite temperature. The unperturbed states are now the Landau states \(|N, X\rangle\) from Eq. (2.19) with discrete eigenenergies \( \varepsilon_N = \hbar \omega_c (N + 1/2) \). Without loss of generality, the direction of \( q \) is taken
along the $y$-axis. The matrix element between initial and final state is then

$$
\langle N', X' \mid e^{i \theta y} \mid N, X \rangle = J_{N', N}(q) \delta_{N', X, -l_B \theta y},
$$

$$
J_{N', N}(q) = \langle N', l_B^2 q \mid N, 0 \rangle.
$$

(2.31)

The functions $J_{N', N}(q)$ are known from literature [75]. In contrast to the free-electron case treated above, the $J_{N', N}(q)$ are smooth functions of $q$, with a typical extension given by the magnetic length, $J_{N', N}(q) \sim \exp(-q^2 l_B^2/2)$ (see Fig. 2.8), and reflect that the Landau states do only have definite values of the momentum in $y$-direction, but not of the total momentum. After evaluating the sum over the orbit coordinates $X$ and $X'$ of initial and final states, the DSF in the ideal, unperturbed case becomes finally

$$
S_{se}(q, \omega) = \frac{\hbar S}{4\pi l_B^2 N_s} \sum_{N', N} f(\varepsilon_N) |J_{N', N}(q)|^2 \delta(\varepsilon_{N'} - \varepsilon_N - \hbar \omega),
$$

(2.32)

where $f(\varepsilon) \propto \exp(-\varepsilon/\kappa_B T)$ is a Maxwellian with the normalization $\sum_{N, X} f(\varepsilon_N) = N_s$. As the possible excitations of the particle clearly can only correspond to transitions between Landau levels, the transferred energy $\hbar \omega$ can here only take discrete values, equal to integer multiples of the Landau level separation.

**Landau level broadening** However, Eq. (2.32) is of limited applicability to a real system, where the Landau levels are broadened by the interaction with the scatterers, as discussed in Sec. 2.3.1. This evidently must also influence the DSF. A strict treatment [7] is beyond the scope of this discussion, but the final result corresponds, as might be expected, to just replacing in Eq. (2.32) the dependence on a discrete energy value $\varepsilon_N$ by an average over the broadened level, $g(\varepsilon_N) \rightarrow \int \, d\varepsilon \, g(\varepsilon) A_N(\varepsilon)/\pi$, with the Landau level shape $A_N(\varepsilon)$ given by Eq. (2.21) or Eq. (2.23) [for the ideal system with $A_N(\varepsilon) = \pi \delta(\varepsilon - \varepsilon_N)$, Eq. (2.32) stays unchanged]. This yields

$$
S_{se}(q, \omega) = \frac{\hbar S}{2\pi l_B^2 N_s} \sum_{N', N} |J_{N', N}(q)|^2 \int \, d\varepsilon \, f(\varepsilon) A_N(\varepsilon + \hbar \omega) A_N(\varepsilon),
$$

(2.33)

and the normalization condition is now written as $\int \, d\varepsilon \, f(\varepsilon) D(\varepsilon) = N_s$, where the density of states $D(\varepsilon)$ is given by Eq. (2.20) with a broadened $A_N(\varepsilon)$. Using the Gaussian shape for $A_N(\varepsilon)$ [Eq. (2.23)], this DSF can be put in the analytic form [7]

$$
S_{se}(q, \omega) = 2\sqrt{\pi} \hbar \left(1 - e^{-\omega_c/\kappa_B T}\right) \sum_{N, N'} e^{-N\omega_c/\kappa_B T} \frac{|J_{N, N'}(q)|^2}{\Gamma_{N, N'}^2} \times \exp \left( -\frac{\hbar^2 [\omega - (N' - N) \omega_c - \Gamma_{N, N'}^2/4\kappa_B T \omega_c]^2}{\Gamma_{N, N'}^2} \right),
$$

(2.34)

where the average broadening of two Landau levels $\Gamma_{N, N'} = \sqrt{\Gamma_{N}^2 + \Gamma_{N'}^2}/2$ has been introduced. In the noninteracting case is $\Gamma_N = \Gamma_{se}$ [Eq. (2.25)], thus also $\Gamma_{N, N'} = \Gamma_{se}$.

Eq. (2.34) is very general, it takes into account thermal occupation of higher Landau levels and fulfills the detailed balance condition. Still, at first sight it is not very transparent, and it is instructive to consider the following limitation to the quantum limit.
DSF in the magnetic quantum limit (QL) For a SE, in a large range of experimental conditions (around $B \sim 1-10$ T, $T \lesssim 2$ K) one has

$$\Gamma_0 \ll k_B T \ll \hbar \omega_c.$$  

The right inequality means that the electron is in the quantum limit (QL), and occupies in equilibrium essentially only the ground Landau level $N = 0$, and the left means that the ground level width is small enough compared to $k_B T$ that the occupation is essentially uniform across the level (see Fig. 2.8), and one can neglect the detailed balance condition on this scale. With these assumptions, and the known form of $J_{0,N}(q)$, the DSF of Eq. (2.34) approaches the simple, temperature-independent analytic form [7]

$$S_{se}(q, \omega) = 2\sqrt{\pi} \hbar \sum_{N=0}^{\infty} \frac{x_q^N \exp(-x_q)}{N!} \exp\left(-\frac{\hbar^2(\omega - N\omega_c)^2}{\Gamma_0^2}\right),$$

(2.35)

with the dimensionless parameter $x_q \equiv q^2 \ell_B^2 / 2$. This quantum limit DSF is graphically represented in Fig. 2.8. The possible excitations in this case are simply transitions of the electron inside the ground level [term $N = 0$ in Eq. (2.35)], or from the ground to higher levels (terms with $N = 1, 2, \ldots$), leading to maxima of the DSF at $\omega = N\omega_c$, $N = 0, 1, \ldots$. The shape of these maxima vs. $\omega$ is directly given by the Landau level shape.

As mentioned, this DSF does not fulfill the detailed balance condition at very small frequencies $\omega \ll \Gamma_0 / \hbar$. In principle, this could matter in DC transport, but the detailed balance condition is here explicitly evaluated in the course of the general treatment anyway (see Sec. A.1.2), the above DSF can then be used in the final expressions.

Figure 2.8: Upper graph: Quantum limit for SE: shown are the DOS $D(\varepsilon)$ (solid curve), the distribution function $f(\varepsilon) \propto e^{-\varepsilon / k_B T}$ (dashed), and the density of occupied states $D(\varepsilon)f(\varepsilon)$ (hatch-filled curve). Nondegenerate SE have $f(\varepsilon) \ll 1$ and occupy only few states at any $\varepsilon$. In the QL with $\Gamma_0 \ll k_B T \ll \hbar \omega_c$, only the lowest Landau level is uniformly occupied. Lower graph: Density plot of the single-electron DSF $S_{se}(q, \omega)$ in the QL [Eq. (2.35)], exhibiting a series of maxima at $\omega = N\omega_c$: the electron being initially in the ground level (upper graph), its possible excitations are transitions inside the same (maximum at $\omega = 0$), or transitions to higher levels (maxima at $\omega = \omega_c, 2\omega_c, \ldots$). The maxima are broadened vs. $\omega$ by the collision broadening of the levels, and extend vs. $q$ due to the Landau states having a momentum uncertainty of order $l_B^{-1}$. The peak values of $S_{se}(q, \omega)$ lie along the line $\hbar \omega = \varepsilon_q$ (cf. Fig. 2.7).
2.4.2 DSF at finite electron density

Non-interacting particles The results obtained above for a single particle remain still useful for a system comprising a finite number $N_s > 1$ of particles. Again, a strict treatment is beyond the scope of this work. However, it is clear that for a system of noninteracting particles, where the many-particle wave functions are just products of single-particle ones, the only difference can be in the statistical occupation of states.

For a noninteracting system with classical statistics, the DSF does not depend on the particle density, and all expressions for the DSF are the same as in Sec. 2.4.1. For a degenerate electron system, it is necessary to take into account the Pauli principle. In this case, the corresponding general expressions for zero and nonzero magnetic field differ from Eqs. (2.29) and (2.33) in the previous section only in that the Maxwellian distribution $f(\varepsilon)$ should be replaced by $f_F(\varepsilon)\left[1 - f_F(\varepsilon')\right]$, where $\varepsilon'$ is the respective energy of the final states, and $f_F(\varepsilon)$ is now the Fermi distribution $f_F(\varepsilon) \propto \left(e^{(\varepsilon - \mu)/k_B T} + 1\right)^{-1}$ [7, 73]. The additional factor $\left[1 - f_F(\varepsilon')\right]$ reflects that transitions to already occupied final states are forbidden for fermions. The analytic final expressions Eqs. (2.30), (2.34), and (2.35) in the previous section apply to a nondegenerate system only, where $f(\varepsilon) \ll 1$ and only a small part of the states is occupied at any energy.

Interacting particles It is substantially more difficult to treat a system of interacting particles. Here, the many-particle wave functions often are not known exactly, and one needs to rely on approximations for the DSF. The treatment depends then very much on the specific properties of the system.

For a qualitative discussion, it is useful to recall that the DSF can also be written as the space and time Fourier transform of the density-density correlation function, or generalized pair correlation function $G(r, t) \equiv \sum_{s} \langle n_s(r, t) n_s(0) \rangle_T$, which gives the probability of finding a particle at point $r$ at time $t$ when there is one at $r = 0, t = 0$ [76]:

$$S(q, \omega) = \frac{1}{N_s} \int_{-\infty}^{\infty} dt e^{i\omega t} \langle n_q(t) n_{-q}(0) \rangle_T = \int dt e^{i\omega t} \int dr e^{-iqr} G(r, t), \quad (2.36)$$

were $n_q(t)$ is a time-dependent operator in the Heisenberg representation, and $\langle A \rangle_T \equiv \sum_j \rho_j \langle j | A | j \rangle$ is the statistical average of the expectation value of the operator $A$. The DSF is therefore also a measure of the typical amplitude of density fluctuations with wave vector $q$ and frequency $\omega$ in the system in thermal equilibrium, and external perturbations excite the system by coupling to these fluctuations.

For the case of a nondegenerate 2D electron system interacting by Coulomb forces, well-known approaches exist for the case of very low or very high plasma parameter $\Gamma_p$. In a weakly coupled system with $\Gamma_p \ll 1$, a long-distance average is a good approximation. The Coulomb interaction can be treated by self-consistent-field methods, such as the random-phase approximation, equivalent to the Debye-Hückel approximation in a classical plasma [5, 32]. Here, the Coulomb interaction influences the individual electron mainly via a mean field produced by many other electrons together. Local fluctuations of the Coulomb field around the average can be neglected. Differences to the excitation spectrum of an entirely noninteracting system arise then mainly at low wave vectors $q$, or large distances, as the mean field will screen an external perturbation on large enough distance, and also give rise to a long-wavelength collective excitation, the plasmon.
In the other extreme, in a strongly coupled system in the solid phase with \( \Gamma_p > 130 \), a short-distance linear expansion can be used. The individual particles perform only small, essentially harmonic oscillations around their equilibrium lattice positions, and the system can be described in terms of phonons as collective elementary excitations \([7, 77]\).

However, the conditions for the experimental part of this thesis correspond to an intermediate range \( 10 < \Gamma_p < 100 \), where the system is in a liquid-like state, and in principle, none of the above approaches can be applied. In transport theory (see Sec. 3.3), the DSF is employed to calculate the momentum loss rate of the electron system due to collisions with scatterers. Here, the range of high \( q \)-vectors is most important, as they correspond to a high value of the momentum exchange in scattering. Moreover, the interaction with the scatterers is a short-range one. In this range, also a strongly correlated system can be described in terms of the behavior of the single electron, as may be seen by the following consideration \([76]\): For a nondegenerate system, the generalized pair correlation function \( G \) splits into a part \( G_s \) describing the correlation between positions of one and the same particle at different times, i.e., corresponding to a DSF depending only on single-electron properties, and a part \( G_d \) referring to pairs of distinct particles. \( G_s \) varies in space with the characteristic wavelength of the electron, of the order of the magnetic length \( l_B \) (or the thermal de Broglie wavelength at zero field). In contrast, the typical length of spacial variation of \( G_d \) corresponds to that of the conventional pair correlation function \( g(r) \), i.e., to the typical width of its first peak \( \delta_p \) [Eq. (1.6)]. For SE, in a large parameter range, extending even into the Wigner solid regime, one has

\[
l_B^{-1} \gg \delta_p^{-1},
\]

and the range of high \( q \)-vectors that is most relevant for transport is dominated by the single-electron quantum excitations. The correlation between the motion of different electrons is instead rather (semi-)classical, and contributes only a very low \( q \). This is also a result of numerical simulations \([78]\).

However, the properties of the single electron may still be influenced by the Coulomb field of the others. Rather than a mean field, which is ineffective at short distance, the relevant influence here is due to a local, fluctuational electric field (FEF), resulting from thermal density fluctuations in the presence of strong interaction (see Sec. 4.1). The FEF is the origin of Coulomb effects in magnetotransport, and their theoretical description in chapter 4 will finally reduce to understanding how the FEF influences the behavior of the single electron, and taking the modified electron behavior into account in the DSF.

Lastly, it should be emphasized that the Wigner crystallization, being a long-range-order phenomenon, does not immediately affect short-range properties and the validity range of condition (2.37). In fact, the latter extends well into the solid regime, and the theoretical description of Coulomb effects developed for the liquid is then still valid \([7]\).
Chapter 3

Magnetotransport theory

This chapter introduces the theoretical treatments of magnetotransport of 2D electrons relevant for the discussion of the experimental data. A short classification is given below.

The classical Drude theory employs the classical equation of motion of a single particle in crossed magnetic and electric fields with a magnetic-field-independent collision rate for scattering (Sec. 3.1). It disregards the Landau quantization, and also electron-electron interaction. It applies therefore only when the Landau quantization is of minor importance, like at very low magnetic field or very high temperature. Interestingly, the same structure of the conductivity tensor is found in the many-electron quantum treatment, only in the latter the collision rate depends on the magnetic field (see below).

There are various ways to treat the regime of pronounced Landau quantization. The most relevant ones are for our purpose conveniently grouped into two general categories:

(1) Those focusing on properties of the single electron, in the following called the single-electron approach, like the SCBA and Ando cyclotron resonance theory (Sec. 3.2).

(2) Those focusing on integral properties of the electron system as a whole, in the following called the many-electron approach (but this does not require electron-electron interaction), like the balance equation theory or the memory function theory (Sec. 3.3).

Essential aspects and differences of the two approaches are outlined below. Both use quantum linear response theory, which calculates the expectation value of some physical quantity of a system that is subject to a weak external perturbation. In the linear regime, the response of the system is entirely determined by its equilibrium properties.

In the single-electron approach, the quantum system considered is basically a single electron in magnetic field, the external perturbation is an applied electric driving field (DC or AC), and one calculates the expectation value of the single-electron current operator. The result depends directly on the single-electron density of states (DOS). Here, a major problem was to adequately describe the equilibrium quantum states of the single electron in the presence of scatterers. This was eventually solved by the SCBA in terms of the Landau level broadening (see Sec. 2.3.1). However, this approach has shortcomings. Of special importance for SE is the conceptual problem in taking into account strong electron-electron interaction in such a single-electron picture, as then single-electron quantum states may no longer be a good description of the system.

The many-electron approach focuses on collective properties of the system, which is best seen in the force balance formalism. It starts by considering the collective motion of the electrons as a whole under the action of the driving field (again AC as well as DC). Using simple physical arguments, one can easily see that the collective motion is essentially classical, and the conductivity tensor has a quasi-Drude form. Linear response
theory is used here in a rather different aspect, namely to calculate the expectation value of the total force on the system due to the interaction with the scatterers, which eventually gives a magnetic-field-dependent, effective momentum collision rate to be inserted in the Drude-like conductivity tensor. The external perturbation as well as the physical quantity whose expectation value is calculated are thus both different from the single-electron approach, but the most important difference is that here the electron system as a whole is considered. The effective collision rate is expressed in terms of the system’s excitation spectrum, i.e., its dynamic structure factor (DSF), which in principle includes also effects of electron-electron interaction.

Still, the second approach does not really require interaction, and it may be employed together with a single-electron DSF, which depends again directly on the single-electron DOS (see Sec. 2.4.1). Interestingly, the result is is not quite identical to that of the first, “pure” single-electron approach. For SE, in the DC case, the difference is between the extended SCBA and the original SCBA, see Sec. 3.3.2. In the AC case (cyclotron resonance), one particular difference is the appearance of subresonances, see Sec. 3.3.1.

In the second approach, the inclusion of Coulomb effects in magnetotransport clearly reduces to finding the appropriate many-electron DSF. This is the subject of Sec. 4.4.

### 3.1 Drude formula

The classical equation of motion for a 2D particle in presence of a perpendicular magnetic field \( B \) and an in-plane electric driving field \( E \) is

\[
m_e \dot{\mathbf{v}} = -e (\mathbf{E} + \mathbf{v} \times \mathbf{B}) - m_e \nu_0 \mathbf{v}
\]

(\( \mathbf{v} \) is the particle velocity). The last term is a frictional term describing the effect of collisions with scatterers\(^1\), and \( \nu_0 \) is usually taken as the zero-field collision rate. For SE in the vapor scattering regime, \( \nu_0 = \nu_a \) [Eq. (2.11)].

For a harmonic driving field \( \mathbf{E}(t) = E_0 \exp(-i\omega t) \), the solution of Eq. (3.1) leads to the usual Drude equations for magnetotransport. With the definition of the current density \( \mathbf{j} = e n_s \mathbf{v} \) one obtains for the conductivity tensor \( \sigma \), defined by \( \mathbf{j} = \sigma \cdot \mathbf{E} \),

\[
\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix},
\]

\[
\sigma_{xx} = \sigma_{yy} = \frac{e^2 n_s}{m_e} \frac{\nu_0 - i\omega}{\omega^2 + (\nu_0 - i\omega)^2},
\]

\[
\sigma_{yx} = -\sigma_{xy} = \frac{e^2 n_s}{m_e} \frac{\omega_c}{\omega^2 + (\nu_0 - i\omega)^2}.
\]

It is convenient to summarize the components of the conductivity tensor in the form

\[
\sigma_{\pm} = \sigma_{xx} \pm i\sigma_{xy} = \frac{e^2 n_s}{m_e} \frac{i}{\omega \pm \omega_c + i\nu_0}.
\]

For comparison with experiments, the diagonal component \( \sigma_{xx} \) is most important. In the DC limit \( \omega \to 0 \) (more exactly, \( \omega \ll \nu_0 \)), one obtains from Eq. (3.2) the usual

\(^1\)In the AC case, the scatterers may additionally lead to a dynamic response force proportional to the electron position, which is here less important and neglected in Eq. (3.1), see Sec. 3.3 for more detail.
3.2 Quantum transport of noninteracting electrons

A general way of treating electric transport quantum-mechanically is given by the Kubo formula \[79\]. Ando \[68, 80\] evaluated them for the case of noninteracting electrons in a magnetic field, and obtained expressions which depend essentially only on the single-electron density of states (DOS), i.e., on the Landau level shape. The latter he calculated using the self-consistent Born approximation (SCBA), for the results see Sec. 2.3.1.

3.2.1 Ando cyclotron resonance theory

In the AC case, or cyclotron resonance (CR) at high frequency \(\omega\) and with short-range scattering, the single-electron theory gives the following result for the real part of the dynamic conductivity, or CR absorption in the quantum limit \(\hbar \omega_c \gg k_B T\) \[80\]:

\[
\text{Re} \sigma_{xx}(\omega) = \frac{e^2 \omega_c}{4\pi^2} \int \frac{d\varepsilon}{\sqrt{\pi \nu_0}} \left[ f(\varepsilon) - f(\varepsilon + \hbar \omega) \right] A_0(\varepsilon) A_1(\varepsilon + \hbar \omega),
\]

where \(A_N(E)\) is the shape of the \(N\)-th Landau level (see Sec. 2.3.1). The results both for a semielliptic and a Gaussian level shape are shown in Fig. 3.1. The physical picture is
here that the AC electric field induces optical transitions of single electrons from ground to first excited Landau level. For SE is usually the level broadening $\Gamma_N \ll k_B T$, and Eq. (3.5) is then a simple convolution of the shapes of ground and first excited level. For a Gaussian $A_N(E)$ [Eq. (2.23)], this gives in turn a Gaussian CR absorption line shape:

$$\text{Re} \sigma_{xx}(\omega) = \frac{\sqrt{\pi} e^2 n_s \hbar}{4 m_e \Gamma_{0,1}} \exp \left[ -\frac{\hbar^2 (\omega - \omega_c)^2}{\Gamma_{0,1}^2} \right],$$

where $\Gamma_{N,N'} \equiv \sqrt{(\Gamma_N^2 + \Gamma_N'^2)/2}$ is the average broadening of two Landau levels. The CR linewidth $\gamma_{\text{CR}}$ (full width at half height of $\text{Re} \sigma_{xx}$) in frequency units is here given by the widths of ground and first Landau level (the linewidth in tesla is then $m_e \gamma_{\text{CR}}/e$):

$$\gamma_{\text{CR}} = 2\sqrt{\ln 2 \Gamma_{0,1}/\hbar}.$$  

For pure vapor atom scattering, $\Gamma_N = \Gamma_a$ [Eq. (2.22)], and $\gamma_{\text{CR}} = 2\sqrt{\ln 2 \Gamma_a/\hbar} \simeq 1.67 \Gamma_a/\hbar$.

### 3.2.2 DC magnetoconductivity—self-consistent Born approximation (SCBA)

In the DC limit $\omega \to 0$ with short-range scattering and at sufficiently high magnetic field $B$ to neglect mixing of Landau levels, the single-electron theory gives the result [32, 68]

$$\sigma_{xx}(B) = \frac{e^2}{\pi \hbar^2} \sum_N \left( N + \frac{1}{2} \right) \int d\varepsilon \left( -\frac{\partial f}{\partial \varepsilon} \right) \frac{\Gamma_N^2}{4} A_N(\varepsilon)^2,$$  

and with a Gaussian Landau level shape $A_N(E)$ [Eq. (2.23)], and a level width $\Gamma_N = \Gamma_a$ [Eq. (2.22)] for pure vapor atom scattering, this results in the analytic expression

$$\sigma_{xx}(B) = \frac{e^2 n_s \sqrt{\pi} \Gamma_a}{m_e \omega_c 4 k_B T} \exp \left[ -\left( \frac{\Gamma_a}{4 k_B T} \right)^2 \right] \coth \left( \frac{\hbar \omega_c}{2 k_B T} \right).$$  

Alternatively, Eq. (3.8) may be evaluated with a semielliptic level shape [81]. Same as in CR (see Fig. 3.1), numerically the result is almost the same, and is not employed here.

A graphic representation of Eq. (3.9) is given in Fig. 3.4, where it is directly compared with the DC result of the balance equation/memory function theory.

### 3.3 Balance equation method

The balance equation approach to transport theory (see Ref. [82] for a review) bases on the separation of the motion of the electron system as a whole from the relative motion of the individual electrons. The conductivity tensor has a quasi-Drude form, only the Drude collision rate is replaced by an effective momentum collision rate, which is generally a property of the electron system as a whole and depends on its quantum excitation spectrum, or dynamic structure factor (DSF), introduced in Sec. 2.4. In the AC case, similar results were previously obtained by the memory function theory [83, 84, 85].

The balance equation method has the advantages of directly giving a clear physical picture and enabling a very elementary treatment of all relevant cases in the same general formalism. Below is first given a general discussion of the derivation of the conductivity
tensor from the momentum balance equation, and a general outline of the calculation of the effective momentum collision rate (details are given in appendix A). The results for the AC and DC conductivities of SE are then presented in Secs. 3.3.1 and 3.3.2.

The method can be extended to treat heating of electrons by means of the energy balance equation. Here, the energy loss of the electrons due to scattering is described through an effective energy collision rate; this allows to determine the electron temperature. This is treated in Sec. 3.3.3, and results are presented for 2D and for 3D electrons.

### Momentum balance equation

One considers a large 2D system of \( N_s \) electrons at the positions \( \mathbf{r}_i \) in a vertical magnetic field \( \mathbf{B} \) and horizontal electric field \( \mathbf{E}(t) \), interacting with a system of scatterers by an interaction Hamiltonian \( H_{\text{int}} \). In the balance method, a distinction is made between the motion of the electron system as a whole, described by center-of-mass (CM) variables, e.g., \( \mathbf{R} \equiv N_s^{-1} \sum_i \mathbf{r}_i \) for the CM position, and the relative motion of the individual electrons inside the CM reference frame, described by relative variables, e.g., \( \mathbf{r}'_i \equiv \mathbf{r}_i - \mathbf{R} \).

The CM behaves then like a separate particle of large mass \( N_s m_e \) and charge \( N_s e \), whose motion will essentially be classical. The general equation of motion for the time-dependent expectation value of the CM position \( \mathbf{R}(t) \equiv \langle \mathbf{R} \rangle \) can be written as [82]

\[
N_s m_e \ddot{\mathbf{R}}(t) = -N_s e \left[ \mathbf{E}(t) + \dot{\mathbf{R}}(t) \times \mathbf{B} \right] + \mathbf{F}.
\]  

(3.10)

This is the momentum balance equation. The right side is the expectation value of the total force acting on the electron system. Here, the first two contributions, from electric and magnetic field, are like in the case of a single particle and unproblematic. The last contribution, \( \mathbf{F} \), comes from the interaction with the scatterers \( H_{\text{int}} \), whose form is left unspecified for the moment. An important point is that electron-electron interaction does not directly contribute to the total force, which reflects Kohn’s argument [86].

The difficulty lies in evaluating \( \mathbf{F} \), which generally depends both on the CM motion and on the relative motion of the individual electrons inside the CM frame. This is treated in more detail below and in appendix A. The final results are, as general linear response theory lets expect [7], similar to the classical case. For instance, under steady-state AC conditions with \( E(t) \), \( \dot{R}(t) \propto \exp(-i\omega t) \), one finds that \( \mathbf{F} = i N_s m_e M(\omega) \dot{\mathbf{R}}(t) \) with some function \( M(\omega) \). The average current being \( \mathbf{j} = e n_s \dot{\mathbf{R}}(t) \) (\( n_s \equiv N_s / S \), \( S \) is the system area), the conductivity tensor has then a quasi-Drude form (cf. Sec. 3.1):

\[
\sigma_{\pm} = \sigma_{xx} \pm i \sigma_{xy} = \frac{e^2 n_s}{m_e} \left( \frac{i}{\omega - \omega_c} + M(\omega) \right)
\]  

(3.11)

Unlike classical Drude theory, \( M(\omega) \), called the memory function [83, 84, 85], is generally frequency dependent, and has imaginary and real parts, corresponding to in-phase and out-of-phase components of \( \mathbf{F} \) relative to the CM velocity. \( \Re M(\omega) \) and \( \Im M(\omega) \) are related by Kramers-Kronig relations, so it suffices to know one of them. In cyclotron resonance, they determine shift and width of the resonance line, respectively.

For SE in the vapor scattering regime, the line shift is usually much smaller than the linewidth; moreover, Coulomb effects mainly manifest themselves in the latter [7]. The focus is therefore on \( \Im M(\omega) \equiv \nu_{\text{eff}}(\omega) \), also called the effective momentum collision rate. It corresponds to the dissipative or frictional part \( \mathbf{F}_{\text{fr}} \) of the force \( \mathbf{F} \) [\( \mathbf{F}_{\text{fr}} = -N_s m_e \nu_{\text{eff}}(\omega) \dot{\mathbf{R}}(t) \)], which must always be present to enable steady-state conditions.
The DC case $\omega \ll \nu_{\text{eff}}$ can be treated by taking the limit $\omega \to 0$ in the AC expressions, or more extensively starting from $R(t) \propto t$, which to lowest order gives the same result. Re $M(0)$ clearly must vanish, and only Im $M(0)$ stays finite. Generally more relevant for transport is therefore Im $M(\omega) \equiv \nu_{\text{eff}}(\omega)$. Its calculation is outlined below.

**Effective momentum collision rate**

The force $\mathbf{F}$ in Eq. (3.10) has been calculated using different versions of linear response theory [82, 87]. Still, there is a more elementary approach [88], applied to DC magnetotransport of SE in the so-called extended SCBA [4, 89]. The following is a variant of this method, developed by the author, which allows to extend it to the AC case.

In the absence of interaction, $|j\rangle$ denotes an unperturbed state of the electron system with energy $\varepsilon_j$, and $|\mathbf{m}\rangle$ an unperturbed state of the scatterer system with energy $\varepsilon_{\mathbf{m}}^{(b)}$. The states $|j\rangle$ are left unspecified. The unperturbed scatterers are assumed to be free bosons with simple and well-known properties. Their system states $|\mathbf{m}\rangle = |m_1, ..., m_q, ...\rangle$ are entirely specified by the number $m_q$ of bosons in each plane-wave state $|q\rangle$.

A weak interaction $H_{\text{int}}$ will induce transitions between the unperturbed states of the whole system $|j, \mathbf{m}\rangle = |j\rangle |\mathbf{m}\rangle$, electrons and scatterers. In elementary first-order time-dependent perturbation theory [67], the transition rate, or probability per time $P_{\langle k,n\rangle(j,m)}$ for the total system to make a transition from initial state $|j, \mathbf{m}\rangle$ to final state $|k, \mathbf{n}\rangle$ under the action of $H_{\text{int}}$ is

$$P_{\langle k,n\rangle(j,m)} = \frac{d}{dt} \left| \frac{1}{i\hbar} \int_{-\infty}^{t} \langle k, n | H_{\text{int}} | j, m \rangle \exp \left( i \omega_{\langle k,n\rangle(j,m)} \tau \right) d\tau \right|^2,$$

where $\hbar \omega_{\langle k,n\rangle(j,m)} \equiv \varepsilon_k - \varepsilon_j + \varepsilon_{\mathbf{n}}^{(b)} - \varepsilon_{\mathbf{m}}^{(b)}$ is the energy difference between initial and final state.

In each such transition, momentum is transferred from the electron system to the system of scatterers, equivalent to a force acting between the systems. Even without knowledge about the electron states, the momentum transfer can be determined from the well-known properties of the scatterers to be $\mathbf{P}_n^{(b)} - \mathbf{P}_m^{(b)}$, where $\mathbf{P}_m^{(b)}$ is the total 2D momentum of the scatterer system in state $|\mathbf{m}\rangle$. The average force $\mathbf{F}$ on the electron system, i.e., its average momentum change rate, is then obtained as the product of momentum transfer times transition rate, summed over all possible transitions:

$$\mathbf{F} = - \sum_{k,n,j,m} \rho_j \rho_m (\mathbf{P}_n^{(b)} - \mathbf{P}_m^{(b)}) P_{\langle k,n\rangle(j,m)}, \quad (3.12)$$

where $\rho_j$ and $\rho_m$ are the occupation probabilities of the initial states of electrons and scatterers, respectively. The unperturbed scatterer system is naturally in statistical equilibrium in the fixed laboratory reference frame. In contrast, the unperturbed electron system is rather in internal equilibrium inside the moving CM reference frame. For SE, this assumption usually stays valid even in the presence of interaction with the scatterers due to the strong electron-electron interaction, as discussed in Sec. 2.2.3.

One can evaluate Eq. (3.12) using different strategies. In Ref. [88], for the DC case, the evaluation is at first performed entirely in the laboratory frame, i.e., the equilibrium frame of the scatterers. This leads to an expression depending on the electron system’s DSF in this frame, which is a nonequilibrium one. As in the DC case the CM moves with constant velocity, the transformation rule Eq. (2.28) can then employed to express this
3.3 Balance equation method

nonequilibrium DSF in terms of the DSF in the CM frame. The latter can be assumed to have the known equilibrium form, as noted above.

However, this method is limited to the DC case. To also treat the AC case, one can take another approach, namely, describe scatterers and electrons in their respective equilibrium frames from the beginning: The interaction Hamiltonian $H_{\text{int}}$ is reformulated in terms of CM variable and relative electron variables, $r_i = \mathbf{R} + r_i'$. The important point is that, due to the large mass of the CM “particle”, the CM motion is essentially classical, i.e., quantum fluctuations can here be neglected [90]. To evaluate $P_{(k,n)(j,m)}$, one can then replace the CM variable $\mathbf{R}$ by its expectation value $\langle \mathbf{R} \rangle$, i.e., treat it as a time-dependent parameter entering $H_{\text{int}}$, which determines the time-dependence of $P_{(k,n)(j,m)}$. This leaves essentially only the evaluation with respect to the relative variables inside the CM frame, which is trivial due to the internal equilibrium.

Details of the calculations can be found in Sec. A.1. For instance, the AC result for $\nu_{\text{eff}}(\omega)$ (imaginary part of the memory function) for elastic vapor atom scattering is

$$\nu_{\text{eff}}^{(a)}(\omega) = \frac{1 - e^{-\hbar \omega / k_B T}}{\omega} \frac{\hbar^2 \nu \omega}{4m_e^2S} \sum q q^2 S(q, \omega).$$

Here, the DSF $S(q, \omega)$ describes the (still unspecified) equilibrium excitation spectrum at frequency $\omega$ of the electron system in the CM frame. The possible excitations of the system at this frequency are summed up with weight $q^2$, which strongly favors high values of the momentum exchange. Clearly the system can lose momentum most efficiently in excitations with a high value of the momentum exchange, i.e., they contribute most to the friction force. Most important is therefore the high-$q$ range in the DSF, which corresponds generally to the single-electron part of the excitation spectrum (see Sec. 2.4.2). The factor $1 - \exp(-\hbar \omega / k_B T)$ originates from the detailed balance condition [Eq. (2.27)]: Seen from the laboratory frame, in a collision the electron system loses CM momentum and kinetic energy by creating an excitation that increases its internal energy by $\hbar \omega$. This process is also possible in the opposite direction, the ratio between loss and gain contributions to the friction force is given by the detailed balance condition.

So far, no assumptions have been made concerning the equilibrium DSF, so the description is still general, and applies at zero and nonzero magnetic field. To be more specific, one must consider what conditions the electron system is subject to in the CM frame: The force from the uniform external electric field does not depend on electron variables and acts only on the CM motion, it cannot additionally accelerate an individual electron inside the CM frame. In contrast, the Lorentz force depends linearly on the electron velocity, $\dot{\mathbf{r}}_i = \dot{\mathbf{R}} + \dot{r}'_i$, and acts both on the CM motion and on the relative motion of the electrons inside the CM frame. The unperturbed electron system in the CM frame is thus subject only to the magnetic field, if present.

However, a problem arises here from the fact that, formally, the unperturbed electron system does not interact with the scatterers at all, and in a magnetic field, it should exhibit no collision broadening of the Landau levels, i.e., the single-electron DOS should be singular. This leads to unphysical results. It is therefore customary to include the level broadening in the DSF [83, 84, 85]. A justification for this may be found by a more strict derivation of the memory function theory [91, 92].

In summary, in the balance equation formalism the uniform external electric field couples only to the center-of-mass motion of the whole electron system, the CM motion couples in turn to the internal, quantum structure of the electron system by modulating the interaction with the scatterers.
3.3.1 AC conductivity (cyclotron resonance)

In cyclotron resonance (CR), one measures the AC conductivity \( \sigma_{xx}(\omega) \) in magnetic field at high frequency \( \omega \gtrsim \nu_{\text{eff}} \) of the driving field. In the experiment, one either keeps \( \omega \) fixed and varies the magnetic field, i.e., the cyclotron frequency \( \omega_c \), or vice versa. For a small range of variation around resonance \( \omega = \omega_c \), the behavior of \( \sigma_{xx}(\omega) \) is essentially the same in both cases. For a general discussion, it is instructive to also consider large variations, in the following these are variations of magnetic field at fixed \( \omega \), as this corresponds to the experimental procedure used here. For SE is usually \( \omega \gg \nu_{\text{eff}} \), and from Eq. (3.11) one obtains then \( \sigma_{xx} = (\sigma_+ + \sigma_-)/2 \approx \sigma_+ / 2 \), with the real and imaginary parts

\[
\begin{align*}
\text{Re} \sigma_{xx}(\omega) &\approx \frac{\nu_{\text{eff}}(\omega)}{2m_e (\omega - \omega_c)^2 + \nu_{\text{eff}}(\omega)^2}, \\
\text{Im} \sigma_{xx}(\omega) &\approx \frac{e^2 n_s}{2m_e (\omega - \omega_c)^2 + \nu_{\text{eff}}(\omega)^2}.
\end{align*}
\]

(3.13)

Here, \( \text{Re} \sigma_{xx}(\omega) \) describes the in-phase, dissipative part of the average current with respect to the driving electric field, and \( \text{Im} \sigma_{xx}(\omega) \) the out-of-phase, reactive part. \( \text{Re} \sigma_{xx}(\omega) \) is usually called the absorption, \( \text{Im} \sigma_{xx}(\omega) \) the dispersion. For SE, the total AC effective momentum collision rate \( \nu_{\text{eff}}(\omega) \) is the sum of the respective rates due to vapor atom and ripplon scattering, \( \nu_{\text{eff}}(\omega) = \nu_{\text{eff}}^{(a)}(\omega) + \nu_{\text{eff}}^{(r)}(\omega) \), where (Sec. A.1.1, [17, 77])

\[
\begin{align*}
\nu_{\text{eff}}^{(a)}(\omega) &= \frac{1 - e^{-\hbar \omega/k_B T}}{\omega} \frac{\hbar^2 \nu_a}{4m_e^2 S} \sum_q q^2 S(q, \omega), \\
\nu_{\text{eff}}^{(r)}(\omega) &= \frac{1 - e^{-\hbar \omega/k_B T}}{\omega} \frac{1}{4\hbar m_e S} \sum_q q^2 V_q^2 Q_q^2 \\
&\quad \times \left[ N_q^{(r)} S(q, \omega + \omega^{(r)}_q) + (N_q^{(r)} + 1) S(q, \omega - \omega^{(r)}_q) \right].
\end{align*}
\]

(3.14)

Here, \( \nu_a = 3\gamma m_e U_e^2 n_s / 8\hbar^3 \) is the zero-field collision rate [Eq. (2.11)], and inelastic effects in \( \nu_{\text{eff}}^{(a)}(\omega) \) have been neglected. At \( \hbar \omega \gg k_B T \), one can also neglect the term \( \exp(-\hbar \omega/k_B T) \). Still, this term is relevant for the transition to the DC case, same as the usually negligible inelastic terms in \( \nu_{\text{eff}}^{(r)}(\omega) \).

In the following is discussed CR in the noninteracting, single-electron case. The expressions (3.14) for \( \nu_{\text{eff}}(\omega) \) are then evaluated with the single-electron DSF \( S_{\text{se}}(q, \omega) \) in magnetic field [Eq. (2.34)]. For a general discussion, it is sufficient to consider the result for vapor atom scattering, \( \nu_{\text{se}}^{(a)}(\omega) \), the behavior of \( \nu_{\text{se}}^{(r)}(\omega) \) is qualitatively similar.

The main characteristic of \( \nu_{\text{se}}^{(a)}(\omega) \) is its strong variation with magnetic field, see Fig. 3.2(a). This is also the main difference to classical Drude theory. \( S_{\text{se}}(q, \omega) \) is characterized by the Landau quantization, and exhibits sharp maxima at \( \omega = N\omega_c, N = 0, 1, 2, ..., \), whose form is directly determined by the Landau level shape (see Fig. 2.8). These maxima then also appear in \( \nu_{\text{se}}^{(a)}(\omega) \) (note that Fig. 3.2 corresponds to variation of \( \omega_c \) at fixed \( \omega \), contrary to Fig. 2.8). Physically, this can be understood as follows: The CM motion modulates the interaction between electrons and scatterers with frequency \( \omega \), and dissipation can then in magnetic field only occur if this modulation can induce single-electron transitions between Landau levels. Only then the friction force, and thus \( \nu_{\text{se}}^{(a)}(\omega) \), are finite. Note that in this picture the quantum transitions are not induced by a direct interaction with the electric field, and are not restricted by selection rules.
3.3 Balance equation method

However, the (classical) CM motion has a large amplitude only at $\omega = \omega_c$, where it is in resonance with the driving electric field, as described by the Lorentzian conductivity tensor [Eq. (3.13)]. This gives a single main resonance in $\sigma_{xx}(\omega)$ at $\omega = \omega_c$, see Fig. 3.2(b). Only the corresponding maximum in $\nu_{se}^{(a)}(\omega)$ at $\omega = \omega_c$ is relevant here, i.e., dissipation occurs here by single-electron transitions between adjacent Landau levels only (mostly to the next higher, if $\hbar \omega \gg k_B T$). Apart from this, the CM motion is out of resonance at all other maxima of $\nu_{se}^{(a)}(\omega)$ and has there only small amplitude, leading only to small subresonances in $\sigma_{xx}(\omega)$ at lower magnetic field.

**Figure 3.2:** (a): Single-electron AC effective momentum collision rate $\nu_{se}^{(a)}(\omega)$ vs. magnetic field (solid curve), numerically calculated from Eq. (3.14) and the single-electron DSF [Eq. (2.34)]. Same as the DSF in Fig. 2.8, $\nu_{se}^{(a)}(\omega)$ shows strong maxima at $\omega = N \omega_c$, contrary to the constant zero-field collision rate $\nu_a$ of Drude theory (dotted). (b): CR absorption Re $\sigma_{xx}(\omega)$ (solid curve, normalized to $\sigma_0 = e^2 n_e / m_\epsilon \nu_a$), obtained by inserting $\nu_{se}^{(a)}(\omega)$ from graph (a) in Eq. (3.13). Re $\sigma_{xx}(\omega)$ has a main resonance at $\omega = \omega_c$, determined by the corresponding maximum in $\nu_{se}^{(a)}(\omega)$ alone, and small subresonances at $\omega_c < \omega$ from the other maxima in $\nu_{se}^{(a)}(\omega)$. The strong variation of $\nu_{se}^{(a)}(\omega)$ around $\omega = \omega_c$ gives the main resonance a non-Lorentzian shape, similar to the Gaussian of Ando theory [Eq. (3.6), dashed], see also Fig. 3.3.

If $\omega = \omega_c$ lies in the quantum limit $\hbar \omega_c \gg k_B T$, one obtains from the quantum limit DSF [Eq. (2.35)] a simple analytic expression for the main resonant maximum in $\nu_{se}^{(a)}(\omega)$ around $\omega = \omega_c$. It is determined by the $N = 1$ term in Eq. (2.35) alone, which corresponds to SE transitions from ground to first excited Landau level:

$$\nu_{se}^{(a)}(\omega) \approx \sqrt{\pi} \Gamma_a^2 \omega_c \exp \left( -\frac{\hbar^2 (\omega - \omega_c)^2}{\Gamma_{se}^2} \right) \text{ for } \omega \sim \omega_c. \quad (3.15)$$

Here, $\Gamma_a = \hbar \sqrt{2 \omega_c \nu_a / \pi}$ is just introduced to conveniently summarize parameters and should be distinguished from the true single-electron Landau level width $\Gamma_{se} = \sqrt{\Gamma_a^2 + \Gamma_a'}^2$.

It is instructive to compare the result of Eq. (3.15) with the simpler Ando quantum CR theory (see Sec. 3.2.1). The collision rate $\nu_{se}^{(a)}(\omega)$ enters the Lorentzian expression of Eq. (3.13) as a linewidth parameter, and for pure vapor scattering ($\Gamma_{se} = \Gamma_a$), its peak value at $\omega = \omega_c$ corresponds to a CR linewidth of $2 \nu_{se}^{(a)}(\omega_c) = \sqrt{\pi} \Gamma_a / \hbar \approx 1.77 \Gamma_a / \hbar$. This differs only by 6% from the Ando result of $1.67 \Gamma_a / \hbar$ [Eq. (3.7)]. Moreover, the strong variation of $\nu_{se}^{(a)}(\omega_c)$ on a scale determined by the Landau level broadening, i.e., also by $\Gamma_a$ for pure vapor scattering, gives after insertion in Eq. (3.13) a main resonance of Re $\sigma_{xx}(\omega)$ that is not a Lorentzian, but rather similar to the Gaussian shape of Ando CR theory, as shown in Fig. 3.2 and Fig. 3.3. The remaining discrepancies can be ascribed to the approximations involved in any theory.
Figure 3.3: Comparison of different CR absorption shapes $\text{Re} \sigma_{xx}(\omega)$ in the quantum limit, normalized to the same height: single-electron Ando theory Eq. (3.6) (dashed); single-electron memory function theory Eqs. (3.13) and (3.15) (solid); and a pure Lorentzian (dash-dotted) obtained from Eq. (3.13) with a constant collision rate $\nu_{\text{eff}} = \sqrt{\pi} \Gamma_a/2\hbar$ [peak value of Eq. (3.15)]. Note that all curves have almost the same width (dotted line), and that the memory function curve is closer to the Gaussian from Ando theory than to the pure Lorentzian.

