Tuning the nonlinear dispersive coupling of nanomechanical string resonators

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We investigate nonlinear dispersive mode coupling between the flexural in- and out-of-plane modes of two doubly clamped, nanomechanical silicon nitride string resonators. As the amplitude of one mode transitions from the linear response regime into the nonlinear regime, we find a frequency shift of two other modes. The resonators are strongly elastically coupled via a shared clamping point and can be tuned in and out of resonance dielectrically, giving rise to multimode avoided crossings. When the modes start hybridizing, their polarization changes. This affects the nonlinear dispersive coupling in a nontrivial way. We propose a theoretical model to describe the dependence of the dispersive coupling on the mode hybridization.

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I. INTRODUCTION

Nonlinear mechanical mode coupling affects the dynamics of driven systems of coupled nanomechanical resonators, even in situations where the modes are tuned far from resonance or from internal resonances [1,2]. It is apparent as a dispersive eigenfrequency shift of the coupled mode which depends on the oscillation amplitude of the driven mode. Nonlinear mode coupling has been demonstrated between the harmonics of doubly clamped beams or strings, between the orthogonal flexural modes of a singly clamped nano- or microcantilever, and between the vibrational modes of a torsional resonator and one of its suspension springs [2–9].

Here we present nonlinear dispersive mode coupling between the fundamental flexural modes of two adjacent, intrinsically coupled nanomechanical resonators. The resonators under investigation are doubly clamped, strongly prestressed silicon nitride string resonators with fundamental eigenfrequencies in the range of 6 MHz. A simple theoretical model is developed to describe the behavior of the coupled nonlinear system under eigenfrequency tuning. The model reveals that both nonlinear coefficients, the Duffing nonlinearity and the dispersive coupling, depend on the hybridization of the underlying modes. The prospect of controlling the nonlinear coefficients of a system of two or more resonators may be of interest for future applications of nonlinear nanomechanical networks.

II. ACTUATION AND DETECTION

The two freely suspended silicon nitride strings forming the resonator pair are depicted in Fig. 1. They share one clamping point, which has been engineered to enable strong linear mechanical coupling between the two string resonators [10]. The strings have a nominal length of 55 μm, a width of 180 nm, and a thickness of 100 nm. The mechanical (flexural) modes have Q factors in the range of 200 000 as a result of the high tensile stress (1.46 GPa) in the silicon nitride nanostrings. One of the two resonators (resonator 1 in Fig. 1) is dielectrically controlled [11,12]. Actuation is achieved by applying a rf drive tone as well as a dc voltage \( U_{dc} \) to one of the electrodes. At the same time, the dc voltage allows one to tune the resonator’s eigenfrequencies. The second gold electrode is connected to a λ/4 microwave cavity for cavity-assisted, dielectric displacement detection [13]. To minimize dielectric crosstalk, the rf and dc voltages are applied to the outer electrode, whereas the microwave-cavity-assisted displacement detection is performed on the inner electrode situated between the two string resonators. All measurements are done at a pressure below \( 10^{-4} \) mbar and at room temperature.

III. CHARACTERIZATION IN THE LINEAR REGIME

The two lowest flexural modes of a nanostring correspond to its fundamental out-of-plane (OP) and in-plane mode (IP). For the case of two nanostrings, one has to deal a priori with four bare, intrinsic fundamental modes corresponding to the out-of-plane and in-plane mode of the first resonator (OP1 and IP1) and the out-of-plane and in-plane mode of the second resonator (OP2 and IP2).