Unlike Ando CR theory, one cannot obtain a simple and exact expression for the CR linewidth $\gamma_{\text{CR}}$ in the balance equation/memory function theory, at least in the single-electron quantum limit. But from Fig. 3.3, one can conclude that as a rule, it is usually in good approximation given by just two times the peak value of $\nu_{\text{eff}}(\omega)$ at $\omega = \omega_c$:

$$\gamma_{\text{CR}} \approx 2\nu_{\text{eff}}(\omega_c)$$

(the linewidth in tesla is then $m_e\gamma_{\text{CR}}/e$). This rule applies also with Coulomb effects, and simplifies linewidth evaluations in more complex cases.

3.3.2 DC conductivity—extended SCBA

The DC magnetoconductivity of SE is usually measured at low frequency $\omega \ll \nu_{\text{eff}}$ over a large range of magnetic field $B$. In our case, the experimental setup has circular (Corbino-) geometry, and only $\sigma_{xx}(B)$ is measured. From Eq. (3.11), one has

$$\sigma_{xx}(B) = \frac{e^2 n_s}{m_e} \frac{\nu_{\text{eff}}(B)}{\omega_c^2 + \nu_{\text{eff}}(B)^2},$$  \hspace{1cm} (3.16)

where the DC effective momentum collision rates $\nu_{\text{eff}}(B)$ for vapor atom and ripplon scattering are, neglecting inelastic effects for vapor atoms (Sec. A.1.2, [60]),

$$\nu_{\text{eff}}^{(a)}(B) = \frac{1}{k_B T} \frac{\hbar^2 \nu_a}{4m_e^2 S} \sum_q q^2 S(q, -q \cdot \mathbf{V}),$$

$$\nu_{\text{eff}}^{(r)}(B) = \frac{1}{k_B T} \frac{1}{2m_e S} \sum_q q^2 V_q^2 Q_q N_q S(q, \omega^{(r)}_q - q \cdot \mathbf{V}).$$  \hspace{1cm} (3.17)

Here, $\mathbf{V}$ is the drift or CM velocity of the electrons. For $\mathbf{V} \to 0$ the above expressions are identical to the limit $\omega \to 0$ in the AC expressions [Eq. (3.14)], which illustrates the universality of the balance equation/memory function theory.

The above expressions correspond to the case $|\hbar q \cdot \mathbf{V}| \ll k_B T$, which enables the introduction of $\nu_{\text{eff}}(B)$ in the first place (see Sec. A.1.2). However, this does not yet allow to neglect $q \cdot \mathbf{V}$ in the DSF entering $\nu_{\text{eff}}(B)$ itself, because in magnetic field, the DSF depends on another small parameter, namely the Landau level width $\Gamma_{\text{se}} \ll k_B T$. If $q \mathbf{V}$ becomes of the order of $\Gamma_{\text{se}}$, the so-called cold nonlinear effect occurs [60], visible as decrease of $\nu_{\text{eff}}(B)$ at increase of $\mathbf{V}$. This phenomenon is closely related to Coulomb effects [7], see Sec. 4.4. No corresponding effect is present in the AC expressions [Eq. (3.14)].
3.3 Balance equation method

In the following, $\nu^{(a)}_{\text{eff}}(B)$ for the noninteracting case and $V \rightarrow 0$ is discussed in more detail. The single-electron DSF $S_{se}(q, \omega)$ [Eq. (2.34)] is then evaluated at $\omega = 0$ in Eq. (3.17), this corresponds to pure intra-level transitions [$N = N'$ in Eq. (2.34)]. If the broadening is the same for all levels, one obtains a simple analytic expression for the single-electron DC effective collision rate for vapor scattering in the whole range $\omega_c > \nu_a$:

$$\nu^{(a)}_{\text{se}}(B) = \frac{\sqrt{\tau \omega_c \Gamma^2_a}}{4 \Gamma_{se} k_B T} \exp \left[ -\left( \frac{\Gamma_{se}}{4 k_B T} \right)^2 \right] \coth \left( \frac{\hbar \omega_c}{2 k_B T} \right). \tag{3.18}$$

Same as in Eq. (3.15), $\Gamma_a = \hbar \sqrt{2 \omega_c \nu_a/\pi}$ is here only introduced for abbreviation, while $\Gamma_{se} = \sqrt{\Gamma_a^2 + \Gamma_T^2}$ is the real Landau level broadening, although $\Gamma_{se} = \Gamma_a$ for pure vapor scattering. Unlike the AC case, $\nu^{(a)}_{\text{se}}(B)$ is essentially a smooth function of the magnetic field $B$. In Eq. (3.18), the $\coth$-term describes the effect of thermal occupation of higher Landau levels at low $B$, when $\hbar \omega_c < k_B T$; this term becomes constant in the quantum limit $\hbar \omega_c > k_B T$ (corresponding to $B \gtrsim 1$ T at $T \sim 1$ K). The $\exp$-term is noticeably different from one only in the ultra-quantum limit $\Gamma_{se} > k_B T$ at very high $B$, when only the lower part of the ground Landau level is occupied; it is unimportant for this work.

The $B$-dependence of $\nu^{(a)}_{\text{se}}(B)$ is best discussed in the ratio $\omega_c/\nu^{(a)}_{\text{se}}(B) = \tan(\varphi_H)$, where $\varphi_H$ is the Hall angle, or angle between the directions of driving field and resulting current, shown in Fig. 3.4(a). Note that $\varphi_H$ is a nonmonotonic function of $B$. At low $B$, when higher Landau levels are thermally occupied, it increases with $B$, but as soon as the quantum limit $\hbar \omega_c > k_B T$ is reached, the behavior reverses to a decrease with $B$.

Figure 3.4: (a): Magnetic field dependence of $\omega_c/\nu^{(a)}_{\text{se}}(B) = \tan(\varphi_H)$ (solid curve), where $\nu^{(a)}_{\text{se}}(B)$ is the DC collision rate from Eq. (3.18). The dotted line shows the Drude case of constant zero-field collision rate $\nu_a$. (b): The inverse DC conductivity $\sigma^{(a)}_{xx}(B)$ from Eqs. (3.16) and (3.18) (solid curve), normalized to the zero-field conductivity $\sigma_0 = e^2 n_s/m_e \nu_a$. In the regime $\omega_c > \nu_a$, starting below 0.1 T, the Landau quantization leads to a strong increase of $\nu^{(a)}_{\text{se}}$ in (a), and to a strong decrease of $\sigma^{(a)}_{xx}$ compared to Drude theory [Eq. (3.4), dotted]. At low $B$ is usually $\omega_c \gg \nu^{(a)}_{\text{se}}$, and $\sigma^{(a)}_{xx} \propto \omega_c^2/\nu^{(a)}_{\text{se}}$ is identical to the SCBA [Eq. (3.9), dashed]. Note that $\varphi_H$ first increases with $B$, then decreases in the quantum limit. This finally invalidates at high $B$ the condition $\omega_c \gg \nu^{(a)}_{\text{se}}$, and $\sigma^{(a)}_{xx}$ deviates from the SCBA.

The nonmonotonic behavior of $\varphi_H$ can be physically explained as follows: In the balance equation method, $\nu^{(a)}_{\text{eff}}(B)$ describes the momentum loss rate of the electron system due to scattering. In the single-electron case, $\nu^{(a)}_{\text{se}}(B) \propto \sum_{\mathbf{q}} q^2 S_{se}(q, 0)$ [Eq. (3.17)], where the sum essentially adds up the number of possible elastic single-electron transitions in magnetic field, weighted by $q^2$ according to their contribution to the total momentum loss of the system. The number of possible transitions is determined by the density of final
states, and thus proportional to \( \omega_c/\Gamma_{se} \). In the quantum limit, the characteristic value of the momentum exchange \( q \) is given by the inverse magnetic length \( l_B^{-1} \). Together, one estimates \( \nu_{se}^{(a)}(B) \propto l_B^{-2} \omega_c/\Gamma_{se} \propto B^{3/2} \) in the quantum limit, which agrees with Eq. (3.18), and is stronger than \( \omega_c \propto B \). This explains the decrease of \( \varphi_H \) vs. \( B \) in the quantum limit.

The contrary behavior at low \( B \) is due to the occupation of higher levels, as elastic transitions of electrons in higher levels with index \( N > 0 \) have a higher characteristic value of the momentum exchange, \( q \sim \sqrt{2N + 1}l_B^{-1} \) (see Sec. 2.3). The occupation of higher levels being thermal, one can estimate \( N \sim k_BT/\hbar \omega_c \). Together, this gives now \( q \sim \sqrt{k_BT/\hbar \omega_c l_B^{-1}} \), which is independent of \( B \) as a result of two opposite effects. Thus, \( \nu_{se}^{(a)}(B) \) is now only proportional to the density of states, \( \nu_{se}^{(a)}(B) \propto \omega_c/\Gamma_{se} \propto B^{3/2} \), which agrees also with Eq. (3.18) (\( \cosh x \approx x^{-1} \) for \( x \ll 1 \)) and is weaker than \( \omega_c \propto B \). (Similar estimates apply to the peak values of \( \nu_{se}^{(a)}(\omega) \) in the AC case, but are not very useful here as one usually measures CR in a very narrow \( B \) range.)

Experimental DC data are usually presented in inverse form, i.e., as \( 1/\sigma_{xx}(B) \), as shown in Fig. 3.4(b). For our typical conditions (\( T \approx 1.3 \text{ K}, B \approx 10 \text{ T} \)), the high-cyclotron-frequency condition \( \omega_c \gg \nu_{eff}(B) \) is approximately fulfilled, and Eq. (3.16) can be approximated by \( \sigma_{xx}(B) \approx e^2 \nu_{eff}(B)/m_e \omega_c^2 \), which for \( \nu_{eff}(B) = \nu_{se}^{(a)}(B) \) is identical to the SCBA result of Eq. (3.9). However, note that the condition \( \omega_c \gg \nu_{eff}(B) \) eventually fails at high \( B \) due to \( \nu_{se}^{(a)}(B) \propto B^{3/2} \) in the quantum limit [see Fig. 3.4(b)].

### 3.3.3 Electron heating

Usually, one tries to measure transport while keeping the electron system close to thermal equilibrium with the scatterer system. In this linear case, one only needs the AC and DC momentum collision rates of Secs. 3.3.1 and 3.3.2 to describe transport. However, a very strong electric driving field may also drive the electron system away from thermal equilibrium with the scatterers, which can lead to interesting nonlinear effects.

In the following, this situation is treated using the effective temperature approximation. The electron system is assumed to be always in internal equilibrium, same as before for the momentum collision rates, only its effective internal temperature \( T_e \) may now be different from that of its environment \( T \). This assumption is usually justified for SE because the strong electron-electron interaction establishes internal equilibrium sufficiently fast (see Sec. 2.2.3), except possibly at very low electron density (see Sec. 7.3).

#### Energy balance equation and effective energy collision rate

To determine the effective electron temperature \( T_e \), one can use the energy balance treatment, which is analogous to the momentum balance treatment outlined at the beginning of Sec. 3.3. The electron system’s total internal energy inside the CM frame is \( E_e = \sum_j \epsilon_j \rho_j \), where \( \epsilon_j \) and \( \rho_j \) are energy and occupation probability of the \( j \)-th system state in the CM frame. The driving field moves the system as a whole against the force \( \mathbf{F} \) from the interaction with the scatterers [see Eq. (3.10)], and increases \( E_e \) with rate \( \dot{E}_e = -\dot{R}(t) \cdot \mathbf{F} \). On the other hand, the electrons lose excess energy by inelastic effects in collisions with scatterers at some rate \( W \). Thus, the equation for the rate of change of \( E_e \) with time, the energy balance equation, can be written as [82]

\[
\dot{E}_e = P + W.
\]
3.3 Balance equation method

In the effective temperature approximation is \( \rho_j \propto e^{-\nu_j/k_BT_e} \), and the time-derivative of \( E_e \) can be written as \( \dot{E}_e = C_e \dot{T}_e \), where \( C_e = \partial E_e/\partial T_e \) is the specific heat of the electron system \[82\]. For SE, \( C_e \) is of order \( N_s k_B \), the value at zero magnetic field.

For a real AC driving field \( \textbf{E}(t) = E_0 \cos(\omega t) \) acting on the system, and with the conductivity tensor \( \sigma_{\pm} \) and the memory function \( M(\omega) \) defined by Eq. (3.11), one finds

\[
P(t) = -\dot{\textbf{R}}(t) \cdot \textbf{F} = S \text{Re} \left[ \sigma_{xx} - i\sigma_+ M(\omega) e^{-2i\omega t} \right] E_0^2/2.
\]

In cyclotron resonance with \( \omega \gg \nu_{\text{eff}} \), near the resonance \( \sigma_{xx} \) is very small and thus \( P \approx S \text{Re} \sigma_{xx} E_0^2/2 \). In the DC limit \( \omega \ll \nu_{\text{eff}} \), on the other hand, \( P(t) \approx S \text{Re} \sigma_{xx} E_0^2 \cos^2(2\omega t) \).

The energy change rate \( W \) due to collisions with scatterers can be expressed in the same way as the momentum change rate in Eq. (3.12), i.e., as the product of transition rate times energy transfer, summed over all possible transitions:

\[
W = - \sum_{k,n,j,m} \rho_j \rho_m \left( \varepsilon^{(b)}_n - \varepsilon^{(b)}_m \right) P_{k,n}(j,m).
\]

The only difference to Eq. (3.12) is that the total momentum \( \textbf{P}^{(b)}_m \) is here replaced by the total energy \( \varepsilon^{(b)}_m \) of the scatterer system in state \( |m\rangle \). Details of the calculation are found in Sec. A.2. One obtains that in lowest order, \( W \) depends only on the difference in temperature between electron system and scatterer system, and can be written as

\[
W = -N_s k_B (T_e - T) \nu_{\text{en}}(T_e),
\]

where \( \nu_{\text{en}}(T_e) \) has the dimension s\(^{-1}\) and is called the effective energy collision rate.

One can in the energy relaxation also distinguish AC and DC cases, \( \omega \gg \nu_{\text{en}} \) and \( \omega \ll \nu_{\text{en}} \) respectively\(^2\). For \( \omega \gg \nu_{\text{en}} \), the electron temperature is only sensitive to the time-average of \( P(t) \), while for \( \omega \ll \nu_{\text{en}} \), it is only sensitive to the momentary value of \( P(t) \). Most important for this work is the AC case, here one obtains for vapor atom and ripplon scattering respectively (see Sec. A.2, and for ripplon scattering also Ref. [60])

\[
\nu^{(a)}_{\text{en}}(T_e) = \frac{1}{k_B T_e} \frac{\hbar^3 \nu_a}{2 m_e M S} \sum_q \left( q^2 + \frac{4}{3} \gamma^2 \right) S(q,0),
\]

\[
\nu^{(r)}_{\text{en}}(T_e) = \frac{1}{k_B T_e} \frac{1}{2 \rho_{He} S} \sum_q |q| V_q^2 S(q,\omega_q^{(r)}).
\]

Here, \( S(q,\omega) \) is the equilibrium DSF of the electron system at temperature \( T_e \). In \( \nu^{(a)}_{\text{en}}(T_e) \), in a magnetic field of 1.3 T is \( \gamma/q \sim \gamma_0 \ell_B \simeq 3 \) [\( \gamma_0 \) is the vertical wavefunction parameter, see Eq. (1.3)] and the second term in the round brackets is then most important. It reflects that vertically quantized SE can transfer large values of momentum and kinetic energy to a vapor atom in vertical direction. In \( \nu^{(r)}_{\text{en}}(T_e) \), only one-ripplon processes are taken into account, as two-ripplon processes are in magnetic field strongly suppressed by the narrow Landau levels with \( T_{\text{se}} \ll k_B T \) [60]. The total energy collision rate is additive, \( \nu_{\text{en}} = \nu^{(a)}_{\text{en}} + \nu^{(r)}_{\text{en}} \), same as the total momentum collision rate. The DC expressions for \( \nu_{\text{en}} \) are essentially the same as Eq. (3.22) (see Sec. A.2), except that they may show an equivalent to the cold nonlinear effect (cf. Sec. 3.3.2).

\(^2\)Note that the AC and DC cases with respect to \( \nu_{\text{en}} \) are not always the same as those with respect to the momentum collision rate \( \nu_{\text{eff}} \). For instance, for vapor scattering is \( \nu_{\text{en}}/\nu_{\text{eff}} \sim 10^{-3} \).
In Sec. 7.3, the energy balance is used to determine \( T_e \) in heating by CR excitation with \( \omega \gg \nu_{en} \) and in steady-state conditions \( \dot{E}_e = 0 \). In this case, Eq. (3.19) becomes

\[
P = N_e k_B (T_e - T) \nu_{en}(T_e),
\]

and \( P \approx \text{Re} \sigma_{xx} \int_S (E_0^2/2) dS \) (\( S \) is the system area)\(^3\) can be directly determined from the measured CR signal using Eq. (6.10).

### Three-dimensional electrons

So far, the discussion of transport and heating of SE assumed the system to be purely two-dimensional. This is no longer correct if the electrons are heated to a temperature \( T_e \) high enough that a significant amount of them occupy higher surface states. It is difficult to treat this situation rigorously, as this requires to determine wave functions and energies of all levels in the effective vertical potential [Eq. (1.8)], and the excitation spectrum must then include intra- and inter-level transitions for each occupied surface level. Such a rigorous treatment of hot electron transport has been done to date only in a semi-classical approach neglecting correlation effects \([93, 94, 95]\). It is unclear how far this approach applies to our case, where heating occurs by CR excitation and the magnetic quantization is certainly important in the early stages of heating.

However, the situation is again simplified if \( T_e \) becomes so high that the electrons are distributed over a large number of higher surface levels. The electrons are then effectively three-dimensional and scatter essentially only at vapor atoms. Moreover, in this work heating heating was measured in magnetic fields below 2 T, where the 3D electrons generally populate also a large number of higher Landau levels. In this case, it is indeed justified \([90, 95]\) to treat the electrons semi-classically, and neglect the Landau quantization. The balance equations treatment, together with the 3D free-electron DSF, lead then to the following simple results for the 3D momentum collision rate \( \nu_{\text{eff}}^{(3D)} \) and energy collision rate \( \nu_{\text{en}}^{(3D)} \) for vapor atom scattering (see Sec. A.3):

\[
\nu_{\text{eff}}^{(3D)} = \frac{8 A n_a}{3} \sqrt{\frac{2 k_B T_e}{\pi m_e}},
\]

\[
\nu_{\text{en}}^{(3D)} = \frac{8 m_e A n_a}{M} \sqrt{\frac{2 k_B T_e}{\pi m_e}}.
\]

(\( A = m_e^2 U_a^2 / \pi \hbar^2 \) is the electron-vapor atom scattering cross section). The above is the DC expression for \( \nu_{\text{eff}}^{(3D)} \), but for our typical conditions, the AC result is numerically the same (see Sec. A.3). The same results are obtained using the Boltzmann equation with the effective temperature approximation \([93]\).

\(^3\)The integral takes into account that the driving field \( E_0 \) may vary across the SE layer. In contrast, \( T_e \) and therefore \( \sigma_{xx} \) can usually be taken as homogeneous because energy is usually distributed so fast among the SE that they heat as a whole (see the discussion of the cold nonlinear effect in Ref. [7]).
Chapter 4

Theory of Coulomb effects

In the preceding section 3.3, the balance equation/memory function method was shown to provide a very universal theoretical description of magnetotransport, valid both at high frequency and in the DC limit. It was evaluated for the case of a noninteracting electron system, using the single-electron DSF $S_{se}(q, \omega)$ in magnetic field from Sec. 2.4.1. To include Coulomb effects in this formalism, it is necessary to find a proper approximation for the many-electron DSF $S_{me}(q, \omega)$ of the interacting electron system in magnetic field.

As discussed in Sec. 2.4.2, for SE in the liquid state ($10 < \Gamma_p < 100$), no well-established general approach to calculate the DSF exists, but it is clear that in transport with short-range scattering, most important are the high values of the momentum exchange $q$. Here, the DSF is mainly determined by the single-electron quantum properties, even at strong interaction, if only the condition $l_B^{-1} \gg \delta_p^{-1}$ [Eq. (2.37)] holds. For SE, this is true in a large parameter range, extending even into the solid phase. The question is then how the Coulomb interaction affects the quantum properties of the single electron.

A first important insight into this question came from Dykman [10, 96]: the relevant parameter is the local electric field acting on the single electron due to the presence of the others (Sec. 4.1). This field is usually called fluctuational electric field (FEF), as its origin are thermal density fluctuations, which prevent the electrons to arrange themselves in an ideal Wigner lattice, where the field on each electron would be zero. The important point is that the FEF is usually quasi-uniform, i.e., it varies negligibly over the distance of the electron wavelength. This means that the FEF is essentially all the single electron “knows” about the other electrons in a typical short-range scattering event, which leads again to the conclusion that the treatment reduces to an effective single-electron one. Also, the condition for the quasi-uniformity of the FEF is the same as above: $l_B^{-1} \gg \delta_p^{-1}$.

Hence, the interacting system can formally be treated as a collection of noninteracting electrons, each only influenced by its local, quasi-uniform FEF. In a normal magnetic field, the FEF induces a fast Hall drift motion of the single electron’s cyclotron orbit center. According to Monarkha [7, 17, 18], the derivation of the many-electron DSF consists then essentially of two parts: first, describe how the FEF influences the quantum states of the single electron (Sec. 4.3), and secondly, find how the single electron contributes to the DSF of the system as a whole (Sec. 4.4). Both tasks are greatly simplified by a simple “trick”, namely to describe the single-electron quantum states not in the center-of-mass (CM) reference frame of the whole system, but to describe each electron in a local reference frame moving along with the electron’s cyclotron orbit center on its FEF-induced Hall drift (Sec. 4.2).

Once the many-electron DSF is found, the description of Coulomb effects in transport
essentially reduces to inserting the DSF in the formula derived by the balance equation method. The results are discussed in Sec. 4.5.

One should remark that, in principle, the single electron is in transport not only influenced by the internal FEF, but also by the external driving field. In the following discussion of Coulomb effects, the driving field is assumed to be negligible compared to the FEF. This also means that one can neglect the motion of the CM reference frame of the electron system with respect to the laboratory reference frame, which is the rest frame of the scatterer system. However, basically the same effects as induced by the FEF can also be induced by a strong enough DC driving field, as already mentioned in Sec. 3.3.2. This is known as the cold nonlinear effect in magnetotransport [7].

4.1 Fluctuational electric field (FEF)

The characteristic properties of the local, fluctuational electric field $E_f$ acting on the individual electron can be understood in very simple terms. In a 2D electron system, the most important contribution to the local field comes from short-range interaction, as is shown below. SE are usually a strongly correlated system, where even the liquid state has a high degree of short-range order that is very similar to a Wigner solid. In an ideal hexagonal lattice, the restoring electric field acting on a particular electron at a small displacement $\delta_f$ from its equilibrium position $r_i$ (see Fig. 4.1) is in linear approximation

$$E_f \simeq (\delta_f \cdot \nabla r_i) \sum_{j \neq i} \frac{-e}{4\pi\varepsilon_0} \frac{(r_i - r_j)}{|r_i - r_j|^3} \bigg|_{eq} = \frac{1}{2} \delta_f \cdot \frac{e}{4\pi\varepsilon_0} \sum_{j \neq i} \frac{1}{|r_i - r_j|^3} \bigg|_{eq} \approx 4.45 \frac{e}{4\pi\varepsilon_0} n_s^{3/2} \delta_f. \quad (4.1)$$

The derivative was here evaluated for equilibrium lattice positions of the other electrons. Obviously, most important is the contribution of nearest neighbors.

**Figure 4.1:** Schematic picture for the origin of Coulomb effects in magnetotransport of SE. Thermal fluctuations displace the individual electron from its equilibrium position in the strongly correlated system by a distance $\delta_f$ and result in a finite restoring electric field $E_f$ acting on the electron. Typically, $E_f$ is approximately uniform over the electron wavelength, i.e., over the magnetic length $l_B$ in a magnetic field with $\hbar\omega_c > k_BT$.

In a simple Einstein picture, each electron performs then small harmonic oscillations about its equilibrium position with a characteristic frequency $\omega_0 \simeq 2.11e n_s^{3/4} / \sqrt{4\pi\varepsilon_0 m_e}$, and using classical statistics, the probability for one electron to be displaced by a distance $\delta_f$ is given by the Boltzmann factor $\exp(-\varepsilon_\delta / k_BT)$, where $\varepsilon_\delta = m_e \omega_0^2 \delta_f^2 / 2$ is the potential energy of the electron at displacement. The mean-square values of the displacement $\delta_f$ and of the field $E_f$ acting on an electron are then

$$\langle \delta_f^2 \rangle = 4F^{-1} \frac{4\pi\varepsilon_0 k_B}{e^2} n_s^{-3/2}T, \quad \langle E_f^2 \rangle = F \frac{k_B}{4\pi\varepsilon_0} n_s^{3/2}T \equiv \left( E_f^{(0)} \right)^2. \quad (4.2)$$
4.2 Introduction of local reference frames

and their distribution is Gaussian, i.e., the absolute value of the FEF has the distribution

\[ P(E_f) = (\pi \langle E_f^2 \rangle)^{-1} \exp\left(-E_f^2 / \langle E_f^2 \rangle\right). \]

(4.3)

The above simple estimate gives \( F \simeq 8.9 \) for the coefficient in Eq. (4.2). A strict treatment of the thermal vibrations of a hexagonal lattice in the harmonic approximation [6, 97] gives \( F = 8.91 \). A Monte-Carlo study [98] found the above results to be valid in a surprisingly large parameter range both in the solid and in the liquid phase of the electron system. For instance, the coefficient \( F \) is found to depend on the plasma parameter \( \Gamma_p \), but it differs from the asymptotic value \( F = 8.91 \) (for \( \Gamma_p \rightarrow \infty \)) at most by about 10% in the whole range \( 10 < \Gamma_p < \infty \), although in this range the structure of the system changes from a liquid, where correlations decay within twice the mean particle distance, to a Wigner solid. This behavior of \( F \) is a consequence of the field \( E_f \) being determined primarily by the short-range order in the system, as can be readily seen from Eq. (4.1).

As discussed at the beginning of this chapter, the notion of the FEF is especially useful if it is approximately uniform over the electron wavelength, i.e., the magnetic length \( l_B \) in a magnetic field with \( \hbar \omega_c > k_B T \) (or the thermal wavelength at zero field). Then it completely describes the influence of electron-electron interaction on short-range electron scattering, and the transport treatment of the interacting system reduces to an effective single-electron one. The characteristic distance over which the FEF varies is given by \( \sqrt{\langle \delta_f^2 \rangle} \) from Eq. (4.2), so that the condition for the FEF to be quasi-uniform is

\[ l_B \ll \sqrt{\langle \delta_f^2 \rangle}, \]

(4.4)

This is also the condition for the relative motion of the electrons to be semi-classical in a magnetic field, which is necessary for the above results to be valid. Otherwise, the motion of a single electron becomes quantized in the field of the others [97]. As \( \sqrt{\langle \delta_f^2 \rangle} \) is almost the same as the peak width \( \delta_p \) of the pair distribution function [Eq. (1.6)], the above condition is the same as the condition (2.37) for the single-electron part of the excitation spectrum to be the most important in transport.

4.2 Introduction of local reference frames

In a normal magnetic field \( \mathbf{B} \), the quasi-uniform FEF \( \mathbf{E}_f \) acting on the single electron induces a sideways Hall drift motion of the electron’s cyclotron orbit center [see Fig. 4.2(a)]. With the field \( \mathbf{E}_f \), also the Hall drift velocity \( \mathbf{u}_f = \mathbf{E}_f \times \mathbf{B} / B^2 \) (between scattering events) can be regarded as quasi-uniform for short-range scattering, and contains the same information as the FEF itself. In fact, \( \mathbf{u}_f \) may even be considered as the more relevant parameter for the treatment of transport, as is discussed in the following.

A strict general treatment of the quantum motion of an electron in crossed magnetic and electric fields \( \mathbf{B} \) and \( \mathbf{E}_f \) is rather difficult. However, the treatment is much simplified by describing each electron in a local reference frame that moves along with the drift velocity \( \mathbf{u}_f \) of its cyclotron orbit center [16, 17, 18], and not in the center of mass (CM) reference frame of the whole system, as shown in Fig. 4.2(b). In this local frame, the orbit center is at rest, and the electric field \( \mathbf{E}_f^{\text{(local)}} \) acting on the electron vanishes due to the usual transformation of electromagnetic fields at a change of the reference frame,

\[ \mathbf{E}_f^{\text{(local)}} = \mathbf{E}_f^{\text{(CM)}} - \mathbf{B} \times \mathbf{u}_f = 0 \]
(for nonrelativistic velocities), so that the electron is here only under the influence of the magnetic field, which does not change noticeably in strength for nonrelativistic drift velocities. The usual discrete Landau spectrum of the electron is thus always restored in the local frame, regardless of the strength of the field $E_f$ in the CM frame, which simplifies the treatment considerably. The origin of Coulomb effects in magnetotransport is in this approach rather the fast relative motion of the electron and the system of scatterers.

**Figure 4.2:** (a) View from CM reference frame: The single electron is subject to magnetic and electric fields $B$ and $E_f$. Its resulting motion is a superposition of a quantum cyclotron one and a $E_f$-induced Hall drift of the cyclotron orbit center with velocity $u_f = E_f \times B / B^2$. The scatterers (here, $^4$He vapor atoms) are at rest, and the electron’s kinetic energy is conserved in a collision. (b) View from local reference frame moving along with the electron’s orbit center with the drift velocity $u_f$: Here, the orbit center is at rest and the electric field is zero. The usual discrete Landau spectrum is thus restored. Scatterers move here with $-u_f$, and the electron’s kinetic energy is no longer conserved in a collision.

The perturbation of the basic Landau states of the electron due to the scatterers can now conveniently be treated in a sort of generalization of the well-known SCBA (see Sec. 2.3.1). It must only be taken into account that the scatterers are in the local frame not static, but move as a whole with $-u_f$. Scattering becomes then effectively inelastic due to the usual Doppler effect, which reduces scattering and the usual collision broadening of the Landau levels. This is discussed in the following section 4.3.

Moreover, as clearly each single electron has in its local frame a DSF of the usual single-electron form [Eq. (2.33)], it is now straightforward to construct the many-electron DSF of the whole system in the CM frame by simple application of the Doppler formula Eq. (2.28). The inclusion of Coulomb effects in the DSF reduces then to a simple one-line equation, presented in Sec. 4.4.

### 4.3 Coulomb narrowing of Landau levels

According to Sec. 4.2, a single electron has in its local frame moving with velocity $u_f$ the usual discrete Landau spectrum, only perturbed by the scatterers, assumed to be $^4$He vapor atoms in the following. This is basically the same situation as treated by the SCBA in terms of the collision broadening of the Landau levels (Sec. 2.3.1). Still, the SCBA treats only the special case of completely static scatterers, where a collision generally changes the electron’s momentum by a finite value $\hbar q$, but conserves its kinetic energy (this is usually a good approximation for vapor atoms due to their large mass).

In the electron’s local frame, however, the scatterers move as a whole with $-u_f$. The consequences of this can be physically understood as follows: In this case, a collision with momentum exchange $\hbar q$ also changes the electron’s kinetic energy by a finite value $-\hbar q \cdot u_f$ due to the usual Doppler effect, and collisions become effectively inelastic.

This strongly affects the scattering probability, because the possible values for the energy exchange are restricted by the Landau energy spectrum, as schematically shown in Fig. 4.3. Basically, only values that correspond to transitions inside or between Landau levels are allowed. Collisions with $\Gamma_a < |\hbar q \cdot u_f| < \hbar \omega_c$, where $\Gamma_a$ is the Landau level width in the SCBA [Eq. (2.22)], are strongly suppressed. In such a collision, the electron
makes a transition to a final state lying between levels, which means a vanishingly small density of final states and thus a very low transition probability. For SE, there is usually a sufficiently large parameter difference, $\Gamma_a \ll h\omega_c$, to make this effect rather pronounced.

Changes in the scattering probability will in turn affect the Landau level width, which for short-range scatterers is given by the collision broadening, and basically a measure of the (inverse) collision-limited lifetime of the Landau states (see Sec. 2.3.1). Suppression of collisions increases the electron’s lifetime and reduces the collision broadening, i.e., the Landau levels become narrower than in the SCBA.

For quantitative considerations, the ground Landau level, for instance, has a typical mean value of the momentum exchange at intra-level scattering of $\hbar q \sim \sqrt{2}\hbar/l_B$, as can be estimated from the squared matrix element for an electron transition from ground Landau state to ground Landau state $|J_{0,0}(q)|^2 \propto \exp(-q^2l_B^2/2)$ [Eq. (2.31)]. Hence, the typical value of the energy exchange for the single electron can be estimated as

$$|\hbar q \cdot \mathbf{u}_f| \sim \sqrt{2}eE_fl_B$$

and depends on the value of the FEF acting on this electron, and on the magnetic field.

At an increase of $eE_fl_B$, one can then distinguish three different regimes of Coulomb effects on the ground Landau level width $\Gamma_0$ (see Fig. 4.4):

(A) For $\sqrt{2}eE_fl_B \ll \Gamma_a$, the energy exchange is always negligibly small, and the situation is essentially the same as in the SCBA, so the ground level width is given by the usual self-consistent collision broadening [Eq. (2.22)]: $\Gamma_0 = \Gamma_a = h\sqrt{2}\omega_c/\pi$.

(B) For $h\omega_c \gg \sqrt{2}eE_fl_B \gg \Gamma_a$, an increasing part of the collisions become suppressed and the ground level width decreases to $\Gamma_0 < \Gamma_a$.

(C) At still larger $\sqrt{2}eE_fl_B \gg h\omega_c$, the electron can make transitions to other Landau levels; this re-stimulates the scattering probability and slows down the level narrowing. In the limit $\sqrt{2}eE_fl_B \gg h\omega_c$, the electron scatters into an effective continuum of final states, self-consistency is no longer required, and the scattering probability and the Landau level width asymptotically approach a value essentially determined by the zero-field collision frequency, $\Gamma_0 \sim h\nu_a$, independent of the magnetic field.
Higher Landau levels behave qualitatively similar to the ground level, only they change faster with increasing $eE_lB$ (see Fig. 4.4) because they have a higher typical value of the momentum exchange, and thus of the Doppler energy exchange at scattering. For instance, the squared matrix element for an electron transition from first excited state to first excited state is $|J_{1,1}(q)|^2 \propto (1 - q^2 l_B^2/2)^2 \exp(-q^2 l_B^2/2)$; this gives a typical value $\hbar q \sim \sqrt{2} \hbar l/B$, which is larger than the value for the ground level $\hbar q \sim \sqrt{2} \hbar l/B$.

In principle, it is possible to calculate the Landau level shape $A_N(\varepsilon)$ by a sort of generalization of the SCBA. However, the inelasticity makes a strict solution difficult. The problem can be simplified by fixing the level shape to a known form; this is justified because the exact shape usually is of minor importance (see Sec. 2.3.1 and Fig. 3.1). This leaves only the level width to be calculated. For $\Gamma_N \ll \hbar \omega_c$, one can find [17, 18]:

$$\left( \frac{\Gamma_N}{\Gamma_a} \right)^2 = \frac{1}{A_N(\varepsilon_N)} \sum_{N'} \int_0^{2\pi} \frac{d\varphi}{2\pi} \int_0^{\infty} dx_q |J_{N,N'}(x_q)|^2 A_{N'}(\varepsilon_N + \hbar q \cdot \mathbf{u}_f)$$  \hspace{1cm} (4.6)

$(x_q \equiv q^2 l_B^2/2)$. For $u_f = 0$, this reproduces the SCBA result $\Gamma_N = \Gamma_a$ [Eq. (2.22)]. Using a Gaussian level shape, a simple expression is found for the ground level width [17, 18]:

$$(\Gamma_0/\Gamma_a)^2 = \sqrt{[1 + C_0(\delta)]^2 + \lambda^4 - \lambda^2},$$  \hspace{1cm} (4.7)

where the parameter $\lambda \equiv \sqrt{2} e E_l l_B/\Gamma_a$ describes the suppression of collisions for $\lambda \gtrsim 1$, and $\delta \equiv \sqrt{2} e E_l l_B/\hbar \omega_c$ describes the re-stimulation of collisions for $\delta \gtrsim 1$. The function $C_0(\delta)$ is analytically approximated by $C_0(\delta) \simeq (3\delta/\sqrt{\pi} - 0.6 + 1/\delta^2) \exp(-1/\delta^2)$. The solutions for higher levels cannot be put in such a simple analytic form (see Refs. [17] and [7]) and are not presented here. Typical results for $\Gamma_0$ and $\Gamma_1$ are shown in Fig. 4.4.

**Figure 4.4:** The Coulomb reduction of the collision broadening of the Landau levels: shown are the ground level width $\Gamma_0$ [Eq. (4.7), solid curves] and the width of the first excited level $\Gamma_1$ (dashed curves) for a single electron in its local frame versus the energy exchange parameter $\sqrt{2} e E_l l_B$ at three different values of the magnetic field $B$. The values at $\sqrt{2} e E_l l_B = 0$ are identical to the SCBA result $\Gamma_N = \Gamma_a = \hbar \sqrt{2} \omega_c \nu_a/\pi$ [Eq. (2.22)]. $\nu_a$ is the zero-field collision rate.

It should be emphasized that the above discussion strictly only applies to the Landau spectrum of one particular electron in its local frame. In the whole system, the electrons differ in their FEF due to the Gaussian distribution of the latter [Eq. (4.3)], and therefore also differ in their degree of Coulomb level narrowing in their respective local frames.

Still, from the mean value of the FEF, $E^{(0)}_l \propto n_s^{3/4} T^{1/2}$ [Eq. (4.2)], one can obtain a mean value of the Doppler energy exchange for the whole electron system [cf. Eq. (4.5)],

$$|\hbar q \cdot \mathbf{u}_f| \sim \sqrt{2} e E^{(0)}_l l_B \equiv \Gamma_C.$$  \hspace{1cm} (4.8)
Inserting $\Gamma_C$ in Eqs. (4.6) or (4.7), one in turn obtains a mean level width for the whole system in dependence of density $n_s$ (or temperature $T$). The result for the saturation case is shown in Fig. 4.5, including also holding field effects and the level broadening due to ripplon scattering as follows: the ripplon broadening should be affected by Coulomb effects similarly to the vapor atom broadening and is also for this work of minor importance, therefore it is in the following included only as a correction to the single-electron broadening, i.e., $\Gamma_a$ in Eqs. (4.6) and (4.7) is replaced by $\Gamma_{se} = \sqrt{\Gamma_a^2 + \Gamma_r^2}$ [Eq. (2.25)].

![Figure 4.5: The mean ground Landau level width $\Gamma_0$ for the whole electron system due to vapor atom and ripplon scattering vs. the electron density $n_s$ at saturation for three values of the magnetic field $B$, normalized to the single-electron level width in the limit of zero holding field: the dashed curves shows the single-electron result $\Gamma_0 = \Gamma_{se}$ [Eq. (2.25)], influenced only by holding field effects, and the solid curves show the result including the Coulomb narrowing, calculated from Eq. (4.7) for $E_f = E_f^{(0)} \propto n_s^{3/4}T^{1/2}$ [Eq. (4.2)] and $\Gamma_a$ replaced by $\Gamma_{se}$ (see text).]

### 4.4 Many-electron DSF of the interacting electron liquid

As discussed at the beginning of this chapter and in Sec. 4.1, if the condition $\frac{l_B^1}{\delta_p^1}$ is fulfilled, which means that the FEF acting on the single electron is quasi-uniform and the relative motion of the electrons is semi-classical, then most important for the treatment of transport with short-range scattering are the single-electron quantum excitations. The excitation spectrum of the whole electron system is then obtained as the classical average of the independent quantum excitation spectra of the individual electrons.

In its local reference frame moving with velocity $u_f$, the FEF acting on the individual electron vanishes and the usual discrete Landau spectrum is restored (see Sec. 4.2). The excitation spectrum of this electron in its local frame is then also just the usual DSF of a single electron in magnetic field, $S_{se}(q, \omega)$ from Eq. (2.33), as shown in Fig. 4.6(a) for the case of the quantum limit. To be exact, one must of course also take into account the Coulomb narrowing of the Landau levels in the local frame (see Sec. 4.3), but this does not change the general form of the DSF and is of minor importance for the following.

The important point is that in the transcription into the CM reference frame of the whole electron system, the DSF of the single electron changes according to the Doppler formula Eq. (2.28) to the form $S^{(CM)}(q, \omega) = S_{se}(q, \omega - q \cdot u_f)$, as shown in Fig. 4.6(b) and (c). The many-electron DSF of the whole system in the CM frame is then simply obtained as the average of the contributions of the individual electrons [7, 17],

$$S_{me}(q, \omega) = \langle S_{se}(q, \omega - q \cdot u_f) \rangle_f,$$  

(4.9)
Figure 4.6: Construction of the the many-electron DSF of the interacting electron system.

(a): A single electron has in its local frame moving with $u_f$ a DSF of the usual single-electron form $S_{se}(q, \omega)$ [Eq. (2.35)], same as in Fig. 2.8, only affected by the Coulomb narrowing of the Landau levels. (b, c): Transformed into the CM frame, the DSF from (a) becomes $S^{(CM)}(q, \omega) = S_{se}(q, \omega - q \cdot u_f)$ due to the Doppler effect [Eq. (2.28)], shown in (b) for $q$ parallel $u_f$, and in (c) for $q$ antiparallel $u_f$. The many-electron DSF of the whole system is then obtained as the average of the contributions of the individual electrons, $S_{me}(q, \omega) = \langle S_{se}(q, \omega - q \cdot u_f) \rangle_f$ [Eq. (4.9)], shown both in (b) and in (c) for the case $\hbar \omega_c \gg \Gamma_C \gtrsim \Gamma_{se}$. The averaging broadens and reduces the sharp maxima of $S_{se}(q, \omega)$ from Fig. 2.8. (d): Same as (c) for the case of very strong interaction $\Gamma_C \gtrsim \hbar \omega_c$. The maxima in $S_{me}(q, \omega)$ now start to overlap at high $q$.

where $\langle \rangle_f$ denotes the average over the distribution of $u_f$, i.e., the Gaussian distribution of $E_f$ [Eq. (4.3)]. This simple one-line equation summarizes all effects of the Coulomb interaction on the DSF, and thus on transport. They are here the result of Doppler effects induced by the fast drift velocities of the individual electrons, similar to the Coulomb narrowing of the Landau levels. Same as for the latter [cf. Eq. (4.8)], the mean value of the Doppler shift term in Eq. (4.9) for the whole electron system can be estimated as

$$|\hbar q \cdot u_f| \sim \sqrt{2} e E^{(0)}_f l_B \equiv \Gamma_C,$$

(4.10)

where $E^{(0)}_f \propto n^{3/4}_s T^{1/2}$ is the mean value of the FEF [Eq. (4.2)].

Because the Landau level shape enters $S_{se}(q, \omega)$, a rigorous evaluation of Eq. (4.9) must take into account the dependence $\Gamma_N = \Gamma_N(\sqrt{2} e E_f l_B)$ [Eqs. (4.6) and (4.7)] of the individual electron’s Landau level width on the FEF, shown in Fig. 4.4. This leads to a rather complicated expression. However, it is a quite good approximation to set the level width for all electrons to the same mean value $\Gamma_N = \Gamma_N(\Gamma_C)$, same as done for Fig. 4.5. This leaves only the average over the Doppler shift term in the frequency argument of the DSF in Eq. (4.9), which is the main effect on the DSF anyway.

The final result can then be presented in an analytical form and corresponds simply to making in the single-electron DSF $S_{se}(q, \omega)$ from Eq. (2.34) the following substitution for the average Landau level broadening term $\Gamma_{N,N'}^2$:

$$\Gamma_{N,N'}^2 \to \Gamma_{N,N'}^2 + x_q \Gamma_C^2,$$

(4.11)
where \( x_q = q^2 l_B^2 / 2 \) and \( \Gamma_C = \sqrt{2eE_i^{(0)}a} \) [Eq. (4.10)], and where on the right side now \( \Gamma_{N,N'}^{2,\nu} \equiv [\Gamma_{N}^{2} C_{\nu} C_{\nu}^{2} (\Gamma_{C}) + \Gamma_{N'}^{2} C_{\nu} C_{\nu}^{2} (\Gamma_{C})] / 2 \) also includes the mean Coulomb narrowing of the levels.

For instance, in the quantum limit, where \( S_{se}(q, \omega) \) has the simpler form of Eq. (2.35), the substitution of Eq. (4.11) gives explicitly

\[
S_{me}(q, \omega) = 2 \sqrt{\pi} \hbar \sum_{N=0}^{\infty} \frac{x_q^N \exp(-x_q)}{N!} \frac{1}{\sqrt{\Gamma_{0,N}^{2} + x_q \Gamma_{C}^{2}}} \exp\left(-\frac{\hbar^2 (\omega - N \omega_c)^2}{2 \Gamma_{0,N}^{2} + x_q \Gamma_{C}^{2}}\right). \tag{4.12}
\]

Coulomb effects act thus towards an effective broadening and reduction of the maxima vs. frequency \( \omega \) in the many-electron DSF; this is described in Eq. (4.12) by the parameter \( x_q \Gamma_{C}^{2} \), which enters in the same way as the level broadening \( \Gamma_{0,N}^{2} \) and strongly outweighs the Coulomb narrowing of the latter (which only becomes relevant when already \( x_q \Gamma_{C}^{2} > \Gamma_{0,N}^{2} \) and thus only gives rise to a minor correction). This can be seen in Fig. 4.6(b–d), which shows Eq. (4.12) for different \( \Gamma_C \).

Analogous to the Coulomb narrowing of the Landau levels in Sec. 4.3, one can in the many-electron DSF distinguish three different regimes at an increase of \( \Gamma_C \):

(A) For \( \Gamma_C \ll \Gamma_{se} \), the many-electron DSF \( S_{me}(q, \omega) \) is identical to the single-electron one \( S_{se}(q, \omega) \) [Fig. 2.8], and exhibits sharp maxima vs. \( \omega \) at \( \omega = N \omega_c, N = 0, 1, \ldots \) [terms with \( N = 0, 1, \ldots \) in Eq. (4.12)], whose width is given by \( \Gamma_{N,N'} = \Gamma_{se} \).

(B) For \( \hbar \omega_c \gg \Gamma_C \gg \Gamma_{se} \), the sharp maxima vs. \( \omega \) of \( S_{se}(q, \omega) \) become strongly reduced and broadened in \( S_{me}(q, \omega) \) [Fig. 4.6(b, c)]. However, as long as \( \Gamma_C \ll \hbar \omega_c \) the maxima are still well separated and distinct.

(C) For \( \Gamma_C \gg \hbar \omega_c \), the individual maxima in the DSF start to overlap, mainly at high values of \( q \) [Fig. 4.6(d)].

Interestingly, the same influence that leads to a narrowing of the single-electron Landau levels and DSF in the local frames results thus in a broadening and reduction of the many-electron DSF in the CM frame. Both effects are ultimately related to the same changes in the scattering behavior of the electrons, although one must be careful at comparing single-electron properties with those of the whole system: The Coulomb level narrowing corresponds to a reduced equilibrium scattering rate of the single electron, similarly the reduction of the many-electron DSF maxima ultimately is equivalent to a reduced effective scattering rate of the whole system in transport (see Sec. 4.5).

The main physical differences between single-electron DSF and many-electron DSF in the above model may be summarized as follows: In the noninteracting case, the electrons are basically identical in their respective excitation spectra, and all of them are always excited together in the same way. In a magnetic field, the system as a whole exhibits then the sharp maxima vs. frequency of the single-electron DSF, which correspond to single-electron transitions between Landau levels [case (A) above]. In the interacting case, the electrons move relative to each other with high velocity, which shifts the frequency dependence of their respective excitation spectra by the Doppler effect. The electrons differ now in their respective conditions for excitation, and cannot be excited together anymore. Compared to the noninteracting case, now only a part of the electrons can be excited at any frequency, but this over a much broader frequency range, resulting in the reduction and broadening of the maxima in the many-electron DSF. As long as the maxima are well separated, one still excites at any frequency only electrons that
make transitions over the same number $\Delta N$ of Landau levels. In the quantum limit, for instance, near $\omega = 0$ one still only excites transitions with $\Delta N = 0$, or near $\omega = \omega_c$ only transitions with $\Delta N = 1$ [case (B)]. But if the Doppler shifts are so large that the maxima overlap, one can excite electrons with different $\Delta N$ at the same frequency. In the quantum limit, at $\omega = 0$ one now also excites transitions with $\Delta N = 1, 2$, or even higher, and at $\omega = \omega_c$ also transitions with $\Delta N = 0$ and 2, 3, or higher [case (C)].

A similar effect is well-known in atomic spectroscopy as the Doppler broadening of the spectral lines of gases. Here, the different thermal velocities of the individual gas atoms also shift their respective excitation frequencies, same as the FEF-induced drift velocities in the case of SE.

An important point to remember is that in the balance equation formalism (Sec. 3.3), the DSF ultimately determines the relaxation behavior of the driving-field-induced CM motion of the whole system via the interaction with the scatterers. This leads, for instance, to a quite opposite behavior of DSF and measured CR signal. The decrease and broadening of the DSF maxima transforms directly into a decrease and broadening of the maxima in the effective collision rate, and thus ultimately into a narrowing of the measured CR signal. This is discussed in more detail in the following Sec. 4.5.

### DSF of the Wigner crystal

Although the experimental part of this thesis only deals with SE in the liquid phase, it is instructive to also shortly consider the solid phase. Here, the DSF can be calculated in a strict way from the phonon spectrum of the Wigner crystal in magnetic field, using well-established methods of neutron scattering theory [77]. This treatment is entirely independent from the model for the DSF of the liquid discussed above. The important point is that in the relevant parameter range, both treatments finally give identical results, in agreement with the discussion in Sec. 2.4.2. The only difference is that the treatment of the solid formally does not include any broadening of the Landau levels, which is usually very small in the Wigner crystal regime anyway.

### 4.5 Coulomb effects in magnetotransport

The two preceding sections treated the changes induced by the strong Coulomb interaction in the fundamental system properties of SE, like the Landau level width and the DSF. The strength of Coulomb effects is characterized by the single parameter $\Gamma_C = \sqrt{2} e E_{\perp}^{(0)} l_B$, which determines both the Coulomb narrowing of the Landau levels (Sec. 4.3) and the broadening and reduction of the maxima in the many-electron DSF (Sec. 4.4).

It remains to discuss how these changes affect the magnetotransport of SE. Formally, one simply has to insert the many-electron DSF into the expressions for the effective collision rates derived by the balance equation method (Sec. 3.3) to obtain final expressions both for the cyclotron resonance (discussed in Sec. 4.5.1) and the DC magnetoconductivity (discussed in Sec. 4.5.2) of the interacting system. The interaction mainly changes the frequency dependence of the DSF and leads to a broadening and reduction of its maxima, basically the same changes occur then in the effective collision rates.
4.5 Coulomb effects in magnetotransport

4.5.1 Cyclotron resonance

Quantum CR in the balance equation/memory function formalism was discussed for the single-electron case in Sec. 3.3.1. The tensor for the AC magnetoconductivity [Eq. (3.13)] has a Lorentzian structure, as in classical Drude theory, but with a frequency- and magnetic-field-dependent AC effective momentum collision rate \( \nu_{se}(\omega) \) (width parameter of the Lorentzian), which is determined by the DSF of the electron system [Eq. (3.14)]. As a rule, ultimately the CR linewidth \( \gamma_{CR} \) is determined solely by the value of \( \nu_{se}(\omega) \) at resonance \( \omega = \omega_c \) (see Fig. 3.3): \( \gamma_{(MF)}^{(CR)} = 2\nu_{se}(\omega_c) \), but the CR line shape may be strongly affected by the variation of \( \nu_{se}(\omega) \) around the resonance (see Fig. 3.2).

The effects of Coulomb interaction on CR are described by inserting the many-electron DSF \( S_{me}(q, \omega) \) from Sec. 4.4 into Eq. (3.14) to calculate the many-electron AC effective collision rate \( \nu_{me}(\omega) \). As in the Coulomb effects on the DSF, discussed in Sec. 4.4 and shown in Fig. 4.6, one can then also in CR generally distinguish three different regimes at an increase of the parameter \( \Gamma_C = \sqrt{2eE^{(0)}_iI_B} \), which is taken to be due to an increase of the electron density \( n_s \) in the following (\( E^{(0)}_i \propto n_s^{3/4}T^{1/2} \)).

For illustration, Fig. 4.7 shows in the upper row graphs of \( \nu_{me}(\omega) \) vs. magnetic field \( B \) for three values of \( n_s \) at saturation, each corresponding to one of the three characteristic regimes (A, B, C) discussed in more detail below. The corresponding CR absorption \( \text{Re} \sigma_{xx}(\omega) \) is shown in the lower row of graphs. For completeness, the curves in Fig. 4.7 include both vapor atom and ripplon scattering; they were calculated numerically using Eq. (2.34) modified according to Eq. (4.11) to approximate the many-electron DSF. The following detailed discussion gives explicit analytic expressions for the main resonance in the quantum limit, where the single-electron DSF \( S_{se}(q, \omega) \) is given by Eq. (2.35) and the many-electron DSF \( S_{me}(q, \omega) \) can be approximated by Eq. (4.12), and it treats only vapor atom scattering. Ripplon scattering behaves similarly and is mostly negligible in Fig. 4.7, except at very high \( n_s \). It is included in the CR linewidth later on.

(A) In the noninteracting case, corresponding to the limit \( \Gamma_C \ll \Gamma_{se} \), the many-electron DSF reduces to the single-electron one, \( S_{me}(q, \omega) = S_{se}(q, \omega) \), same as the collision rate, \( \nu_{me}(\omega) = \nu_{se}(\omega) \). This regime has already been discussed in Sec. 3.3.1: \( S_{se}(q, \omega) \) is characterized by pronounced maxima vs. \( \omega \) at \( \omega = N\omega_c, \; N = 0, 1, 2, \ldots \) (see Fig. 2.8), these maxima appear then also in \( \nu_{se}(\omega) \), as shown in Fig. 4.7(A), or Fig. 3.2. The CR absorption \( \text{Re} \sigma_{xx}(\omega) \) vs. \( B \) has a main resonance at \( \omega = \omega_c \) and a series of subresonances at lower \( B \) (better visible in Fig. 3.2). The linewidth \( \gamma_{(MF)}^{(CR)} = 2\nu_{se}(\omega_c) \) of the CR main resonance is here rather large. The strong variation of \( \nu_{se}(\omega) \) around \( \omega = \omega_c \) strongly affects the CR line shape and renders it close to a Gaussian, as in Ando CR theory. The resonant maximum in \( \nu_{se}(\omega) \) around \( \omega = \omega_c \) is determined solely by the \( N = 1 \) term in \( S_{se}(q, \omega) \) from Eq. (2.35), which corresponds to SE transitions between adjacent Landau levels. It is given by the simple analytic Eq. (3.15), repeated here for direct comparison:

\[
\nu_{se}^{(a)}(\omega) \simeq \frac{\sqrt{\pi} \Gamma_{se}^2 \omega_c}{2 \Gamma_{se} \hbar \omega} \exp\left(-\frac{\hbar^2(\omega - \omega_c)^2}{\Gamma_{se}^2}\right) \quad \text{for} \quad \omega \sim \omega_c.
\]

(B) For \( \hbar \omega_c \gg \Gamma_C \gtrsim \Gamma_{se} \), the sharp maxima vs. \( \omega \) of \( S_{se}(q, \omega) \) become strongly reduced and broadened in \( S_{me}(q, \omega) \) due to the Doppler shifts of the individual electrons’

\[1\]In principle, an increase of temperature \( T \) leads to similar effects, but in our case this would also significantly increase the occupation of higher Landau levels, which complicates the situation.
excitation spectra, but are still well separated and distinct [Fig. 4.6(b, c)]. Consequently, also $\nu_{\text{me}}(\omega)$ is significantly reduced and varies over a broader range of frequency, or magnetic field, as seen in Fig. 4.7(B). As mentioned in Sec. 4.4, this can be understood as the Doppler shifts resulting in only a part of the electrons contributing to scattering at any frequency, but over a broader frequency range. This leads in turn to a narrower and higher main resonance of $\text{Re } \sigma_{xx}(\omega)$, which also tends more towards a Lorentzian form. In contrast, the subresonances, which are directly proportional to $\nu_{\text{me}}(\omega)$, become less pronounced. From the approximation Eq. (4.12) for $S_{\text{me}}(q, \omega)$, one obtains now the following approximation for the resonant maximum in $\nu_{\text{me}}^{(a)}(\omega)$ around $\omega = \omega_c$, which for $\Gamma_C \ll \hbar \omega_c$ is still determined solely by the $N = 1$ term in $S_{\text{me}}(q, \omega)$ [17]:

$$
\nu_R(\omega) \simeq \frac{\sqrt{\pi} \Gamma_0^2 \omega_c}{4 \hbar \omega} \int_0^\infty \frac{x^2 \exp(-x)}{\sqrt{\Gamma_{0,1}^2 + x \Gamma_C^2}} \exp\left(-\frac{\hbar^2 (\omega - \omega_c)^2}{\Gamma_{0,1}^2 + x \Gamma_C^2}\right) dx \quad \text{for} \quad \omega \sim \omega_c. \tag{4.14}
$$

This expression reduces to Eq. (4.13) for $\Gamma_C \ll \Gamma_{\text{se}}$. Although an approximation, Eq. (4.14) is usually sufficient for a quantitative evaluation (a more strict evaluation includes the dependence $\Gamma_N = \Gamma_N(E_f)$ in the average for the DSF, see Ref. [17]). For qualitative discussions, it is instructive to further simplify Eq. (4.14) by the replacement $\Gamma_{0,1}^2 + x \Gamma_C^2 \to \Gamma_{\text{se}}^2 + \Gamma_C^2$; this gives the even more transparent form [7]

$$
\nu_R(\omega) \simeq \frac{\sqrt{\pi} \omega_c \Gamma_0^2}{2 \hbar \omega \sqrt{\Gamma_{\text{se}}^2 + \Gamma_C^2}} \exp\left(-\frac{\hbar^2 (\omega - \omega_c)^2}{\Gamma_{\text{se}}^2 + \Gamma_C^2}\right), \tag{4.15}
$$

Figure 4.7: The many-electron AC effective momentum collision rate $\nu_{\text{me}}(\omega)$ (upper three graphs) and the resulting CR absorption $\text{Re } \sigma_{xx}(\omega)$ (lower three graphs) vs. magnetic field for three different saturation electron densities $n_s$. Here, $\nu_{\text{me}}(\omega)$ was calculated numerically from Eq. (3.14) using Eq. (2.34) modified according to Eq. (4.11) to approximate the many-electron DSF (analytic expressions for the quantum limit are given in the text). Increase of $n_s$ increases the Coulomb effect parameter $\Gamma_C = \sqrt{2e F_i^{(0)}_t} B$ due to $E_i^{(0)} \propto n_s^{3/2} T^{1/2}$. The employed values of $n_s$ correspond to the three cases discussed in the text: (A) noninteracting, single-electron limit with $\Gamma_C \ll \Gamma_{\text{se}}$ (same as Fig. 3.2); (B) beginning Coulomb effects with $\hbar \omega_c \gg \Gamma_C \gtrsim \Gamma_{\text{se}}$; (C) strong Coulomb effects with $\Gamma_C \gtrsim \hbar \omega_c$. 

\[\text{Graphs:}\]

- **(A)** $n_s = 10^7 \text{ cm}^{-2}$
- **(B)** $n_s = 10^8 \text{ cm}^{-2}$
- **(C)** $n_s = 10^9 \text{ cm}^{-2}$

\[\omega = 2\pi \times 56 \text{ GHz} \quad T = 1.3 \text{ K} \]

\[\text{magnetic field } B [\text{T}] \]

\[\text{Figure 4.7:}\hspace{1cm}\text{The many-electron AC effective momentum collision rate } \nu_{\text{me}}(\omega) \text{ (upper three graphs) and the resulting CR absorption } \text{Re } \sigma_{xx}(\omega) \text{ (lower three graphs) vs. magnetic field for three different saturation electron densities } n_s. \text{ Here, } \nu_{\text{me}}(\omega) \text{ was calculated numerically from Eq. (3.14) using Eq. (2.34) modified according to Eq. (4.11) to approximate the many-electron DSF (analytic expressions for the quantum limit are given in the text). Increase of } n_s \text{ increases the Coulomb effect parameter } \Gamma_C = \sqrt{2e F_i^{(0)}_t} B \text{ due to } E_i^{(0)} \propto n_s^{3/2} T^{1/2}. \text{ The employed values of } n_s \text{ correspond to the three cases discussed in the text: (A) noninteracting, single-electron limit with } \Gamma_C \ll \Gamma_{\text{se}} \text{ (same as Fig. 3.2); (B) beginning Coulomb effects with } \hbar \omega_c \gg \Gamma_C \gtrsim \Gamma_{\text{se}}; \text{ (C) strong Coulomb effects with } \Gamma_C \gtrsim \hbar \omega_c.}\]
which illustrates well the broadening and reduction of the resonant maximum in \( \nu_{\text{me}}(\omega) \) as soon as \( \Gamma_C \gtrsim \Gamma_{\text{se}} \), and is used for discussion later on.

(C) For \( \Gamma_C \gtrsim \hbar \omega_c \) the individual maxima start to overlap in \( S_{\text{me}}(q, \omega) \), mainly at high values of \( q \) [Fig. 4.6(d)]. \( \nu_{\text{me}}(\omega) \) is now almost continuous vs. frequency or magnetic field, as shown in Fig. 4.7(C). The main resonance of \( \text{Re} \sigma_{xx}(\omega) \) has now a pure Lorentzian shape and the subresonances have vanished completely. The width of the CR main resonance increases again with \( \Gamma_C \) in this regime, for the following reasons: Partly, this is due to holding field effects, if \( n_s \) is increased at saturation. But more importantly, the main resonance at \( \omega = \omega_c \) is in this regime not only determined by the \( N = 1 \) term in \( S_{\text{me}}(q, \omega) \) from Eq. (4.12) alone, but due to the strong Doppler broadening of the DSF, also nonresonant terms with \( N \neq 1 \) become relevant at \( \omega = \omega_c \). As mentioned in Sec. 4.4, this corresponds to an increasing part of the electrons making now transitions to increasingly higher levels; thus, the number of electrons partaking in scattering no longer decreases. Moreover, as transitions to higher levels have a higher typical value of the momentum exchange \( q \), additionally the mean value of \( q \) increases in this regime, i.e., the electrons scatter more efficiently, too. The efficiency is taken into account in the general formula Eq. (3.14) for \( \nu_{\text{eff}}(\omega) \) by the weighting with \( q^2 \), as discussed in Sec. 3.3.

To evaluate the nonresonant terms, one can use that they are almost constant vs. \( \omega \) in the relevant range around \( \omega = \omega_c \), and that the Landau level broadening can usually be neglected altogether for \( \Gamma_C \gtrsim \hbar \omega_c \) due to \( \Gamma_{\text{se}} \ll \hbar \omega_c \). They can then be summarized to a nonresonant contribution \( \nu_{\text{NR}} \) to be added to the resonant contribution \( \nu_R(\omega) \) from Eq. (4.14), so that the complete expression for \( \nu_{\text{me}}(\omega) \) around \( \omega_c = \omega \) for the full range of Coulomb effects is finally given by [7, 17]

\[
\nu_{\text{me}}^{(a)}(\omega) \simeq \nu_R(\omega) + \nu_{\text{NR}}, \quad (4.16)
\]

\[
\nu_{\text{NR}} = \frac{\sqrt{\pi} \Gamma_0^2}{4 \hbar^2 \omega_c} A \left( \Gamma_C \sqrt{\Gamma_0 \hbar \omega_c} \right),
\]

with the approximation \( A(x) \simeq 3.2 x^{-0.94} \exp[-(0.3/x)^3] \). For \( \Gamma_C \ll \hbar \omega_c \), the contribution of \( \nu_{\text{NR}} \) is negligible compared to \( \nu_R(\omega) \), but for \( \Gamma_C \gtrsim \hbar \omega_c \), it becomes dominant and leads to a strong re-increase of the CR linewidth (see also Fig. 4.8).

**CR linewidth**

The CR linewidth \( \gamma_{\text{CR}} \) is a convenient single parameter for a quantitative comparison between theory and experiment, and in the balance equation/memory function formalism (Sec. 3.3.1) generally given by the collision rate \( \nu_{\text{eff}}(\omega) \) at \( \omega = \omega_c \) alone: \( \gamma_{\text{CR}}^{(\text{MF})} = 2 \nu_{\text{eff}}(\omega_c) \).

The many-electron linewidth due to vapor atom scattering in the quantum limit is directly obtained from Eq. (4.16). The restriction to \( \omega = \omega_c \) also allows a stricter evaluation of the resonant contribution \( \nu_R(\omega) \) including the dependence \( \Gamma_N(E_f) \) in the average Eq. (4.9) for the many-electron DSF [17], but usually Eq. (4.14) is quite sufficient.

The many-electron AC effective collision rate due to ripplon scattering \( \nu_{\text{me}}^{(\ell)}(\omega) \) is more difficult to calculate due to the complicated form of the interaction, but behaves similarly to that of vapor atom scattering. In our relevant parameter range (\( T > 1 \) K), ripplon scattering mainly gives a contribution at high \( n_s \), where \( \Gamma_{\text{se}} \ll \Gamma_C \) and the Landau level broadening can be neglected in the DSF. Using thus Eq. (4.12) with \( \Gamma_{0,N} = 0 \) for the
many-electron DSF in the quantum limit, one can find the following expression for the ripplon contribution to the CR linewidth at $\omega = \omega_c$ [7, 77]:

$$
\nu^{(r)}_{\text{me}}(\omega_c) = \frac{(eE \perp)^2 k_B T}{4\sqrt{\pi} \hbar \alpha_{He} \Gamma_C} \sum_{N=0}^{\infty} \frac{1}{N!} F_N(\hbar \omega_c | N - 1 | / \Gamma_C),
$$

(4.17)

$$
F_N(y) = \int_0^\infty x^{N-1/2} V_S(x) \exp(-x - y^2/x) dx,
$$

$$
V_S(x) = 1 + \Lambda x eE \perp l_B^2 B \omega_c^2 x^2 \gamma^2 l_B^2,
$$

where the function $w(y)$ is the same as in Eq. (2.9). Equation (4.17) contains also a resonant contribution (term with $N = 1$) and nonresonant ones (terms with $N \neq 1$), the first is dominant at low $\Gamma_C \ll \hbar \omega_c$, it decreases with increasing $\Gamma_C$, and the latter become then relevant at $\Gamma_C \gtrsim \hbar \omega_c$, the same as in vapor atom scattering.

**Figure 4.8:** The influence of Coulomb effects on the behavior of the CR linewidth $\gamma_{\text{CR}}$ vs. electron density $n_s$ according to the balance equation/memory function theory [Eq. (4.18), solid curve], and according to Ando CR theory [Eq. (4.19), dashed curve]. The circles mark the values of $n_s$ corresponding to the three cases (A), (B), and (C) of the preceding Fig. 4.7 and the discussion in the text.

Equation (4.17) is to be included at high enough $\Gamma_C \gg \Gamma_{\text{se}}$ (high $n_s$) only, it diverges at low $\Gamma_C$ due to the neglect of the Landau level broadening. The total many-electron CR linewidth (full width at half height) in the balance equation/memory function formalism due to vapor atom and ripplon scattering together is then

$$
\gamma_{\text{CR}}^{(MF)} = 2 \nu^{(a)}_{\text{me}}(\omega_c) + 2 \nu^{(r)}_{\text{me}}(\omega_c),
$$

(4.18)

with $\nu^{(a)}_{\text{me}}(\omega_c)$ from Eq. (4.16) and $\nu^{(r)}_{\text{me}}(\omega_c)$ from Eq. (4.17). The density dependence of $\gamma_{\text{CR}}^{(MF)}$ is shown in Fig. 4.8. One easily recognizes the three characteristic regimes discussed already in more detail above:

(A) At low $n_s$ ($n_s \lesssim 10^7 \text{ cm}^{-2}$ in Fig. 4.8), one is in the noninteracting, single-electron limit with $\Gamma_C \ll \Gamma_{\text{se}}$. All electrons have the same excitation spectrum and contribute equally to scattering at $\omega = \omega_c$. For $T > 1 \text{ K}$, one can here neglect ripplon scattering, and the CR linewidth is then $\gamma_{\text{CR}}^{(MF)} \simeq \sqrt{\pi \Gamma_a / \hbar}$ [Eq. (4.13)].

(B) Above $n_s \sim 10^7 \text{ cm}^{-2}$ in Fig. 4.8, the regime of Coulomb effects begins with $\hbar \omega_c \gg \Gamma_C \gtrsim \Gamma_{\text{se}}$, here the CR linewidth shows strong Coulomb narrowing vs. $n_s$. The Doppler shifts of the individual electrons’ excitation spectra result now in only a part of them contributing to scattering at $\omega = \omega_c$, which reduces the resonant contributions both of vapor atom and ripplon scattering to $\gamma_{\text{CR}}^{(MF)}$. 
(C) At high \(n_s (n_s \gtrsim 3 \times 10^8 \text{ cm}^{-2})\) in Fig. 4.8, the interaction is so strong that \(\Gamma_C \gtrsim \hbar \omega_c\), and the Coulomb narrowing of the CR linewidth reverses into a strong Coulomb broadening. The Doppler shifts are here strong enough to let the electrons scatter to higher Landau levels, and the nonresonant terms become important in \(\gamma_{\text{CR}}^{(MF)}\).

### Ando CR theory with density-dependent Landau level width

It is instructive to compare the above results of the memory function formalism with the simpler Ando CR theory (Sec. 3.2.1), where the CR linewidth in the quantum limit is simply given by the average width of ground and first excited Landau level,

\[
\gamma_{\text{CR}}^{(Ando)} = 2 \sqrt{\ln 2} \Gamma_{0,1}/\hbar. \tag{4.19}
\]

If in \(\Gamma_{0,1}\) is included vapor atom and ripplon scattering and the Coulomb level narrowing as for Fig. 4.5 in Sec. 4.3, \(\gamma_{\text{CR}}^{(Ando)}\) has indeed a similar behavior vs. \(n_s\) as the linewidth \(\gamma_{\text{CR}}^{(MF)}\) from memory function theory, as shown in Fig. 4.8. Still, a significant quantitative difference appears at high \(n_s\) in the regime (C), where \(\Gamma_C \gtrsim \hbar \omega_c\), here \(\gamma_{\text{CR}}^{(MF)}\) becomes significantly larger than \(\gamma_{\text{CR}}^{(Ando)}\) (up to 50% in Fig. 4.8).

This difference can be attributed to the memory function theory taking also into account the increase of the typical momentum exchange \(q\) at scattering to higher Landau levels, as discussed on page 61. Physically, the Landau level width can be regarded as a measure of the number of collisions alone, while the effective momentum collision rate of memory function theory also takes into account their efficiency for momentum relaxation.

Moreover, in Ando CR theory, a strict calculation of the many-electron CR line shape, i.e., Landau level shape, is rather difficult, as discussed in Sec. 4.3. This is not a problem in memory function theory, which predicts a change from a Gaussian at low \(n_s\) to a pure Lorentzian at high \(n_s\). Generally, the balance equation/memory function method should give a better description of an interacting system, as discussed at the beginning of chapter 3. This is confirmed in the comparison with the experimental results later on.

### 4.5.2 DC magnetoconductivity

In the balance equation method, the DC case is formally essentially just the limit \(\omega \to 0\) \((\omega \ll \nu_{\text{eff}})\) of the general AC expressions (see Sec. 3.3.2). In a magnetic field \(B\), the DC conductivity \(\sigma_{xx}(B)\) is given by the quasi-Drude expression of Eq. (3.16), with a DC effective momentum collision rate \(\nu_{\text{eff}}(B)\) [Eq. (3.17)] determined by the DSF at \(\omega = 0\).

Like in the AC case, Coulomb effects in DC transport are described by inserting the many-electron DSF from Sec. 4.4 in Eq. (3.17) to calculate the many-electron DC collision rate \(\nu_{\text{me}}(B)\). The effects are basically the same as for the CR linewidth, discussed in Sec. 4.5.1. This is already seen from the formal similarity of the general expressions for \(\nu_{\text{eff}}(B)\) [Eq. (3.17)] and the CR linewidth \(\gamma_{\text{CR}}^{(MF)} = 2\nu_{\text{eff}}(\omega_c)\) [Eq. (3.14)]: In both cases, the DSF is integrated over \(q\) along the peak of one of its maxima, only for \(\gamma_{\text{CR}}^{(MF)}\), it is the maximum at \(\omega = \omega_c\), and for \(\nu_{\text{eff}}(B)\), the one at \(\omega = 0\). It is then clear that, apart from numerical factors, \(\gamma_{\text{CR}}^{(MF)}\) and \(\nu_{\text{eff}}(B)\) are affected in the same way by the broadening and reduction of the maxima in the many-electron DSF.

For vapor scattering and in the quantum limit \(\hbar \omega_c > k_B T\), the result for \(\nu_{\text{me}}(B)\) can
be presented as a correction to the single-electron collision rate $\nu_{\text{se}}(B)$ of Eq. (3.18) [18],

$$\frac{\nu_{\text{me}}^{(a)}(B)}{\nu_{\text{se}}^{(a)}(B)} = \left\langle \frac{\Gamma_{\text{se}} \left[ \Gamma_0^2 + (eE_t I_B)^2 \right]}{\left[ \Gamma_0^2 + 2 (eE_t I_B)^2 \right]^{3/2}} \right\rangle_f + \left\langle \frac{\Gamma_{\text{se}}}{\sqrt{2eE_t I_B}} D_0 \left( \frac{\hbar \omega_c}{\sqrt{2eE_t I_B}} \right) \right\rangle_f,$$  

(4.20)

where $\langle \rangle_f$ denotes an average over the Gaussian distribution of $E_t$ [Eq. (4.3)], and where \( \Gamma_0 = \Gamma_0(E_t) \) is given by Eq. (4.7). The function $D_0(x)$ is analytically approximated by $D_0(x) \simeq (x^4 + x^2 - 0.05 + 3.5/x^2) \exp(-x^2)$.

Fig. 4.9 shows the $n_s$-dependence of $\nu_{\text{me}}(B)$ at fixed $B$ in the quantum limit. It is qualitatively the same as that of $\gamma_{\text{CR}}^{(\text{MF})}$ in Fig. 4.8, with the same three typical regimes encountered at increase of $n_s$, i.e., increase of the parameter $\Gamma_C = \sqrt{2eE_t^{(0)} I_B} \propto n_s^{3/4}$.

(A) At low $n_s$ ($n_s \lesssim 10^7 \text{ cm}^{-2}$ in Fig. 4.9), one is in the noninteracting regime with $\Gamma_C \ll \Gamma_{\text{se}}$. The many-electron DSF reduces to the single-electron one, $S_{\text{se}}(q, \omega)$ [Eq. (2.35)], same as the DC collision rate, $\nu_{\text{me}}(B) = \nu_{\text{se}}(B)$, where $\nu_{\text{se}}(B)$ is determined by the $N = 0$ term in $S_{\text{se}}(q, \omega)$ (maximum at $\omega = 0$), which corresponds to SE transitions inside Landau levels. All electrons have the same excitation spectrum and contribute equally to scattering at $\omega = 0$.

(B) Above $n_s \sim 10^7 \text{ cm}^{-2}$, Coulomb effects begin with $\hbar \omega_c \gg \Gamma_C \gtrsim \Gamma_{\text{se}}$. The Doppler shifts of the individual electrons’ excitation spectra result now in only part of them contributing to scattering at $\omega = 0$; this corresponds to the reduction of the $N = 0$ term in the many-electron DSF Eq. (4.12) and leads to a reduction of $\nu_{\text{me}}(B)$.

(C) At high $n_s$ ($n_s \gtrsim 3 \times 10^8 \text{ cm}^{-2}$), the interaction is so strong that $\Gamma_C \gtrsim \hbar \omega_c$. The Doppler shifts are now strong enough to allow the electrons to scatter to higher Landau levels, which corresponds to an increasing influence at $\omega = 0$ of higher-order terms with $N > 0$ in the many-electron DSF, and $\nu_{\text{me}}(B)$ increases again.

**Figure 4.9:** The many-electron DC effective momentum collision frequency $\nu_{\text{me}}^{(a)}(B)$ [Eqs. (4.20) or (4.21)], solid curve for vapor atom scattering vs. saturation density $n_s$ at fixed magnetic field $B$ in the quantum limit. Coulomb effects act on $\nu_{\text{me}}^{(a)}(B)$ essentially in the same way as on the CR linewidth shown in Fig. 4.8. Also shown is the single-electron result $\nu_{\text{se}}^{(a)}(B)$ [Eq. (3.18)], dashed curve which is only influenced by holding field effects.

**Magnetic field dependence**

The behavior of $\nu_{\text{eff}}(B)$ vs. $n_s$ is usually not as unambiguously obtained from the experiment as the CR linewidth, therefore Coulomb effects are studied in the DC case rather in the behavior of $\nu_{\text{eff}}(B)$ or rather $\sigma_{xx}(B)$ vs. magnetic field $B$ at a fixed value of $n_s$. 
In contrast to the AC case, where the experimental investigation is usually limited to a rather small range of $B$, the comparatively weak $B$-dependence of $\nu_{\text{eff}}(B)$ and $\sigma_{xx}(B)$ makes it easy to investigate them over a very large range of $B$.

The important point is that at large changes of $B$, the relative strength of Coulomb interaction varies due to the different $B$-dependencies of the characteristic parameters: $\Gamma_C = \sqrt{2} e F_l^{(0)} T_B \propto 1/\sqrt{B}$ decreases with $B$, while both $\Gamma_{se} \propto \sqrt{B}$ (for vapor scattering) and $\hbar \omega_c \propto B$ increase with $B$. Thus, at high enough $B$, one is in the noninteracting limit $\Gamma_C \ll \Gamma_{se}$, and subsequent reduction of $B$ gives rise to Coulomb effects, with the same succession of typical regimes (see below) as at increase of $n_s$ at fixed $B$, discussed above.

One can obtain an approximation for the many-electron effective collision rate in the whole quantum regime $\omega_c > \nu_a$ by combining $\nu_{se}(B)$ from Eq. (3.18) with the quantum limit correction Eq. (4.20). A simple analytic interpolation formula for the result is

$$
\nu_{\text{me}}^{(a)}(B) \approx \frac{\sqrt{\pi} \omega_c \Gamma_a^{2}}{4 \Gamma_s k_B T} \left[ 1 + \frac{\sqrt{\pi} \hbar \omega_c / \Gamma_s}{\exp[(\hbar \omega_c / \Gamma_s)^2]-1} \right] \exp \left[ -\left( \frac{\Gamma_{se}}{4k_B T} \right)^2 \right] \coth \left( \frac{\hbar \omega_c}{2k_B T} \right),
$$

where $\Gamma_s \equiv \sqrt{\Gamma_{se}^2 + b^2 \Gamma_C^2}$ with the numerical parameter $b = 2/\sqrt{\pi}$. It should be pointed out that this approximation may be unsatisfactory at low $B$, when higher Landau levels are occupied, as Eq. (4.20) is derived for the quantum limit. For instance, SE in higher levels have a higher typical value of the momentum exchange at scattering; this leads to a faster development of Coulomb effects, as already noted on page 54. The $B$-dependence of $\nu_{\text{me}}(B)$ should therefore be studied at high enough $n_s$, so that Coulomb effects appear well in the quantum limit; this is assumed in the following discussion. Also, one should compare $\nu_{\text{me}}(B)$ to the single-electron result $\nu_{se}(B)$ to exclude the strong $B$-dependence that is not due to Coulomb effects, but due to the changes in $\Gamma_{se}$ and $\hbar \omega_c$ alone.

For illustration, Fig. 4.10 shows graphs of $\nu_{\text{me}}^{(a)}(B)$ and $\nu_{\text{me}}^{(a)}(B)$ vs. magnetic field $B$, and also graphs of the ratio $\omega_c / \nu_{\text{eff}}(B) = \tan(\varphi_H)$ ($\varphi_H$ is the Hall angle) and graphs of $\sigma_{xx}^{-1}(B)$ for the both cases. At our experimental conditions, the high-cyclotron-frequency condition $\omega_c \gg \nu_{\text{eff}}$ is usually fulfilled, so that approximately $\sigma_{xx}^{-1} \propto \omega_c^2 / \nu_{\text{eff}}$. For a direct comparison, Fig. 4.11 shows a graph of the ratio $\nu_{\text{me}}^{(a)}(B) / \nu_{\text{se}}^{(a)}(B)$ vs. $B$. The same regimes that occur vs. increasing $n_s$ at fixed $B$, discussed above, now occur vs. decreasing $B$:

(A) At high $B$, one is in the single-electron limit $\Gamma_C \ll \Gamma_{se}$, and $\nu_{\text{me}}(B) = \nu_{se}(B)$.

(B) At subsequent decrease of $B$, but still in the quantum limit in Figs. 4.10 and 4.11, Coulomb effects start with $\hbar \omega_c \gg \Gamma_C \gg \Gamma_{se}$. This corresponds to case (B) in the $n_s$-dependence discussed above. The reduction of the $N=0$ term in the many-electron DSF Eq. (4.12) results in the ratio $\nu_{\text{me}}(B)/\nu_{se}(B)$ in Fig. 4.11 becoming increasingly smaller with decreasing $B$, and the many-electron $\varphi_H$ and $\sigma_{xx}^{-1}$ in Fig. 4.10 deviate from the respective single-electron results towards higher values.

(C) At still lower $B$ ($B \lesssim 1$ T in Figs. 4.10 and 4.11), one finally has $\Gamma_C \gg \hbar \omega_c$, and the higher-order terms in the many-electron DSF lead to a strong re-increase of the ratio $\nu_{\text{me}}(B)/\nu_{se}(B)$ vs. decreasing $B$, corresponding to case (C) in the $n_s$-dependence. Interestingly, this almost exactly compensates the $B$-dependence of $\nu_{se}(B)$ and results in $\nu_{\text{me}}(B)$ itself varying very little in this regime. Numerically, $\nu_{\text{me}}(B)$ is here close to the zero-field collision rate $\nu_a$ [Eq. (2.11)], and the many-electron $\varphi_H$ and $\sigma_{xx}^{-1}$ are close to the respective results of classical Drude theory.
It should be pointed out that the remarkable result that \( \nu_{\text{me}}(B) \) is almost constant in (C) does not mean that the system becomes classical. Each individual electron still has a well-defined Landau spectrum in its local frame down to a much lower value of \( B \), as can be seen from the graph of \( \Gamma_0/\hbar \omega_c \) vs. \( B \) in Fig. 4.11. The single-electron condition for the overlap of the Landau levels and the end of the quantum regime is \( \omega_c \lesssim \nu_a \) for vapor atom scattering; this corresponds to very low \( B \lesssim 0.05 \) T in Figs. 4.10 and 4.11, and Coulomb effects even narrow the levels in the local frames (see the graph of \( \Gamma_0/\Gamma_a \) in Fig. 4.11).

**Figure 4.10:** Coulomb effects on the DC magnetoconductivity in the vapor atom scattering regime: the top graph shows the \( B \)-dependencies of the single-electron and many-electron DC effective collision frequencies \( \nu_{\text{se}}(B) \) [Eq. (3.18), dashed curve] and \( \nu_{\text{me}}(B) \) [Eq. (4.21), solid curve]; the middle graph shows the ratio \( \omega_c/\nu_{\text{eff}}(B) = \tan(\phi_H) \) for \( \nu_{\text{se}}(B) \) and \( \nu_{\text{me}}(B) \) (\( \phi_H \) is the Hall angle); and the bottom graph shows the corresponding results for the inverse magnetoconductivity \( \sigma_{xx}^{-1}(B) \propto \omega_c^2/\nu_{\text{eff}}(B) \) [Eq. (3.16)], normalized to the zero-field conductivity \( \sigma_0 = e^2n_s/m_e\nu_a \).

**Figure 4.11:** Magnetic field dependencies of the ratio \( \nu_{\text{me}}(B)/\nu_{\text{se}}(B) \) [Eq. (4.20)], of the ratio \( \Gamma_0/\Gamma_a \) [Eq. (4.7) with \( E_f = E_f^{(0)} \)] with \( \Gamma_a \) being the ground Landau level width in single-electron theory and \( \Gamma_0 \) additionally including the Coulomb narrowing, and of the ratio \( \Gamma_0/\hbar \omega_c \).
Chapter 5

Previous work on Coulomb effects

In the following, a short overview is given of the earlier experimental studies and corresponding theoretical work on Coulomb effects. In cyclotron resonance, only two studies were performed prior to the present one, both in the ripplon scattering regime. Moreover, they appeared to disagree with each other, and only one of them could partially be explained by previous theories. Only in the DC magnetoconductivity, a conclusive picture emerged from previous experimental studies and theories, although the latter treat the transition from the noninteracting to the interacting case somewhat qualitatively.

5.1 Cyclotron resonance

Study by Edel’man

The first extensive study of cyclotron resonance in SE was performed by Edel’man more than two decades ago [8, 9]. Fig. 5.1 shows measurements of the CR half-width vs. holding electric field $E_\perp$ at saturation [i.e., the electron density was varied together with $E_\perp$ according to Eq. (1.7)], conducted at low temperature in the ripplon scattering regime, and in the quantum limit. The data show an interesting nonmonotonic dependence of the linewidth on $E_\perp$. At low $E_\perp$, the linewidth decreases with increasing $E_\perp$, this behavior was already suspected by Edel’man to originate from electron-electron interaction. At high $E_\perp$, the linewidth increases strongly with $E_\perp$, which can qualitatively be explained by holding field effects. However, as the measurements were conducted at saturation, it is difficult to distinguish between holding field effects and effects from electron-electron interaction. No adequate theory for CR in the quantum regime being developed at the time, the data were tentatively compared to a semi-classical calculation, corresponding to Eq. (2.17). This is clearly not justified for the experimental conditions, which correspond to the quantum limit, as was also realized by Edel’man, although some phenomenological agreement can be found. For the vapor atom scattering regime at 1.3 K, Edel’man only reported an almost linear increase of the linewidth with increasing $E_\perp$ [8].

Dykman CR theory

Dykman and Khazan [6, 10] realized the importance of the internal fluctuational electric field for the transport behavior of SE, and were the first to develop a theory for cyclotron resonance of strongly interacting SE. Their result for the CR linewidth due to vapor atom
Figure 5.1: Data of the quantum cyclotron resonance half-width $\Delta H$ vs. holding field $E_\perp$ at saturation; the figure is taken from Ref. [8]. At saturation, a field of $E_\perp = 100 \text{ V/cm}$ corresponds to an electron density of $n_s = 1.1 \times 10^8 \text{ cm}^{-2}$. The measurements were taken at a frequency of 18.5 GHz (resonant magnetic field 0.66 T) and at low temperature $T$ in the ripplon scattering regime. $T = 0.72 \text{ K};$ $T = 0.4 \text{ K}.$

scattering is

$$\gamma_{\text{CR}} \simeq \frac{3\pi \Gamma_n^2}{8h\Gamma_C} \propto \frac{1}{n_s^{3/4}},$$

which means that the interaction alone leads here only to a monotonic decrease of the linewidth. The result for ripplon scattering behaves similarly.

Fig. 5.2 shows the comparison between the theory of Dykman and Khazan and data of Edel’man from the previous figure. The decreasing region of the linewidth at low $E_\perp$ can be well described by the theory. However, in the increasing region at larger $E_\perp$, which according to the theory should be entirely due to holding field effects, significant discrepancies between theory and experiment start to appear. The theory also does not take into account the collision broadening of the Landau levels, and therefore diverges in the noninteracting limit $\Gamma_C \ll \Gamma_{\text{se}}$, rather than reproducing the single-electron result Eq. (3.7). However, in the ripplon scattering range, this is usually negligible, because the level broadening $\Gamma_r$ due to ripplons is extremely small.