By varying the dc voltage we tune the intrinsic frequencies of the bare modes of resonator 1 such that two (or more) of the four modes become resonant. In this case, the intrinsic modes hybridize owing to linear interstring and intrastring mode coupling [10]. This linear coupling gives rise to the hybridized eigenmodes of the system. Figure 2 shows the frequency response of the system as a function of the dc voltage (from −32 V to 32 V) applied to the outer electrode. The three lines correspond to the eigenfrequencies of the hybridized eigenmodes of the system formed by OP1, IP1, and OP2, denoted eigenmodes 1, 2, and 3. The frequency of the bare IP2 mode lies well above all other modes and does not hybridize with the other modes. Therefore, mode IP2 will be disregarded in the following. Multimode avoided crossings are visible between −30 V and −20 V and between 15 V and...
We introduce a voltage dependence for the intrinsic eigenfrequencies $\omega_i$ of the bare modes in the system. As only resonator 1 is surrounded by the gold electrodes, we expect to control only the modes of resonator 1 by our dielectric actuation and tuning technique. However, we find experimentally that mode OP2 of resonator 2 is also slightly affected by the dc voltage. Therefore, we assume $\omega_i(\Delta U_{dc})/(2\pi) = \omega_{i0}/(2\pi) + \kappa_i(\Delta U_{dc} - \Delta U_{i0})^2 + d_i(\Delta U_{dc} - \Delta U_{i0})^3$ with $\omega_{i0} (i = 1, 2, 3)$ being the (theoretical) intrinsic eigenfrequencies of the bare modes in the absence of coupling and vanishing voltage. We have considered a quadratic tuning factor $\kappa_i$ along with a cubic correction $d_i$. To be more general, we have also assumed an offset voltage $U_{i0}$ of the vertices of the tuning parabolas. A voltage offset can arise as a consequence of a buildup of static polarization. Additionally we assume the linear coupling $\gamma_{ij}$ to be independent of the dc voltage, neglecting dielectric effects.

The equations of motion can be rewritten as

$$\ddot{v}_i + \omega_i^2 v_i + \sum_{j \neq i} \kappa_{ij}(v_i - v_j) + \gamma_{ij} v_i^3 + \left(\sum_{j \neq i} \gamma_{ij} v_j^3\right) v_i = 0 \tag{1}$$

($i, j = 1, 2, 3$), where $v_i$ is the vibration amplitude of the bare modes with $i = 1$ (OP1), $i = 2$ (OP2), $i = 3$ (IP1), $\omega_i$ is the eigenfrequency of the $i$th bare mode, and $\kappa_{ij}$ describes the linear coupling between the bare modes $i$ and $j$. The parameters $\gamma_{ij}$ are the Duffing nonlinearities, whereas $\gamma_{ij} (i \neq j)$ are the dispersive coupling coefficients (with the notation $\gamma_{ij} = \gamma_{ji}$). We omit nonlinear interactions of the kind $v_i v_j$ on the basis that they are related to a breaking of symmetry in the suspended nanowire and generally negligible. Furthermore, we ignore the external drive, the damping, and the noise. We will include these terms in the subsequent sections.

In this section we focus on the linear regime by disregarding, for the moment, the nonlinearities,

$$\ddot{v}_i + \omega_i^2 v_i + \sum_{j \neq i} \kappa_{ij}(v_i - v_j) = 0 \tag{2}$$

We introduce a voltage dependence for the intrinsic eigenfrequencies $\omega_i$ of the bare modes in the system. As only resonator 1 is surrounded by the gold electrodes, we expect to control only the modes of resonator 1 by our dielectric actuation and tuning technique. However, we find experimentally that mode OP2 of resonator 2 is also slightly affected by the dc voltage. Therefore, we assume $\omega_i(\Delta U_{dc})/(2\pi) = \omega_{i0}/(2\pi) + \kappa_i(\Delta U_{dc} - \Delta U_{i0})^2 + d_i(\Delta U_{dc} - \Delta U_{i0})^3$ with $\omega_{i0} (i = 1, 2, 3)$ being the (theoretical) intrinsic eigenfrequencies of the bare modes in the absence of coupling and vanishing voltage. We have considered a quadratic tuning factor $\kappa_i$ along with a cubic correction $d_i$. To be more general, we have also assumed an offset voltage $U_{i0}$ of the vertices of the tuning parabolas. A voltage offset can arise as a consequence of a buildup of static polarization. Additionally we assume the linear coupling $\gamma_{ij}$ to be independent of the dc voltage, neglecting dielectric effects.