Study by Wilen and Giannetta

Almost a decade later than Edel’man, Wilen and Giannetta [11] performed another CR study of SE that focussed specifically on interaction effects. They measured in the ripplon scattering regime, and varied the electron density independently of the holding electric
field to separate interaction effects from holding field effects. Their results surprisingly were rather different from those of Edel’man, although they covered a comparable density range, and in contradiction to the theory of Dykman and Khazan [6]. As can be seen in Fig. 5.3, there is no longer a decreasing region of the CR linewidth at all. Instead, it monotonically increases with density, even without holding field effects, while the theory of Dykman and Khazan predicts a monotonic decrease for this case.

Figure 5.2: Comparison of the theory of Dykman and Khazan (solid line) with the CR linewidth data at 0.4–0.43 K of Edel’man from the previous Fig. 5.1 (dashed line). The x-axis represents the holding electric field where $E_H^{(0)} \simeq 100 \text{ V/cm}$, and the y-axis represents the CR half-width in units of $\sim 0.5 \text{ Gauss}$. The figure is taken from Ref. [6].

Figure 5.3: Data of the CR linewidth vs. electron density $n_s$ in units of $10^8 \text{ cm}^{-2}$ at constant holding field in the ripplon scattering regime (figure taken from Ref. [11]). The measurements were taken at a frequency of 2 GHz (resonant magnetic field 0.071 T), a temperature of 0.062 K, and a holding field of 275 V/cm. The solid line shows the result of the theory of Dykman and Khazan [6].
5.2 DC magnetoconductivity

Study by Lea et al.

The most extensive and successful previous study of Coulomb effects focused exclusively on the DC magnetoconductivity. Lea et al. [12, 13] measured the DC magnetoconductivity up to 10 T both in the vapor atom and ripplon scattering regime, and found density-dependent deviations from the single-electron theory of Sec. 3.3.2, as shown in Fig. 5.4. The experimental behavior was in good agreement with a many-electron theory developed by Dykman et al. [14, 15], an extension of the earlier work on cyclotron resonance [6, 10]. The theory describes the electron system in its center of mass reference frame and considers the fluctuational electric field $E_f$ to give a continuous correction $eE_fX$ to the single-electron energy spectrum ($X$ is the electron coordinate along $E_f$). In a qualitative picture, this leads to a characteristic energy uncertainty $eE_f\lambda$ of the single electron, where $\lambda$ is the characteristic electron wavelength (thermal wavelength or magnetic length). At low magnetic field $B$, the experimental data in Fig. 5.4 can be seen to follow the classical Drude theory up to rather high $B$, this can be explained in the above picture by the energy uncertainty “washing out” the Landau quantization for $eE_f\lambda \gg \hbar \omega_c$. The theory disregards the collision broadening $\Gamma_{se}$ of the Landau levels, and can therefore describe magnetotransport in a strict sense only for $eE_f\lambda \gg \Gamma_{se}$, it becomes inadequate at high magnetic field, where eventually the opposite limit is reached (see line m in Fig. 5.4). To achieve good agreement with the experimental data, the many-electron result must therefore be combined with an appropriate single-electron theory using a qualitative self-consistent procedure, which is not unambiguous [12, 13].

**Figure 5.4:** Measurements of the longitudinal DC magnetoconductivity $\sigma_{xx}$ in the vapor scattering regime for two different electron densities (□, ○); the figure is taken from Ref. [13]. The single-electron theory for the experimental data (□) is shown as the dashed curve s. For the same data, the pure many-electron theory according to Refs. [14] and [15] is shown as the dash-dotted curve m. For both densities, the solid lines t and t1 show the results of the combination of pure many-electron theory with single-electron theory, and the solid lines d and d1 show the results of classical Drude theory.
Chapter 6

Description of the experiment

The employed experimental setup allows to measure the conductivity of surface electrons (SE) simultaneously at high frequency, 40–60 GHz, and at low frequency in the kHz range, equivalent to the AC and DC limits of the conductivity respectively, as the momentum collision rate of SE is typically a few GHz. The measurements were performed at temperatures around 1.3 K, i.e., in the vapor scattering regime, and in magnetic fields up to 10 T. The main focus was here on the AC conductivity in magnetic field, or cyclotron resonance (CR), where a combination of high measurement frequency and high sensitivity allowed to cover a more extended and a previously not investigated parameter range. The low-frequency part of the setup is generally an important help in the control of SE, even in a CR measurement. Moreover, it enables complementary measurements of the DC magnetoconductivity for direct comparison with the CR data.

In the following, first are described the features of the measurement cell containing the SE sample in Sec. 6.1, and the general setup and proceedings in an experiment in Sec. 6.2. More detailed descriptions of the specific methods for the AC and DC measurements and the respective signal analysis follow in Secs. 6.3 and 6.4 respectively. In both cases, one ultimately obtains the longitudinal conductivity in absolute units.

6.1 Measurement cell

The measurement cell, depicted in Fig. 6.1, consists of a metallic cavity in form of an upright cylinder of radius $a = 5.4$ mm, partly immersed in liquid helium. The liquid layer inside the cavity supports the SE.

For AC measurements, the cavity is used as a microwave (MW) resonator, operated in reflection for best resolution. By a small hole in the thin copper-beryllium plate forming its top part, the cavity is coupled to a rectangular waveguide about 1 m long, which leads up to the rest of the MW setup at room temperature. The coupling can be adjusted in-situ by means of a vertically movable teflon piece in the waveguide, this changes the effective size of the coupling hole [99]. The bottom of the cavity consists of a disc cut from raw circuit board with a 35 $\mu$m thick copper coating on the top side. This disc is mounted on a vertically movable support (plunger); this enables to tune the resonance frequency of the cavity in-situ from 40 to 60 GHz by adjusting its height $l$ between 7 and 3 mm. The cavity side wall is made of stainless steel with rather low conductivity to minimize heating by eddy currents in measurements with magnetic field modulation. Cavity side wall and top plate are thinly plated with silver and gold respectively, to
improve the surface conductivity and minimize losses of the MW field. Details of the MW technique and the data analysis are described in Sec. 6.3.

**Figure 6.1:** Schematic (top) and perspective (bottom) drawing of the low-temperature part of the experimental setup. The SE layer is located inside a metallic cavity in form of an upright cylinder. For measurements of the high-frequency conductivity, the cavity is used as a microwave resonator, tunable between 40 and 60 GHz by means of a vertically movable plunger. The bottom part of the cavity consists of an array of electrodes in circular Corbino geometry. Using the so-called Sommer-Tanner technique, low-frequency measurements are performed by measuring in the kHz range the admittance between two of the bottom electrodes, each of them coupling capacitively to the electron layer.

The DC conductivity of the SE is measured using the so-called Sommer-Tanner technique of capacitive coupling at low frequency [100]. For this, the copper top-layer of the circuit board disc forming the cavity bottom is divided into four sections in the form of a center disc (C) surrounded by three concentric rings (M, A, G), separated by narrow grooves (∼ 60 µm wide) cut with a special tool on a lathe (see Fig. 6.1). Each section is connected to a lead from below and forms thus a separate electrode. Between two of these bottom electrodes, usually the center disc C and the second concentric ring A, the admittance is measured in the kilohertz range. Each of these electrodes couples capacitively to the electron layer located above, and the low-frequency admittance gives information both on the density profile and the DC conductivity of the electron layer.
The first concentric ring $M$ is usually grounded to reduce direct cross-talk between $C$ and $A$. Details of the Sommer-Tanner technique and its analysis are described in Sec. 6.4.

Also, top plate and side wall of the cavity are separate parts, mounted electrically insulated and separately connected to leads. By applying DC potentials to the different parts of the cavity, one can adjust the electrostatic field configuration and the radial density profile of the SE layer. Generally, negative potentials are applied to top plate, side wall and the outermost bottom electrode $G$, the electron layer is then mainly confined to the region above the three inner bottom electrodes $C$, $M$, and $A$ held at DC ground, with a small AC voltage applied to $C$ for the admittance measurements.

The electrons are produced by thermal emission from the tungsten filament of a miniature light bulb without glass cover. The filament is positioned in a small opening at the upper edge of the cavity, where it least disturbs the MW electric field.

The cavity is surrounded by a capacitive level meter consisting of two concentric stainless steel cylinders separated by a gap of about 0.5 mm. By measuring the capacitance between inner and outer cylinder, the height of the helium liquid filling the gap is monitored. To calibrate the level meter against the liquid level in the cavity, the latter is separately monitored by measuring the capacitance between cavity top-plate and central bottom electrode $C$.

6.2 General setup and experimental procedure

Generally, all low-frequency admittances, between the Corbino electrodes, cavity top and bottom, and the level meter cylinders, are measured by the three-terminal method using shielded cables to reduce noise and stray capacitance as far as possible. For better accuracy, bridge configurations are used, usually an automatic capacitance bridge (Andeen-Hagerling AH2500A) for the cylindrical level meter, and a manual capacitance bridge (General Radio GR1616) together with a two-phase lock-in amplifier for the admittance between the Corbino electrodes or cavity top and bottom, which is more versatile.

For insertion into the cryostat, the measurement cell described in Sec. 6.1, together with its support structure and room-temperature connections, is fitted into a stainless steel tube about 35 mm wide and 1.5 m long and closed at the bottom (see Fig. 6.2). The insert tube is joined vacuum-tight to a room-temperature head piece which carries the sealed electrical and mechanical feedthroughs and waveguide flanges, and flanges for connection to a gas handling system. The tube is inserted into a $^4$He bath cryostat featuring a long narrow bottom part (tail) that fits into the room-temperature bore of a 10 T superconducting magnet. The magnet bore also contains a water-cooled copper coil surrounding the cryostat tail for CR measurements with magnetic field modulation. A separate modulation coil outside the cryostat is used because the CR lines are rather wide (up to 0.4 T), and modulation requires a large amplitude of more than hundred Gauss to be effective. This would lead to intolerable mechanical and temperature perturbations if an internal coil were used. Also, the modulation frequency must be kept as low as $\sim 5$ Hz, which is not optimal for suppression of $1/f$-noise; but nevertheless the noise is observed to increase at higher frequency, probably because a cavity partly filled with superfluid $^4$He is especially sensitive to mechanical perturbations and heating. Moreover, the helium level in the insert shows a behavior resembling fountain effect at higher modulation frequencies; this is another indication for heating by induced eddy currents.

The cryostat is pumped to a vapor pressure of a few millibar, which corresponds to
Figure 6.2: Schematic representation of the low-temperature setup, and a block diagram of the 40–60 GHz microwave spectrometer.

a $^4$He bath temperature $T$ of about 1.3 K. The pressure is then stabilized by a feedback loop: A Baratron capacitive pressure gauge measuring the vapor pressure is read out at a rate of about 10 Hz by a computer running a PID control program which in turn adjusts a stepper motor activated butterfly valve in the pumping line. This method is insensitive to magnetic fields and achieves a very homogeneous temperature in the whole cryostat which is also highly stable as the $^4$He vapor pressure depends exponentially on $T$. This is essential, as the large cross section of the $^4$He vapor column in the insert tube renders the liquid level in the cavity very sensitive to temperature or pressure changes. Additionally, the temperature is measured by a germanium resistor close to the cavity.

To adjust the helium level inside the cavity, $^4$He vapor is condensed into the insert tube; the rising liquid level is monitored by the change in capacitance $C_{\text{cyl}}$ of the cylindrical level meter (see Fig. 6.1). Simultaneously, the capacitance between central Corbino disc and cavity top plate is measured to determine the value of $C_{\text{cyl}}$ when liquid enters the cavity. The cylindrical level meter can be calibrated by determining this value in-situ for different positions of the cavity bottom, the cavity height being always known with high accuracy from the MW resonance frequency [Eq. (6.3)]. With the calibration, the liquid level $h$ inside the cavity is known in absolute values. Usually, $h$ is adjusted to a rather small value ($h \sim 0.7$ mm) to have a large capacitance between the SE layer and the Corbino electrodes; moreover, this seems to reduce surface oscillations of the liquid which generate noise in the MW dispersion signal (see Sec. 6.3.2). If the liquid surface is charged with SE, $h$ is somewhat reduced by electrostatic pressure (see Sec. 1.2).

To charge the helium surface to saturation, top plate, side walls, and outermost bottom electrode $G$ are set on the same negative DC potential $-V_{\text{dc}}$, while Corbino inner
6.3 Measurement of AC conductivity

6.3.1 The microwave spectrometer

The microwave (MW) setup (see Fig. 6.2) works continuously at frequencies \( f \) between 40 and 60 GHz, and employs a phase-sensitive heterodyne detection system that allows to simultaneously measure absorption and dispersion of a sample with high resolution. The two MW sources, main oscillator (MO) and local oscillator (LO), base on broadband 10–15 GHz yttrium iron garnet oscillators, whose frequency is quadrupled. The LO continuously follows the MO with a constant frequency offset of \(-300 \text{ MHz}\), this is achieved by a phase locked loop circuit. The MO frequency is measured by a MW frequency counter (XL Microwave 3201); this counter also features a source-locking option which employs a feedback loop to lock the MO frequency to a user-set value. The entire MW setup was originally built for measurements of electron spin resonance \(^2\) (ESR) \([99, 101]\).

A part of the output of both MW sources is mixed directly (Mixer A); this provides a reference signal at the intermediate frequency (IF) of 300 MHz. The rest of the MO output is directed to the resonator cavity containing the sample. The MW power \( P_{\text{MW}} \) reaching the cavity can be varied with a calibrated attenuator; it is about 1 mW (0 dBm) at 0 dB attenuation. At the cavity, a part of the incident wave is reflected back along the same waveguide. The reflected wave is separated from the incident one in a directional coupler and then mixed with the rest of the LO output in a second mixer (Mixer B); this gives the IF measurement signal at 300 MHz which is directly amplified in a low-noise amplifier attached to the mixer. IF reference and IF measurement signal are led to a two-channel phase sensitive detector, which separately outputs the in-phase (\( I \)) and out-of-phase (\( Q \)) components of the measurement signal with respect to the reference. The phase of the detection can be adjusted by a phase shifter in the IF reference path.

As the mixing conserves the phase information, the detector output in complex notation \((I + iQ)\) is directly proportional to the complex amplitude ratio of reflected wave to incident wave at a point in the waveguide near the cavity, the so-called complex reflection coefficient \( r_{\text{MW}} = r_{\text{MW}}^r + i r_{\text{MW}}^i \)[102], which is a convenient parameter for the signal analysis. At first, the normalization factor between \( I + iQ \) and \( r_{\text{MW}} \), which is generally complex, is unknown. It can be determined from the known behavior of \( r_{\text{MW}} \) vs. frequency \( \omega = 2\pi f \) around a resonance frequency \( \omega_0 \) of the cavity, the so-called

\[ r_{\text{MW}}(\omega) = r_{\text{MW}}^0(1 + iQ) \]

1An alternative is to apply a magnetic field near CR and monitor the increase of MW absorption. However, the signal is then proportional to the electron density \( n_s \) and generally too weak at low \( n_s \), and also stronger perturbed by the charging process, as it is sensitive to electrons in the whole cavity.

2For free electrons the ESR signal is smaller than the CR signal by a factor \( \hbar \omega / 2m_e c^2 \approx 4 \times 10^{-21} \text{ fs} \) for comparable filling factors and linewidths, and probably not observable in SE in the present setup.
cavity locus, shown in Fig. 6.3. Ideally, the locus describes a circle starting and ending at $r_{MW} = -1$, which corresponds to total reflection, for $\omega$ far below and above $\omega_0$, and with a minimum of $|r_{MW}|$ on the real axis at $\omega = \omega_0$, where the power dissipation inside the cavity is highest. The minimum depends on the coupling; the case $|r_{MW}| = 0$ at $\omega = \omega_0$ is called critical coupling. In practice, the cavity locus is usually somewhat distorted due to additional phase shifts and parasitic interferences between different parts of the MW setup, which also change with frequency. This introduces some uncertainty in the normalization, especially at high frequency because then the cavity locus covers a larger frequency range due to the lower Q-factor of the cavity [see Eq. (6.6)], and at high attenuation (low $P_{MW}$) because then the parasitic interferences are relatively larger.

Figure 6.3: The spectrometer output in terms of the complex reflection coefficient $r_{MW}$ at the cavity, the so-called Smith chart, in the usual convention $r'_{MW} = \text{Re} r_{MW}$ to the left and $r''_{MW} = \text{Im} r_{MW}$ downwards $[102]$. The dashed circle represents the signal of the empty cavity vs. frequency $\omega$ around a resonance frequency $\omega_0$, the so-called cavity locus, shown here for the case of critical coupling, when $r_{MW} = 0$ at $\omega = \omega_0$ (see text). Also shown is an example of a CR measurement on SE, performed with the frequency locked to $\omega = \omega_0$.

In the experiment, first the cavity is tuned to resonance at the desired frequency; then the coupling is adjusted to critical, which gives the best resolution for the sample measurements. Both can be done in-situ. Next, the detection phase is adjusted so that at $\omega = \omega_0$, the detector $I$ and $Q$ outputs separately contain the real and imaginary part of $r_{MW}$, and the measured data can directly be evaluated in terms of $r_{MW}$ (see Sec. 6.3.2). For the sample measurements, the MO frequency is locked to $\omega = \omega_0$, where the signal background is almost zero. An example of a CR measurement on SE is shown in Fig. 6.3.

Cavity resonance mode

In our case, the transverse electric resonance mode $TE_{011}$ is employed to measure CR of SE. It has cylindrical symmetry where only the azimuthal component $E\varphi$ of the electric field is nonzero and only depends on the radial and vertical coordinates $r$ and $z$ $[103]$:

$$E\varphi(r, z) = A_0 \sin\left(\frac{\pi z}{l}\right) J_1\left(\frac{pr}{a}\right).$$

(6.1)

Here, $l = 3–7$ mm and $a = 5.4$ mm are the cavity height and radius, $J_1$ is the first Bessel function, and $p \simeq 3.832$ is the first root of $J_1$. The mode pattern is shown in Fig. 6.4. The total energy stored in the cavity volume $V = \pi a^2 l$ is then

$$W_0 = \frac{\epsilon_0}{2} \int_V E\varphi(r, z)^2 \, dV = \frac{J_1^2(p) \epsilon_0}{2} A_0^2 V \simeq 0.0811 \frac{\epsilon_0}{2} A_0^2 V.$$  

(6.2)
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The resonance frequency $\omega_0 = 2\pi f_0$ of the $TE_{011}$ mode depends on the cavity dimensions according to the relation

$$\left(\frac{\omega_0}{c}\right)^2 = \left(\frac{\pi}{l}\right)^2 + \left(\frac{p}{a}\right)^2,$$

which is graphically represented in Fig. 6.5.

### 6.3.2 Signal analysis

If a sample is inside the cavity, it interacts with the standing electromagnetic wave of the resonance mode, and induces changes of the measured reflection coefficient $r_{MW}$. Inside the sample, the amplitude of the wave is damped, and its wavelength is changed. Both effects can be described by complex material parameters, as the complex electric or magnetic susceptibility, or the complex conductivity of the sample.

In CR of SE, the relevant parameter is the complex longitudinal conductivity $\sigma_{xx}$, where $\text{Re}\sigma_{xx}$ describes the damping and is also called the absorption, while $\text{Im}\sigma_{xx}$
describes the wavelength change and is called the dispersion\(^3\). In the following is outlined how \(\sigma_{xx}\) is obtained from the measured reflection coefficient \(r_{MW}\).

**Cavity admittance** If the SE induce only small changes of \(r_{MW}\), these are approximately proportional to changes of \(\sigma_{xx}^* = \text{Re}\sigma_{xx} - i\text{Im}\sigma_{xx}\). However, this is no longer true for large changes of \(r_{MW}\). The maximum value of \(\sigma_{xx}\) being proportional \(n_s/\nu_{\text{eff}}\) [cf. Eq. (3.13)], this may occur in the limit of low collision rate \(\nu_{\text{eff}}\) or high electron density \(n_s\), which is relevant for us because for the study of Coulomb effects, \(n_s\) is varied over two orders of magnitude. The spectrometer output does then no longer represent the true CR line shape, and it is necessary to use the exact relation as follows \(^{102}\): To describe the properties of the cavity (with or without sample), one better introduces its dimensionless or reduced complex admittance \(Y = Y' + iY''\), related to the reflection coefficient by

\[
   r_{MW} = \frac{1 - Y}{1 + Y}, \quad Y = \frac{1 - r_{MW}}{1 + r_{MW}} \tag{6.4}
\]

and it is actually \(Y\) that is linearly related to \(\sigma_{xx}^*\) [the exact relation is given in Eq. (6.9)] and must be calculated from the directly measured \(r_{MW}\) using Eq. (6.4) to obtain the correct CR line shape, an example is shown in Fig. 6.6. It should be mentioned that this is a possible source of error in measurements at high values of \(n_s/\nu_{\text{eff}}\), because the larger the changes of \(r_{MW}\), the more important is it to accurately normalize the spectrometer output to \(r_{MW}\), otherwise the nonlinearity is not entirely corrected by Eq. (6.4).

**Figure 6.6:** The CR signal from Fig. 6.3, measured at \(f = 40\,\text{GHz}\), plotted vs. magnetic field. Due to a high electron density \(n_s = 8 \times 10^8\,\text{cm}^{-2}\) and low temperature \(T = 1.28\,\text{K}\) (i.e., low collision rate), the signal is here rather large. The measured reflection coefficient \(r_{MW}\) (thin curves) is then no longer proportional to the conductivity \(\sigma_{xx}\); moreover, absorption \(\text{Re}\sigma_{xx}\) and dispersion \(\text{Im}\sigma_{xx}\) are mixed in \(r_{MW}\). To obtain the correct CR line shape, \(r_{MW}\) is transformed to the admittance \(Y\) (heavy curves) by Eq. (6.4). The transformation also removes noise from the absorption curve, this noise is due to oscillations of the \(^4\text{He}\) liquid and in principle only affects the dispersion.

**Background and noise effects** Apart from the nonlinearity of the spectrometer output discussed above, measurements with a high value of \(n_s/\nu_{\text{eff}}\) pose at first no special problems. In the opposite limit of low \(n_s\) and high \(\nu_{\text{eff}}\), where the measurement signal is very small, the output nonlinearity is negligible, but parasitic effects become important.

The spectrometer itself is slightly sensitive to magnetic field\(^4\), moreover, it also mea-

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\(^3\)One could also use a complex electric susceptibility, which also ultimately describes the interaction with the electric field, we will only use \(\sigma_{xx}\) for simplicity and connection with theory.

\(^4\)Probable reasons are conductivity changes in the low-temperature parts, mechanical deformation of magnetic parts, or stray-field influence on the room-temperature MW sources and detection system.
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Measures ESR signals of any residual contamination present on the inner surface of the cavity, where the magnetic MW field is highest. The rather broad CR lines of SE in the vapor scattering phase “collect” a lot of this magnetic-field-dependent background, which also varies most around the resonant field for CR of SE, coinciding with the resonant field for ESR with a $g$-factor of 2, an example is shown in Fig. 6.7. Fortunately, this kind of background is quite stable, it can be measured separately with high resolution at a higher MW power as for SE (which are rather sensitive to heating) and then be eliminated from the measured CR data by subtraction. As a reliability check, the background was usually measured both before and after a series of measurements with SE, with a positive voltage applied to the cavity walls and top-plate to remove all SE from the bulk $^4$He surface. This should also have revealed any effects due to accumulation of SE on the $^4$He film covering cavity walls and top-plate, which were never observed.

**Figure 6.7:** An example of a CR measurement at rather low $n_s = 4.6 \times 10^7$ cm$^{-2}$ ($f = 60$ GHz, $T = 1.28$ K). The SE signal is here very small, and the original data of $r'_\text{MW}$ and $r''_\text{MW}$ (topmost curves in both graphs) are strongly distorted by the background signal of the empty cavity. Measuring the background separately at higher power, i.e., better resolution (middle curves, $r''_\text{MW}$ is additionally smoothed by removing higher Fourier components), and subtracting it from the original data gives the signal of the SE alone (bottom curves, all curves are adjusted in offset for orientation). This gives reasonable accuracy at least for the absorption $r'_\text{MW}$. The dispersion $r''_\text{MW}$ is still strongly affected by oscillations and level drifts of the liquid, see also Fig. 6.8(c).

Another kind of background are time-dependent fluctuations and drift effects which only occur if liquid helium is inside the cavity, and only affect the dispersion, see Figs. 6.6, 6.7 and 6.8 for examples. It can be concluded that these are due to surface oscillations and level drifts of the liquid, which has a small but finite dielectric susceptibility. This noise cannot be removed entirely, only the fast oscillations can be reduced somewhat by time-averaging, but this is limited by the sweep rate of the magnetic field, which cannot be reduced without increasing the drift effects. However, in principle, this background noise only affects the dispersion, and even when at low $n_s$ the noise may be so large that the dispersion CR signal is unrecognizable (see Figs. 6.7 and 6.8(c) for examples), the absorption can then still be measured accurately. In practice, the dispersion noise will always be seen to some extent in the absorption, too, because it is rather difficult to adjust the detection phase in the experiment so precisely that absorption and dispersion are completely separated on the detector outputs. An error of about 5° can be enough to visibly affect the absorption if the noise is large compared to the signal. (This should be distinguished from the mixing of absorption and dispersion in 5°The electric field is practically zero at the cavity walls, apart from the skin effect, so it is unlikely to have any parasitic CR signal, like from SE on helium films.
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$r_{\text{MW}}$ due to the nonlinearity of the relation Eq. (6.4), visible in Fig. 6.6, which depends on the signal strength and is negligible at low $n_s$.

To deal with this problem, care is taken to sample the detector $I$ and $Q$ outputs almost simultaneously, so that the level oscillation noise in both is well correlated. The computer program used for data taking directly indicates the correlation and enables thus a rather good phase adjustment during the measurement itself. Moreover, an even better phase adjustment can then be performed afterwards on the measured data, the criterion being simply to minimize the noise in the absorption data. Thus, the level oscillation noise can be almost entirely removed from the absorption, moreover, this procedure ensures almost perfect separation of absorption and dispersion. In all figures presented here, the detection phase has been corrected by this procedure.

Magnetic field modulation  Even in the absorption, the resolution of a direct measurement of the detector output becomes unsatisfactory at very low $n_s$. The residual noise level is in this case mainly the intrinsic noise of the detector, which means that the signal/noise ratio could in principle be improved by a stronger MW amplitude, but the sensitivity of SE to heating forbid this for equilibrium measurements. The latter are usually conducted at 60 dB attenuation, i.e., about $-60$ dBm (1 nW) incident MW power at the cavity. The resolution is then usually still high enough to obtain the CR linewidth with sufficient accuracy down to the noninteracting limit at low $n_s$, but for a detailed study of the CR line shape, a higher resolution is desirable.

For this reason, a few measurements were performed with magnetic field modulation and lock-in detection, using an external modulation coil as described in Sec. 6.2. Fig. 6.8(a) shows an example of measured modulation data, which are proportional to $-dr'_{\text{MW}}/dB$ and $-dr''_{\text{MW}}/dB$ (of course, one must here also correct the detection phase and separately measure and subtract the background signal, this has already been done for all data shown in Fig. 6.8). For evaluation and comparison with theory, the measured data are numerically integrated, the result is shown in Fig. 6.8(b). To avoid line shape distortions due to overmodulation, the modulation amplitude is always kept below a rms value of $10^{-2}$ T, which is below 10\% of the minimum CR linewidth ($\sim$ 0.1 T) obtained in any of the measurements. Occasionally, spurious peaks occur in the modulation data, mainly in the dispersion [an example is indicated by the thick arrow in Fig. 6.8(a)]. They are disregarded in the integration, i.e., replaced by a linear interpolation.

The direct part of the detector output is always simultaneously measured, too, and also shown in Fig. 6.8(b). The direct data are always necessary for the phase correction (the modulation data are essentially unaffected by the liquid surface oscillations), moreover, as long as $n_s$ is sufficiently high to still have a good resolution in the direct data, too, they serve as an additional check for the reliability of the measurement and the integration, and to easily obtain the normalization for the integrated modulation data without exact knowledge of the modulation amplitude. The latter can be determined only approximately in an independent way. The noise increase in the direct signal due to the modulation was negligible compared to the intrinsic detector noise in our case.

The gain in resolution due to the modulation can be appreciated from Fig. 6.8(c), where the integrated modulation signal is compared to the simultaneously measured direct signal for the case of a very low $n_s$. 
Quantitative analysis of the cavity admittance

If one is only interested in the CR line shape and the CR linewidth, one only needs data that are proportional to the conductivity $\sigma_{xx}$. Obtaining the admittance $Y$ by the methods described above is then in principle sufficient.

In our case, however, a more detailed analysis is necessary, for instance for the evaluation of the heating experiments. This also enables to obtain $\sigma_{xx}$ in absolute values, and to assess the reliability of the measurement and of the signal analysis.

An exact analysis of cavity plus sample as a MW circuit element bases on the Poynting theorem, it has been performed for instance in Ref. [102]. As in Ref. [102] the focus is on ESR, the final expressions are expressed in terms of the magnetic susceptibility only, but from the general treatment, it is straightforward to reformulate them in terms of $\sigma_{xx}$, so only the results will be given in the following.

Cavity Q-factor  For a detailed analysis, it is first necessary to characterize the empty cavity without sample. If the MW frequency $\omega = 2\pi f$ is close to the frequency $\omega_0$ of a distinct resonance of the cavity, the admittance $Y_0$ of the empty cavity is given by

$$Y_0 = \frac{Q_c^{-1}}{Q_0^{-1}} - i \frac{2(\omega - \omega_0)/\omega_0}{Q_c^{-1}}. \quad (6.5)$$

Here, $Q_0$ is the quality (Q-) factor of the closed cavity (i.e., without coupling), defined as $2\pi \times$ (energy $W_0$ stored in the cavity volume)/(energy lost by dissipation in the cavity
walls during one cycle). $Q_e$ is the external or coupling Q-factor, defined analog to $Q_0$ replacing dissipation losses by radiation losses through the coupling hole. (The total Q-factor of the empty cavity is then $Q_{\text{tot}}^{-1} = Q_0^{-1} + Q_e^{-1}$.)

Eqs. (6.5) and (6.4) correspond to $r_{\text{MW}}$ vs. $\omega$ describing a circle in the Smith chart, as shown in Fig. 6.3 for the case $Q_0 = Q_e$, i.e., critical coupling with $r_{\text{MW}} = 0$ and $Y_0 = 1$ at $\omega = \omega_0$. Critical coupling is always assumed in the following, and the only parameters to determine are then the resonance frequency $\omega_0$, which is trivial, and $Q_e$, which is obtained by fitting a Lorentzian to measured values of $|r_{\text{MW}}|^2$ vs. $\omega$, using the relation

$$1 - |r_{\text{MW}}|^2 = \frac{Q_0^2}{(\omega - \omega_0)^2 + \omega_0^2}.$$  

(6.6)

derived from Eqs. (6.5) and (6.4) for critical coupling $Q_0 = Q_e$. A typical value is $Q_e \sim 13000$ at 40 GHz when the ratio between volume and inner surface of the cavity is comparatively high, the Q-factor decreases significantly towards higher frequencies.

**Mean conductivity and filling factor** To describe the influence of a sample with a finite conductivity on the resonance mode, one introduces a mean conductivity $\langle \sigma \rangle$, averaged over the total cavity volume $V$ according to

$$\frac{\int_V J \cdot E_0 \, dV}{\int_V E \cdot E_0 \, dV} \equiv \langle \sigma \rangle. \tag{6.7}$$

Here, $J$ is the 3D current density inside the sample, $E_0$ is the electric field of the ideal, lossless $TE_{011}$ resonance mode of the cavity, and $E$ is the actual electric field inside the cavity with sample. In the following, it is always assumed that the pattern of $E$ is close to that of the ideal mode $E_0$. The conditions for this are discussed further below.

Eq. (6.7) must be evaluated for a layer of SE at height $h$ horizontally extending over a circular area $S$ with radius $r_S \leq a$. The 3D current density $J$ can be formally related to the 2D one $j$ inside the SE layer by setting $J = \delta(z - h)j$. Obviously, only the longitudinal current component $j_\parallel$ parallel to the in-plane component of $E_0$ contributes to $\langle \sigma \rangle$. For $E$ and $E_0$ given by the $TE_{011}$ mode [Eq. (6.1)] and assuming a homogeneous conductivity $\sigma_{xx}$ of the SE layer, one has $j_\parallel = \sigma_{xx} E_\varphi$. Insertion into Eq. (6.7) finally gives \[103\]

$$\langle \sigma \rangle = \sigma_{xx} \phi, \quad \phi = \frac{\int_S E_0^2 \, dS}{\int_V E_0^2 \, dV} = 2 \frac{\sin^2 \left( \frac{\pi h}{l} \right)}{\left( \frac{\pi h}{l} \right)} \frac{\phi_r \left( \frac{r_S}{a} \right)}{\phi_r \left( \frac{r_S}{a} \right)}, \tag{6.8}$$

$$\phi_r(y) = \frac{y^2 \left( J_0^2(y) - \frac{2}{\pi} J_0(y) J_1(y) + J_1^2(y) \right)}{p^2 J_0^2(p)},$$

where $\phi$ is called the filling factor. For a 2D sample, it has the dimension $m^{-1}$. All parameters entering $\phi$ can be determined with good accuracy, the cavity height $l$ from its resonance frequency $f_0$, the $^4$He level $h$ as described in Sec. 6.2, and the inner radius $r_G \simeq 4.4$ mm of the bottom electrode $G$ gives an estimate for the mean SE pool radius and thus for $r_S$. We typically have $h \sim 0.7$ mm, which gives $\phi \sim 20$ $m^{-1}$ at $f_0 = 40$ GHz.

With the definition of $\langle \sigma \rangle$, one obtains from Ref. \[102\] the following relation for the total admittance of the cavity with sample:

$$Y = Y_0 + Q_e \frac{\langle \sigma \rangle^*}{\varepsilon_0 \omega}. \tag{6.9}$$

\[6\]The mean pool radius is to be distinguished from the radius of the pool edge introduced in Sec. 6.4.2.
where \( Y_0 \) is the admittance of the empty cavity given by Eq. (6.5), and \( Q_e \) is the coupling Q-factor. Assuming critical coupling \( Q_e = Q_0 \), and also that the MW frequency is locked exactly to \( \omega = \omega_0 \), we have simply \( Y_0 = 1 \) for the CR measurements.

For a consistency check, consider, for example, the CR measurement from Fig. 6.6, which should correspond to saturation. The absorption curve has an almost perfect Lorentzian shape, which according to memory function theory (Sec. 3.3.1) implies a magnetic-field-independent effective momentum collision rate, and thus that the area of the curve is proportional to the electron density \( n_s \). Fitting a Lorentzian to \( R e Y \), one obtains from the area of the curve, the values of \( \phi, \omega \), and \( Q_e \) for this measurement, Eq. (6.9), and the memory function formula for \( R e \sigma_{xx} \) [Eq. (3.13)], a value of \( n_s = 7.9 \times 10^8 \text{ cm}^{-2} \). Independently, one obtains from the applied holding voltage the theoretical saturation value \( n_s = 8.3 \times 10^8 \text{ cm}^{-2} \). Both values differ only by 5%, which is surprisingly good, considering that the errors in the normalization of the detector output and in the determination of \( Q_e \) and \( \phi \) are each probably of the same order. Also, the actual \( n_s \) is indeed expected to be slightly below the theoretical saturation value (see Sec. 6.4.3).

One might still ask if the electrons can possibly significantly disturb the field pattern of the resonance mode, considering from Figs. 6.3 and 6.6 that at least the induced changes of \( r_{MW} \) can be comparatively large. In this case the filling factor would itself be a function of \( \sigma_{xx} \), and all of the signal analysis described so far would break down.

However, Eq. (6.9) shows that the typical scale of the changes of \( Y \) or \( r_{MW} \) depends mainly on the Q-factor \( Q_e = Q_0 \), which is a measure of sensitivity. If \( Q_e \) is high, even small changes of \( \sigma_{xx} \) will lead to large signals, without significantly affecting the mode pattern. The influence of the sample on the mode is rather given by the combination \( \langle \sigma \rangle^*/\epsilon_0 \omega \) entering Eq. (6.9). For instance, \( R e \langle \sigma \rangle/\epsilon_0 \omega \) is just \( 1/2\pi \) times the energy dissipated in the sample during one cycle divided by the total energy \( W_0 \), as can be seen from Eq. (6.7), i.e., it describes the sample-induced change of the cavity Q-factor. (The total Q-factor with sample is then \( Q_{tot}^{-1} = Q_0^{-1} + Q_e^{-1} + R e \langle \sigma \rangle/\epsilon_0 \omega \).)

Still, the sample may strongly disturb the mode locally without leading to large overall Q-factor changes, if only the filling factor is low enough. One can estimate the local field distortion by hypothetically placing the sample in the field maximum, i.e., letting it occupy the whole middle plane of the cavity so that \( h = l/2 \) and the filling factor [Eq. (6.8)] has the maximum value \( \phi = 2/l \). Together with the relation \( 2/\omega l \lesssim 1/c \), evident from Eq. (6.3), this finally leads to the dimensionless parameter \( \sigma_{xx}/\epsilon_0 c \) as a measure for the local sample-induced perturbation. The same parameter occurs in the treatment of a simplified one-dimensional resonator consisting of two parallel infinite metallic plates, with a 2D conducting sample occupying an intermediate plane [104]. This resonator model can be solved exactly for any value of \( \sigma_{xx} \), and leads to the same combination \( \sigma_{xx}/\epsilon_0 c \) as a measure for the strength of the coupling between electromagnetic field and sample, and for the local deviation from the unperturbed mode.

For the measurement from Figs. 6.3 and 6.6, one obtains at resonance a maximum value of \( \sigma_{xx}/\epsilon_0 c \simeq 3 \times 10^{-3} \), which is about the largest value ever reached in any of the measurements performed here. The local field perturbation is thus estimated to be always below 1%, and can be considered small.

**Power input and electric MW field acting on sample**

For complete characterization of the conditions the SE are subject to, it remains to quantify the strength of the electric MW field at the sample position and the MW power.
that is dissipated inside the sample. In the experiment, also the nonlinear conductivity behavior of SE was studied, where these are relevant parameters.

The parameter directly controlled is the energy flow $P_{\text{MW}}$ through the waveguide down to the cavity, which is adjusted by a calibrated attenuator (see Fig. 6.2) and is about 1 mW (0 dBm) near the cavity at 0 dB attenuation. The energy flow in the opposite direction is then $P_{\text{out}} = |r_{\text{MW}}|^2 P_{\text{MW}}$. From energy conservation one further has $P_{\text{MW}} = P_{\text{out}} + \omega Q_0^{-1} W_0 + \epsilon_0^{-1} \text{Re} \langle \sigma \rangle W_0$, where the two last terms on the right side are the power dissipated in the cavity walls and in the sample respectively [$W_0$ is the total energy stored in the cavity volume, see Eq. (6.2)]. Together with Eqs. (6.9) and (6.4), and a little manipulation, one derives the following two equations:

$$-2 \left( \text{Re} r_{\text{MW}} + |r_{\text{MW}}|^2 \right) P_{\text{MW}} = \epsilon_0^{-1} \text{Re} \langle \sigma \rangle W_0 \equiv P, \quad (6.10)$$
$$\left(1 + 2 \text{Re} r_{\text{MW}} + |r_{\text{MW}}|^2 \right) P_{\text{MW}} = Q_e^{-1} \omega W_0,$$

valid for for critical coupling $Q_0 = Q_e$. In the first equation, $P$ is the power dissipated in the sample [cf. Eq. (3.23)], which is thus immediately obtained from $P_{\text{MW}}$ and the measured signal $r_{\text{MW}}$. The significant parameter is here the dissipated power per electron, which for instance for the measurement of Fig. 6.6, conducted at $P_{\text{MW}} = -60$ dBm ($10^{-9}$ W), attains a maximum value of about $10^{-18}$ W in the middle of the resonance.

From the second of the above equations, together with Eqs. (6.1) and (6.2) relating $W_0$ to the MW electric field inside the cavity, one obtains the electric field strength acting on the sample. For rough estimates, one may neglect $r_{\text{MW}}$ on the left side, and assume $W_0$ with sample to be approximately the same as that of the empty cavity. A typical value for the maximum field in the SE layer [for its radial distribution see Eq. (6.1)] is about $3 \text{ V m}^{-1}$ at $h = 0.7 \text{ mm}$, $Q_e = 13,000$, $P_{\text{MW}} = -60$ dBm, and $f = f_0 = 40 \text{ GHz}$.

### 6.4 Measurement of DC conductivity

#### 6.4.1 The Sommer-Tanner method

As a direct electrical contact to the SE layer is not feasible, the DC conductivity of SE is measured using a technique of capacitive coupling at low frequency, first employed by Sommer and Tanner [100]. The present setup uses electrodes in circular Corbino geometry, as shown in Fig. 6.1. A small AC voltage $V_{\text{ac}}(t) \propto \exp(i \omega t)$ with $\omega$ in the kHz range is applied to the central electrode $C$ to drive the SE back and forth in radial direction. The resulting alternating perturbation of the SE layer’s radial density profile in turn couples to the outer electrode $A$, and leads to an AC current $I_{\text{ac}}$ flowing between $A$ and ground. The in-phase and out-of-phase components of $I_{\text{ac}}$ with respect to $V_{\text{ac}}$ are detected separately by a two-phase lock-in amplifier with a current preamplifier.

It is advantageous to use a bridge circuit to measure $I_{\text{ac}}$. The present setup uses a General Radio Model 1616 manual capacitance-conductance bridge employing fixed reference capacitances and conductances and a variable ratio transformer for the balancing. The bridge allows to directly determine the complex admittance $I_{\text{ac}}/V_{\text{ac}}$ between electrodes $C$ and $A$, which is a convenient parameter for the signal analysis.

Moreover, the bridge allows to exclude from the measurement offset currents originating from direct coupling between the electrodes and unshielded parts of the leads. The bridge is usually balanced to have $I_{\text{ac}} = 0$ for an uncharged $^4$He surface, and this is also assumed in the following analysis; i.e., $I_{\text{ac}}/V_{\text{ac}}$ is from now on always understood as the
6.4 Measurement of DC conductivity

measured change in admittance due to the presence of the SE with respect to that of the empty cell. It is convenient to write $I_{ac}/V_{ac} \equiv G_x + i\omega C_x$ with an equivalent conductance $G_x$ and capacitance $C_x$ separately describing the in-phase and out-of-phase components. This notation corresponds to a parallel equivalent circuit of $G_x$ and $C_x$, which is not really adequate here (see Sec. 6.4.2) but is still the most convenient presentation of the signal.

By comparison with the reference conductances and capacitances, the bridge allows to correctly adjust the detection phase and to calibrate the lock-in outputs directly in terms of $G_x$ and $C_x$. This also also reduces errors due to currents bypassing the lock-in detector, and avoids having to determine the rather small excitation voltage $V_{ac}$ (which typically has a rms value smaller than 70 mV).

In the Corbino layout, the in-plane driving electric field is radial, which in a vertical magnetic field results in a SE current that has both radial and azimuthal (Hall) components, but the Hall current is shortened and only the radial current leads to a change of the SE density. The measured change of admittance $G_x + i\omega C_x$ is therefore sensitive to the longitudinal conductivity $\sigma_{xx}$ of the SE layer only.

6.4.2 Signal analysis

Different methods exist for relating the measured Corbino admittance $I_{ac}/V_{ac} = G_x + i\omega C_x$ to the DC conductivity $\sigma_{xx}$ of the SE layer. A simple and phenomenological but instructive one is the equivalent circuit model, which in most cases yields $\sigma_{xx}$ with sufficient accuracy, apart from an unknown proportionality factor. Still, the equivalent circuit model treats the system in terms of lumped elements, and greatly simplifies the actual physical situation. A first improvement is, for example, to consider the system rather as a 2D transmission line with distributed capacitance and conductance [105]. However, a 2D approximation can only hold as long as the lateral size of the Corbino electrodes is large compared to their separation from the SE layer. This is usually not fulfilled in the present setup, where the width of the ring-shaped electrodes is 0.9 mm, while the $^4$He layer thickness is typically around 0.7 mm. For a more reliable analysis, and to determine $\sigma_{xx}$ in absolute units, we will therefore mainly use a numerical method, which takes into account the actual 3D electrostatic potential in the measurement cell.

The numerical method also provides a convenient way to control electron density and holding electric field, which is rather important for a systematic study of density-dependent Coulomb effects. This is discussed in detail in Sec. 6.4.3.

Equivalent circuit model

The equivalent circuit model [106] takes the admittance to be equivalent to that of a lumped series circuit of two capacitances $C_1$ and $C_2$ connected by a conductance $G_e$, as shown by the inset in Fig. 6.9. Here, $C_1$ and $C_2$ represent the capacitive coupling of the electrodes $C$ and $A$ to the respective areas of the SE layer located above them and are taken to depend on geometry only, while $G_e$ represents the resistance to the radial SE flow between the two areas and is taken to be also proportional to $\sigma_{xx}$. A direct capacitance between $C$ and $A$ is balanced by the bridge circuit, as described in Sec. 6.4.1, and thus excluded from the measurement.

With the total series capacitance $C_e \equiv (C_1^{-1} + C_2^{-1})^{-1}$, the impedance of this series RC circuit is then $V_{ac}/I_{ac} = G_e^{-1} + (i\omega C_e)^{-1}$. In contrast, the measured admittance
\[ I_{ac}/V_{ac} = G_x + i\omega C_x \] describes rather a parallel equivalent RC circuit. Comparing the two equations leads to the following relations:

\[
G_x + i\omega C_x = \frac{G_e^{-1} + i(\omega C_e)^{-1}}{G_e^{-2} + (\omega C_e)^{-2}} = \frac{\omega C_e}{G_e} + i\frac{\omega C_e}{G_e} \left( \frac{1}{(\omega C_e/G_e)^2 + 1} \right),
\]

\[
G_e^{-1} - i(\omega C_e)^{-1} = \frac{G_x - i\omega C_x}{G_e^2 + (\omega C_x)^2}.
\]

The first equation gives the dependence of the measured admittance change \( G_x + i\omega C_x \) on the parameter \( \omega C_e/G_e \propto \omega/\sigma_{xx} \); it is plotted in Fig. 6.9. For \( \omega C_e/G_e \ll 1 \), the admittance is equal to \( i\omega C_e \) and insensitive to changes of \( G_e \propto \sigma_{xx} \). On the other hand, for \( \omega C_e/G_e \gg 1 \), the admittance goes asymptotically to zero. For best resolution, one therefore adjusts \( \omega \) so that one measures \( \sigma_{xx} \) in the range \( 0 < \omega C_e/G_e < 1 \), which is equivalent to \( 0 < G_x/\omega C_x < 1 \). Also, a more precise analysis (see Fig. 6.11) indicates that the equivalent circuit model starts to fail for \( G_x/\omega C_x > 1 \), although the deviations appear here to be rather small.

![Figure 6.9: The Corbino admittance \( G_x + i\omega C_x \) according to the equivalent circuit model: The left graph shows the dependence of \( G_x \) and \( \omega C_x \) on the parameter \( \omega C_e/G_e \propto \omega/\sigma_{xx} \), and the right graph shows an Argand diagram plot with \( \omega C_e/G_e \) as implicit parameter. The circuit diagram is shown as inset in the left graph \( [C_e \equiv (C_1^{-1} + C_2^{-1})^{-1}] \). By the second equation in Eqs. (6.11), one obtains from the measured \( G_x + i\omega C_x \) the equivalent series conductance \( G_e \) and thus \( \sigma_{xx} \propto G_e \), apart from an unknown dimensionless proportionality factor. This factor can only be guessed at from the system geometry, and is usually fixed by comparison with conductivity theory.

**Numerical analysis in the Corbino geometry**

In following, the numerical method proposed by Wilen and Giannetta [107] is introduced, and applied to the geometry of the present setup. It consists of three steps: first, finding a general relation for the influence of the surface charge layer on the electrostatic potential inside the cell; secondly, using this relation to find the static equilibrium surface charge distribution; thirdly, calculating the time-dependent perturbation of the charge distribution due to the excitation voltage \( V_{ac}(t) \), which gives the admittance.
Potential calculation by the method of finite differences  
To describe the electrostatic potential $\Phi$ inside the cell in the presence of an arbitrary radial charge distribution, one conveniently uses discrete Green’s functions. In circular symmetry, the surface charge layer is approximated by a series of $N$ concentric charge rings of radii $r_i$, and one starts by separately calculating the contribution of each individual ring to the total potential with homogeneous boundary conditions, i.e., zero potential on the cell walls. The total potential of the charge layer as a whole is then obtained by superposition.

A solution of the Poisson equation for a single ring can be obtained in analytical form as an infinite sum of Bessel functions $[26]$. An alternative is to calculate the potential numerically by the method of finite differences $[107]$, which is somewhat more versatile. Here, the potential in $(r, z)$-space is calculated on a 2D grid of discrete values $r_i$ (taken to be equal to the charge ring radii in the following) and $z_j$ by an iterative procedure, called grid relaxation. For convenience, we use grid points that are evenly spaced by $dr$ in radial direction with $r_i = (i - 1/2) dr$ ($i = 1, ..., N$), while the vertical spacing is assumed to have different values $dz^+$ and $dz^-$ below and above the $^4$He surface for better adjustment to the liquid level. $N$ is typically around 80 for a sufficient resolution.

Applying Gauss’ law, $\int_S \varepsilon \nabla \Phi \cdot dS = -\varepsilon_0^{-1} \int_V \rho \, dV$, to an elementary grid cell of volume $V$ and surface $S$ enclosing a single grid point $(r_i, z_j)$ [the cell is a 3D ring lying in the $(r, z)$-plane with rectangular cross section], and writing the potential gradient in its finite difference form, one obtains an expression relating the potential $\Phi_{i,j}$ at point $(r_i, z_j)$ to the potentials at the four nearest neighbor points $(r_{i\pm1}, z_{j\pm1})$. For instance, for a grid point at the $^4$He surface with an area charge density $-en_i \equiv -en_i(r_i)$, one obtains

$$-\frac{-en_i \pi r_i dr}{\varepsilon_0} = \frac{\varepsilon^+ \Phi_{i+1,j} - \Phi_{i-1,j} + \varepsilon^- \Phi_{i,j+1} - \Phi_{i,j-1}}{2} 2\pi r_i dr + \frac{\varepsilon^+ dz^+ + \varepsilon^- dz^-}{2}$$

$$\times \left[ \frac{\Phi_{i+1,j} - \Phi_{i,j}}{dr} 2\pi \left( r_i + \frac{dr}{2} \right) + \frac{\Phi_{i,j+1} - \Phi_{i,j-1}}{dr} 2\pi \left( r_i - \frac{dr}{2} \right) \right].$$

Here, $\varepsilon^-$ and $\varepsilon^+$ are the dielectric constants of $^4$He liquid and vapor respectively. It is trivial to adapt the above expression to the other grid points without charge. For the numerical procedure, it is put in the form $\Phi_{i,j} = f(\Phi_{i-1,j}, \Phi_{i+1,j}, \Phi_{i,j-1}, \Phi_{i,j+1}).$

The calculation starts with essentially arbitrary initial values $\Phi_{i,j}$ for all interior grid points, while those at the cell walls are given by the boundary conditions, that is, $\Phi = 0$ for the homogeneous charge ring solutions. The grid is “relaxed” by repeatedly calculating for all interior grid points new potential values from their nearest neighbor potentials: $\Phi_{i,j}^{(\text{new})} = f(\Phi_{i-1,j}, \Phi_{i+1,j}, \Phi_{i,j-1}, \Phi_{i,j+1})$. One can speed up the convergence of the procedure by using the final solution for one ring to get good initial values for the next ring, and by using successive over-relaxation, where the potential values are updated as $\Phi_{i,j}^{(\text{new})} = w f(\Phi_{i-1,j}, \Phi_{i+1,j}, \Phi_{i,j-1}, \Phi_{i,j+1}) + (1 - w) \Phi_{i,j}$, with $1 < w < 2$.

Once the potentials are known, electric fields are obtained by differentiation. We will mainly need the radial field $E^{(r)}$ along the $^4$He surface generated by a given radial charge distribution described by the set $\{n_i\}_{i=1,...,N}$, where $n_i \equiv n_i(r_i)$. From superposition, the total value of $E^{(r)}$ at $r_i + dr/2$ may be written as $E_i^{(r)} = \sum_{i=1}^N G_{i,l}^{(r)} n_l$, where $G_{i,l}^{(r)}$ is the radial field at $r_i + dr/2$ due to a single ring of unit charge density at $r_l$.

The $N \times N$ matrix $G^{(r)}$ thus defined is a discrete Green’s function. Similarly, one can obtain another matrix $G^{(b)}$ enabling to calculate the vertical electric field $E^{(b)}$ along the cell bottom generated by a given radial charge distribution, to be used later on to calculate the admittance between the Corbino electrodes. Finally, for the evaluation
of experiments at strongly undersaturated conditions, one uses still another matrix to calculate the exact holding field across the SE layer.

**Static charge distribution** Once the $N \times N$ matrix $G^{(r)}$ for the radial field is calculated for a specific cell geometry, one can easily obtain the equilibrium radial charge distribution for an arbitrary set of DC potentials applied to the different parts of the cell from the requirement that the charge pool must be an equipotential surface. Here, thermal motion of the electrons can usually be neglected at temperatures of around 1 K, as the corresponding thermal electron energy is only about $10^{-4}$ eV.

The solution depends then only on the radius of the electron pool edge $r^{(p)}$. Let $\tilde{E}_{i}^{(r)}$ be the radial electric field at $r_{i} + dr/2$ on the $^4$He surface due to the potentials applied to the cell walls alone, calculated also by the method of finite differences, only with inhomogeneous boundary conditions at the cell walls and zero charge inside the cell. If the charge pool extends out to $r^{(p)} = r_M$ (with $M \leq N$), i.e., $n_l = 0$ for $i > M$, the total radial field must then be zero everywhere inside the charged area:

$$\tilde{E}_{i}^{(r)} + \sum_{l=1}^{M} G_{i,l}^{(r)} n_l = 0 \text{ for } 1 \leq i \leq M,$$

which is solved to obtain the $n_l$ for $1 \leq l \leq M$ by inverting the corresponding $M \times M$ submatrix of $G^{(r)}$. The total surface charge is then $Q_s = \sum_{i=1}^{N} A_i n_i$, where $A_i = 2\pi r_i dr$ is the area of the $l$-th charge ring. A solution for a new value of total charge $Q'_s$ with the same pool radius can be obtained by scaling all applied potentials by $Q'_s/Q_s$.

Examples of the static radial charge distribution are shown in Fig. 6.10(a) for the case that the same negative potential $-V_{dc}$ is applied to the cell’s top-plate, side wall and outermost bottom electrode $C$, with electrodes $C$, $M$, and $A$ at DC ground. Saturation (i.e., zero electric field above the charge layer) corresponds in this case to the charge pool covering the whole $^4$He surface with $r^{(p)}$ equal to the cell radius $a$.

**Dynamic calculation** To analyze the admittance measurements, one must proceed to describe the dynamical behavior of the electron layer due to an additional alternating voltage $V_{ac} e^{i\omega t}$ applied to the center Corbino electrode $C$.

Let $\delta E_k^{(r)} e^{i\omega t}$ be the alternating radial electric field at $r_k + dr/2$ on the $^4$He surface due to the applied AC potential $V_{ac} e^{i\omega t}$ alone. For typical measurement frequencies $\omega = 2\pi f$ with $f$ in the kHz range, field retardation effects can be neglected (the characteristic wavelength being on the order of several km and much larger than the cell dimensions) and the time-dependent potentials inside the cell still satisfy the Poisson equation, i.e., the field amplitude $\delta E_k^{(r)}$ can again be calculated by the method of finite differences, like in the static case, an example is shown in Fig. 6.10(b).

One starts by assuming that the charge pool has an equilibrium radius $r^{(p)} = r_{M+1}$ with $M + 1 \leq N$ (using $M + 1$ instead of $M$ simplifies the following expressions) and an equilibrium charge distribution $\{n_l\}_{l=1,...,N}$, with $n_l = 0$ for $i > M + 1$. The $n_l$ are calculated as described above. The alternating external field will induce an alternating density perturbation $\delta n_k e^{i\omega t}$ at radius $r_k$. The $\delta n_k$ can be determined using the continuity equation and Ohm’s law, as is outlined in the following.

A subtle point is that charge conservation sets the condition $\delta Q_s \equiv e \sum_{l=1}^{N} A_i \delta n_l = 0$, which can generally only be fulfilled by considering the pool edge to be mobile and to alternatingly expand and contract slightly beyond the equilibrium position. In the
following, the pool is assumed to contract by exactly one grid point, and the perturbation at \( r_{M+1} \) is determined solely from the condition \( \delta Q_s = 0 \):

\[
\delta n_{M+1} = -\sum_{l=1}^{M} \frac{A_l}{A_{M+1}} \delta n_l. \tag{6.12}
\]

In a strict sense, this can only be a solution for one special value of \( V_{ac} \) or one specific equilibrium charge distribution, as one must have \( \delta n_{M+1} = -n_{M+1} \) for the pool to truly contract by exactly one grid point, but in the linear regime of the admittance, and for a sufficiently fine grid, this is of minor importance. For checking, the case of expansion by one grid point was always treated, too, and finally the average value was used.

The integral form of the continuity equation \( \int_S \mathbf{j} \cdot d\mathbf{S} = -i \omega \int_V \rho \, dV \) (or rather its 2D version) can now be used to relate the induced radial surface current density \( j_k^{(r)} e^{i\omega t} \) at \( r_k + dr/2 \) to the change of the total surface charge inside the area bounded by \( r_k + dr/2 \):

\[
j_k^{(r)} 2\pi (r_k + dr/2) = i\omega e \sum_{l=1}^{k} A_l \delta n_l \text{ for } 1 \leq k \leq M. \tag{6.13}
\]

On the other hand, Ohm’s law can be used to relate \( j_k^{(r)} \) to the amplitude of the total alternating radial field at \( r_k + dr/2 \), which is the sum of the external field \( \delta \tilde{E}_k^{(r)} \) and the field from the density perturbation \( \{\delta n_l\}_{l=1,\ldots,N} \), obtained by the Green’s function \( G^{(r)} \):

\[
j_k^{(r)} = \sigma_k \left( \delta \tilde{E}_k^{(r)} + \sum_{l=1}^{M+1} G_{k,l}^{(r)} \delta n_l \right) \text{ for } 1 \leq k \leq M. \tag{6.14}
\]

Here, \( \sigma_k \) is the value of \( \sigma_{xx} \) at \( r_k \). In magnetic field, one may use \( \sigma_k = e^2 m^{-1} n_k \nu_{eff} / (\omega_c^2 + \nu_{eff}^2) \), which takes into account the radial variation of the static charge density \( n_k = n_s(r_k) \). For measurements in the vapor scattering regime, one can neglect the influence of the radial variation of \( n_s \) and the holding electric field on the effective collision rate \( \nu_{eff} \), and use here a single value corresponding to the conditions at the center of the SE pool.

Using Eq. (6.12) to express \( \delta n_{M+1} \) on the right side of Eq. (6.14), and combining Eqs. (6.13) and (6.14) to eliminate \( j_k^{(r)} \), one finally obtains a direct relation between the external field and the resulting density perturbation:

\[
\delta \tilde{E}_k^{(r)} = \sum_{l=1}^{M} H_{k,l} \delta n_l \text{ for } 1 \leq k \leq M, \tag{6.15}
\]

\[
H_{k,l} = -G_{k,l}^{(r)} + G_{k,M+1}^{(r)} \frac{A_l}{A_{M+1}} + i\times \begin{cases} \omega e A_l [2\pi (r_k + dr/2) \sigma_k]^{-1} & \text{if } l \leq k \\ 0 & \text{if } l > k \end{cases}.
\]

By inverting the complex \( M \times M \)-matrix \( H \), one obtains the values \( \delta n_l \) for \( 1 \leq l \leq M \), with \( \delta n_{M+1} \) given by Eq. (6.12), an example is shown in Fig. 6.10(b). Note that the density perturbation becomes especially large at the pool edge due to the motion of the latter, this corresponds to an increase of the capacitance per area in a transmission line description [108]. This effect can render the admittance especially sensitive to the position of the edge, see also the discussion in Sec. 6.4.3.

In the limit \( \omega / \sigma_{xx} \longrightarrow 0 \) the matrix \( H \) is real, and the dynamic calculation becomes equivalent to the static one, i.e., the density perturbation has only a real (in-phase)
Description of the experiment

Figure 6.10: (a) Cross-section of measurement cell (cf. Fig. 6.1) with usual configuration of DC potentials at walls (see text), combined with graphs of the static SE density \( n_s(r) \) along the \(^4\text{He}\) surface for three SE pool radii \( r^{(p)} \), normalized to the density \( n_s^{(\text{sat})} \) at the pool center \( r=0 \) at saturation. Saturation corresponds here to \( r^{(p)} \) being equal to the cell radius \( a=5.4 \text{ mm} \).

(b) Same as (a), with a small additional AC voltage \( V_{ac}e^{i\omega t} \) applied to the center Corbino disc \( C \), resulting in a radial driving field \( \delta E^{(r)}(r) \) and a perturbation \( \delta n(r) \) of the static surface charge density \( n_s(r) \). The perturbation \( \delta n(r) \) was here calculated for the case \( \omega/\sigma_{xx} \to 0 \) corresponding to a purely capacitive admittance \( I_{ac}/V_{ac} \), and for contraction of the charge pool.

component and completely screens the radial driving field. This corresponds to the case \( \omega C_e/G_e \to 0 \) in the equivalent circuit model, and the admittance is then purely capacitive. For a finite \( \omega/\sigma_{xx} \), the density perturbation also has an imaginary (out-of-phase) component corresponding to a resistive part of the admittance.

To determine the admittance \( I_{ac}/V_{ac} = G_x + i\omega C_x \), it remains to calculate the current \( I_{ac} \) flowing between Corbino pick-up electrode \( A \) and ground due to the surface charge perturbation \( \delta n = \{\delta n_t\}_{t=1,\ldots,N} \). By use of the Green’s function \( G^{(b)} \), one obtains the vertical electric field \( \delta E^{(b)} \) along the cell bottom generated by \( \delta n \), and thus the charge density perturbation \( \epsilon \delta n^{(b)} = -\epsilon_0 \epsilon_0 \delta E^{(b)} \) induced on the bottom electrodes. The current \( I_{ac} \) is then obtained by integration of \( i\omega \epsilon \delta n^{(b)} \) over the area of the pick-up electrode.

A typical example of the admittance calculated for the present setup as a function of \( \omega/\sigma_{xx} \) is shown in Fig. 6.11. The behavior is qualitatively very similar to that found in the equivalent circuit model. For a more detailed comparison, the equivalent series conductance \( G_e \) has been calculated with Eq. (6.11) from the numerically obtained \( G_x \) and \( C_x \), and the plot of \( \omega/G_e \) vs. \( \omega/\sigma_{xx} \) has been included in Fig. 6.11. This plot is essentially linear in the range \( G_x/\omega C_x < 1 \), so the equivalent circuit model (which assumes \( G_e \propto \sigma_{xx} \)) agrees with the numerical analysis in this range. Visible deviations from the linear behavior only start to appear for \( G_x/\omega C_x > 1 \).

For a comparison with experiment, Fig. 6.12(a) shows measured data of the admittance in dependence of magnetic field \( B \), and the corresponding theoretical curves obtained from the numerical method. In the numerical calculation, the effective collision rate \( \nu_{\text{eff}} \) from single-electron theory [Eq. (3.18)] has been used; all other parameters are
determined by direct measurement, except the charge pool radius, or rather, the electron density \( n_s \). This is controlled in the experiment by a method described in Sec. 6.4.3.

In Fig. 6.12(a), numerical calculation and experiment agree very well concerning \( G_x \), where the small differences in the low-\( B \) range can be safely attributed to Coulomb effects (see Sec. 7.2.1). In \( C_x \), the experimental and theoretical curves agree well concerning the dependence on magnetic field, but there is an offset of about 8%, which occurs rather regularly, and whose origin is not quite clear. It may be due to irregularities in the setup, like the meniscus of the \(^4\)He level, or a slight tilt of the cell.

It also seems that generally the experimental behavior is less satisfactorily described by the numerical method at high \( n_s \) (high holding field), which may be due to the SE-induced deformation of the \(^4\)He surface (see Sec. 1.2). For example, at \( n_s = 10^9 \text{ cm}^{-2} \), the deformation depth is \( \Delta h \simeq 0.1 \text{ mm} \) [Eq. (1.11)], which is quite large, and the sensibility of the admittance to the conditions at the pool edge, discussed above, might lead to noticeable effects even at distinctly lower \( n_s \). The experimental study of the DC conductivity was therefore limited to lower \( n_s \) (\( n_s \lesssim 10^8 \text{ cm}^{-2} \)) than the CR studies.

By inverting the numerically calculated dependence of \( G_x \) on \( \sigma_{xx}^{-1} \), one can convert the measured \( G_x \) to \( \sigma_{xx}^{-1} \); the result is shown in Fig. 6.12(b). Also shown is the result of an evaluation of the measured data by the equivalent circuit model Eq. (6.11). The unknown proportionality factor between equivalent series conductance \( G_e \) and \( \sigma_{xx} \) was here adjusted for agreement with the result of the numerical method at high \( B \). Generally, it is about one in this setup. The two evaluation methods give very close results, as expected from Fig. 6.11 in the range \( G_x/\omega C_x < 1 \). Also, the experimental data agree well with single-electron theory, apart from the small Coulomb effects at low \( B \).

However, good agreement between numerical method and equivalent circuit model is not always achieved; there appear for instance deviations if saturation is approached too closely, probably because in this case the edge effects become more important. Moreover, the helium level inside the cell tends sometimes to change with magnetic field, probably due to temperature changes or mechanical effects\(^7\) to which this setup is particularly sensitive due to the use of an insert tube, as described in Sec. 6.2. These level changes affect predominantly \( C_x \), and while the evaluation by the equivalent circuit model be-

\(^7\)Spacial inhomogeneity of the magnetic field can also lead to level changes [109], but should rather lead to an increase, and should also have occurred in other setups operated in the same conditions.
Figure 6.12: (a) Measured data of $G_x$ (○) and $\omega C_x$ (□) vs. magnetic field $B$, and the corresponding theoretical curves (solid) obtained by the numerical method together with single-electron theory Eq. (3.18). The conditions are the same as for Figs. 6.10(b) and 6.11. (b) Data of $\sigma_{xx}^{-1}$ (○) obtained from the measured $G_x$ in (a) by the numerical method. Also shown are the corresponding data of $G_e^{-1}$ (+) obtained from the measured $G_x$ and $\omega C_x$ in (a) by the equivalent circuit model [Eq. (6.11)] and adjusted to the numerical data at high $B$. The solid curve shows the single-electron theory for $\sigma_{xx}(B)$ [Eqs. (3.16) and (3.18)].

comes then unreliable, the numerical inversion of $G_x$ alone still appears to give a correct result, as can be checked by comparison with single-electron theory at high magnetic field, an example is shown in Fig. 6.13.

Figure 6.13: The same as Fig. 6.12 for another measurement. Here, the experimental $\omega C_x$ [(□) in (a)] deviates from the numerical curve (solid), probably due to a decrease of the $^4$He level with $B$. This affects the evaluation by the equivalent circuit model, whose result [(+) in (b)] here visibly deviates from theory (solid curve). In contrast, the result of the evaluation by the numerical method [(○) in (b)], which relies only on $G_x$ [(○) in (a)], still well agrees with theory.
6.4.3 Control of electron density and holding electric field

A problem frequently encountered in experiments on SE is how to reliably determine the electron density $n_s$. Determining $n_s$ via the CR line shape, as described on page 83, is rather cumbersome, moreover it requires comparison with a theoretical line shape, which is often an approximation. Another important motivation for the numerical analysis described in the previous section is that it provides a convenient method to control $n_s$.

Generally, the method consists of varying the DC potentials applied to the cell walls at nonsaturated conditions with constant total surface charge $Q_s$, which changes the charge pool radius $r^{(p)}$. This in turn changes the admittance, and by comparison with the result of the numerical analysis, $n_s$ can be determined.

It is here best to have a very low value of $\omega/\sigma_{xx}$, so that $\xi_{k,l} \rightarrow 0$ and the matrix $H$ in Eq. (6.15) is real. The admittance is then purely capacitive, $I_{ac}/V_{ac} = i\omega C_x$, with $C_x$ depending solely on geometry, i.e., on the $^4$He liquid level $h$ and the SE pool radius $r^{(p)}$.

The usual configuration is to have the same negative potential $-V_{dc}$ applied to the cell’s top-plate, side wall, and outermost bottom electrode $G$, and to measure $C_x$ between center Corbino disk $C$ and second Corbino ring $A$ (see Fig. 6.10). At an increase of $V_{dc}$, the pool radius $r^{(p)}$ always decreases, but $C_x$ depends nonmonotonically on $V_{dc}$, with a maximum that corresponds in this configuration roughly to $r^{(p)}$ being equal to the outer radius $r^{(A)}$ of electrode $A$, see Fig. 6.14: At sufficiently high $V_{dc}$, when $r^{(p)} < r^{(A)}$, decrease of $V_{dc}$, i.e., increase of $r^{(p)}$, leads to increase of $C_x$, because the capacitive coupling between SE layer and electrode $A$ increases. But at $r^{(p)} > r^{(A)}$, increase of $r^{(p)}$ rather increases the coupling between SE layer and outermost electrode $G$ and cell wall, which bypasses current to AC ground and leads to decrease of $C_x$. This effect is probably enhanced by the density perturbation being especially large at the pool edge, as discussed on page 89 and visible in Fig. 6.10(b). The decrease becomes very pronounced when $r^{(p)}$ approaches the cell wall radius $a$. At $r^{(p)} = a$, the SE layer is at saturation, and the electron density $n_s^{(sat)}$ at the pool center is for $h \ll r^{(A)}$ in good approximation

$$en_s^{(sat)} \simeq \epsilon H \epsilon_0 \frac{V_{dc}}{h},$$

(6.16)
derived assuming a uniform vertical electric field between SE layer and cell bottom. At further decrease of $V_{dc}$, charge must leave the bulk $^4$He surface. In the experiment, charge loss even seems to occur somewhat earlier, at least at low electron density, as can be seen in Fig. 6.14, where the theoretical $C_x(V_{dc})$ is compared to measured data.

The maximum value of $C_x$ depends only on the $^4$He level $h$, and can in principle also be used for its determination. In our case, $h$ is rather determined independently, which gives a slight offset between theory and experiment in $C_x$ (see Fig. 6.12) but an overall better description. In Fig. 6.14, this offset is corrected for better comparison, apart from this the only adjustment is the scaling of the $V_{dc}$-axis for the theoretical $C_x$ to fit it to the measured one, which determines the total charge.

For our setup, $C_x$ attains its maximum value when the density $n_s$ at the pool center is about 25% below the saturation value $n_s^{(sat)}$ for the same $V_{dc}$, as can be seen from Fig. 6.14. For practical purposes, it is therefore sufficient to just ensure that one is well on the low-$V_{dc}$ side of the maximum to know $n_s$ to be about 10% below saturation, with an acceptable error of less than $\pm 10\%$. This provides an easy way to control the density in the experiment without any calculation. It is also better not to approach saturation too closely, as otherwise spontaneous charge loss may occur during the measurement.
indicates that saturation can be approached more closely in this case. Starting far below saturation (high $V_{dc}$), first $V_{dc}$ is decreased until near saturation charge begins to leave the surface. The theoretical $C_x$ has been fitted to this branch by scaling the $V_{dc}$-axis, this determines the total charge $Q_s = 1.65 \times 10^7 e$. Saturation, corresponding to $r^{(p)} = a$, is in theory reached at $V_{dc} = 2.15 V \equiv V^{(sat)}$, with $n_s^{(sat)} = 2.67 \times 10^7 cm^{-2}$. After the charge loss, $V_{dc}$ is again increased, the measured $C_x$ then follows a curve corresponding to a lower total charge $Q_s' = 0.88 Q_s$ (thus $V^{(sat)} = 0.88 V^{(sat)}$), $n_s^{(sat)} = 0.88 n_s^{(sat)}$, determined by a new fit of the theoretical $C_x$ (not shown).

Figure 6.14: Dependence on holding voltage $-V_{dc}$ of: SE pool radius $r^{(p)}$ (dashed), SE density at the pool center $n_s$ (dash-dotted), and capacitance $C_x$ (solid), all calculated numerically for constant total surface charge $Q_s$. Here, $r^{(p)}$ is normalized to the cell radius $a$, and $n_s$ to the saturation density corresponding to the momentary value of $V_{dc}$. The dotted vertical line indicates $r^{(p)}$ being equal to the outer radius $r^{(A)}$ of Corbino electrode $A$; this roughly coincides with the maximum of $C_x$ (see text). The circles show experimental data of $C_x$ vs. $V_{dc}$, the sense of variation of $V_{dc}$ is indicated by the arrows. Starting far below saturation (high $V_{dc}$), best accuracy requires a numerical treatment.

At high electron density, surface deformation effects (see Sec. 1.2) may also be important, but are difficult to account for in the numerical analysis beyond a uniform reduction of $h$. The CR measurement analyzed on page 83 indicates that saturation can be approached more closely in this case.

Note that not too far below saturation, the holding field $E_\perp$ is still almost equal to the saturation value $E_\perp^{(0)} = en_s/2\varepsilon_0$ [Eq. (1.7)]: Assuming uniform electric fields between SE layer and cell top and bottom, one estimates $(E_\perp/E_\perp^{(0)} - 1) \sim 2(h/l)(n_s^{(sat)}/n_s - 1)$. In our cell, typically $h/l \sim 0.1$, so that for instance if $n_s$ is 10% below $n_s^{(sat)}$, one estimates $E_\perp$ to be only about 2% above $E_\perp^{(0)}$. Better accuracy requires a numerical treatment anyway, because the field between SE layer and cell top is not really uniform. Uniformity requires $l \ll a$, which is not fulfilled at low MW resonance frequency of the cell.

The setup also allows to perform measurements at constant $n_s$ in dependence of the holding electric field in undersaturated conditions. In this case, the top-plate potential $V_{top}$ is varied independently of the potential $V_{guard}$ of cell wall and bottom electrode $G$ (see Fig. 6.1). The desired value of the holding field is adjusted via $V_{top}$, while $V_{guard}$, which mainly determines the radial electric field, is used to keep the SE pool radius constant, this is controlled by observing $C_x$. The electron density stays then also approximately constant, apart from a usually small, radial redistribution of charge within the pool. As noted above, for the geometry of our cell ($l/a \sim 1$), the exact value of the holding field must then also be calculated by the matrix method outlined in Sec. 6.4.2.
Chapter 7

Experimental results and discussion

In the following, first are presented the new experimental results in quantum cyclotron resonance (CR) in the vapor scattering regime (Sec. 7.1), and compared to previous theories and the new theory for Coulomb effects outlined in chapter 4. Also, they are compared to previous CR experiments (Sec. 7.1.5). Complementary measurements of the DC magnetoconductivity in the vapor scattering regime are presented and discussed in Sec. 7.2. Finally, Sec. 7.3 discusses experiments on the nonlinear conductivity at heating.

To experimentally investigate Coulomb effects in magnetotransport of SE, both in the AC case (CR) and DC case, the key experiment is to study the dependence on the electron density $n_s$, as the latter only affects the strength of Coulomb interaction, i.e., the mean value of the fluctuational electric field (FEF) $E_i^{(0)} \propto n_s^{3/4} T_e^{1/2}$ [Eq. (4.2), $T_e$ is the electron temperature], which enters the parameter $\Gamma_C = \sqrt{2e E_i^{(0)} l_B}$ describing both the Doppler narrowing of Landau levels (Sec. 4.3) and, more importantly, the Doppler broadening of the dynamic structure factor (DSF) (Sec. 4.4). For convenience, these measurements (see Sec. 7.1.1 for the AC, and Sec. 7.2.1 for the DC case) were performed at saturation, where changes of $n_s$ also change the holding electric field, this gives an additional contribution at high $n_s$ (see Sec. 7.1.2) but does not change the general picture.

Coulomb effects may also be studied by varying the magnetic field $B$, which changes the parameter $\Gamma_C$, the Landau level width, and the level separation with different dependencies. Generally, Coulomb effects depend on the size of $\Gamma_C$ compared to Landau level width and level separation. In DC transport, it is rather easy to cover a sufficiently large range of $B$ in a single measurement (see Sec. 7.2). In CR, one is usually limited to a single value of $B$, but the variable microwave (MW) frequency of the present setup allowed to study for the first time the $B$-dependence in a limited range here, too (see Sec. 7.1.4).

Also investigated was the influence of the strength of the interaction with the scatterers. In the vapor scattering regime, this can be done by varying the temperature $T$ of the measurement cell, which changes the density of scatterers and thus scattering rate and Landau level width, but essentially nothing else, see Sec. 7.1.3 for the AC, and Sec. 7.2.2 for the DC case. Another possibility is to vary the holding field (Sec. 7.1.2). As the FEF not only depends on the electron density but also on the electron temperature $T_e$, it is also of interest to study variations of $T_e$ at fixed $T$. In the present setup, a simple way to only heat the SE is to increase the MW power under CR conditions, see Sec. 7.3. The resulting nonlinear effects were studied both in the AC (Sec. 7.3.1) and the DC conductivity (Sec. 7.3.2). Moreover, these experiments allow to study also the energy relaxation of SE, and the population of higher surface levels (Sec. 7.3.3).
7.1 Linear quantum cyclotron resonance (CR)

7.1.1 Dependence on electron density

Fig. 7.1 shows a series of CR measurements on SE in the vapor scattering regime, conducted at varying electron densities \( n_s \sim 10^7 - 10^9 \) cm\(^{-2} \) at saturation. To have the best signal/noise ratio down to the lowest \( n_s \), this series was measured at the lowest microwave (MW) frequency of the setup, 40 GHz, where the cavity Q-factor is highest (see Sec. 6.3.2), at rather low temperature (1.28 K) so that the collision rate is low and the CR amplitude is high, and with magnetic field modulation (most other CR data presented in the following were measured without modulation). The MW power \( P_{MW} \) reaching the cavity was kept at \( \sim 1 \) nW to ensure being in the linear regime, where the electrons are in thermal equilibrium with the scatterers, \( T_e = T \) (cf. Sec. 7.3.1).

With the normalization of the MW spectrometer output, and the determination of the cavity Q-factor and the filling factors for the measurements (see Sec. 6.3), one obtains from the measured data and Eq. (6.9) the complex AC conductivity \( \sigma_{xx} \) in absolute units. In Fig. 7.1, the results are normalized by \( n_s \) to have all curves of comparable size.

Figure 7.1: Cyclotron resonance behavior in dependence of the electron density \( n_s \) at temperature \( T = 1.28 \) K and microwave frequency \( f_{MW} = 40 \) GHz (resonant magnetic field \( B_{res} = 1.43 \) T). The left graph shows data of the CR absorption \( \text{Re} \sigma_{xx} \) and the right graph the corresponding dispersion \( \text{Im} \sigma_{xx} \). The measurements were performed with magnetic field modulation and lock-in detection (see page 80) at frequency \( f_{mod} = 5.3 \) Hz and amplitude \( B_{mod} = 10^{-2} \) T rms, and in this case with decreasing \( B \), which shifts the CR data slightly to lower \( B \). The selection of curves drawn in heavier style are compared with memory function theory in Fig. 7.4.
We will first shortly discuss the CR data shown in Fig. 7.1 phenomenologically. Within the resolution of the measurement, the CR lines at the three lowest densities \( n_s \leq 1.2 \times 10^7 \text{ cm}^{-2} \) are essentially identical in Fig. 7.1, indicating that they correspond to the noninteracting, single-electron limit where Coulomb effects are negligible. Towards higher density \( n_s \gtrsim 10^7 \text{ cm}^{-2} \) one can observe at first a pronounced narrowing of the linewidth (by about a factor of three, see Fig. 7.3), accompanied by a change in the line shape, which is better visible in the following Fig. 7.2. At very high density \( n_s \gtrsim 3 \times 10^8 \text{ cm}^{-2} \), the linewidth narrowing reverses to a broadening in Fig. 7.1.

Fig. 7.2 compares the absorption line shape in Fig. 7.1 in the low-\( n_s \) limit to that at higher \( n_s \). For a more objective comparison, a Gaussian and a Lorentzian model curve have been fitted to each of the experimental data sets. The only restriction in the fitting was that the baseline of the theoretical curves should be determined by the experimental data points farthest from the resonance (at 0.8 T and 2.1 T for these data). It is evident that the shape of the low-\( n_s \) curve is very close to a Gaussian, while the high-\( n_s \) curve has an almost ideal Lorentzian shape.

For a quantitative evaluation of the CR linewidth, the experimental absorption data were therefore fitted with Gaussian model curves at low \( n_s \), and with Lorentzian curves at high \( n_s \), and with both in the intermediate \( n_s \)-range, where the line shape is somewhat ambiguous. The results are presented in Fig. 7.3 and discussed below.

**CR linewidth**

Fig. 7.3 shows the CR linewidth vs. electron density \( n_s \) obtained from Fig. 7.1, together with the results of the theoretical treatments of Coulomb effects in CR presented in Sec. 4.5.1, i.e., the balance equation/memory function theory with the many-electron DSF, where the linewidth is determined by the AC effective momentum collision rate [Eq. (4.18)], and the Ando CR theory, where the linewidth is directly determined by
the Landau level width [Eq. (4.19)]; these two theoretical approaches were previously compared in Fig. 4.8. In both cases, vapor atom as well as ripplon scattering was included, for the relative importance of these two, see Fig. 7.3. Also shown is the earlier theory of Dykman and Khazan [Eq. (5.1)].

Figure 7.3: CR linewidth vs. electron density $n_s$ at temperature $T = 1.28$ K and resonant magnetic field $B = 1.43$ T. The data were obtained from the absorption measurements shown in Fig. 7.1 (and the corresponding measurements vs. increasing $B$, which were not shown) by Gaussian fitting (⊙) and Lorentzian fitting (△). Also shown are the results of memory function theory with the many-electron DSF [Eq. (4.18), solid curve], Ando CR theory with the Coulomb narrowing of the Landau levels [Eq. (4.19), dashed], and Dykman CR theory [Eq. (5.1), dotted], cf. Fig. 4.8.

Increase of $n_s$ increases the Coulomb interaction, i.e., the FEF $E_i(0) \propto n_s^{3/4} T^{1/2}$ which enters the parameter $\Gamma_C = \sqrt{2eE_i(0)}l_B$ describing both the Doppler narrowing of the Landau levels (Sec. 4.3), relevant for Ando theory, and the Doppler broadening of the many-electron DSF (Sec. 4.4), relevant for memory function theory.

In Fig. 7.3, both memory function theory and Ando theory well describe the experimental CR linewidth in the noninteracting regime $\Gamma_C \ll \Gamma_{se}$ ($\Gamma_{se}$ is the single-electron Landau level width, see Sec. 2.3.1), corresponding here to $n_s \lesssim 10^7$ cm$^{-2}$, and in the regime of beginning Coulomb effects $\hbar \omega_c \gg \Gamma_C \gg \Gamma_{se}$, corresponding here to $10^8$ cm$^{-2} \gtrsim n_s \gtrsim 10^7$ cm$^{-2}$. In the former regime, the linewidth is essentially constant, while in the latter, the Doppler effects due to the fast FEF-induced Hall drift velocities of the SE reduce the rate of electron-scatterer collisions, and the linewidth shows Coulomb narrowing with increasing $n_s$, as discussed in Sec. 4.5.1. In memory function theory, this is described by the reduction of the resonant contribution Eq. (4.14) to the AC effective collision rate, which corresponds to SE scattering between adjacent Landau levels.

In the regime of strong Coulomb effects $\Gamma_C \gtrsim \hbar \omega_c$, corresponding to $n_s \gtrsim 3 \times 10^8$ cm$^{-2}$ in Fig. 7.3, very strong Doppler effects make the SE scatter between different Landau levels, and the Coulomb narrowing of the CR linewidth reverses into a Coulomb broadening vs. $n_s$. In memory function theory, this is described by the nonresonant contribution in Eq. (4.16). In this regime, significant quantitative differences appear between the linewidth according to memory function theory and that of Ando theory. The former is much larger, by about 50% at the highest $n_s$-value in Fig. 7.3, which can be understood as the memory function approach taking additionally into account that scattering to higher Landau levels has a higher value of the momentum exchange, as discussed in Sec. 4.5.1. This gives indeed a better description of the experimental data in Fig. 7.3.

Fig. 7.3 also shows the result of the theory of Dykman and Khazan, which describes the data only in the Coulomb narrowing regime at intermediate densities. It is equivalent to taking into account in the memory function treatment of Sec. 4.5.1 only the resonant contribution Eq. (4.14) to the linewidth and to neglecting the collision broadening $\Gamma_{se}$ altogether. It is therefore valid only if $\hbar \omega_c \gg \Gamma_C \gg \Gamma_{se}$, and deviates from the experi-
mental data both at high \( n_s \), when the omitted nonresonant contribution in Eq. (4.16) becomes important (which describes scattering to higher Landau levels and is necessary to explain the Coulomb broadening of the linewidth), and at low \( n_s \), when Coulomb effects are negligible compared to the single-electron collision broadening.

It should be remarked that at high \( n_s \), ultimately the condition \( l_B \ll \sqrt{\langle \delta f^2 \rangle} \) [Eq. (4.4)] is no longer well fulfilled \( (l_B/\sqrt{\langle \delta f^2 \rangle} \approx 0.5 \) for the highest \( n_s \) in Fig. 7.3), and the non-uniformity of the fluctuational electric field may become relevant. However, at least within the \( n_s \)-range covered by Fig. 7.3, no indication of this is yet apparent, the experimental data are well described by memory function theory up to the highest \( n_s \).

**CR line shape**

Another feature to compare with theory is the line shape change from Gaussian to Lorentzian observed in the experimental CR data in Figs. 7.1 and 7.2, which occurs simultaneously with the Coulomb narrowing of the CR linewidth. This transformation is difficult to explain in Ando CR theory, but it can easily be explained in the memory function approach by the Doppler broadening of the maxima in the many-electron DSF and AC effective collision rate \( \nu_{\text{eff}} \), discussed in Sec. 4.5.1 and illustrated in Fig. 4.7: In the noninteracting limit, the maxima in \( \nu_{\text{eff}} \) are quite sharp and lead in combination with the Drude-like conductivity tensor to a CR line shape that is very close to a Gaussian [Fig. 4.7(A)]. Increasing Coulomb interaction broadens the maxima, and the line shape ultimately becomes a pure Lorentzian, like in classical Drude theory [Fig. 4.7(B, C)].

![Figure 7.4: A selection of the CR data from Fig. 7.1 (○), together with the corresponding results of many-electron memory function theory from Sec. 4.5.1 (solid curves), cf. Fig. 4.7.](image)

Fig. 7.4 show a rigorous comparison between measured AC conductivity and memory function theory in absolute units. Here, the only adjustments were to arrange the baselines of the curves for better orientation and to have identical baselines for each pair of experimental and corresponding theoretical curve, and the resonance position of the
theoretical curves has been adjusted to the slight $B$-shift of the experimental data due to the modulation. Apart from this, Fig. 7.4 contains no adjustable parameter.

Even for these very strict conditions, one obtains reasonable agreement between theory and experiment, moreover, the most visible discrepancies concern the amplitude of the curves, mainly in the low-$n_s$ range, which may be entirely due to errors in the amplitude of the experimental curves, originating from the uncertainties in the normalization of the MW spectrometer output, in the determination of the cavity filling factor, or in the electron density. Also, even the theoretical curves do not necessarily represent the correct amplitude; there is, for example, a noticeable difference in amplitude between memory function theory and Ando CR theory even in the noninteracting case, as is visible in Fig. 3.2, although the line shape and linewidth of both is very similar, as shows Fig. 3.3.