The equations of motion can be rewritten as

$$\ddot{v}_i = -\Theta \dddot{v}_i, \tag{3}$$

with the mode matrix $\Theta$ defined as

$$\Theta = \begin{bmatrix} \omega_1^2 + \kappa_{12} + \kappa_{13} & -\kappa_{12} & -\kappa_{13} \\ -\kappa_{12} & \omega_2^2 + \kappa_{21} + \kappa_{23} & -\kappa_{23} \\ -\kappa_{13} & -\kappa_{23} & \omega_3^2 + \kappa_{31} + \kappa_{32} \end{bmatrix}. \tag{4}$$

The eigenvalues $\Omega_i$ of the matrix $\Theta$ in Eq. (4) denote the eigenfrequencies of the three hybridized modes. They depend on the dc voltage, the dc tuning strength, and the eigenfrequencies of the bare modes. We use a genetic fit [10] to obtain the parameters entering in the frequencies $\Omega_i$, which we extract from the experimental data in Fig. 2. The result of the fitting yields the following parameters: $\kappa_{12}/(2\pi)^2 = 4.138 \text{ MHz}^2$, $\kappa_{13}/(2\pi)^2 = 1.9098 \text{ MHz}^2$, $\kappa_{23}/(2\pi)^2 = 2.546 \text{ MHz}^2$, $\omega_{10}/(2\pi) = 5.8606 \text{ MHz}$, $\omega_{20}/(2\pi) = 5.9897 \text{ MHz}$, $\omega_{30}/(2\pi) = 6.1117 \text{ MHz}$, $U_{10} = 0.656 \text{ V}$, $U_{20} = -1.1 \text{ V}$, $U_{30} = 0.485 \text{ V}$, $c_1/(2\pi) = 165.59 \text{ Hz/V}^2$, $c_2/(2\pi) = 1 \text{ Hz/V}^2$, $c_3/(2\pi) = -282.5 \text{ Hz/V}^2$, $d_{1}/(2\pi) = 1.368 \text{ Hz/V}^3$, and $d_{2}/(2\pi) = -2.2 \text{ Hz/V}^3$.

In Fig. 3(a) we show the three calculated hybridized eigenfrequencies $\Omega_i$ (blue solid line, red short dashed line, and green long dashed line) based on the above parameters for voltages between 0 V and 32 V. As a comparison we show the extracted data from Fig. 2 (blue, red, and green dotted lines). As the dc voltage dependence of the eigenfrequencies is (almost) symmetric for positive and negative voltages, we further only consider the positive voltage spectrum.

Using the transformation

$$v_i = \sum_{j=1}^{3} e_{ji} (V_{dc}) q_j, \tag{5}$$

we set the variables $q_j$ as the vibration amplitude of the hybridized eigenmodes. Figures 3(b), 3(c) and 3(d) show the (voltage dependent) components $e_{ji}$ of the calculated eigenvectors of the three hybridized eigenmodes, respectively. One can observe that the three bare modes are always partially hybridized even in the limit of vanishing dc voltage, since $e_{ji}$ remains nonzero even for $j \neq i$. On the other hand, in

FIG. 1. Scanning electron micrograph (false color) of a series of resonator pairs. Resonator 1 is dielectrically controlled via a pair of gold electrodes. A rf drive tone and a dc voltage $U_{dc}$ are applied to the outer (upper) electrode. A microwave cavity is connected to the inner (lower) electrode situated between the two resonators for cavity-assisted displacement detection of the modes of resonator 1.

28 V. The white dash-dotted line indicates a voltage of 15 V, which is used for the measurements shown in Fig. 4.

We consider the simplest theoretical model formed by three linearly coupled harmonic oscillators (corresponding to the bare modes). Theoretically the bare modes are described by

$$\ddot{v}_i + \omega_i^2 v_i + \sum_{j \neq i} \kappa_{ij}(v_i - v_j) + \gamma_{ij} v_i^3 + \left(\sum_{j \neq i} \gamma_{ij} v_j^3\right) v_i = 0 \tag{1}$$

(i, j = 1, 2, 3), where $v_i$ is the vibration amplitude of the bare modes with $i = 1$ (OP1), $i = 2$ (OP2), $i = 3$ (IP1), $\omega_i$ is the eigenfrequency of the $i$th bare mode, and $\kappa_{ij}$ describes the linear coupling between the bare modes $i$ and $j$. The parameters $\gamma_{ij}$ are the Duffing nonlinearities, whereas $\gamma_{ij} (i \neq j)$ are the dispersive coupling coefficients (with the notation $\gamma_{ij} = \gamma_{ji}$). We omit nonlinear interactions of the kind $v_i v_j$ on the basis that they are related to a breaking of symmetry in the suspended nanowire and generally negligible. Furthermore, we ignore the external drive, the damping, and the noise. We will include these terms in the subsequent sections.