In Fig. 7.5, experiment and theory are therefore compared concerning the line shape only, here the amplitude of the theoretical curves has been adjusted to be the same as that of the experimental curves for the CR absorption $\text{Re}\sigma_{xx}$. From Figs. 7.4 and 7.5, one can see that the memory function theory almost perfectly reproduces the Lorentzian line shape experimentally observed in the high-$n_s$ limit, and is a good approximation for the Gaussian line shape observed in the low-$n_s$ limit. Discrepancies are mainly visible in the intermediate density range (see the curves to $n_s = 3.7 \times 10^7$ cm$^{-2}$ in Fig. 7.4), where the line shape is somewhat ambiguous, but considering the approximations involved in any theory, the results of memory function theory agree remarkably well with the experimental data.

![Figure 7.5: Two of the measurements from Fig. 7.4 (○), together with the corresponding results of memory function theory (solid curves), adjusted in amplitude to the experimental data.](image)

In all, the memory function theory with the many-electron DSF very well describes the CR behavior vs. density $n_s$ in all relevant aspects, both regarding the CR linewidth showing Coulomb narrowing and broadening, and the transformation of the line shape from Gaussian to Lorentzian. In the following, we will only use this theory to further discuss the experimental behavior.
7.1 Linear quantum cyclotron resonance (CR)

7.1.2 Holding field dependence

So far, the behavior of CR vs. electron density $n_s$ was discussed only in terms of Coulomb effects, which affect only the excitation spectrum, i.e., the DSF of the SE system. In principle, this is not entirely correct, as changes of $n_s$ at saturation of course also change the holding electric field $E_\perp$, which affects the coupling between electrons and scatterers (see Sec. 2.2). This does not change the Coulomb parameter $\Gamma_C$ describing the Doppler broadening of the many-electron DSF (Sec. 4.4); it only affects the interaction parameters entering the general expression for the effective collision rate $\nu_{\text{eff}}$ [Eq. (3.14)] and the single-electron Landau level collision broadening $\Gamma_{\text{se}}$ [Eq. (2.25)] entering the DSF.

Concerning the interaction with vapor atoms, which is the dominant one in the present experiments, $E_\perp$ leads to compression of the vertical SE wave function, i.e., increase of the parameter $\gamma$ [Eq. (1.4)], which enters the zero-field collision rate $\nu_a$ [Eq. (2.11)], and thus $\nu_{\text{eff}}$ and $\Gamma_{\text{se}}$. At high $E_\perp$, also the interaction with ripplons may become noticeable, although it is generally less important in the present case. It depends much stronger on $E_\perp$ entering directly the interaction matrix element $V_q$ [Eq. (2.8)].

To investigate the contribution of holding field effects to the CR linewidth, measurements were performed under non-saturated conditions. Here, $E_\perp$ was increased at a fixed electron density according to the procedure described in Sec. 6.4.3. The results are shown in Fig. 7.6, together with the results of memory function theory [Eq. (4.18)]. It seemed here the best choice to work at the lowest frequency of the MW setup (40 GHz), where the cavity Q-factor and thus the measurement resolution are highest. In this case, however, the cavity is rather high, and a rather high voltage must be applied to the cavity top plate to have a noticeable increase of $E_\perp$ in excess of the saturation value $E^{(0)}_\perp = n_s/2\varepsilon_0$. Moreover, the cavity side wall starts then to screen the field of the top plate; i.e., the vertical electric field above the SE layer is not vertically uniform but significantly higher near the top plate than near the SE layer, which is close to the bottom of the cavity. This further reduces the excess field acting on the SE and makes it necessary to also calculate the actual value of $E_\perp$ by the numerical method described in Sec. 6.4.2.

![Figure 7.6:](image)

From memory function theory, it is easily seen that increase of $E_\perp$ alone can only lead to a monotonic increase of the CR linewidth $\gamma_{\text{CR}}$, as shown by the dashed theoretical curves in Fig. 7.6 and confirmed by the experimental data. For pure vapor atom scattering, at fixed $n_s$ we have $\gamma_{\text{CR}} \propto \nu_a/\Gamma_a \propto \sqrt{\nu_a}$ in the single-electron limit at low
$n_s$, and the stronger dependence $\gamma_{\text{CR}} \propto \nu_a$ at sufficiently strong interaction at high $n_s$, when in the many-electron DSF the Landau level collision broadening $\Gamma_a$ is negligible compared to the Doppler broadening (for a more thorough discussion, see the following Sec. 7.1.3). Ripplon scattering behaves similarly.

Thus, holding field effects can clearly not explain the pronounced narrowing of the CR linewidth vs. $n_s$ observed at low $n_s$ at saturation (cf. Fig. 7.3), moreover, they are rather weak in this range, as can be seen from the low-$n_s$ data in Fig. 7.6. Still, holding field effects noticeably contribute to the re-increase of the CR linewidth observed at high $n_s$ at saturation, as is seen from the high-$n_s$ data in Fig. 7.6, although they alone cannot explain the reversal of the linewidth narrowing to broadening vs. $n_s$ seen in Fig. 7.3. Also, it should be noted that the stronger $E_{\perp}$-dependence at high $n_s$ is actually also a many-electron effect, as it is due to the many-electron transformation of the DSF.

7.1.3 Dependence on scatterer density

The particle density of $^4$He vapor atoms $n_a$, which are the dominant scatterers for the present experiments, depends exponentially on temperature [Eq. (2.1)]. This provides an easy way to experimentally investigate the influence of $n_a$ (or generally, the influence of the coupling to the scatterers) on the CR behavior, while keeping all other parameters almost constant. The interest lies here in the fact that according to the theory of Coulomb effects (chapter 4), the dependence of the effective collision rate on $n_a$ is different in the single-electron limit and the strongly interacting limit, which reflects again the many-electron transformation of the DSF. This effect can directly be compared in the AC and the DC conductivity (cf. Sec. 7.2.2). $n_a$ enters the theoretical expressions only via the zero-field collision rate $\nu_a \propto n_a$ [Eq. (2.11)], which conveniently summarizes all parameters of the interaction with vapor atoms, so the following we refer to $\nu_a$.

The single-electron DSF of the noninteracting system [Eqs. (2.33)–(2.35)] depends directly on the single-electron density of states. The height of the maxima of the DSF is proportional to the inverse single-electron Landau level width $1/\Gamma_{\text{se}}$, with $\Gamma_{\text{se}} \simeq \Gamma_a \propto \sqrt{n_a}$ in the vapor scattering regime (Sec. 2.3.1). But in the many-electron DSF (Sec. 4.4), at sufficiently strong interaction, $\Gamma_{\text{se}}$ is negligible compared to the Doppler broadening $\Gamma_C$.

For calculation of the CR linewidth $\gamma_{\text{CR}}$, the DSF must be inserted in the general equation (3.14) for the AC effective collision rate, which also depends directly on $\nu_a$. One obtains then at weak interaction, when $\Gamma_C \ll \Gamma_{\text{se}}$, a square-root relation $\gamma_{\text{CR}} \propto \nu_a/\Gamma_{\text{se}} \propto \sqrt{\nu_a}$ for pure vapor scattering [Eq. (4.13)], the same as in Ando CR theory, where $\gamma_{\text{CR}} \propto \Gamma_{\text{se}}$ (Sec. 3.2.1). But at strong enough interaction, when $\Gamma_C \gg \Gamma_{\text{se}}$, the DSF is independent of $\Gamma_{\text{se}}$ and one obtains a linear relation: $\gamma_{\text{CR}} \propto \nu_a$ [Eqs. (4.14)–(4.16)]. (For our experimental conditions, at $\Gamma_C \gg \Gamma_{\text{se}}$ is usually also $\Gamma_C \gtrsim \hbar \omega_c$; however, the latter is not really necessary to change the dependence on $\nu_a$.)

Fig. 7.7 shows measurements of the CR linewidth $\gamma_{\text{CR}}$ vs. saturation electron density $n_s$ at two different bath temperatures $T_1 = 1.23$ K and $T_2 = 1.36$ K. The difference in temperature itself is only 10% and negligible for the following, while the vapor atom density $n_a$, and thus $\nu_a$, increases by almost exactly a factor of 2 from $T_1$ to $T_2$. All other parameters are the same for $T_1$ and $T_2$. At both temperatures, the data in Fig. 7.7 show vs. $n_s$ the typical behavior already familiar from Fig. 7.3, and agree well with the corresponding result of many-electron memory function theory for vapor atom and ripplon scattering together [Eq. (4.18)]. At the lower temperature $T_1$, ripplon scattering gives already a distinct contribution at high $n_s$. 
7.1 Linear quantum cyclotron resonance (CR)

Figure 7.7: Experimental data of the CR linewidth vs. saturation density at two temperatures: \( T_1 = 1.23 \text{ K} \) (full symbols, \(^4\text{He} \text{ vapor atom density } n_a = 6.2 \times 10^{24} \text{ m}^{-3} \) ) and \( T_2 = 1.36 \text{ K} \) (open symbols, \( n_a = 12.3 \times 10^{24} \text{ m}^{-3} \)). The symbols have the same meaning as in Fig. 7.3. The respective results of memory function theory are shown as dashed curves for vapor atom scattering alone [Eq. (4.16)], and as solid curves for vapor atom and ripplon scattering together [Eq. (4.18)]. Also shown is the ratio of the theoretical linewidths \( \gamma_{\text{CR}}(T_2)/\gamma_{\text{CR}}(T_1) \), as dotted curve for vapor scattering alone, and as dash-dotted curve for vapor and ripplon scattering together.

Comparing the \( T_1 \) and \( T_2 \) data in Fig. 7.7, one observes indeed at high \( n_s \), i.e., strong interaction, that \( \gamma_{\text{CR}} \) increases by approximately the same factor of 2 as \( \nu_a \). Small deviations from the linear relation between \( \nu_a \) and \( \gamma_{\text{CR}} \) arise here from the contribution of ripplon scattering which depends only weakly on temperature and is relatively more important at \( T_1 \); this leads to the difference between the dotted and dash-dotted theoretical curves in Fig. 7.7. On the other hand, at low \( n_s \) the increase of \( \gamma_{\text{CR}} \) is much weaker, and corresponds indeed to about a factor of \( 1.4 \simeq \sqrt{2} \), as expected for weak interaction.

For a more precise comparison, in Fig. 7.8 the data from Fig. 7.7 have been normalized to the single-electron CR linewidth for vapor atom scattering \( \sqrt{\pi \Gamma_a}/\hbar \propto \sqrt{\nu_a} \) [Eq. (4.13)]. This makes the data for the two temperatures coincide in the low \( n_s \) limit but not in the high \( n_s \) limit, which clearly shows the different dependencies on \( \nu_a \) (ripplon scattering has been omitted in the normalization, as it would in any case only increase the difference between the \( T_1 \) and \( T_2 \) data at high \( n_s \)). This normalization also allows a direct comparison of Coulomb effects in the AC and DC collision rate, cf. Fig. 7.18.

Figure 7.8: The data for the CR linewidth \( \gamma_{\text{CR}} \) from the previous Fig. 7.7, normalized to the theoretical single-electron linewidth due to vapor atom scattering \( \sqrt{\pi \Gamma_a}/\hbar \propto \sqrt{\nu_a} \). Shown are the experimental data and the results of memory function theory for vapor atom and ripplon scattering together.
7.1.4 Magnetic field dependence

According to theory, Coulomb effects depend on the interplay of the three characteristic parameters $\Gamma_C = \sqrt{2} e E_t^{(0)} l_B$ (describing the Doppler broadening of the many-electron DSF, see Sec. 4.4), $\Gamma_{\text{se}}$ (single-electron Landau level broadening, see Sec. 2.3.1), and $\hbar \omega_c$ (level separation). All of these depend on the magnetic field $B$, and each differently: $\Gamma_{\text{se}} \propto \sqrt{B}$ (for vapor scattering, as is assumed in the following), $\hbar \omega_c \propto B$, and $\Gamma_C \propto n_s^{3/4} / \sqrt{B}$. For a thorough understanding of Coulomb effects, especially with respect to the conflicting results of the earlier CR studies (see the following Sec. 7.1.5), it is therefore instructive to compare the CR behavior at different resonant magnetic fields.

In the present MW setup, the cavity resonance frequency can be tuned between 40 GHz and 60 GHz, and Fig. 7.9 shows two sets of measurements of the CR linewidth vs. electron density $n_s$ at the corresponding resonant magnetic fields $B_1 = 1.43$ T (the same data as in Fig. 7.3) and $B_2 = 2.15$ T. All other parameters are the same for the two data sets. A first general observation is that increase of $B$ shifts the development of the CR linewidth vs. $n_s$, both Coulomb narrowing and broadening, to higher values of $n_s$. The reason is that increase of $B$ generally weakens Coulomb effects, as this reduces $\Gamma_C$, while both $\Gamma_{\text{se}}$ and $\hbar \omega_c$ increase. This already gives a qualitative explanation for the conflicting results of the earlier CR studies (see Sec. 7.1.5 for details).

If we look in more detail, we can also see that the CR linewidth behaves differently at low $n_s$ and at high $n_s$. The noninteracting and the Coulomb narrowing regime $\Gamma_C \ll \hbar \omega_c$ at low $n_s$ are mainly determined by the resonant contribution to the effective collision rate $\nu_{\text{eff}}$, for which one finds from Eq. (4.15) the approximate relation $\nu_R(\omega_c) \propto \Gamma_{\text{se}} / \sqrt{1 + (\Gamma_C / \Gamma_{\text{se}})^2}$. Here, the Coulomb narrowing of $\nu_R$ depends on the ratio $\Gamma_C / \Gamma_{\text{se}} \propto n_s^{3/4} / B$ in the denominator and shifts to higher $n_s$ at an increase of the resonant $B$-field, as noted before. Additionally, $\nu_R$ increases then independently of Coulomb effects due to the increase of $\Gamma_{\text{se}} \propto \sqrt{B}$ in the numerator. In the low-$n_s$ range, the data for different $B$ can therefore be normalized to a single curve by normalizing the linewidth to $\Gamma_{\text{se}}$ and plotting the result vs. $\Gamma_C / \Gamma_{\text{se}}$, as is demonstrated in Fig. 7.10.

On the other hand, in the strongly interacting, Coulomb broadening regime $\Gamma_C \gtrsim \hbar \omega_c$ at high enough $n_s$, the CR linewidth eventually is dominated by the nonresonant contribution to $\nu_{\text{eff}}$, for which one can approximately write $\nu_{\text{NR}}(\omega_c) \propto \Gamma_C / \hbar \omega_c$ according to Eq. (4.16). In this regime, $\Gamma_{\text{se}}$ is unimportant, and the CR linewidth is only determined...
by the Coulomb broadening, which depends on the ratio $\Gamma_C/\hbar \omega_c \propto n_s^{3/4}/B^{3/2}$ and shifts also to higher $n_s$ for an increase of $B$, like the Coulomb narrowing, but with a different $B$-dependence. Ultimately, one obtains thus at high $n_s$ a crossing of the CR linewidth data from different $B$, which can be seen in Fig. 7.9 at $n_s \sim 6 \times 10^8 \text{ cm}^{-2}$. Moreover, $\nu_{NR}$ can clearly not be normalized in the same way as $\nu_R$, which leads in Fig. 7.10 to the discrepancies visible at high $n_s$ between the $B_1$ and $B_2$ data.

### 7.1.5 Comparison to previous CR studies

It is instructive to review now again the earlier CR studies summarized in chapter 5, and show that their partly unexplained and even conflicting results are in entire agreement with the picture emerging from the present work.

In Fig. 7.11, two representative data sets from the CR study of Edel’man shown in Fig. 5.1 are compared with the results of the new memory function theory for Coulomb effects. The earlier experiments measured almost exclusively in the ripplon scattering regime, where the Landau level broadening $\Gamma_{se} \simeq \Gamma_r$ is very small, and can usually be neglected. They did not approach the noninteracting limit $\Gamma_C \ll \Gamma_{se}$ where the CR linewidth becomes constant, while this limit is indeed reached in the measurements presented in the preceding sections, conducted in the vapor atom scattering regime.

Apart from this, the Coulomb effects in the ripplon regime seen in Fig. 7.11 are essentially the same as those in the vapor scattering regime seen in Fig. 7.3, and in entire agreement with the new theory from Sec. 4.5.1: At low $n_s$ (low holding field) in Fig. 7.11 is $\hbar \omega_c \gg \Gamma_C \gg \Gamma_{se}$, and the data show strong Coulomb narrowing of the CR linewidth, because the Doppler effects due to the fast FEF-induced Hall drift velocities of the electrons reduce the rate of electron-scatterer collisions. This regime can be described by the resonant contribution to the effective collision rate $\nu_{eff}$ alone [term $N = 1$ in Eq. (4.17) for ripplon scattering], which corresponds to the approximation of the earlier many-electron CR theory of Dykman and Khazan [6], cf. Fig. 5.2. The strong re-increase of the linewidth at high $n_s$, previously unexplained, can in the new theory be explained by the scattering of electrons to higher Landau levels, occurring for $\Gamma_C \gtrsim \hbar \omega_c$ and giving rise to an additional nonresonant contribution to $\nu_{eff}$ [terms with $N \neq 1$ in Eq. (4.17)]. Taking into account both contributions results in good quantitative
agreement between theory and measured data.

Figure 7.11: Two representative data sets at different temperatures (○, ○) from Fig. 5.1 (taken from Ref. [8]), together with the results of memory function theory for Coulomb effects from Sec. 4.5.1 (solid curves). The data were taken at saturation, when a holding field of \( E_{\perp} = 100 \text{ V/cm} \) corresponds to an electron density of \( n_s = 1.1 \times 10^8 \text{ cm}^{-2} \). At both temperatures, ripplon scattering [Eq. (4.17)] is dominant; at the higher temperature, vapor atom scattering has also been included in the theoretical curves according to Eq. (4.18). The dashed curves show the resonant contribution to the CR linewidth alone [term \( N = 1 \) in Eq. (4.17) for ripplon scattering], which corresponds to the approximation of Ref. [6].

The scattering of electrons to higher levels can also explain the conflicting results obtained by Wilen and Giannetta shown in Fig. 5.3, of which the low \( n_s \) part is compared in Fig. 7.12 with the result of the new many-electron memory function theory for ripplon scattering [Eq. (4.17)]. Here, the magnetic field is very low (0.071 T), which results in the Coulomb narrowing regime \( \hbar \omega_c \gg \Gamma_c \gg \Gamma_{se} \) being shifted to very low \( n_s \), outside the measured range. This shifting effect can be directly seen in Fig. 7.9. The measured data in Fig. 7.12 correspond always to \( \Gamma_c \gg \hbar \omega_c \), when the nonresonant contribution to \( \nu_{\text{eff}} \) is dominant, and thus only Coulomb broadening of the CR linewidth is observed.

It should be noted that for these data actually the condition (4.4) for the quasi-uniformity of the FEF \( l_{B} \ll \sqrt{\delta_f^2} \) is no longer fulfilled (in fact, \( l_{B} \gg \sqrt{\delta_f^2} \) for Fig. 5.3), and in a strict sense, the treatment of chapter 4 is no longer grounded. Still, the data can be well described by Eq. (4.17) at least within the \( n_s \) range shown in Fig. 7.12, while at higher \( n_s \) discrepancies arise.

Figure 7.12: The low-\( n_s \) part of the data (○) from Fig. 5.3 (taken from Ref. [11]), together with the result of many-electron memory function theory for ripplon scattering, Eq. (4.17) (solid curve). The dashed curve shows the resonant term \( N = 1 \) in Eq. (4.17) alone, which corresponds to the approximation of Ref. [6].
7.2 Linear DC magnetoconductivity

In the balance equation/memory function formalism (Sec. 3.3), AC and DC conductivity are treated on the same footing, and they should be affected very similarly by Coulomb effects, as discussed in Sec. 4.5.2. A complete theory of Coulomb effects should therefore describe the DC conductivity as well as the AC one. In the following, it is shown that this is indeed the case for the new theory described in chapter 4, i.e., the balance equation method together with the many-electron DSF of Sec. 4.4, and that thus Coulomb effects in the AC and DC case can be understood in the same general picture.

7.2.1 Dependence on magnetic field and electron density

Fig. 7.13 shows two measurements of the inverse DC longitudinal conductivity $1/\sigma_{xx}(B)$ versus magnetic field $B$ at two different electron densities $n_s$ at saturation. The measured Corbino admittance was here analyzed by the numerical method described in Sec. 6.4.2 to obtain $1/\sigma_{xx}$ in absolute units.

According to the balance equation method discussed in Sec. 3.3.2, one has $\sigma_{xx}(B) \simeq e^2 n_s \nu_{\text{eff}}(B)/m_e \omega_c^2$ [Eq. (3.16)], where $\nu_{\text{eff}}(B)$ is the DC effective collision rate. For a more detailed and comparative view, in Fig. 7.14 is plotted the ratio $\omega_c/\nu_{\text{eff}}(B)$ or tangent of the Hall angle obtained from the two measurements in Fig. 7.13; this gives better resolution at low $B$. In the following, it is shown that the DC collision rate is affected by Coulomb effects very similarly to the AC one (i.e., the CR linewidth), and that both are equally well described by the Coulomb effect theory of chapter 4.

Figure 7.13: Experimental data of the inverse DC magnetoconductivity $1/\sigma_{xx}$ vs. magnetic field in the vapor scattering regime at two different SE densities $n_s$ at saturation (○, □). Also shown are the respective results of balance equation theory for $\sigma_{xx}$ [Eq. (3.16)] with different DC effective collision rates for vapor scattering $\nu_{\text{eff}}^{(a)}(B)$ (ripplon scattering is here negligible): the zero-field collision rate $\nu_a$ [Eq. (2.11), dotted curves] corresponding to Drude theory; the single-electron collision rate $\nu_{\text{se}}^{(a)}(B)$ [Eq. (3.18), dashed curves]; and the many-electron collision rate $\nu_{\text{me}}^{(a)}(B)$ from the quantum limit correction Eq. (4.20) for the higher $n_s$ [solid curve in (b)].

As discussed in Sec. 4.5.2, contrary to CR, a single measurements of the DC magnetoconductivity at fixed $n_s$ usually covers a large range of $B$, so that the strength of Coulomb effects varies within one measurement due to the different $B$-dependencies of...
Figure 7.14: The tangent of the Hall angle $\omega_c/\nu_{\text{eff}}(B)$ [$\nu_{\text{eff}}(B)$ is the DC effective collision rate] of the experimental data from Fig. 7.13 in the low $B$ range ($\bigcirc$, $\square$), together with theoretical results for $\nu_{\text{eff}}(B)$: the zero-field (Drude) result $\nu_a$ [Eq. (2.11), dotted curve]; the single-electron result $\nu_{\text{se}}^{(a)}(B)$ [Eq. (3.18), dashed curve]; both $\nu_a$ and $\nu_{\text{se}}^{(a)}(B)$ are here approximately the same for the two $n_s$; the many-electron results $\nu_{\text{me}}^{(a)}(B)$ from the quantum limit correction Eq. (4.20); and the results of numerical calculations of $\nu_{\text{me}}^{(a)}(B)$ from Eq. (3.17) for the two $n_s$ (dash-dotted curves); and the results of numerical calculations of $\nu_{\text{me}}^{(a)}(B)$ from Eq. (3.17) for the two $n_s$ (solid curves) and the results of numerical calculations of $\nu_{\text{me}}^{(a)}(B)$ from Eq. (3.17) for the two $n_s$ (dash-dotted curves), using Eq. (2.34) with Eq. (4.11) to approximate the many-electron DSF with higher populated Landau levels.

The characteristic parameters: the Coulomb parameter $\Gamma_C = \sqrt{2\epsilon E_t(0)}l_B \propto 1/\sqrt{B}$ (describing the Doppler broadening of the many-electron DSF) decreases with $B$, while the single-electron Landau level broadening $\Gamma_{\text{se}} \propto \sqrt{B}$ (for vapor scattering) and the level separation $\hbar\omega_c \propto B$ both increase with $B$ (cf. Sec. 7.1.4, which investigates the $B$-dependence in a limited range in CR, too).

At high enough $B$, one therefore always attains the noninteracting limit $\Gamma_C \ll \Gamma_{\text{se}}$, where the DC effective collision rate $\nu_{\text{eff}}(B)$ has its single-electron value $\nu_{\text{se}}^{(a)}(B)$ [Eq. (3.18) for pure vapor scattering], represented by the dashed curve(s) in Figs. 7.13 and 7.14.

Subsequent reduction of $B$ increases the strength of Coulomb effects. One then first enters the regime $\hbar\omega_c \gg \Gamma_C \gtrsim \Gamma_{\text{se}}$, where Doppler effects due to the fast FEF-induced Hall drift velocities of the SE reduce the rate of collisions with scatterers, i.e., reduce the many-electron effective collision rate $\nu_{\text{me}}(B)$ compared to $\nu_{\text{se}}^{(a)}(B)$, which increases both $1/\sigma_{xx}$ in Fig. 7.13 (well visible here only in the high $n_s$ data) and the Hall angle in Fig. 7.14 (here visible at both $n_s$). This corresponds to the Coulomb narrowing of the CR linewidth discussed in Sec. 7.1.1\textsuperscript{1}. With further decrease of $B$, one enters the regime of strong Coulomb effects $\Gamma_C \gtrsim \hbar\omega_c$, where SE scattering between different Landau levels leads to re-increase of $\nu_{\text{me}}(B)$ (i.e., re-decrease of $1/\sigma_{xx}$ and the Hall angle in Figs. 7.13 and 7.14 respectively, below about 1 T), this corresponds to the Coulomb broadening of the CR linewidth. These Coulomb effects vs. decreasing $B$, previously discussed in Sec. 4.5.2, are well visible in both experimental data sets in Fig. 7.14.

The value of $n_s$ here mainly determines at what $B$-field (with decreasing $B$) Coulomb effects become noticeable, due to the $n_s$-dependence of the FEF $E_t(0) \propto n_s^{3/4} T^{1/2}$ entering $\Gamma_C$. For the data measured at low density $n_s \simeq 10^7 \text{ cm}^{-2}$ in Fig. 7.13(a), Coulomb effects and the resulting deviations from the single-electron behavior become relevant only at very low $B$, where on the typical scale of Fig. 7.13(a), which rather focuses on the quantum limit, we have essentially no resolution. Overall, these data are in Fig. 7.13(a) well described by single-electron theory alone [dashed curve in Fig. 7.13(a)]. By pre-

\textsuperscript{1}In the noninteracting and beginning Coulomb effect regimes, DC and AC collision rates differ in memory function theory formally essentially only in that the first is determined by SE scattering inside Landau levels, and the second by scattering between adjacent Landau levels (cf. Secs. 4.5.1 and 4.5.2).
senting the same data in form of the Hall angle in Fig. 7.14, the deviations from the single-electron behavior (dashed curve in Fig. 7.14) are better visible, and can be compared with many-electron theory: Using the quantum limit correction Eq. (4.20) to obtain the many-electron effective collision rate $\nu_{\text{me}}(B)$ (lower solid curve in Fig. 7.14) appears for the low-$n_s$ data not a very good approximation, most likely because for these data Coulomb effects develop only at such low $B$ that the population of higher Landau levels is already quite important and Eq. (4.20) is then no longer strictly valid, as already noted in Sec. 4.5.2. However, a numerical calculation of $\nu_{\text{me}}(B)$ from the general expression Eq. (3.17) (lower dash-dotted curve in Fig. 7.14), using Eq. (2.34) modified according to Eq. (4.11) to approximate the many-electron DSF, which treats Coulomb effects in higher populated levels more strictly, can correct the deviations.

In contrast, for the data at high density $n_s \approx 10^8\,\text{cm}^{-2}$ in Fig. 7.13(b), Coulomb effects become relevant and visible at much higher $B$, well inside the quantum limit, and the resulting deviations from the single-electron behavior are for these data well described by the quantum limit many-electron correction Eq. (4.20) [solid curve in Fig. 7.13(b), and upper solid curve in Fig. 7.14]. The numerical calculation of $\nu_{\text{me}}(B)$ (upper dash-dotted curve in Fig. 7.14) gives here only a comparatively small correction; in fact, it is less appropriate in the quantum limit, as the substitution Eq. (4.11) in the DSF neglects the dependence $\Gamma_0 = \Gamma_0(E_I)$ of the ground Landau level width on the distribution of the FEF, as discussed in Sec. 4.4.

Interestingly, the identical, almost linear behavior of both experimental data sets at very low $B$ in Fig. 7.14, which is close to classical Drude theory, is at least approximated by the $\nu_{\text{me}}(B)$ from the quantum limit correction (solid curves), and almost exactly reproduced by the numerical calculations (dash-dotted curves), and is here not due to the system becoming classical, but the result of electron scattering between different Landau levels, as already noted in Sec. 4.5.2 (see Fig. 4.10).

A customary way to present the DC magnetoconductivity is to normalize $\sigma_{xx}$ by the zero-field conductivity $\sigma_0 = e^2n_s/m_e\nu_a$, where $\nu_a$ is the zero-field collision rate [Eq. (2.11)]. This is done in Fig. 7.15 for the two experimental data sets from Fig. 7.13. In this representation, the two measurements cross at $B \approx 7\,\text{T}$. This is due to the holding field dependence of $\nu_a$, which gives at saturation a larger value of $\nu_a$ for the high-$n_s$ data, and thus a smaller value of $\sigma_0/\sigma_{xx} \propto \nu_a^{-3/2}$ in the single-electron limit at high $B$ (see the single-electron curves in Fig. 7.15).

Figure 7.15: The two measurements at different electron densities $n_s$ (○, □) and the theoretical curves from Fig. 7.13, normalized to the respective theoretical zero-field DC conductivity $\sigma_0 = e^2n_s/m_e\nu_a$, where $\nu_a$ is the zero-field DC collision rate [Eq. (2.11)]. Shown are again the results of single-electron theory [Eq. (3.18), dashed curves] for the two $n_s$, and the result of the many-electron correction [Eq. (4.20), solid curve] for the higher $n_s$. The Drude theory (dotted curve) is in this representation almost the same for the two $n_s$. 
7.2.2 Dependence on scatterer density

Same as in CR (Sec. 7.1.3), one can observe in the DC conductivity different dependencies on the density of vapor atoms $n_a$—or rather, on the zero-field collision rate $\nu_a \propto n_a$—in the noninteracting and the strongly interacting limit, which reflects the many-electron transformation of the DSF.

Fig. 7.16 shows another two measurements of $1/\sigma_{xx}$ vs. $B$, conducted at comparable electron densities $n_s$ but at a higher temperature than those in Fig. 7.15. Again, the low-density and high-density data are well described by single-electron theory [Eq. (3.18)] and the many-electron correction Eq. (4.20) respectively.

**Figure 7.16:** Another two measurements of the static magnetoconductivity $(\bigcirc, \blacksquare)$, presented in the same way as those in Fig. 7.15 and conducted at approximately the same electron densities $n_s$ but at a higher temperature $T = 1.36 \text{ K}$ (higher scatterer density). The meaning of the theoretical curves is the same as in Figs. 7.15 and 7.13: the dotted curve shows the result of Drude theory (approximately the same for both $n_s$), the dashed curves the respective results of single-electron theory [Eq. (3.18)] for the two $n_s$, and the solid curve shows the result of the many-electron correction Eq. (4.20) for the higher $n_s$.

Fig. 7.17 shows the tangent of the Hall angle $\omega_c/\nu_{\text{eff}}(B)$ of the high-$n_s$ data both from Fig. 7.15 and Fig. 7.16. As in the case of CR in Sec. 7.1.3, the difference in temperature between the two data sets in Fig. 7.17 corresponds to a change in $n_a$ and $\nu_a$ by almost exactly a factor of 2.

**Figure 7.17:** The tangent of the Hall angle $\omega_c/\nu_{\text{eff}}(B)$ of the high-density data ($n_s \approx 10^8 \text{ cm}^{-2}$) from Fig. 7.15 ($T = 1.23 \text{ K}, ^4\text{He vapor atom density } n_a = 6.2 \times 10^{24} \text{ m}^{-3}$) and Fig. 7.16 ($T = 1.36 \text{ K}, n_a = 12.3 \times 10^{24} \text{ m}^{-3}$). The dashed curves show the respective results of single-electron theory [Eq. (3.18)] for the two temperatures, and the solid curves the respective results of the many-electron correction Eq. (4.20).

In the DC conductivity, the noninteracting and the strongly interacting limit are encountered within a single measurement with fixed $n_s$ at high and low magnetic field respectively. Comparing the two data sets in Fig. 7.17, it can be observed that their difference is relatively smaller in the high-$B$ limit at 10 T than in the low-$B$ limit at,
7.3 Nonlinear conductivity

The results presented so far correspond to the linear regime of the conductivity, where the external AC or DC driving fields are very small and the SE are approximately in thermal equilibrium with the scatterers. Additionally were studied the nonlinear effects induced by a strong enough microwave (MW) driving field under cyclotron resonance (CR) conditions, when the electrons are effectively heated to a temperature $T_e$ higher than that of the scatterers $T$. The effects were observed both in the CR signal itself (Sec. 7.3.1) and in the simultaneously measured DC magnetoconductivity (Sec. 7.3.2).
The heating behavior is of interest with respect to Coulomb effects, as the fluctuational electric field (FEF) depends not only on electron density but also on electron temperature: \( E^{(0)}_f \propto n_{e}^{3/4} T_{e}^{1/2} \). Moreover, it allows to study not only the momentum relaxation but also the energy relaxation of the electrons. Lastly, heating can drastically change the electrons’ state from a two-dimensional system in the magnetic quantum limit to a quasiclassical, three-dimensional one; this is in detail discussed in Sec. 7.3.3.

### 7.3.1 Cyclotron resonance

Fig. 7.19 shows a series of CR measurements at low electron density \( n_s \) at saturation. Between each measurement, the MW power \( P_{MW} \) reaching the cavity was increased by 3 dB, corresponding to a factor of 2, starting from \( P_{MW} \sim 1 \text{nW} \) used for measurements in the linear regime. Another series of measurements, conducted similarly but at a higher \( n_s \), is shown in Fig. 7.20. In Fig. 7.21(a) the linewidth values of the measurements from Figs. 7.19 and 7.20 are plotted versus \( P_{MW} \), together with those from two other measurement series at higher values of \( n_s \).

**Figure 7.19:** A series of CR measurements at increasing microwave powers \( P_{MW} \) fed to the cavity, conducted at low density \( n_s = 1.3 \times 10^7 \text{ cm}^{-2} \) at saturation. All other conditions were the same as for the measurements of Fig. 7.1 (temperature \( T = 1.28 \text{K} \), MW frequency \( f_{MW} = 40 \text{GHz} \), resonant magnetic field \( B = 1.42 \text{T} \)). Starting at \( P_{MW} \sim -60 \text{dBm} \) (1 nW) for the lowest curves, \( P_{MW} \) was increased by 3 dB (about a factor of two) between each measurement. The letters marking absorption curves are for comparison with the DC measurements in Fig. 7.24.

For completeness, it is first pointed out that at all densities in Figs. 7.19–7.21(a), the CR line shows essentially no evolution vs. \( P_{MW} \) in the low-power range from \(-60 \text{dBm} \).
Figure 7.20: Another series of CR measurements at increasing MW powers $P_{MW}$, conducted in the same conditions as those in Fig. 7.19, but at a higher density $n_s = 4.2 \times 10^7$ cm$^{-2}$. The letters marking absorption curves are for comparison with the DC measurements in Fig. 7.25.

Figure 7.21: (a) CR absorption linewidth vs. MW power $P_{MW}$ at different electron densities $n_s$ at saturation (temperature $T = 1.28$ K, MW frequency $f_{MW} = 40$ GHz, resonant magnetic field $B = 1.42$ T). The data were obtained by fitting with Lorentzians from the measurements of Figs. 7.19 and 7.20, and similar measurements at two higher densities. (b) The linewidth values in the linear regime at $P_{MW} \approx -60$ dBm from (a) (full symbols) included in the linear linewidth data of Fig. 7.3 (open symbols).
up to almost $-50 \text{ dBm}$. This confirms that the CR results presented previously in Sec. 7.1 represent the linear regime of the conductivity, where electron heating is negligible. Once $P_{\text{MW}}$ is increased above $-50 \text{ dBm}$, strong nonlinear effects become visible in Figs. 7.19–7.21. In the following, these effects are first shortly discussed phenomenologically, then a quantitative comparison with theory is presented for the AC momentum collision rate.

At low $n_s$, the CR linewidth in Fig. 7.21(a) shows successive strong narrowing and broadening with increasing $P_{\text{MW}}$ (in the following called power narrowing and power broadening respectively), which is phenomenologically rather similar to the Coulomb effects observed vs. $n_s$ in the linear regime in Fig. 7.3. This is as might be expected from the dependence of the FEF (which is the origin of Coulomb effects, see Sec. 4.1) on electron temperature: $E_l^{(0)} \propto n_s^{3/4} T_e^{-1/2}$. Moreover, at higher $n_s$, when the CR linewidth is already reduced by Coulomb effects vs. $n_s$ in the linear regime (cf. Sec. 7.1.1), the power narrowing becomes successively less pronounced until at the highest $n_s$ in Fig. 7.21(a), only the power broadening is observed. For comparison, in Fig. 7.21(b) the linear linewidth data at $P_{\text{MW}} \simeq 1 \text{ nW}$ of all the measurements from Fig. 7.21(a) are plotted vs. $n_s$, together with the other linear data from Fig. 7.3. This behavior is further suggestive of a relationship between the Coulomb effects observed in dependence of $n_s$ in the linear regime and the behavior at heating, at least for the power narrowing.

Still, the heating behavior may have alternative explanations (like population of higher Landau or surface levels), and a more quantitative analysis and comparison with theory is required before drawing any conclusions. A problem is here that the details of the nonlinear CR line shape are different from those of the linear one, and its analysis requires some caution. For instance, in a single measurement vs. $B$, the power dissipated in the SE, and thus the electron temperature $T_e$, vary strongly across the resonance following the local value of the CR absorption itself [see Eq. (6.10)]. This is visible, for example, in the fact that the power narrowing in Figs. 7.19 and 7.20 (see also Fig. 7.26 for a close-up) starts in the form of a narrow peak developing in the middle of an otherwise almost unchanged CR absorption line (in the figures best visible at $P_{\text{MW}} = -39 \text{ dBm}$): the SE are still in the linear regime at the sides of the line, and only in the middle the power absorption, and thus $T_e$, are high enough to induce noticeable nonlinear effects. Additionally, the effect is self-energizing, as the linewidth narrowing of course induces an additional increase of the power absorption in the middle of the resonance.

In the linear case, the most convenient parameter for a quantitative analysis is the CR linewidth $\gamma_{\text{CR}}$, which in memory function theory (Sec. 3.3.1) corresponds to the AC effective collision rate $\nu_{\text{eff}}(\omega)$ at resonance $\omega = \omega_c$: $\gamma_{\text{CR}} = 2 \nu_{\text{eff}}(\omega_c)$. In the nonlinear case, the significance of the linewidth is less clear. Strictly speaking, it does not correspond to an effective collision rate at a single electron temperature, as $T_e$ varies across the resonance. Still, also in the linear case, $\nu_{\text{eff}}$ may vary strongly around $\omega = \omega_c$ without invalidating the above relation (see Figs. 3.2 and 3.3). We will therefore assume that also at heating, the CR linewidth approximately corresponds to the AC collision rate at $\omega = \omega_c$ alone, i.e., at a single electron temperature.

For checking, additionally an alternative method is used to obtain $\nu_{\text{eff}}(\omega_c)$, namely from the peak value of the CR absorption $\text{Re} \sigma_{xx}(\omega)$ at $\omega = \omega_c$ (marked by the vertical dotted line in Figs. 7.19 and 7.20). In memory function theory, the peak value in absolute units is directly related to the AC collision rate at this point: $\text{Re} \sigma_{xx}(\omega_c) \simeq e^2 n_s/2 m_e \nu_{\text{eff}}(\omega_c)$ [Eq. (3.13)]. Essentially the same method is used to obtain the DC collision rate in Sec. 7.3.2. In the AC case, this method should be rather reliable at high MW power, but unfortunately it is probably much less so at low MW power. The reason is
that the normalization of the spectrometer output (see Sec. 6.3.1) can only be determined with sufficient accuracy at rather high MW power, for low power measurements it is then scaled according to the setting of the MW attenuator, whose calibration becomes increasingly less accurate at high attenuation. This is probably the reason that even in the very low power range when the linewidth data clearly indicate equilibrium in Figs. 7.21(a) and 7.22, the absorption peak data in Fig. 7.23 still tend to show a spurious slight decrease vs. increasing power. In the following, the values of $\nu_{\text{eff}}(\omega_c)$ obtained from the CR linewidth and from the absorption peak value will therefore both be used for a more detailed discussion.

Also, one needs to convert the MW power $P_{\text{MW}}$ reaching the cavity to a quantity significant for the electron system. By inserting $P_{\text{MW}}$ and the normalized spectrometer signal $r_{\text{MW}}$ into Eq. (6.10), one can directly determine the power $P$ dissipated in the electron system in absolute units. The latter is an unambiguous and convenient parameter for a quantitative discussion. In the following, the behavior of $\nu_{\text{eff}}(\omega_c)$ vs. $P$ at $\omega = \omega_c$ is compared with quantitative theoretical calculations.

### AC momentum collision rate at cyclotron resonance

The upper graph in Fig. 7.22 replots the experimental linewidth data from Fig. 7.21(a) in terms of the AC effective momentum collision rate at resonance $\nu_{\text{eff}}(\omega_c)$ vs. the power $P/N_s$ dissipated per electron at this point ($N_s$ is the total number of electrons). The same plot is done in Fig. 7.23 for the $\nu_{\text{eff}}(\omega_c)$ data obtained from the CR absorption peak values, as explained above. Except at very low power, the two methods to obtain $\nu_{\text{eff}}(\omega_c)$ differ by not more than about 30%. At high power, the $\nu_{\text{eff}}$ data from the CR absorption peak values should be rather reliable, while at low power, one should probably rather rely on the $\nu_{\text{eff}}$ data from the CR linewidth, as discussed above.

Also shown in Figs. 7.22 and 7.23 are the results of theoretical calculations using the balance equations method (Sec. 3.3): Assuming the effective temperature approximation (see Sec. 3.3.3) to be valid, and using the energy balance equation [Eq. (3.23)] together with calculations of the effective energy collision rate $\nu_{\text{en}}$, one can establish a relation between $P$ and the effective electron temperature $T_e$. The latter given, one can in turn calculate the effective momentum collision rate $\nu_{\text{eff}}$, both for the AC and DC case (the latter is discussed in Sec. 7.3.2). The theoretical treatment reduces thus to calculations of $\nu_{\text{en}}$ and $\nu_{\text{eff}}$ in dependence of $T_e$, which requires essentially only the knowledge of the correct form of the dynamic structure factor (DSF) of the electron system. Explicit expressions for the DSF can be found for two limiting cases, at low and high $T_e$:

- At low $T_e$, one can treat the system as strictly two-dimensional, and use the DSF of an electron system in magnetic field to calculate $\nu_{\text{en}}$ and $\nu_{\text{eff}}$ according to Eq. (3.22) and Eq. (3.14) respectively [or Eq. (3.17) for $\nu_{\text{eff}}$ in the DC case]. The 2D calculations were employed in the low $T_e$ range up to 2.5 K (see the solid curves in the lower graph in Fig. 7.22), corresponding to the low power range up to about 0.4 fW. In this temperature range, it seems reasonable to still expect a predominant population of the surface ground state (see Sec. 7.3.3 for more detail). The calculations were performed numerically, as it is necessary to take into account higher populated Landau levels. Shown in Figs. 7.22 and 7.23 are the results of a single-electron calculation (marked SE) using the single-electron DSF [Eq. (2.34)], and the results of many-electron calculations for the four different electron densities $n_s$ (marked □, ○, △, ◇) using the many-electron DSF obtained by modifying
Eq. (2.34) according to Eq. (4.11). Ripplon scattering turned out to be always negligible in $\nu_{en}$ and $\nu_{eff}$, except for $\nu_{eff}$ at the highest electron density.

- At high enough $T_e$, one can assume the electrons to populate mostly high quasi-free surface states and treat the system as effectively three-dimensional, and also neglect the Landau quantization as well as Coulomb effects (see Sec. 3.3.3). In this case, $\nu_{en}$ and $\nu_{eff}$ are given by the simple analytic expressions of Eq. (3.24). In Figs. 7.22 and 7.23, the resulting curves (marked 3D) are plotted in the high $T_e$ range above 10 K, corresponding to the high power range above 0.5 fW.