In this section we focus on the linear regime by disregarding, for the moment, the nonlinearities,

$$\ddot{v}_i + \omega_i^2 v_i + \sum_{j \neq i} \kappa_{ij}(v_i - v_j) = 0 \tag{2}$$

FIG. 2. Frequency response of hybridized eigenmodes 1 (lowest frequency branch), 2 (intermediate frequency), and 3 (highest frequency branch), showing multimode avoided crossings between −30 V and −20 V and between 15 V and 28 V. The white dash-dotted line indicates a dc voltage of 15 V used for the measurements shown in Fig. 4.
For example, (b) shows the components of eigenmode 1, with $e_{11}$, $e_{12}$, and $e_{13}$ being the contributions of the bare modes OP1, OP2, and IP1, respectively.

Finally, the bare modes strongly hybridize in the area of the avoided crossings between 15 V and 28 V. As the drive tone approaches the eigenfrequency, $\omega_d \approx \Omega_2$ (viz. $\Omega_2 - \omega_d$ becomes comparable to the linewidth), one starts to drive eigenmode 2. This is clearly visible in Fig. 4(b), which displays the measured frequency response of the driven eigenmode 2, namely its vibration amplitude as a function of the drive frequency. The vibrational state of eigenmode 2 is clearly in the nonlinear regime and the vibration amplitude shows the characteristic shape of the Duffing oscillator (with negative, i.e., softening, Duffing nonlinearity). In our high quality resonators operating at room temperature, the nonlinear regime of the driven eigenmode manifests in the power spectrum (Fig. 4(a)) as the formation of two satellite peaks [14] around the drive tone (almost a delta peak), clearly resolved in the frequency range $\omega_d \approx \Omega_2$.

Here we focus our attention on the other two undriven hybridized eigenmodes which are apparent in the power spectrum as a result of the presence of the noise-enhanced thermal fluctuations. Figure 4(a) reveals a clear frequency shift for both undriven eigenmodes 1 and 3, as the drive frequency approaches the eigenfrequency of eigenmode 2. This occurs when the vibration amplitude of the driven eigenmode strongly increases as we cross the resonant condition $\omega_d \approx \Omega_2$. Far away from any internal resonance, the minimal horizontal lines correspond to the (enhanced) thermal motion of the three hybridized eigenmodes at the dc voltage 15 V. As the drive tone approaches the eigenfrequency, $\omega_d \approx \Omega_2$ (viz. $\Omega_2 - \omega_d$ becomes comparable to the linewidth), one starts to drive eigenmode 2. This is clearly visible in Fig. 4(b), which displays the measured frequency response of the driven eigenmode 2, namely its vibration amplitude as a function of the drive frequency. The vibrational state of eigenmode 2 is clearly in the nonlinear regime and the vibration amplitude shows the characteristic shape of the Duffing oscillator (with negative, i.e., softening, Duffing nonlinearity). In our high quality resonators operating at room temperature, the nonlinear regime of the driven eigenmode manifests in the power spectrum (Fig. 4(a)) as the formation of two satellite peaks [14] around the drive tone (almost a delta peak), clearly resolved in the frequency range $\omega_d \approx \Omega_2$.

Finally, the bare modes strongly hybridize in the area of the avoided crossings between 15 V and 28 V. Note that the microwave cavity-assisted readout technique is sensitive only to the modes of resonator 1, which is situated between the electrodes. Therefore, a given hybridized eigenmode can only be detected if it has a sufficiently strong contribution from a bare mode of resonator 1.

IV. DISPERSIVE NONLINEAR COUPLING

We now fix the dc voltage to 15 V and apply a sinusoidal drive tone $\omega_d$ close to the eigenfrequency $\Omega_2$ of the intermediate hybridized eigenmode. In addition, white noise (power of $-5$ dBm; bandwidth of 7 MHz) is applied to enhance the thermal motion of the resonators. We sweep the drive frequency $\omega_d$ and measure the power spectrum which is reported in Fig. 4(a). Far away from the resonant drive, i.e., $\omega_d \neq \Omega_2$, the three horizontal lines correspond to the (enhanced) thermal motion of the three hybridized eigenmodes at the dc voltage 15 V. As the drive tone approaches the eigenfrequency, $\omega_d \approx \Omega_2$ (viz. $\Omega_2 - \omega_d$ becomes comparable to the linewidth), one starts to drive eigenmode 2. This is clearly visible in Fig. 4(b), which displays the measured frequency response of the driven eigenmode 2, namely its vibration amplitude as a function of the drive frequency. The vibrational state of eigenmode 2 is clearly in the nonlinear regime and the vibration amplitude shows the characteristic shape of the Duffing oscillator (with negative, i.e., softening, Duffing nonlinearity). In our high quality resonators operating at room temperature, the nonlinear regime of the driven eigenmode manifests in the power spectrum (Fig. 4(a)) as the formation of two satellite peaks [14] around the drive tone (almost a delta peak), clearly resolved in the frequency range $\omega_d \approx \Omega_2$.

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eigenmode \((\omega_i \sim \Omega_i)\) we set
\[
\ddot{q}_i(t) = -\Omega_i^2 q_i(t) - \Gamma_i \dot{q}_i(t) - \gamma_{22} q_3(t) + F_0 \cos(\omega_i t) + \delta F_i(t),
\]
whereas for the other two eigenmodes
\[
\ddot{q}_1(t) = -(\Omega_1^2 + \gamma_{23} q_3(t)) q_1(t) - \Gamma_1 \dot{q}_1(t) + \delta F_1(t),
\]
\[
\ddot{q}_3(t) = -(\Omega_3^2 + \gamma_{23} q_3(t)) q_3(t) - \Gamma_3 \dot{q}_3(t) + \delta F_3(t).
\]
In Eq. (6) and Eqs. (7), (8) \(\Gamma_i\) represents the damping of the \(i\)th eigenmode and \(\delta F_i(t)\) the (enhanced) thermal noise. The parameter \(\gamma_{23}\) is the Duffing nonlinearity of the driven eigenmode at the dc voltage 15 V. Since the two undriven eigenmodes are weakly fluctuating only due to the (enhanced) thermal noise, we neglect their effect of nonlinear coupling on the driven eigenmode (this term should correspond to a noise effect in the frequency of the driven eigenmode) [15].

As the driven mode is oscillating at the drive frequency, far away from the eigenfrequency \(\Omega_1\) and \(\Omega_3\) of the other two eigenmodes, we have to take only the time average \(q_3(t)\) of the driven mode to account for its effect on the other two modes. We set the latter time average as \(q_3(t) = (\overline{\gamma_{23}} \langle q_3^2 \rangle)^{1/2}\), with \(\overline{\gamma_{23}}\) the measured signal and \(\langle q_3^2 \rangle\) a calibration factor. Then we can write the renormalized eigenfrequencies of eigenmodes 1 and 3 as a function of the amplitude of the driven eigenmode 2 as
\[
\tilde{\Omega}_1^2 = \Omega_1^2 + \frac{1}{2} \overline{\gamma_{23}} \overline{\gamma_{23}} x_2^2,
\]
\[
\tilde{\Omega}_3^2 = \Omega_3^2 + \frac{1}{2} \overline{\gamma_{23}} \overline{\gamma_{23}} x_2^2.
\]

As mentioned above, the dispersive coupling manifests itself in a frequency shift of one mode, if the vibration amplitude of another mode is changed.

Figures 4(c) and 4(d) show the shift of the frequencies \(\tilde{\Omega}_1^2\) of eigenmode 1 (blue dots) and \(\tilde{\Omega}_2^2\) of eigenmode 3 (green dots) as a function of the amplitude of the driven eigenmode. We carry out a fit of Eq. (9) and Eq. (10) to the data (black lines) to obtain the dispersive coupling (including a calibration factor \(\overline{\gamma_{23}}\) which accounts for the displacement sensitivity of eigenmode 2) and find \(\overline{\gamma_{23}} = -6.694 \times 10^{-19} \text{Hz}^2/\text{V}^2\) and \(\overline{\gamma_{23}} = -2.368 \times 10^{-19} \text{Hz}^2/\text{V}^2\).