**Figure 7.22:** Upper graph: The symbol+line data show the AC momentum collision rates $\nu_{eff}(\omega)$ at $\omega = \omega_c$ obtained from the CR linewidth data in Fig. 7.21(a) vs. the power dissipated per electron at this point (magnetic field $B = 1.42$ T, bath temperature $T = 1.28$ K, $n_s$ is the electron density at saturation). The dashed curves show theoretical results for $\nu_{eff}(\omega)$, obtained from: a 2D single-electron calculation (marked SE); 2D many-electron calculations including Coulomb effects for the four $n_s$ values (□, ○, △, ◊); the analytical expressions for 3D free electrons (3D). Details of the calculations are explained in the text. Lower graph: The theoretical energy collision rates $\nu_{en}$ (dashed curves) and electron temperatures $T_e$ (solid curves) corresponding to the different theoretical curves in the upper graph.

One can see in Figs. 7.22 and 7.23 that these purely 2D and 3D calculations can reasonably well explain most of the experimental behavior (without adjustable parameter), except at the lowest density (□), where the decrease of $\nu_{eff}$ (the power narrowing of the CR linewidth) begins almost an order of magnitude in power earlier than predicted from the 2D many-electron calculation for this case. But for the three higher densities (○, △, ◊), the 2D many-electron calculations comparatively well reproduce the decrease of $\nu_{eff}$ observed in the experiment, both concerning its power dependence and the relative change of $\nu_{eff}$. In the theoretical curves, this decrease is entirely due to Coulomb effects, i.e., due to the temperature dependence of the fluctuational electric field $E_f(0) \propto n_s^{3/4}T_e^{1/2}$. In contrast, the single-electron calculation (marked SE) shows almost no variation of $\nu_{eff}$.
in the $T_e$ range up to 2.5 K. The theoretical curves also reproduce the vanishing of the decrease of $\nu_{\text{eff}}$ (vanishing of the power narrowing) observed at the highest density ($\diamond$).

At high power, $\nu_{\text{eff}}$ shows a re-increase (power broadening of the CR linewidth), and the experimental data for the different densities coincide in Figs. 7.22 and 7.23, except some minor deviation for the lowest density in Fig. 7.23. In this power range, the data are very well described by the theory for 3D free electrons (marked 3D), which disregards the Landau quantization and Coulomb effects. The agreement is especially good for the data in Fig. 7.23, which should in this power range be more reliable. In the 3D theoretical curves, the re-increase of $\nu_{\text{eff}}$ is entirely due to the dependence of the 3D single-electron momentum collision rate [Eq. (3.24)] on the electrons’ kinetic energy.

An interesting feature is that the theoretical 2D and 3D curves in Figs. 7.22 and 7.23 almost overlap, although they are plotted for rather different temperature ranges, which indicates that the transition from the 2D to the 3D system state occurs within a very narrow power range, with a fast change in temperature (solid curves in lower graph in Fig. 7.22). This can be explained by the large difference of the 2D and 3D energy collision rates for vapor atom scattering, the 2D one being at low $T_e$ almost an order of magnitude larger than the 3D one at 10 K (dashed curves in lower graph in Fig. 7.22).

In all, the theoretical results presented in Figs. 7.22 and 7.23 seem able to give a rather consistent picture of the experimental behavior, except the early power narrowing at the lowest density. However, they are still incomplete as long as the transition between the 2D and 3D limiting cases is not treated in a more rigorous way. This is rather difficult, as it requires in principle to determine wave function and energy of each higher surface state, and its population, taking into account the effective vertical electric field due to correlation effects (see Sec. 1.1.1), and the excitation spectrum should then also include intra- and inter-level transitions for each occupied surface level.

Such a strict treatment of hot electron transport has been done to date only in a semiclassical approach using the Boltzmann equation [93, 94, 95]. This treatment assumed a constant vertical holding field, but considering the long-range nature of the effective vertical field even at saturation, this should be of minor importance. Interestingly, the authors also found a non-monotonic behavior of the collision rate at heating, similar to that in Figs. 7.22 and 7.23, but only in the ripplon scattering regime. Here, it is entirely due to the population of higher surface states. This effectively decouples the electrons

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**Figure 7.23:** The same as the upper graph of Fig. 7.22, only here the data of the AC effective collision rate $\nu_{\text{eff}}(\omega)$ at $\omega = \omega_c$ (symbol+line data) were obtained from the peak values of the CR absorption.
Experimental results and discussion

from the ripplons and leads thus in the first stage of heating to a drastic decrease of the collision rate at heating, which corresponds in CR to a power narrowing of the linewidth. In later stages, with all electrons effectively 3D, the temperature dependence of the 3D collision rate at vapor atoms leads to a re-increase of the latter, which is the same as the present 3D treatment. The power narrowing due to the population of higher surface states was indeed observed in earlier CR studies in the ripplon scattering regime [8, 110].

However, the phenomenological similarity to the present results is at least partly misleading. First of all, one cannot entirely apply the results of a semi-classical treatment to the present case, where the magnetic quantization is certainly still important at least in the early stages of heating. In fact, in the vapor atom scattering regime, the semi-classical treatment essentially only obtained a monotonic increase of the collision rate at heating, which was indeed observed in experiments at zero magnetic field [111]. Thus, it cannot directly explain the power narrowing in Figs. 7.22 and 7.23. Moreover, from the graphs published in Ref. [93], one can estimate that at least for the holding fields corresponding to the three higher densities in Figs. 7.22 and 7.23, the population of higher surface states should become relevant only when the effective electron temperature is already close to the energy difference of ground and first excited surface state, which is at least 5.7 K (see Sec. 1.1). In the $T_e$ range up to 2.5 K, the purely 2D calculation employed here should still be a reasonable approximation, and it can rather well explain the power narrowing observed in Figs. 7.22 and 7.23 by Coulomb effects.

For completeness, the population of higher surface levels is treated in more detail in Sec. 7.3.3, but as it will indeed turn out that the deviations from the purely 2D and 3D treatments employed so far should be important only for the lowest density in Figs. 7.22 and 7.23, the 2D and 3D results will first be used to also discussed the DC case.

7.3.2 DC magnetoconductivity

Figs. 7.24 and 7.25 shows two series of measurements of the inverse DC magnetoconductivity $1/\sigma_{xx}(B)$ at two electron densities $n_s$, recorded simultaneously with the CR measurements of Figs. 7.19 and 7.20 at various values of the MW power $P_{MW}$ incident on the cavity. The CR-induced heating of the electron system, whose effect on the AC collision rate was discussed in the preceding section, naturally also affects the DC collision rate, and leads to the $P_{MW}$-dependent features seen around the resonant magnetic field in Figs. 7.24 and 7.25. Previous studies [112, 113, 114] investigated CR heating phenomenologically in the DC magnetoconductivity alone, without considering the CR behavior, and disregarding Coulomb effects. In the following, it is demonstrated that the AC and DC collision rates behave essentially in the same way, and that both agree equally well with the quantitative theoretical description given in the preceding section.

Still, AC and DC behavior differ in one specific feature that only appears at very low $n_s$. In Fig. 7.24, the heating effects start at low $P_{MW}$ between $-60$ and $-45$ dBm with a distinct decrease of $1/\sigma_{xx}(B)$ in form of a broad dip developing around the CR position. Due to $1/\sigma_{xx}(B) \propto \omega_c^2/\nu_{eff}(B)$ [Eq. (3.16)], this corresponds to an increase of the DC effective momentum collision rate $\nu_{eff}(B)$ (see also Fig. 7.27). No corresponding feature is observed in the simultaneously measured CR signal in Figs. 7.19 and 7.22/7.23.

This dip in $1/\sigma_{xx}(B)$ has approximately the same shape and width as the CR absorption signal in the linear regime, as shows Fig. 7.26, and it vanishes with increasing

2The DC field gives only $\sim 10^{-18}$ W per electron, estimated by the numerical analysis of Sec. 6.4.2.
7.3 Nonlinear conductivity

Figure 7.24: Upper graph: A series of measurements of the inverse DC magnetoconductivity $1/\sigma_{xx}(B)$ at different MW powers $P_{MW}$, recorded simultaneously with the CR data shown in Fig. 7.19 at low electron density $n_s = 1.3 \times 10^7$ cm$^{-2}$ at saturation (temperature $T = 1.28$ K, MW frequency $f_{MW} = 40$ GHz, resonant magnetic field $B = 1.42$ T). Also shown is an equilibrium measurement (without CR heating) of $1/\sigma_{xx}(B)$ in the range 0–3 T.

Lower graph: A selection of measurements from the upper graph, shown in close-up. The letters marking curves here and in Fig. 7.19 indicate pairs of DC and AC measurements taken together.

Figure 7.25: Another series of measurements of the inverse DC magnetoconductivity $1/\sigma_{xx}(B)$ at different MW powers $P_{MW}$, recorded simultaneously with the CR data shown in Fig. 7.20 at electron density $n_s = 4.2 \times 10^7$ cm$^{-2}$ at saturation. Apart from the higher $n_s$, all parameters are the same as in Fig. 7.24. The letters marking curves here and in Fig. 7.20 indicate pairs of DC and AC measurements taken together. Note that the broad dip visible at the lower $n_s$ of Fig. 7.24 in the low $P_{MW}$ range (see, for example, curve c in Fig. 7.24) is now absent.
electron density \( n_s \), being already absent in Fig. 7.25. A possible explanation is that at low \( n_s \), the effective temperature approximation is no longer fully valid, and the electrons interact not strongly enough to establish internal equilibrium sufficiently fast, i.e., redistribute energy gained in one specific excitation over all available degrees of freedom. The CR excitation could then induce a slight selective overpopulation of higher Landau levels (assuming initially quantum limit conditions), without noticeably changing the electron distribution inside the levels, the latter not being directly influenced by the CR excitation. Electrons in higher levels have a higher typical value of the momentum exchange at scattering, which acts towards increasing the effective collision rate in balance equation theory (Sec. 3.3). In CR, this effect is compensated for by the electrons in higher levels being able to also make transitions to lower ones, which gives a gain contribution to the center of mass (CM) motion of the whole system and acts towards decreasing the AC effective collision rate. In the DC case, loss and gain contributions to the CM motion come rather from transitions inside the levels, and if the electron distribution inside the levels does not change, a net increase of the DC effective collision rate should result.

All other features observed at heating in the AC and DC case correspond to each other. Above \( P_{\text{MW}} = -45 \text{ dBm} \), in the same power range when the CR absorption in Figs. 7.19 and 7.20 shows strong power narrowing in form of a narrow peak appearing inside the linear signal, the DC signal in Figs. 7.24 and 7.25 develops a similar peak at the CR position. In either case, his corresponds to a decrease of the AC and DC collision rate respectively at \( \omega = \omega_c \). Upon further heating, when the CR power narrowing reverses into a power broadening, the DC signal develops a second dip around the CR position. This corresponds to a re-increase of the AC and DC collision rate respectively at \( \omega = \omega_c \).

For a quantitative analysis, one can proceed in essentially the same way as in CR

**Figure 7.26:** Comparison between the CR absorption line shape \( \text{Re} \sigma_{xx}(\omega) \) (dashed curves) and the simultaneously measured DC magnetoconductivity \( 1/\sigma_{xx}(B) \) (solid curves) for the low-density data \((n_s = 1.3 \times 10^7 \text{ cm}^{-2})\) of Figs. 7.19 and 7.24 at two different MW input powers \( P_{\text{MW}} \). The DC magnetoconductivity is normalized to the equilibrium conductivity \( \sigma_{xx}^{(eq)}(B) \) measured in the absence of heating and to the MW power \( P_{\text{MW}} \). Also shown is the inverted CR absorption line \(-\text{Re} \sigma_{xx}(\omega)\) (dotted curves) for better comparison with the dip in the DC magnetoconductivity. Note that this dip has the same shape as the CR absorption line in the linear regime.
in Sec. 7.3.1. From the values of $\sigma_{xx}(B)$ at the CR position $\omega = \omega_c$ (dotted vertical line in Figs. 7.24 and 7.25) one can determine in absolute units the DC collision rate $\nu_{\text{eff}}(B)$ at this point, using $\sigma_{xx}(B) \simeq e^2 n_s \nu_{\text{eff}}(B)/m_e \omega_c^2$ [Eq. (3.16) for $\omega_c \gg \nu_{\text{eff}}(B)$]. This corresponds to the determination of the AC collision rate from the CR absorption peak in Sec. 7.3.1, whose results were shown in Fig. 7.23, but with the important difference that this method is much more reliable in the DC case due to the absence of scaling.

In the following, the behavior of $\nu_{\text{eff}}(B)$ at heating is compared with the DC results of the theoretical calculations previously employed in Sec. 7.3.1 for the AC case.

**DC momentum collision rate at cyclotron resonance**

Fig. 7.27 shows in the upper graph for the same four densities $n_s$ whose AC data were presented in Figs. 7.22/7.23 the measured DC effective momentum collision rate $\nu_{\text{eff}}(B)$ at the CR position $\omega = \omega_c$ vs. the power $P/N_s$ dissipated per electron at this point. Also shown are the theoretical results for $\nu_{\text{eff}}(B)$ from the 2D and 3D calculations described on page 115f. The results for the energy collision rate $\nu_{\text{en}}$ and electron temperature $T_e$ (lower graph in Fig. 7.27) stay the same as in the lower graph in Fig. 7.22.

**Figure 7.27:** The symbol+line data in the upper graph show the DC momentum collision rates $\nu_{\text{eff}}(B)$ at CR $\omega = \omega_c$ ($B = 1.42$ T, $T = 1.28$ K) vs. the power dissipated per electron at this point for the same four densities $n_s$ whose AC collision rates were shown in Figs. 7.22 and 7.23. The data were obtained from the measurements of Figs. 7.24 and 7.25 and similar measurements for the other densities. The dashed curves in the upper graph show the theoretical $\nu_{\text{eff}}(B)$ obtained by the same 2D and 3D calculations as employed for the AC case in Figs. 7.22 and 7.23, details are explained the text and on page 115f in Sec. 7.3.1. The theoretical energy collision rates $\nu_{\text{en}}$ and electron temperatures $T_e$ in the lower graph are the same as in Fig. 7.22.

Comparing Fig. 7.27 to Figs. 7.22/7.23, one can see that the measured collision rates behave at heating very much the same in the AC and DC case, except the initial
increase of the DC rate at the lowest \( n_s \), which corresponds to the dip in \( 1/\sigma_{xx}(B) \) discussed above. Also, both the AC and the DC data agree equally well with theory.

Like in the AC case in Figs. 7.22/7.23, the 2D calculation was in Fig. 7.27 employed for the low \( T_e \) range up to 2.5 K, corresponding to a power range up to about 0.4 fW, only with the DC formula [Eq. (3.17)] for the momentum collision rate instead of the AC one. As in Figs. 7.22/7.23, the 2D many-electron calculations for the different densities in Fig. 7.27 (marked \( \Box, \bigcirc, \triangle, \diamond \)) rather well reproduce the decrease of the momentum collision rate observed experimentally, concerning both the power dependence and the relative change of \( \nu_{\text{eff}}(B) \), except at the lowest density (\( \Box \)), where, like in the AC case, the decrease occurs at much smaller power than expected from theory. At the three higher \( n_s \), the agreement with theory is here even better than in the AC case.

The decrease of \( \nu_{\text{eff}}(B) \) is somewhat more pronounced in the DC case, both in the experimental data and the theoretical curves, due to the stronger dependence of Eq. (3.17) on electron temperature. As a result, already the 2D single-electron DC calculation (marked SE) shows a distinct decrease vs. \( T_e \) in Fig. 7.27, which is entirely absent from the AC result in Figs. 7.22/7.23. A similar difference can be seen both in the experimental data and the 2D many-electron calculations at the highest density (\( \diamond \)). Here, the power narrowing has completely vanished in the AC case in Figs. 7.22/7.23, while there is still a distinct decrease of \( \nu_{\text{eff}}(B) \) visible in the DC case in Fig. 7.27.

In the high \( T_e \) range above 10 K (power range above 0.5 fW), the 3D analytic expressions for \( \nu_{en} \) and \( \nu_{\text{eff}} \) [Eq. (3.24)] were used again in Fig. 7.27, these are the same for the AC and DC case. Again, the experimental data are reasonably well described by the single 3D curve, although the experimental data at the different \( n_s \) do not coincide as well as in the AC case in Figs. 7.22/7.23. A possible explanation is that the strong heating (\( T_e \) reaches tens of kelvin here) leads to a small spatial redistribution of the electrons. The Sommer-Tanner method (Sec. 6.4) is generally rather sensitive to geometry, and should be stronger affected by such a spatial redistribution than the MW measurement.

In all, the 2D and 3D calculations can well explain most of the experimental behavior at heating, both in the AC and DC case, without adjustable parameter. It remains to discuss if the respective \( T_e \) ranges assumed are reasonable, this is done in the next section.

### 7.3.3 Population of higher surface states

By the WKB method, as outlined in Sec. 1.1.2, one can calculate the energy levels in the effective vertical potential and determine their relative population in dependence of temperature. This allows to treat the population of higher surface states at heating in a more quantitative way, and to verify whether the validity ranges assumed for the 2D and 3D theoretical treatments described on page 115\( f \) are reasonable.

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<td>(&lt; 10^{-3})</td>
<td>0.02</td>
<td>0.11</td>
<td></td>
</tr>
</tbody>
</table>

**Table 7.1:** Ground state fraction \( \Delta_1 \) [Eq. (1.9)] at different saturation electron densities \( n_s \) and electron temperatures \( T_e \), calculated by the WKB method (see Sec. 1.1.2).
7.3 Nonlinear conductivity

Table 7.1 shows the calculated fraction of electrons in the ground state $\Delta_1$ [Eq. (1.9)] at different electron temperatures $T_e$ for the four densities $n_s$ whose AC and DC collision rates at heating were shown in Figs. 7.22/7.23 and 7.27. According to Table 7.1, the fraction of electrons in higher levels $(1 - \Delta_i)$ at $T_e = 2.5$ K should be less than 11% for the three higher $n_s$ in Figs. 7.22/7.23 and 7.27, which appears rather small. However, for the lowest $n_s$, the fraction in higher levels reaches already 30% at $T_e = 2.5$ K.

To discuss the possible resulting effects in more detail, in the following the population of higher surface levels is integrated in the transport treatment at least in an approximative way. To do this, one can for instance simply combine the purely 2D and 3D treatments described on page 115f by treating all electrons in higher levels as three-dimensional and quasi-free (without Landau quantization), i.e., consider the system to consist of a 2D quantized fraction of size $\Delta_1$, and a quasi-free 3D fraction of size $(1 - \Delta_1)$, similarly to the model used in Ref. [115]. The treatment of the 2D fraction includes Coulomb effects in the following, as described on page 115f.

Using the balance equation method (Sec. 3.3), there are essentially two possibilities to combine the transport properties of the 2D and the 3D fraction in this integrated treatment. If electron-electron interaction is sufficiently strong throughout the whole electron system, one can assume both fractions to move together as a single body and to have a single effective temperature. The momentum and energy balance equations must then be applied to the system as a whole. In this case, the total friction force on the whole system due to the interaction with the scatterers is the sum of the forces acting separately on the two fractions. The same applies the total energy loss. This means that all collision rates (AC and DC momentum collision rates as well as the energy collision rate) of the whole system are given by the sum of the contributions of the two fractions:

$$\nu^{(\text{tot})} = \nu^{(2D)} \Delta_1 + \nu^{(3D)} (1 - \Delta_1).$$

(7.1)

This is called the strongly interacting case in the following. The 3D collision rates being smaller than the 2D ones at low $T_e$, one can immediately estimate the population of higher surface levels to give in this case at most a correction of the order $(1 - \Delta_1)$ to the purely 2D results. Thus, the correction should here be less than 11% for the three higher densities up to 2.5 K according to Table 7.1.

The strongly interacting case is essentially equivalent to the complete control approximation discussed on page 22f in Sec. 2.2.3. In analogy to the latter, one may estimate the strength of electron-electron interaction by comparing the momentum collision rate due to scatterers $\nu_{\text{eff}}$ with the typical frequency of short-wavelength oscillations in a Wigner crystal $\omega_0 \approx 2.11en_s^{3/4}/\sqrt{4\pi\epsilon_0m_e}$ (see Sec. 4.1). The complete control approximation is only valid for $\omega_0 > \nu_{\text{eff}}$. From Table 7.2, one can see that this is the case for the two highest $n_s$, but for the lowest $n_s$, the value of $\omega_0$ is already a factor of three smaller than $\nu_{\text{eff}}^{(2D)}$, so the strongly interacting case is here not necessarily a good approximation.

The other limiting case is to assume that the 3D and 2D fractions move independently of each other. In this case, they form two separate parallel currents adding up to the total current, and one should rather add the conductivities of the two fractions. In the DC magnetoconductivity, $\sigma_{xx}(B) \propto \nu_{\text{eff}}(B)/\omega_c^2$ for $\omega_c \gg \nu_{\text{eff}}(B)$ [Eq. (3.16)], so the equivalent total DC momentum collision rate is then still given by Eq. (7.1). In CR, however, one has $\text{Re} \sigma_{xx}(\omega_c) \propto 1/\nu_{\text{eff}}(\omega_c)$ at resonance [Eq. (3.13)], so the equivalent total AC momentum collision rate is now given by the relation [115]

$$\frac{1}{\nu_{\text{eff}}^{(\text{tot})}} = \frac{\Delta_1}{\nu_{\text{eff}}^{(2D)}} + \frac{1 - \Delta_1}{\nu_{\text{eff}}^{(3D)}}.$$

(7.2)
Experimental results and discussion

<table>
<thead>
<tr>
<th>$n_s$ [cm$^{-2}$]</th>
<th>1.3 $\times 10^7$</th>
<th>4.2 $\times 10^7$</th>
<th>13 $\times 10^7$</th>
<th>41 $\times 10^7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_0$ [GHz]</td>
<td>7.3</td>
<td>18</td>
<td>41</td>
<td>97</td>
</tr>
<tr>
<td>$\nu_{\text{eff}}^{(2D)}$ [GHz]</td>
<td>25</td>
<td>18</td>
<td>10</td>
<td>9.5</td>
</tr>
<tr>
<td>$\nu_{\text{en}}^{(2D)}$ [MHz]</td>
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<td>44</td>
<td>38</td>
<td>28</td>
</tr>
<tr>
<td>$\nu_{\text{eff}}^{(3D)}$ [GHz]</td>
<td>3.8</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\nu_{\text{en}}^{(3D)}$ [MHz]</td>
<td>1.5</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 7.2: Comparison of typical 2D and 3D momentum and energy collision rates of SE (AC case for $\nu_{\text{eff}}^{(3D)}$) at different saturation densities $n_s$ (temperature $T_e = T = 1.28$ K; magnetic field $B = 1.42$ T). $\omega_0$ is the typical frequency of short-wavelength oscillations in a Wigner crystal.

This is called the weakly interacting case in the following. For $\nu_{\text{eff}}^{(3D)} < \nu_{\text{eff}}^{(2D)}$, here the correction to the purely 2D result is larger than that obtained with Eq. (7.1).

However, this still leaves the question how to treat the energy relaxation of the two fractions in the weakly interacting case. In the author’s opinion, one can then still assume each fraction to be in internal equilibrium, and even to have both approximately the same effective electron temperature, same as in the strongly interacting case. The point is that concerning the interaction with vapor atoms, the momentum collision rate of SE is about three orders of magnitude larger than their energy collision rate (see table 7.2) due to the large mass difference of electron and vapor atom. In other words, electron collisions with vapor atoms are very much less efficient for the energy relaxation than for the momentum relaxation. In contrast, in electron-electron collisions both collision partners have the same mass, and the momentum and energy collision rates are about the same, and can both be estimated by $\omega_0 \approx 2.11en_s^{3/4}/\sqrt{4\pi\varepsilon_0m_e}$. So, even if electron-electron interaction is too weak to couple both fractions to the same mean momentum, it can still be strong enough to couple them to the same effective temperature.

A more quantitative estimate for the difference in temperature of 3D and 2D fraction (each in internal equilibrium) can be made as follows: Basing on Eq. (3.23), together with $\text{Re} \sigma_{xx}(\omega_c) \approx \varepsilon^2 n_s/2m_e\nu_{\text{eff}}(\omega_c)$ for the CR absorption at resonance [Eq. (3.13)], one may make the following ansatz for the coupled energy balance equations for the 3D and 2D fraction respectively, including electron-electron scattering:

$$\frac{e^2 E_0^2}{4m_e} \frac{(1 - \Delta_1)}{\nu_{\text{eff}}^{(3D)}} = (1 - \Delta_1) k_B \left( T_e^{(3D)} - T_e^{(2D)} \right) \nu_{\text{en}}^{(e-e)} + s,$$

$$\frac{e^2 E_0^2}{4m_e} \frac{\Delta_1}{\nu_{\text{eff}}^{(2D)}} = \Delta_1 k_B \left( T_e^{(2D)} - T \right) \nu_{\text{en}}^{(2D)},$$

where $E_0$ is the MW driving field, $T_e^{(3D)}$ and $T_e^{(2D)}$ are the effective temperatures of the 3D and 2D fractions respectively, and $\nu_{\text{en}}^{(e-e)}$ is the effective energy collision rate for electron-electron scattering. These equations assume the two fractions to be entirely decoupled in the momentum relaxation, which is a worst-case assumption. The first equation means that the 3D fraction gains energy from the driving field (left side), and loses energy to the 2D fraction for $T_e^{(3D)} > T_e^{(2D)}$ (first term on right side) and to the scatterers. Concerning the scatterers, one needs here only to take into account that they give a positive contribution $s > 0$. The second equation reflects that the 2D fraction gains energy from the 3D fraction (first term on left side) and from the driving field (second term on left side), and loses energy to the scatterers (right side). With $s > 0$, one obtains
then
\[
\frac{T_e^{(3D)} - T_e^{(2D)}}{T_e^{(2D)} - T_e^{(2D)}} < \frac{\nu_{\text{en}}^{(e-e)}}{\nu_{\text{en}}^{(e-e)}} \left( \nu_{\text{eff}}^{(3D)} + \frac{1 - \Delta_1}{\Delta_1} \nu_{\text{eff}}^{(2D)} \right).
\]

Taking \(\omega_0\) as an estimate for \(\nu_{\text{en}}^{(e-e)}\), and using the values of table 7.2, one obtains that \((T_e^{(3D)} - T_e^{(2D)})/(T_e^{(2D)} - T_e^{(2D)}) < 0.04\) for the lowest \(n_s\). With increasing \(T_e\), the difference becomes even less due to the \(T_e\) dependencies of the different collision rates. Thus, it is reasonable to assume the two fractions to be always strongly interacting as far as the energy relaxation is concerned.

Figure 7.28: Results of the two-fraction model for integrating the population of higher surface levels in transport for the two lowest densities \(n_s\) from Figs. 7.23 and 7.27. The upper two graphs show the results for the AC and DC momentum collision rates \(\nu_{\text{eff}}(\omega)\) and \(\nu_{\text{eff}}(B)\) respectively, with the solid curves representing the strongly interacting case [Eq. (7.1)], and the dotted curve in the upper graph the weakly interacting AC case [Eq. (7.2)] (see text). Also shown are the results of the pure 2D and 3D treatments as dashed curves, and the experimental data in symbol+line style, same as in Figs. 7.23 and 7.27. The bottom graph shows the results for electron temperature \(T_e\) and ground state fraction \(\Delta_1\) of the integrated treatment (solid and dotted curves respectively), and the \(T_e\) obtained in the pure 2D and 3D treatments (dashed curves, same as in Fig. 7.27). The small circles show the values of \(\Delta_1\) when \(T_e = 2.5\) K.

In Fig. 7.28 the results of the integrated treatment are shown both for the strongly interacting and the weakly interacting case (the latter affecting only the AC effective collision rate, as discussed above), and compared with the purely 2D and 3D results and the experimental data for the two lowest densities \(n_s^{(1)} = 1.3 \times 10^7 \text{ cm}^{-2}\) and \(n_s^{(2)} = 4.2 \times 10^7 \text{ cm}^{-2}\) from Figs. 7.23 and 7.27.

A first important observation is that for the higher density \(n_s^{(2)}\), both cases of the integrated treatment give almost the same results for the momentum collision rates
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(upper two graphs in Fig. 7.28), and deviate only very little from the pure 2D and 3D results, thus the latter two should indeed be a good approximation at higher density, as expected from table 7.1. In fact, due to $\omega_0$ being about the same size as $\nu_{eff}^{(2D)}$ for $n_s^{(2)}$ (see table 7.2), this density corresponds probably already to the strongly interacting case in the integrated treatment.

At the lower density $n_s^{(1)}$, the corrections due to the integrated treatment in the upper two graphs of Fig. 7.28 are more pronounced, and can partly explain the early decrease of $\nu_{eff}$ at heating observed at low $n_s$ by the population of higher surface levels. Still, significant differences between theory and experimental data remain. It could also be that the early decrease of $\nu_{eff}$ is related to the dip feature in the DC inverse conductivity discussed on page 118, as the two vanish together with increasing $n_s$. In analogy to the qualitative explanation for the dip proposed on page 120, the early onset of heating might then be caused by the CR excitation directly acting against at internal equilibrium in the SE system.

For both densities, $n_s^{(1)}$ and $n_s^{(2)}$, the results for $\Delta_1$ and $T_e$ in the integrated treatment (lowest graph in Fig. 7.28) show a fast transition from the 2D to the quasi-3D limit occurring in a very narrow power range at $P_{MW} \sim 0.2–0.3$ fW and accompanied by a strong increase in electron temperature. This is due to the large difference of the 2D and 3D energy collision rates, as already noted on page 117 and visible in Fig. 7.22 or Fig. 7.27 (the results for the energy collision rates in the integrated treatment are omitted in Fig. 7.28, they are qualitatively similar to those of the momentum collision rates).

A qualitatively new feature is that the theoretical curves for $\Delta_1$ and $T_e$ in Fig. 7.28 even exhibit an S-shape, which indicates an instability. A similar behavior was found in the strict semi-classical treatment of Ref. [93] in the effective temperature approximation in the ripplon scattering regime. However, this result should be taken with care, in view of the approximative nature of the integrated treatment. Moreover, the S-shape appears to become less pronounced with increasing density (this is also qualitatively similar to the results of Ref. [93]) and is already rather small at the higher density $n_s^{(2)}$ in Fig. 7.28.

Ultimately, the exact role of the different effects discussed so far should be determined by more extensive experimental studies. For instance, it would be very instructive to perform heating measurements at non-saturated conditions with a stronger holding electric field. This would reduce the population of higher surface states but not affect Coulomb effects, and one should thus be able to unambiguously distinguish the two. Unfortunately, this possibility was not realized in the present study. However, an earlier study contains measurements of the heating behavior in the DC magnetoconductivity in dependence of the holding field [114]. These measurements were taken without a simultaneous CR measurement, so one cannot directly determine the power dissipated per electron and cannot directly compare the results to the present data. One of the measurements was taken at a density of $n_s = 3.2 \times 10^7$ cm$^{-2}$, which lies between $n_s^{(1)}$ and $n_s^{(2)}$, but at a higher temperature of 1.44 K, which should increase the population of higher surface states but rather reduce Coulomb effects due to the increased scatterer density. In all, the conditions probably correspond approximately to those of $n_s^{(1)}$ in Fig. 7.28, moreover, the heating effects also start with a pronounced dip in these data, same as the $n_s^{(1)}$ data in Fig. 7.24. The data of Ref. [114] show that the narrow peak developing inside the dip at further heating, visible in the $n_s^{(1)}$ data at $P_{MW}$ between $-42$ dBm and $-39$ dBm in Fig. 7.24 and corresponding to the decrease of the DC collision rate in Fig. 7.28, can be suppressed almost entirely (by 80% at least) by an increase of the holding field. This
is strong evidence that the decrease of the collision rate is here predominantly due to
the population of higher surface states. The dip is essentially unaffected by the holding
field, and should thus have a different origin (for instance, a CR-induced selective over-
population of higher Landau levels, as proposed on page 120). In contrast, in another
series of measurements in Ref. [114], conducted in approximately the same conditions,
but at a higher density $n_s = 13.8 \times 10^7 \text{cm}^{-2}$, an even stronger increase of the holding
field reduces the height of the peak only by about 30%. Moreover, while the holding
field data at the lower density were taken only at a single MW power in Ref. [114], those
at the higher density were also taken at different MW powers, and at lower MW power,
the increase of the holding field appears here not to change the peak height at all. This
is strong evidence that at the higher density the peak is predominantly not due to the
population of higher surface states, and thus is probably rather due to Coulomb effects.
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Summary and conclusions

This thesis reports investigations of Coulomb effects in the quantum magnetotransport of surface electrons on liquid $^4$He (SE). SE form a nondegenerate two-dimensional (2D) electron liquid where the Coulomb interaction can become extremely strong, which gives rise to strong many-electron or Coulomb effects that can be observed in quantum cyclotron resonance and DC magnetoconductivity.

Experimental techniques Basing on an existing microwave (MW) spectrometer featuring heterodyne detection and a tunable cavity resonator, an experimental setup was developed which enables to simultaneously measure the AC magnetoconductivity (cyclotron resonance) of SE with high resolution and at variable high frequency (40–60 GHz, corresponding to a resonant magnetic field of 1.4–2.1 T) and the DC magnetoconductivity. For both the AC and DC measurements, methods of signal evaluation were employed that ultimately yield the conductivity in absolute units. In the DC case, the measurement method bases on low-frequency capacitive coupling to the SE layer and the signal analysis involves a numerical treatment, this provides also a method to control the electron density and the configuration of the static electric fields acting on the SE layer. The measurements were performed at temperatures between 1.2 and 1.4 K, where the SE scatter predominantly at $^4$He atoms in the vapor phase.

Quantum cyclotron resonance In cyclotron resonance (CR), the most important part of this thesis, the reported experiments constitute the first systematic investigation of Coulomb effects in SE in the vapor atom scattering regime. The key experiment in CR is to vary the electron density $n_s$, which directly changes the strength of the Coulomb interaction. Here, the combination of high measurement frequency (high resonant magnetic field) and high resolution allowed to investigate for the first time the full range of Coulomb effects in quantum CR, from the noninteracting, single-electron limit at low $n_s$ to the strongly interacting limit at high $n_s$. In the low $n_s$ limit, the observed CR absorption line shape is Gaussian, and the linewidth approaches an equilibrium value, in quantitative agreement with single-electron theory. With increasing $n_s$, Coulomb effects first manifest themselves in form of a pronounced narrowing of the CR linewidth (Coulomb narrowing), accompanied by a change of the absorption line shape from Gaussian to Lorentzian, a feature that was not reported previously. At further increase of $n_s$, the narrowing of the CR linewidth reverses into a strong broadening (Coulomb broadening).

The observed CR behavior, both the nonmonotonic dependence of the linewidth on $n_s$ and the line shape change, are in quantitative agreement with a new theory for Coulomb effects developed by Yu. P. Monarkha. It bases on the balance equation/memory function treatment of transport, which is valid at all frequencies and where the relaxation behavior
of the electron system due to collisions with scatterers is described by an effective collision rate. The latter is calculated by means of the quantum excitation spectrum, or dynamic structure factor (DSF) of the electron system. In a magnetic field, the DSF reflects the Landau quantization.

An important achievement of the new theory is that it shows the many-electron DSF of the interacting system to be related to the single-electron DSF of the noninteracting system by a simple expression, valid from the noninteracting to the strongly interacting limit, even into the Wigner solid regime. Thus all single-electron effects can be naturally and strictly incorporated in the treatment of the interacting many-electron system, which has not been the case in previous approaches. Same as in previous theories, the starting point is the notion of a strong internal fluctuational electric field (FEF) acting on the individual electron. This FEF is quasi-uniform and leads in a magnetic field to a fast Hall drift of the electron’s cyclotron orbit center. The central new idea is to eliminate the FEF locally by describing the individual electron in its own local reference frame moving along with the electron’s orbit center. This restores the usual discrete Landau spectrum of the individual electron in the local frame. The new theory explains Coulomb effects as a sort of inelastic effect, as the fast Hall drift of the individual electron leads to an additional energy exchange in collisions with scatterers, formally a Doppler correction. The additional energy exchange changes the conditions for excitation of electrons between Landau levels, and leads to a Doppler broadening of the DSF and thus to a reduction of the effective rate of electron-scatterer collisions once the typical value of the energy exchange becomes larger than the Landau level broadening. This explains the Coulomb narrowing observed in the CR linewidth, which corresponds to the AC effective momentum collision rate (or momentum relaxation rate) in the balance equation formalism. However, once the typical value of the energy exchange becomes larger than the Landau level separation, electrons can be excited to higher Landau levels, this resstimulates the effective collision rate. The latter effect had not been included in previous theoretical approaches, but it is necessary to also explain the Coulomb broadening of the CR linewidth observed experimentally.

The variable frequency of the CR setup allowed also for the first time to systematically study the influence of the strength of the resonant magnetic field on Coulomb effects. At a higher resonant field, the general nonmonotonic behavior of the CR linewidth vs. $n_s$ (Coulomb narrowing and broadening) is observed to be shifted toward higher $n_s$. This demonstrates that increase of the magnetic field weakens Coulomb effects, and qualitatively explains the controversial results obtained in the previous CR studies mentioned in the introduction. It is also found that the change of the resonant field affects the CR linewidth differently in the regime of the Coulomb narrowing and in the regime of the Coulomb broadening. This demonstrates that the two regimes depend on different parameters, namely on Landau level width and Landau level separation, respectively. Here, too, the observed behavior is in quantitative agreement with the new theory, which can now also quantitatively explain the results of the earlier CR studies.

Also investigated was the influence of the strength of the interaction with the scatterers. It is observed that the CR linewidth depends stronger on the coupling to the scatterers and on the scatterer density in the strongly interacting limit at high $n_s$ than in the noninteracting limit at low $n_s$. This is also in quantitative agreement with the new theory and reflects that the collision rate without interaction directly depends on the Landau level broadening, but loses this dependence at strong enough interaction.
**DC magnetoconductivity**  In addition to the CR experiments, measurements of the DC magnetoconductivity were performed; this enabled a more thorough testing of the new theory and a comparison of the AC and DC conductivity under the same conditions. Unlike CR, Coulomb effects are studied in the DC magnetoconductivity rather in the magnetic field dependence of a single measurement at fixed $n_s$ over a large range of magnetic field $B$ (10 T in the present case), thus the studies of CR and DC conductivity complement each other. At high $B$, one is in the noninteracting limit, and with decreasing $B$ Coulomb effects develop due to the different $B$-dependencies of the relevant parameters Doppler energy exchange, Landau level width, and Landau level broadening, according to the new theory outlined on the previous page. Phenomenologically, one observes vs. decreasing $B$ first a decrease of the DC effective momentum collision rate compared to the behavior in the noninteracting case, and later on a re-increase of the DC collision rate. Both effects shift towards higher $B$ at an increase of $n_s$.

In the DC case too the experimental results are in good quantitative agreement with the new theory; thus the latter is shown to give for the first time a consistent quantitative description for all observed Coulomb effects in the AC as well as in the DC conductivity: a) both the Coulomb narrowing of the CR linewidth vs. increasing $n_s$ and the many-electron decrease of the DC collision rate vs. decreasing $B$ are due to the Doppler energy exchange suppressing electron transitions between Landau levels once it becomes larger than the level width, and b) both the Coulomb broadening of the CR linewidth and the many-electron re-increase of the DC collision rate are due to the energy exchange inducing electron transitions to higher Landau levels once it becomes larger than the level separation.

An especially intriguing feature in the DC conductivity is the fact that Coulomb effects phenomenologically extend the Drude regime (in principle limited to the range below 0.1 T in the present experiments), where the collision rate is constant, to significantly higher values of $B$. This feature was previously interpreted as the strong interaction “washing out” the Landau quantization. However, it is also quantitatively reproduced by the new theory and is here rather due to excitations of electrons to higher Landau levels, with the individual electron still having a well-defined Landau spectrum in its local frame.

In the DC conductivity too was investigated the influence of the strength of the interaction with the scatterers. Here, the results can almost directly be compared with the results in CR, and the same behavior is found, also in good agreement with theory.

**Nonlinear conductivity**  Besides the investigations of the linear AC and DC conductivity summarized above, this thesis also reports complementary studies of the nonlinear conductivity, when the electron system is heated by an increased MW power input under CR conditions. This can also induce Coulomb effects because the strength of the fluctuational electric field depends also on the electron temperature. Moreover, these experiments allow to investigate not only the momentum relaxation of SE, as in the linear case, but also their energy relaxation.

The heating behavior was measured at different electron densities $n_s$ under saturation conditions simultaneously in CR and the DC conductivity. In both the AC and DC case, it is observed that the heating first leads to a reduction of the effective momentum collision rate, and later on to a re-increase. The reduction of the collision rate depends strongly on $n_s$. The results were compared with theoretical calculations and it is demonstrated that except for very low $n_s$, the heating data can be quantitatively
explained as follows: The reduction of the momentum collision rate in the early stages of heating is mainly due to Coulomb effects, with the SE populating predominantly the ground surface level. In contrast, the re-increase of the momentum collision rate in the later stages of heating is due to the electrons now populating a sufficient number of both higher surface levels and higher Landau levels to be effectively three-dimensional (3D) and semi-classical. This explanation is consistent with quantitative calculations of the population of the surface levels using the WKB method. The transition of the electron system from the ground surface state to quasi-3D states appears to occur within a relatively small increase of input MW power; this can be explained by the large difference between the 2D and the 3D energy collision rate leading to an abrupt increase in electron temperature.

Conclusions  Surface electrons are of great interest for the study of many-electron behavior in 2D quantum magnetotransport, as a) the Coulomb interaction can become extraordinarily strong and b) it is possible to make accurate quantitative comparisons between experimental results and first-principles calculations. This thesis presents an extensive study of many-electron or Coulomb effects in the AC and DC, linear and non-linear magnetoconductivity of surface electrons in the liquid phase, most importantly the first experimental investigation of Coulomb effects in quantum cyclotron resonance in the vapor scattering regime. This can be regarded as a substantial extension of earlier experimental work, insofar as it specifically addresses important unresolved issues of previous cyclotron resonance studies. It is demonstrated that essentially all observed behavior can be quantitatively (no adjustable parameters) described by the balance equation/memory function theory together with a simple approximation for the many-electron dynamic structure factor, including the previously controversial cyclotron resonance data.
CR studies of SE are generally technically more difficult than DC studies, and have been conducted in fewer number and over a smaller parameter range. Still, this does not mean that additional CR studies are obsolete even in areas already investigated by other means, as this thesis has hopefully shown. An example is the CR line shape change observed for the first time in the present work due to the possibility to reach the noninteracting regime. In fact, this regime may still contain unobserved features: According to the memory function theory, the CR should here exhibit small subresonances at $\omega = N\omega_c$ ($N > 1$) due to the periodic structure of the single-electron AC effective collision rate. This is a qualitatively new phenomenon which does, for instance, not occur in single-electron Ando CR theory. A verification should be worthwhile, in view of the remarkable success of the balance equation/memory function theory in describing all known experimental results in magnetotransport of SE so far. Unfortunately, the relevant magnetic field range was not systematically investigated in the present work, but it can be estimated that the resolution should have been sufficient to observe at least the first subresonance.

Moreover, Coulomb effects should affect the subresonances in just the opposite way as the main resonance: The subresonances are directly proportional to the AC effective collision rate and should therefore only exhibit a broadening and reduction with increasing interaction.

In the ripplon scattering regime, the noninteracting limit could contain even more structure. The Landau level broadening due to ripplon scattering can become rather small, smaller than the inelastic effect inherent in ripplon scattering ($\Gamma = \hbar\omega_q^{(r)}$). In this case, the periodic peaks in the AC effective collision rate split into double peaks, which should in CR lead to a reduction of the width of the main resonance and be directly visible in the subresonances. The corresponding effect was observed in the DC conductivity in studies of the edge magnetoplasmon [7, 60].

The nonlinear behavior at CR heating was in the general line of work in this thesis rather a side issue, and some questions remain unsettled. However, as Coulomb effects seem now rather well understood in the linear behavior, it would be interesting to complete the picture also with regard to the nonlinear behavior. This is even more so as according to the performed analysis even the nonlinear conductivity, which at first sight appears rather complex, is in principle accessible to a quantitative theoretical description. More detailed experimental investigations of the CR heating behavior would therefore constitute an important additional test for theory. It seems here of foremost interest to better separate the influence of Coulomb effects from that of the population of higher surface levels; this could most easily be done by separately varying the holding electric field (which only affects the population of the surface levels), or in a more complicated but also more unambiguous way by employing a tilted magnetic field to separate the cyclotron resonance of 2D from that of 3D electrons.
The possibility to simultaneously perform AC and DC measurements, as in the setup employed in this thesis, could also enable another kind of nonlinear study, namely in dependence of increased DC excitation. In heating experiments, this would have the advantage that the electron temperature varies much less across the CR line, and the CR linewidth has then a well-defined meaning.

No quantitative theoretical description has been found for the broad dip structure observed in the DC magnetoconductivity at low electron density in the early stages of CR heating. If the proposed qualitative explanation is correct, further study of this feature might enable to gain direct information about the electron-electron energy collision rate. The dip could also be of practical interest: If it is indeed proportional to the linear CR absorption, it could be used as a preliminary means to study linear CR in conditions where usual techniques to measure CR absorption become rather difficult, for instance, at very high magnetic field. In this case the weakly interacting regime also extends to higher density, so the apparent limitation of the effect to this regime should be less of a problem.
Appendix A

Calculation of effective momentum and energy collision rates

In the balance equations method, most important in the momentum balance (Sec. 3.3) and the energy balance (Sec. 3.3.3) are the terms describing the contributions due to the interaction with the scatterers. To evaluate them, one starts from the single-electron interaction Hamiltonians for vapor atom and ripplon scattering, Eqs. (2.5) and (2.8). The interaction Hamiltonians for a system with a finite number $N_s$ of surface electrons are then obtained by a simple summation over the electron positions in the fixed laboratory frame $r_i$. Reformulated in terms of the CM coordinate $R = N_s^{-1} \sum_i r_i$ and relative electron coordinates inside the CM frame $r'_i$ by substituting $r_i = R + r'_i$, the system interaction Hamiltonians for vapor atom and ripplon scattering can finally be written as

$$H_{e-a} = \frac{U_a}{V} \sum_{K,Q} \eta(q_z) a_{K-Q}^{+} a_{K} n'_{-q} e^{iqR},$$

$$H_{e-r} = \frac{1}{\sqrt{S}} \sum_{q} V_q Q_q (b_q + b_{-q}^{+})_{K} n'_{-q} e^{iqR},$$

where $\eta(q_z) = \langle 1 | \exp(iq_z z) | 1 \rangle = (1 - iq_z/2\gamma)^{-3}$, and $n'_q = \sum_i e^{-iqr'_i}$ is the Fourier transform of the electron density operator in relative coordinates.

Without interaction, we denote by $|j\rangle$ an unperturbed state of the electron system in the CM frame with energy $\varepsilon_j$, and by $|m\rangle$ an unperturbed state of the scatterer system in the fixed laboratory frame with energy $\varepsilon_{m}^{(b)}$. The electron states $|j\rangle$ are left unspecified. The unperturbed vapor atoms or ripplons are treated as free bosons. Their system states $|m\rangle = |m_1, ..., m_q, ...\rangle$ are specified by the number $m_q$ of bosons with wave vector $q$.

Interaction induces transitions of the total system, electrons and scatterers, between its different states. To assure adiabatic starting conditions in the past, one turns the interaction on slowly, i.e., writes it as $e^{\lambda t}H_{int}$, where $\lambda > 0$ and $H_{int}$ is one of the above Hamiltonians, and takes the limit $\lambda \to 0$ at the end of the calculations. In elementary first-order time-dependent perturbation theory [67], the probability per time $P_{(k,n)(j,m)}$ for the total system $|j, m\rangle = |j\rangle |m\rangle$ to make a transition from an initial state $|j, m\rangle$ to a final state $|k, n\rangle$ due to the interaction $e^{\lambda t}H_{int}$ is then given by

$$P_{(k,n)(j,m)} = \frac{d}{dt} \frac{1}{|1|} \int_{-\infty}^{t} \langle k, n | H_{int} | j, m \rangle e^{\lambda \tau} \exp(i\omega_{(k,n)(j,m)} \tau) d\tau \bigg|_{\tau=0}^{\tau=t},$$

\[A.2\]
where \( \hbar \omega_{(k,n)\langle j,m \rangle} \equiv \varepsilon_k - \varepsilon_j + \varepsilon^{(b)}_n - \varepsilon^{(b)}_m \) is the energy difference between initial and final states. For vapor atoms, one has \( \varepsilon^{(b)}_m = \sum_K m_K \varepsilon^{(a)}_K \) with \( \varepsilon^{(a)}_K = \hbar^2 K^2 / 2M \) (see Sec. 2.2.1), and for ripplons \( \varepsilon^{(b)}_m = \sum_q m_q \hbar \omega_q^2 \) with \( \omega_q^2 \) from Eq. (2.6).

As the motion of the electron system as a whole is classical, the CM operator \( \mathbf{R} \) can be treated as a time-dependent parameter \( \mathbf{R}(t) \) (see Sec. 3.3), given by the classical average, i.e., \( \mathbf{R}(t) = \mathbf{R}_0 e^{-i\mathbf{q} \cdot \mathbf{a}} \) under steady-state conditions in an AC driving field, and \( \mathbf{R}(t) = \mathbf{V} t \) in a DC driving field. These two cases will be treated separately further below, for the moment the time-dependent terms in \( P_{(k,n)\langle j,m \rangle} \) are left as they are. Inserting the above Hamiltonians in Eq. (A.2), and noting that then in the square of the sum over the wave vectors, only the diagonal elements are nonzero, one obtains straightforwardly the transition rates for vapor atom and ripplon scattering:

\[
P_{(k,n)\langle j,m \rangle}^{(a)}(t) = \frac{U^2}{V^2} \sum_{K,Q} |\eta(Q)|^2 \left| \langle n | a^+_{K-Q} a_K | m \rangle \right|^2 \left| \langle k | n'_{-q} | j \rangle \right|^2 \frac{1}{\hbar^2} J_{\omega_{(k,n)\langle j,m \rangle}}^{(a)},
\]

\[
P_{(k,n)\langle j,m \rangle}^{(r)}(t) = \frac{1}{S} \sum_q |V_q Q|^2 \left| \langle n | a_q + a^+_{-q} | m \rangle \right|^2 \left| \langle k | n'_{-q} | j \rangle \right|^2 \frac{1}{\hbar^2} J_{\omega_{(k,n)\langle j,m \rangle}}^{(r)},
\]

where the abbreviation

\[
J(\tilde{\omega}) \equiv \frac{d}{dt} \left| \int_{-\infty}^{t} e^{\lambda \tau} e^{i \mathbf{q} \cdot \mathbf{R}(\tau)} \exp(i \tilde{\omega} \tau) d\tau \right|^2
\]

stands for the time-dependent terms in the above equations. The \( J(\tilde{\omega}) \) are evaluated later on separately for the AC and DC case.

### A.1 Effective momentum collision rate

With the transition rates \( P_{(k,n)\langle j,m \rangle} \) from Eq. (A.3), the total force on the electron system due to the interaction with the scatterers is given by Eq. (3.12), repeated here:

\[
\mathbf{F} = - \sum_{k,n,j,m} \rho_j \rho_m \left( \mathbf{P}_n^{(b)} - \mathbf{P}_m^{(b)} \right) P_{(k,n)\langle j,m \rangle},
\]

where \( \mathbf{P}_m^{(b)} \) is the total 2D momentum of the scatterer system in state \( |m\rangle \). For vapor atoms is \( \mathbf{P}_m^{(b)} = \sum_K m_K \hbar \mathbf{k} \) with \( \mathbf{K} = (k,k_z) \) and for ripplons \( \mathbf{P}_m^{(b)} = \sum_q m_q \hbar \mathbf{q} \). In both cases, nonzero matrix elements of the boson operators in the scatter states \( \langle n | | m \rangle \) correspond to one-particle processes with \( \mathbf{P}_n^{(b)} - \mathbf{P}_m^{(b)} = -\hbar \mathbf{q} \). For vapor atoms, \( a^+_{K-Q} a_K \) transfers one atom from plane-wave state \( |K\rangle \) to state \( |K-Q\rangle \), and for ripplons, \( a_q + a^+_{-q} \) either creates one in state \( |q\rangle \) or destroys one in \( |q\rangle \). The sum over final scatterer states \( n \) is thus easily evaluated. The statistical averaging over initial scatterer states with weight \( \rho_m \) is then performed in the laboratory frame, where the scatterers are in thermal equilibrium, so that for vapor atoms \( \sum_m \rho_m (1 + m_{K-Q} m_K \simeq f^{(a)}_K \) with the Maxwell distribution \( f^{(a)}_K \) from Eq. (2.4), and for ripplons \( \sum_m \rho_m m_q = N^{(r)}_q \) with the bosonic distribution \( N^{(r)}_q \) from Eq. (2.6). With the abbreviation \( J(\tilde{\omega}) \) defined by Eq. (A.4), the forces due to the interaction with vapor atoms and ripplons are then

\[
\mathbf{F}^{(a)} = \frac{U^2}{\hbar^2} \sum_{K,Q} q |\eta(Q)|^2 f^{(a)}_K \sum_{k,j} \rho_j | \langle k | n'_{-q} | j \rangle |^2 J_{\omega_{k-j} - \omega^{(a)}_{K,Q}}^{(a)},
\]

\[
\mathbf{F}^{(r)} = \frac{1}{\hbar S} \sum_q V_q^2 q^2 |\sum_{k,j} \rho_j | \langle k | n'_{-q} | j \rangle |^2 \left( N^{(r)}_q + 1 \right) J_{\omega_{k-j} + \omega^{(r)}_q}^{(r)} + N^{(r)}_q J_{\omega_{k-j} - \omega^{(r)}_q}^{(r)}).
\]
where \( h\omega_{kj} \equiv \varepsilon_k - \varepsilon_j \), and for vapor atoms \( h\omega_{kQ}^{(a)} \equiv \varepsilon_k^{(a)} - \varepsilon_{k-Q}^{(a)} = h^2 (2K \cdot Q - Q^2) / 2M \). So far the evaluation is just the same as the original one for the DC case only [88]. It remains to evaluate the time-dependent part of the transition rate, \( J(\tilde{\omega}) \) from Eq. (A.4).

A.1 Effective momentum collision rate

A.1.1 AC case

The AC case is most conveniently treated in the complex notation \( R(t) = R_0 e^{-i\omega t} \). However, for a rigorous calculation, the perturbation Hamiltonian must be Hermitian, so one starts with \( R(t) = R_0 e^{-i\omega t} + R_0^* e^{i\omega t} \), and keeps then only the terms proportional to \( R_0 e^{-i\omega t} \) in the final expressions [73]. One usually has \( |q \cdot R_0| \ll 1 \) (see Sec. A.3), so one can approximate \( e^{i\omega t R(t)} \simeq 1 + R_0 e^{-i\omega t} + R_0^* e^{i\omega t} \) and obtains then

\[
J(\tilde{\omega}, t) = \frac{d}{dt} \left| \frac{e^{i(\omega + \lambda)t}}{\tilde{\omega} + \lambda} + iq \cdot \left( R_0 e^{(-i\omega + i\lambda)t} - i\omega R_0 + R_0^* e^{(i\omega + i\lambda)t} \right) \right|^2. \tag{A.6}
\]

In taking the square, one only needs to keep the terms that are linear in \( q \). The other terms either average out in \( F \) or give only negligible higher order corrections. Keeping also only the terms proportional to \( R_0 e^{-i\omega t} \), one obtains then finally

\[
J(\tilde{\omega}, t) = q \cdot R_0 \left[ \left( \frac{1}{\omega + i\lambda} - \frac{1}{\omega + i\lambda} \right) + \left( \frac{1}{\omega - \omega + i\lambda} - \frac{1}{\omega - \omega + i\lambda} \right) \right] e^{(-i\omega + 2\lambda)t}. \tag{A.7}
\]

For elastic collisions with scatterers, i.e., \( \varepsilon_n^{(b)} - \varepsilon_m^{(b)} = 0 \), so that \( \tilde{\omega} = \omega_{kj} \), inserting this result in the force equations (A.5) gives terms of the form \( \chi(q,0) - \chi(q,\omega) \),

\[
\chi(q,\omega) = \frac{1}{\hbar} \sum_{kj} \rho_j \left| \langle k | n_{-q} | j \rangle \right|^2 \left( \frac{1}{\omega - \omega_{kj} + i\lambda} - \frac{1}{\omega + \omega_{kj} + i\lambda} \right)
\]

is the density-density response function of the electrons [73]. The function \( \chi(q,\omega) \) has real and imaginary parts, which result in the real and imaginary parts of the memory function. In the following, only the imaginary part including the small inelastic effects is considered; the real part corresponds to a CR line shift, which is negligible in the conditions relevant for this work. With the symbolic Dirac relation [67, 73]

\[
\lim_{\lambda \to 0} \frac{1}{x - a \pm i\lambda} = P\left( \frac{1}{x - a} \right) \mp i\pi \delta(x - a)
\]

\((P \text{ stands for principal value})\), the imaginary part of Eq. (A.7), which will give the in-phase, dissipative or frictional part \( F_{fr} \) of the force \( F \), is simply

\[
\text{Im } J(\tilde{\omega}, t) = q \cdot R_0 \pi \left[ \delta(\omega - \tilde{\omega}) - \delta(\omega + \tilde{\omega}) \right] e^{-i\omega t}.
\]

Inserting this result in the force equations (A.5), the sums over the electron states give the equilibrium DSF of the electron system in the CM frame at positive and negative frequencies, including the small inelastic effects [in the notation used here, the DSF is \( S(q,\omega) = 2\pi N_e^{-1} \sum_{kj} \rho_j \left| \langle k | n_{-q} | j \rangle \right|^2 \delta(\omega_{kj} - \omega) \), cf. Eq. (2.26)]. Further simplification is then possible with the detailed balance condition \( S(q, -\omega) = e^{-h\omega/\kappa B T} S(q, \omega) \). For vapor atoms, the small inelastic effect is neglected in the balance expression because \( |h\omega_{kQ}^{(a)}| \ll \kappa B T \) and in first order it approximately averages out as the main term in \( \omega_{kQ}^{(a)} \) is uneven in \( K \) and \( Q \). Finally, because the system is isotropic, one can replace \( q \cdot (q \cdot R_0) \)
by \( q^2 R_0 / 2 \), and obtains then expressions of the form \( F_{fr}(t) = -N_a m_a \nu_{eff}(\omega)(-i\omega)R_0 e^{-i\omega t} \), with the AC effective momentum collision rates \( \nu_{eff}(\omega) \) for vapor atom and ripplon being

\[
\nu_{eff}^{(a)}(\omega) \simeq \frac{1}{4\hbar m_\omega} \frac{U_0^2}{V^2} \sum_{q_z} |\eta(q_z)|^2 \sum_q q^2 \sum_K f_K^{(a)} S(q, \omega + \omega_{KQ}^{(a)}) ,
\]

\[
\nu_{eff}^{(r)}(\omega) = \frac{1}{4\hbar m_\omega} \frac{T}{S} \sum_q q^2 V^2 q^2 [N_q^{(r)} S(q, \omega + \omega_{q}^{(r)}) + (N_q^{(r)} + 1) S(q, \omega - \omega_{q}^{(r)})] .
\]

Neglecting inelastic effects for vapor atoms entirely (setting \( \omega_{KQ}^{(a)} = 0 \)), one can further simplify \( \nu_{eff}^{(a)}(\omega) \), using \( \sum_K f_K^{(a)} = V n_a \) with the vapor atom density \( n_a \) [Eq. (2.1)], and \( \sum_{q_z} |\eta(q_z)|^2 = VS^{-1}\gamma/8 \) with the vertical wave function parameter \( \gamma \) [Eq. (1.3)]. This leads to the expressions Eq. (3.14) cited in the discussion of AC transport in Sec. 3.3.1.

### A.1.2 DC case

In the DC case the CM motion is given by \( R(t) = V t \), and one obtains straightforwardly, using \( \lim_{\lambda \to 0} \lambda/(x^2 + \lambda^2) = \pi \delta(x) \),

\[
\frac{d}{dt} \left[ \frac{e^{i[q\cdot V + \tilde{\omega} t]}}{i(q\cdot V + \tilde{\omega}) + \lambda} \right]^2 = 2\pi \delta(q\cdot V + \tilde{\omega}). \tag{A.8}
\]

Inserting this result in Eq. (A.5) gives a DSF with a Doppler shift, and the calculation becomes identical to that of Ref. [88]. The further treatment compares essentially the momentum exchange \( q \parallel \) and antiparallel to \( V \) using the detailed balance condition (where \( \hbar \omega_{KQ}^{(a)} \) is neglected for vapor atom scattering); this results for the friction force in

\[
F_{fr}^{(a)} \simeq \frac{N_a U_0^2}{2\hbar V^2} \sum_q \left[ 1 - \exp\left(\frac{\hbar q\cdot V}{k_B T}\right) \right] \sum_{q_z} |\eta(q_z)|^2 \sum_K f_K^{(a)} S(q, \omega_{KQ}^{(a)} - q\cdot V) ,
\]

\[
F_{fr}^{(r)} = \frac{N_a}{\hbar S} \sum_q \left[ 1 - \exp\left(\frac{\hbar q\cdot V}{k_B T}\right) \right] V^2 q^2 N_q^{(r)} S(q, \omega_{q}^{(r)} - q\cdot V) .
\]

These expressions are still general. Usually, one has \( |\hbar q\cdot V| \ll k_B T \) and can approximate \( 1 - \exp[\hbar q\cdot V / k_B T] \simeq -\hbar q\cdot V / k_B T \). Replacing also \( q\cdot V \) by \( q^2 V/2 \) for the isotropic system, the above expressions acquire the form \( F_{fr} = -N_a m_a \nu_{eff}(B)V \), where the DC effective momentum collision rates \( \nu_{eff}(B) \) for vapor atom and ripplon scattering are

\[
\nu_{eff}^{(a)}(B) \simeq \frac{1}{k_B T} \frac{U_0^2}{4m_e V^2} \sum_{q_z} |\eta(q_z)|^2 \sum_K f_K^{(a)} \sum_q q^2 S(q, \omega_{KQ}^{(a)} - q\cdot V) ,
\]

\[
\nu_{eff}^{(r)}(B) \simeq \frac{1}{k_B T} \frac{T}{2m_e S} \sum_q q^2 V^2 q^2 N_q^{(r)} S(q, \omega_{q}^{(r)} - q\cdot V) .
\]

Same as in the AC case, neglecting \( \omega_{KQ}^{(a)} \) in the DSF, one can further simplify \( \nu_{eff}^{(a)}(B) \), using \( \sum_{q_z} |\eta(q_z)|^2 = VS^{-1}\gamma/8 \) and \( \sum_K f_K^{(a)} = V n_a \). This leads to the expressions Eq. (3.17) cited in the general discussion of DC transport in Sec. 3.3.2.

Interestingly, the behavior may still not be truly linear in quantum DC magnetotransport, as then the DSF depends on the Landau level width \( \Gamma_{se} \), which can be much smaller than \( k_B T \), especially in the ripplon scattering regime, so that in the above equations the \( q\cdot V \)-terms in the frequency argument of the DSF can still be relevant. If these terms are neglected, the DC results are identical to the limit \( \omega \to 0 \) of the AC expressions.
A.2 Effective energy collision rate

Analogously to the total force in Sec. A.1, the total energy change rate $W$ of the SE system due to inelastic interaction with scatterers is given by [Eq. (3.20), repeated here]

$$ W = - \sum_{k,n,j,m} \rho_j \rho_m \left( \varepsilon^{(b)}_n - \varepsilon^{(b)}_m \right) P_{(k,n)(j,m)}, $$

where $\varepsilon^{(b)}_m$ is the energy of the scatterer system in state $|m\rangle$ and $P_{(k,n)(j,m)}$ from Eq. (A.3).

The same calculations that for the total force resulted in Eq. (A.5) give now for the energy change rate due to vapor atom and ripplon scattering

$$ W^{(a)} = \frac{U^2}{h^2 V^2} \sum_{K,Q} \omega_{KQ}^{(a)} |\eta(q_z)|^2 J^{(a)}_K \sum_{k,j} \rho_j |\langle k| n'_q |j\rangle|^2 J\left(\omega_{kj} - \omega_{KQ}^{(a)}\right), \quad (A.9) $$

$$ W^{(r)} = \frac{1}{hS} \sum_q \omega_q^{(r)} V^2 Q^2 \sum_{k,j} \rho_j |\langle k| n'_q |j\rangle|^2 \left[ N_q^{(r)} J\left(\omega_{kj} - \omega_q^{(r)}\right) - \left(N_q^{(r)} + 1\right) J\left(\omega_{kj} + \omega_q^{(r)}\right)\right]. $$

In the evaluation of $J(\tilde{w})$ for the AC case, it is now the first term in Eq. (A.6), which is independent on $R_0$, that gives the main contribution in Eq. (A.9) as long as $|q \cdot R_0| \ll 1$. One obtains then simply $J(\tilde{w}) = 2\pi\delta(\tilde{w})$ for the AC case, and needs thus to consider only the more general DC case where according to Eq. (A.8) one has $J(\tilde{w}) = 2\pi\delta(q \cdot V + \tilde{w})$ (the AC case corresponds then to $V = 0$). Inserted in Eq. (A.9), this gives

$$ W^{(a)} = \frac{N_s U^2}{V^2} \sum_{K,Q} \frac{1}{2M} \left(2K \cdot Q - Q^2\right) |\eta(q_z)|^2 J^{(a)}_K S\left(q, \frac{h(2K \cdot Q - Q^2)}{2M} - q \cdot V\right), $$

$$ W^{(r)} = \frac{N_s}{hS} \sum_q \omega_q^{(r)} V^2 Q^2 \left[ N_q^{(r)} S\left(q, \omega_q^{(r)} - q \cdot V\right) - \left(N_q^{(r)} + 1\right) S\left(q, -\omega_q^{(r)} - q \cdot V\right)\right]. $$

In the case of vapor atom scattering, one can generally neglect the $Q^2$-term in the DSF, which is very small and gives only a vanishing higher-order correction. Assuming the electron system to be in internal equilibrium with an effective temperature $T_e$, which only in the following may be different from that of the scatterers $T$, one can use the detailed balance condition and the transformations $Q \rightarrow -Q$ and $q \rightarrow -q$ respectively (and the form of $N_q^{(r)}$ in ripplon scattering) to put the above equations in the form

$$ W^{(a)} \approx \frac{N_s U^2}{V^2} \sum_{K,Q} \frac{1}{2M} \left\{ K \cdot Q \left[ 1 - \exp\left( -\frac{\hbar^2 K \cdot Q}{M k_B T_e} + \frac{\hbar q \cdot V}{k_B T_e}\right)\right] -Q^2\right\} |\eta(q_z)|^2 J^{(a)}_K $$

$$ \times S\left(q, \frac{\hbar K \cdot Q}{M} - q \cdot V\right), $$

$$ W^{(r)} = \frac{N_s}{hS} \sum_q \omega_q^{(r)} V^2 Q^2 \left\{ 1 - \exp\left[ \frac{\hbar \omega_q^{(r)}}{k_B} \left(\frac{1}{T} - \frac{1}{T_e}\right) + \frac{\hbar q \cdot V}{k_B T_e}\right]\right\} N_q^{(r)} S\left(q, \omega_q^{(r)} - q \cdot V\right). $$

For further evaluation, one can expand the exponential functions in first order, as their arguments are usually small, except at very low temperature or very strong driving field, and one can also omit the $q \cdot V$-terms in their arguments (which average out in lowest
order in the sums over $K$ and $q$ respectively). This results in

\[ W^{(a)} \approx \frac{N_s U_{\text{a}}^2}{V^2} \sum_{K,q} \frac{1}{2M} \left( \frac{\hbar^2}{M k_B T_e} (K \cdot Q)^2 - Q_0^2 \right) |\eta(q_z)|^2 f^{(a)}_K(q, \frac{\hbar K \cdot Q}{M} - q \cdot V), \]

\[ W^{(r)} \approx \frac{N_s}{k_B S} \left( \frac{1}{T_e} - \frac{1}{T} \right) \sum_q \omega_q^2 v_q^2 Q_q^2 N^{(r)}_q(q, \omega_q^{(r)} - q \cdot V). \]

For vapor atom scattering, if one also neglects $K \cdot Q$ in the DSF, the sum over $K$ can be evaluated using $\sum_K f^{(a)}_K = V n_a$ and $\sum_K f^{(a)}_K \hbar^2 K^2 / 2M = V n_a k_B T/2$. For ripplon scattering, at not too low $T$ one can approximate $N^{(r)}_q \approx k_B T / \hbar \omega_q^{(r)}$. This gives

\[ W^{(a)} \approx \frac{N_s U_{\text{a}}^2 n_a}{2MV} \left( \frac{T}{T_e} - 1 \right) \sum_q Q^2 |\eta(q_z)|^2 S(q, -q \cdot V), \]

\[ W^{(r)} \approx \frac{N_s}{2\rho_{He} S} \left( \frac{T}{T_e} - 1 \right) \sum_q |q| V_q^2 S(q, \omega_q^{(r)} - q \cdot V). \]

Using further $\sum_q |\eta(q_z)|^2 = VS^{-1}3\gamma/8$ and $\sum_q q_z^2 |\eta(q_z)|^2 = VS^{-1}\gamma^3/2$ in the case of vapor atom scattering, one obtains finally

\[ W^{(a)} \approx -N_s k_B (T_e - T) \frac{3\gamma U_{\text{a}}^2 n_a}{16M k_B T_e S} \sum_q \left( q^2 + \frac{4}{3} \gamma^2 \right) S(q, -q \cdot V), \]

\[ W^{(r)} \approx -N_s k_B (T_e - T) \frac{1}{2\rho_{He} k_B T_e S} \sum_q |q| V_q^2 S(q, \omega_q^{(r)} - q \cdot V). \]

The above expressions have the form $W = -N_s k_B (T_e - T) \nu_{en}(T_e)$, where $\nu_{en}(T_e)$ is the effective energy collision rate. This leads to Eqs. (3.21) and (3.22) in Sec. 3.3.3. Same as for the DC momentum collision rates, the $q \cdot V$-terms in the frequency argument of the DSF can be relevant in magnetic field, as here the DSF depends on the rather small Landau level width $\Gamma_{se}$. However, in the vapor scattering regime (most important for this work), $\Gamma_{se}$ is usually sufficiently large to neglect these terms altogether, moreover we are mainly interested in the AC case, i.e., heating by cyclotron resonance.

### A.3 Three-dimensional electrons

So far, momentum and energy collision rates were calculated assuming the SE to be strictly two-dimensional. The following treats the case of three-dimensional electrons, which is relevant if the SE system is strongly heated by the driving field and its effective temperature $T_e$ is so high that a large number of higher surface levels is populated.

The electrons scatter in this case essentially only at vapor atoms. The treatment differs from the 2D case only in that in the Hamiltonian $H_{e-a}$ [Eq. (A.1)], the term $\eta(q_z)$ should be omitted and the 2D wave vector $q$ should be replaced by a 3D one $Q$, which further leads to replacing $Q(Q \cdot R_0)$ by $Q^2 R_0 / 3$ in the derivation of the effective momentum collision rate. The general equations for the 3D momentum collision rate
A.3 Three-dimensional electrons

$\nu_{\text{eff}}^{(3D)}$ and 3D energy collision rate $\nu_{\text{en}}^{(3D)}$ for vapor scattering are then easily seen to be

$$\nu_{\text{eff}}^{(3D)}(\omega) = \frac{1 - e^{-\hbar \omega/k_BT_e}}{\omega} \frac{U_a^2 n_a}{6\hbar m_e V} \sum_Q Q^2 S(Q, \omega),$$

$$\nu_{\text{en}}^{(3D)}(T_e) = \frac{U_a^2 n_a}{2Mk_BT_eV} \sum_Q Q^2 S(Q, 0).$$

Here, $\nu_{\text{eff}}^{(3D)}(\omega)$ is the AC collision rate, for the DC case it is sufficient to take the limit $\omega \rightarrow 0$; moreover, below is shown that the result is essentially the same. We are mainly interested in the behavior in magnetic field, where the exact expressions for the DSF are rather complicated. However, we can usually assume that the 3D electrons have a sufficiently high effective temperature that they populate also a large number of higher Landau levels. It is then justified [90, 95] to treat the electrons semi-classically, and neglect the Landau quantization. The DSF has then the simple 3D free-electron form

$$S_0(Q, \omega) = \sqrt{\frac{2\pi m_e}{k_BT_e}} \frac{1}{Q} \exp \left[ -\frac{(\hbar \omega - \varepsilon_Q)^2}{4\varepsilon_Q k_BT_e} \right],$$

with $\varepsilon_Q = \hbar^2 Q^2/2m_e$, the same as the 2D free-electron DSF [Eq. (2.30)]. Inserted in the above equations, one obtains for the 3D momentum collision rate

$$\nu_{\text{eff}}^{(3D)}(\omega) = \frac{16An_a}{3} \sqrt{\frac{2k_BT_e}{\pi m_e}} \left[ \frac{1 - e^{-\hbar \omega/k_BT_e}}{\hbar \omega/(k_BT_e)} \right] \int_0^{\infty} x^3 \exp \left[ -\left( x - \frac{\hbar \omega}{4k_BT_e} x^{-1} \right)^2 \right] dx,$$

where $A = m_e^2 U_a^2 / \pi \hbar^2$ is the vapor atom scattering cross section (Sec. 2.2.1). Numerically, for $\hbar \omega/k_BT_e < 1$ the above expression differs by less than 2% from the DC result $\omega \rightarrow 0$, so the latter is here entirely sufficient. This gives the simple results of Eq. 3.24 in Sec. 3.3.3:

$$\nu_{\text{eff}}^{(3D)} = \frac{8An_a}{3} \sqrt{\frac{2k_BT_e}{\pi m_e}},$$

$$\nu_{\text{en}}^{(3D)} = \frac{8m_eAn_a}{M} \sqrt{\frac{2k_BT_e}{\pi m_e}}.$$

In the experiments presented in this work, the heating is induced by cyclotron resonance excitation. As the excitation can here become rather strong, one should shortly consider in what parameter range is actually fulfilled the condition $|Q \cdot R_0| \ll 1$, or $|Q \cdot R_0| \ll 1$ for the 3D case, on which the whole AC treatment is based (see Sec. 3.1.1). From the general structure of the AC conductivity tensor Eq. (3.11), one can derive the following relation between the amplitude of the driving field $E_0$ and the amplitude of the CR motion $R_0$ in the middle of the resonance at $\omega = \omega_c$ (assuming $\omega \gg \nu_{\text{eff}}$):

$$R_0 = \frac{eE_0}{2m_e\omega \nu_{\text{eff}}}.$$

For typical values of $\omega = 2\pi \times 40$ GHz, $\nu_{\text{eff}} = 30$ GHz, and $E_0 = 3$ V/m for measurements in the linear range (see Sec. 6.3.2), one obtains $R_0 \simeq 0.3$ Å, which is extremely small. At heating, $|Q \cdot R_0|$ becomes then one for instance at approximately $E_0 = 300$ V/m (i.e., increase of the introduced power by four orders of magnitude) and electron temperature $T_e = 50$ K, if $Q_x$ is approximated by the thermal value $\sqrt{m_e k_BT_e}/\hbar$. This is just about the limit of parameters reached in the heating measurements in this work (see Sec. 7.3).
Bibliography


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Other publications of the work presented in this thesis

(ref. [16], abstract at http://dx.doi.org/doi:10.1103/PhysRevLett.82.2772)

(ref. [18], abstract at http://dx.doi.org/doi:10.1103/PhysRevB.59.14884)

(ref. [17], abstract at http://dx.doi.org/doi:10.1103/PhysRevB.62.2593)

(ref. [7], abstract at http://dx.doi.org/doi:10.1016/S0370-1573(02)00136-9)
Zusammenfassung

Diese Arbeit handelt von den so genannten Coulombeffekten in der Magnetoleitfähigkeit von zweidimensionalen (2D) Oberflächenelektronen auf flüssigem Helium. (Im Folgenden sind damit stets Oberflächenelektronen auf dicken Schichten von flüssigem $^4\text{He}$ gemeint, ähnliche Systeme können auch auf anderen kryogenen Substraten realisiert werden, einen Überblick gibt z.B. Ref. [1].) Diese Oberflächenelektronen haben eine Reihe von ungewöhnlichen Eigenschaften, die von großem Interesse für die Grundlagenforschung sind.

Zum Beispiel sind alle ihre Wechselwirkungen mit ihrer Umgebung außergewöhnlich einfach und gut verstanden, dies ermöglicht sehr genaue und gut überprüfbare theoretische Beschreibungen. Außerdem bilden sie—im Gegensatz zu üblichen 2D Elektronensystemen in Halbleiterstrukturen (z.B. MOSFETs)—ein nichtentartetes Elektronensystem, in dem die Fermienergie typischerweise sehr viel kleiner ist als die thermische Energie. Dies bedeutet auch, dass der mittlere Abstand zwischen den Elektronen hier viel größer ist als ihre typische quantenmechanische Wellenlänge.


Bei der Untersuchung von Coulomb- oder Vielelektroneneffekten im Magnetotransport beschäftigt man sich grundsätzlich mit der Frage nach dem Einfluss der Coulombwechselwirkung auf die Quanteneigenschaften der Elektronen, da das Transportverhalten über die Wechselwirkung mit den Streuzentren direkt bestimmt ist durch das quantenmechanische Anregungsspektrum der Elektronen. Im Magnetfeld gewinnen die Quantenaspekte zusätzlich an Bedeutung durch die Ausbildung diskreter Landau niveaus. Dies trifft insbesondere auf Oberflächenelektronen zu, da hier zum einen die Verbreiterung der Landau niveaus besonders gering ist und zum anderen diese nichtentarteten Elektronen bereits bei vergleichsweise kleinen Magnetfeldern das Quantenlimit erreichen und nur noch das unterste Landau niveau besetzen.

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Experimentelle Methoden

Der Messaufbau basiert auf einem vorhandenen Mikrowellenspektrometer mit Heterodyndetektion und einem abstimmmbaren Hohlraumresonator. Es wurde daraus ein Aufbau entwickelt, der gleichzeitige Messung der hochfrequenten (AC-) Magnetoleitfähigkeit bzw. Zyklotronresonanz bei 40–60 GHz (Resonanz bei einem Magnetfeld von 1.4–2.1 T) und der DC-Magnetoleitfähigkeit ermöglicht. Sowohl für die AC- als auch für die DC-Messungen wurden Methoden zur Signalauswertung verwendet, die letztlich die Leitfähigkeit in absoluten Einheiten ergeben. Im DC-Fall beruht die Messmethode auf kapazitiver Kopplung an die Elektronenschicht bei niedriger Frequenz und die Signalanalyse beinhaltet eine numerische Behandlung, dies liefert auch eine Methode zur Kontrolle der Elektronendichte und der elektrostatischen Felder, die auf die Elektronenschicht wirken. Die Messungen wurden durchgeführt bei Temperaturen zwischen 1.2 und 1.4 K, in diesem Temperaturbereich streuen die Oberflächenelektronen vorwiegend an 4He-Atomen in der Gasphase.

Zyklotronresonanz im Quantenlimit

In der Zyklotronresonanz (ZR) stellen die hier beschriebenen Experimente die erste systematische Untersuchung von Coulombefekten in Oberflächenelektronen im Bereich der Gasatomstreuung dar. Das Schlüsselexperiment in der ZR sind Messungen in Abhängigkeit von der Elektronendichte \( n_s \), dies verändert direkt die Stärke der Coulombwechselwirkung. Hierbei erlaubte es die Kombination von hoher Messfrequenz (hohes resonantes Magnetfeld) und hoher Auflösung, zum ersten Mal das gesamte Spektrum von Coulombefekten in der ZR zu untersuchen, vom nicht-wechselwirkenden Einelektronen-Grenzfall bei niedrigem \( n_s \) bis zum stark wechselwirkenden Vielelektronen-Grenzfall bei hohem \( n_s \). Bei niedrigem \( n_s \) hat die beobachtete ZR-Absorptionslinie die Form einer Gausskurve und die Linienbreite nähert sich einem Gleichgewichtswert an, in quantitativer Übereinstimmung mit Einelektronentheorien. Mit zunehmendem \( n_s \) äußern sich Coulombefekte zuerst in Form einer ausgeprägten Verschmälerung der ZR-Linienbreite (Coulombverschmälerung). Dies ist begleitet von einer Änderung der Linienform von einer Gausskurve zur einer Lorentzkurve, ein Effekt über den bislang noch nicht berichtet worden war. Bei weiterer Zunahme von \( n_s \) kehrt sich die Verschmälerung der Linienbreite in eine starke Verbreiterung um (Coulombverbreiterung).

Das beobachtete ZR-Verhalten, sowohl die nichtmonotone Abhängigkeit der Linienbreite von \( n_s \), als auch die Änderung der Linienform, befindet sich in quantitativer
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Ebenfalls untersucht wurde der Einfluss der Stärke der Wechselwirkung mit den Streuzentren. Man beobachtet einen stärkeren Einfluss der Kopplung zu den Streuzentren und deren Dichte auf die ZR-Linienbreite im stark wechselwirkenden Grenzfall bei hohem $n_s$, verglichen mit dem nichtwechselwirkenden Grenzfall bei niedrigem $n_s$. Dies stimmt ebenfalls quantitativ mit der neuen Theorie überein und veranschaulicht, dass die Stoßrate ohne Wechselwirkung unmittelbar abhängt von der Verbreiterung der Landau niveaus, diese Abhängigkeit bei genügend starker Wechselwirkung jedoch verliert.


Ein besonders interessantes Merkmal der DC-Leitfähigkeit ist, dass Coulombeffekte den Drude-Bereich (eigentlich beschränkt auf den Bereich unterhalb 0.1 T in unserem Fall), in dem die Stoßrate konstant ist, phänomenologisch zu deutlich höheren $B$-Werten ausdehnen. Dieses Verhalten wurde bislang interpretiert als ein „Auswaschen“ der Landauquantisierung durch die starke Wechselwirkung. Es wird jedoch auch von der neuen Theorie quantitativ wiedergegeben, und ist hier stattdessen eine Folge der Anregung von Elektronenübergängen zu höheren Landau niveaus sobald der Energieübertrag größer wird als der Niveauabstand.

In der DC-Leitfähigkeit wurde ebenfalls der Einfluss der Stärke der Wechselwirkung mit den Streuzentren untersucht. Hierbei können die Ergebnisse praktisch direkt mit denen in der ZR verglichen werden, und man findet in der Tat das gleiche Verhalten, ebenfalls in guter Übereinstimmung mit der Theorie.

**Nichtlineare Leitfähigkeit** Neben den oben zusammengefassten Untersuchungen der linearen Leitfähigkeit wurden in dieser Arbeit auch einige Untersuchungen der nichtli-
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nearen Leitfähigkeit vorgenommen. Hierbei wird das Elektronensystem aufgeheizt durch Verstärkung der Mikrowellenleistung unter Zyklotronresonanz. Dies kann auch Coulomb-
effekte hervorrufen, da die Stärke des elektrischen Fluktuationsfeldes auch von der Temperatur der Elektronen abhängt. Außerdem kann in diesen Experimenten nicht nur die Impulsrelaxation der Elektronen untersucht werden, wie im linearen Fall, sondern auch ihre Energierelexation.

Das Verhalten bei Aufheizung wurde bei verschiedenen Elektronendichten \( n_s \) unter Sättigung gleichzeitig gemessen in der ZR und der DC-Magnetoleitfähigkeit. Sowohl im AC- als auch im DC-Fall beobachtet man bei zunehmender Aufheizung zuerst eine Abnahme der Stoßrate, und anschließend eine Wiederzunahme. Die Abnahme der Stoßrate hängt dabei stark von \( n_s \) ab. Die Messergebnisse wurden verglichen mit theoretischen Berechnungen, und es zeigt sich, dass außer bei sehr niedrigem \( n_s \) die Daten quantitativ erklärt werden können wie folgt: Die Abnahme der Stoßrate im frühen Stadium der Aufheizung erfolgt vor allem durch Coulombefekte und die Elektronen befinden sich hierbei größtenteils im untersten Oberflächenzustand. Dagegen erfolgt die Wiederzunahme der Stoßrate im späteren Stadium der Aufheizung dadurch, dass die Elektronen nun eine ausreichend große Zahl höherer Oberflächenzustände und auch höherer Landau niveaus besetzen um sich effektiv dreidimensional (3D) und halbklasisch zu verhalten. Diese Erklärung stimmt überein mit quantitativen Berechnungen der Besetzung der Oberflächenzustände mittels der WKB-Methode. Der Übergang des Elektronensystems vom untersten Oberflächenzustand zu quasi-3D-Zuständen erfolgt offenbar während eines vergleichsweise kleinen Anstiegs der Mikrowellenleistung; dies kann erklärt werden durch den großen Unterschied zwischen den Energistoßraten im 2D-Fall und im 3D-Fall, welcher zu einem plötzlichen Anstieg der Elektronentemperatur führen kann.

Schlussfolgerungen

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Acknowledgements

This work was prepared in Grenoble, France at the High Magnetic Field Laboratory of the Max-Planck-Institut für Festkörperforschung and the Centre National de la Recherche Scientifique. Herewith I wish to express my gratitude to all the people who have contributed to its success.

An erster Stelle bedanke ich mich bei Professor Peter Wyder für die freundliche Aufnahme und insbesondere für die unermüdliche Geduld und den unbesiegbaren Optimismus bei der Betreuung.

Professor Paul Leiderer hat sich freundlicherweise als Referent der Dissertation zur Verfügung gestellt. Ihm und seiner Gruppe in Konstanz verdanke ich außerdem meine erste Begegnung mit Oberflächenelektronen sowie viel Unterstützung während der Zeit in Grenoble.

Ich bedanke mich bei Professor Heinz Dehnen für seine Bereitschaft, als Prüfer an der mündlichen Prüfung teilzunehmen.

Without Yuri Monarkha, this work would not exist. Not only did he develop the theoretical description for the experimental results, he also suggested to start investigating Coulomb effects in cyclotron resonance in the first place.

Es würde diese Arbeit ebenfalls nicht geben ohne Martin Seck und „Jacqueline“, das von ihm gebaute Mikrowellenspektrometer.

My senior group members, Gianpaolo Mistura, Oliver Tress, and Frank Penning, provided me with valuable insights into the work with surface electrons.

Valeri Shikin introduced me to many ideas and concepts that have entered this thesis.

Je remercie Walter Joss, Louis Jansen et Gérard Martinez pour leur soutien et leur bons conseils scientifiques et humains.

J’imagine mal les manips sans l’assistance technique de l’équipe de l’infrastructure scientifique, surtout Peter van der Linden, mais aussi Herbert Krath et Jean-Luc Martin, et les grandes et petites merveilles mécaniques réalisées par les gens de l’atelier, comme Hans Dresler, Jürgen Spitznagel, Robert Pankow et Claude Mollard.

J’ai beaucoup apprécié l’assistance informatique reçue de la part de André Plante, Sébastien Buisson et Henk Jongbloets.
D’une aide inestimable dans la lutte contre la paperasse étaient Brigitte Indigo, Ursula Laitenberger, Rachel Graziotti, Sandrine Meilen, Gislaine Meneroud, Ingrid Beyer-Girardet et Christiane Warth.

I profited greatly from the connection to the European network for surface electrons and the INTAS network, run by François Peeters and Mike Leary respectively, and enjoyed the contact to the various distinguished fellow researchers encountered through these.

Herzlichen Dank an Professor Hanns-Ulrich Habermeier, Frank Schartner, Birgit Hammer und Claudia Didschies von der Technologiegruppe des MPI Stuttgart für die viele Unterstützung bei den (leider fehlgeschlagenen) Versuchen mit Elektronen auf Heliumfilmen und strukturierten Substraten.

Für die Hilfe in unzähligen kleinen und großen Dingen und insbesondere für die gute Gesellschaft möchte ich mich bedanken bei allen, die die Zeit am LCMI mit mir geteilt haben, Andreas B., Andreas G., Andreas N., Markus, Cornelius, Roland, Clemens, Georg, Sabine, Anja, Jörg, Roman, Ede, Eckhart, Herbert, Le Hang, Maxim, Camille, Uwe, Enno, Barbaros, Robert, Thomas, Niko, Hans, Martin, Hans-Peter und Arthur.

Und nicht zuletzt, vielen Dank an meine Eltern für die moralische und tatkräftige Unterstützung während der Zeit in Grenoble.