V. VOLTAGE DEPENDENCE OF DUFFING NONLINEARITY AND DISPERSIVE COUPLING

We repeat the evaluation discussed so far for the other modes, namely by driving either eigenmode 1, 2, or 3 and detecting the frequency shifts of the respective other eigenmodes. This allows one to extract all dispersive coupling coefficients \(\overline{\gamma_{ij}}\) and \(\overline{\gamma_{ij}}\) as well as Duffing nonlinearities \(\overline{\gamma_{ij}}\) of the eigenmodes (with \(i, j = 1, 2, 3\) and \(i \neq j\)). Note that we detect different modes, each having a different (and voltage-dependent) calibration factor \(\overline{\gamma_{ij}}\) as a result of the mode-polarization dependent displacement sensitivity of our detection scheme.

We repeat the whole procedure for different dc voltages in a range of 0 V to 22 V to find the dc voltage dependence of the dispersive coupling \(\overline{\gamma_{ij}}, \overline{\gamma_{ij}}, \overline{\gamma_{ij}}, \) and \(\overline{\gamma_{ij}}\). The results are summarized in Fig. 5(a) and Fig. 5(b), which depict the experimentally determined Duffing nonlinearities and dispersive coupling coefficients, both including the appropriate calibration factor.

VI. HYBRIDIZATION-DEPENDENT DUFFING NONLINEARITY AND DISPERSIVE COUPLING

In this section we present a theoretical model that describes the voltage dependence of the Duffing nonlinearities and of the dispersive coupling coefficients. It is based on the dc-voltage dependent hybridization of the modes; see Fig. 3.

We start from the nonlinear potential of fourth order which couples the three bare modes of the system (OP1, OP2, IP1)
\[
\mathcal{V}(\{v_i\}) = \frac{1}{4} \sum_{i=1}^{3} v_i v_i^4 + \frac{1}{2} \left( v_1 v_1 v_1^2 + v_2 v_2 v_2^2 + v_3 v_3 v_3^2 \right)
\]
with amplitudes \(v_i\). Using the transformation Eq. (5), we change the basis and we express the quartic potential \(\mathcal{V}\) of Eq. (11) in terms of the hybridized eigenmodes of the system which are linearly independent. The result reads as follows:
\[
\tilde{\mathcal{V}}(\{q_i\}) = \frac{1}{4} \sum_{i=1}^{3} \tilde{\gamma}_{ij} q_j q_j q_j^4 + \frac{1}{2} \left( \tilde{\gamma}_{13} q_1^2 q_3 + \tilde{\gamma}_{23} q_2^2 q_3 + \tilde{\gamma}_{23} q_2^2 q_3 \right)
\]
\[
+ \sum_{i>j} \psi_i q_i^3 q_j + \sum_{i<j} \epsilon_{ijk} \Phi_i q_i q_j q_k,
\]
with the Levi-Civita antisymmetric tensor \(\epsilon_{ijk}\). The nonlinear potential \(\tilde{\mathcal{V}}\) of Eq. (12) for the amplitudes of the hybridized eigenmodes contains cubic nonlinearities and even three-body interactions. However, far from any internal resonances between the frequencies \(\Omega_i\) of the hybridized eigenmodes, the cubic and the three-body interaction terms can be neglected if we apply a single drive frequency. In other words, we only consider the Duffing nonlinearity of the driven mode, and the dispersive coupling to describe the frequency shift of the nondriven, thermally activated modes. Table I summarizes the notation for the different modes and their interactions.

In Eq. (12), the Duffing nonlinearities \(\tilde{\gamma}_{ij}\) of the hybridized eigenmodes can be related to the Duffing nonlinearities \(\gamma_{ij}\) and the dispersive coupling coefficients \(\gamma_{ij}\) of the bare modes via the components of the eigenvectors of the hybridized
TABLE I. Summary of the notations used for the bare intrinsic modes and the hybridized eigenmodes. The dc-voltage dependence of the coefficients $\kappa_{ij}$ and $\gamma_{ij}$ is not included in the theoretical model.

<table>
<thead>
<tr>
<th>Bare modes</th>
<th>Hybridized modes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amplitude $v_i$</td>
<td>$q_i [e_{ij}(V_{dc})]$</td>
</tr>
<tr>
<td>Linear interaction $\kappa_{ij}$</td>
<td>$\tilde{\gamma}<em>{ij}(V</em>{dc})$</td>
</tr>
<tr>
<td>Nonlinear interaction $\gamma_{ij}$</td>
<td>$\tilde{\gamma}<em>{ij}(V</em>{dc})$</td>
</tr>
</tbody>
</table>

Eigenmodes as follows:

$$\tilde{\gamma}_i = \sum_j e_{ij}^4 \gamma_{jj} + \sum_{j \neq k} e_{ij}^2 e_{ik}^2 \gamma_{jk}. \quad (13)$$

Similarly, we obtain the relation for the dispersive coupling coefficients of the hybridized eigenmodes

$$\tilde{\gamma}_{ij} = 6 \sum_k e_{ik}^2 e_{jk}^2 \gamma_{kk} + \sum_{k \neq m} (e_{ik} e_{jm} + e_{ik} e_{im}) \gamma_{km}$$

$$+ 4 \sum_{k \neq m} e_{ik} e_{jm} e_{jk} e_{im} \gamma_{mk} \quad \text{for } i \neq j. \quad (14)$$

with $i, j, k, m = 1, 2, 3$. By varying the dc voltage, we modify the hybridization of the eigenmodes of the systems, namely the dc voltage-dependent components of the eigenvectors $e_{ij}(V_{dc})$. In consequence, the Duffing nonlinearity of each mode as well as their dispersive coupling to the other modes are altered. In other words, in the model as given by Eqs. (13) and (14), the Duffing nonlinearities and the dispersive coupling coefficients of the hybridized eigenmodes are voltage dependent, $\tilde{\gamma}_{ij}(V_{dc})$, since they depend on the mode polarization. Using the theoretical mode polarization discussed in Fig. 3, we compute the dc-voltage dependence of the Duffing nonlinearities $\tilde{\gamma}_{ij}$ and of the dispersive coupling coefficients $\tilde{\gamma}_{ij}$.

Figures 6(a) and 6(b) display the Duffing nonlinearities ($\tilde{\gamma}_{11}$: blue solid line; $\tilde{\gamma}_{22}$: red short dashed line; $\tilde{\gamma}_{33}$: green long dashed line) as well as the dispersive coupling coefficients ($\tilde{\gamma}_{12}$: blue solid line; $\tilde{\gamma}_{23}$: red solid line; $\tilde{\gamma}_{31}$: green solid line; $\tilde{\gamma}_{32}$: red dashed line; $\tilde{\gamma}_{13}$: green dashed line) of the nanomechanical system formed by three linearly coupled bare eigenmodes of a pair of nanostrip resonators.

We showed that, even if a direct comparison between experimental data and theory was not possible, the theoretical model based on the mode hybridization qualitatively reproduces the nontrivial voltage dependence. From the experimental side, for further investigations on the voltage dependence of the nonlinear coefficients, the displacement detection scheme needs to be improved to allow for a full calibration of the measured data. In addition, a second microwave cavity should be included for resonator 2 to enable the direct detection of all modes. From the theoretical side, the effects of the voltage dependence of the intrinsic parameters of the systems for the bare modes have to be taken into account with a microscopic model for the dielectric modulation of the nonlinearities. Future work will address these directions.

VII. CONCLUSION

We observed nonlinear dispersive mode coupling in a nanomechanical system formed by three linearly coupled bare eigenmodes of a pair of nanostrip resonators.

We also observed a strong voltage dependence of the nonlinear coefficients of the system. Both the Duffing nonlinearities [16] and the dispersive coupling coefficients are modified, particularly in the dc-voltage regime in which strong hybridization occurs. To understand the latter effect, we have analyzed a minimal theoretical model which yields a voltage dependence resulting from the hybridization of the modes.

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