Non-equilibrium Josephson oscillations in Bose-Einstein condensates without dissipation

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We perform a detailed quantum dynamical study of non-equilibrium Josephson oscillations between interacting Bose-Einstein condensates confined in a finite-size double-well trap. We find that the Josephson junction can sustain multiple undamped Josephson oscillations up to a characteristic time scale τ_c without quasipartcles being excited in the system. This may explain recent related experiments. Beyond a characteristic time scale τ_c the dynamics of the junction is governed by fast, quasiparticle-assisted Josephson tunneling as well as Rabi oscillations between the discrete quasiparticle levels. We predict that an initially self-trapped BEC state will be destroyed by these fast dynamics.

One of the striking manifestations of quantum mechanics on a macroscopic level is the particle current induced by the phase difference between two coherent wave functions connected by a weak link, known as the Josephson effect [1]. For tunneling between the macroscopic wave functions of two Bose-Einstein condensates (BEC) trapped in a double-well potential this phenomenon leads to temporal oscillations of the population imbalance z between the two condensates even in the ground state [2, 3]. However, unlike in superconducting Josephson junctions, the interaction between the atoms in the condensates gives rise to regimes of fundamentally new dynamical behavior. If the initial population imbalance exceeds a critical value depending on the interaction strength, the large-amplitude Josephson oscillations (delocalized regime) cease, and the BEC is trapped in one of the two wells with only small-amplitude oscillations of z about the non-zero mean value (self-trapped regime). This complex, non-linear dynamics has been theoretically predicted for the ground state [4, 5, 6] and experimentally observed recently [7]. It is not only interesting in its own right but also relevant for any merging process of BECs, e.g., for quantum engineering or for producing a continuous source of condensed atoms [8, 9].

In the recent experiment [7] the Josephson junction is prepared in a non-equilibrium situation by ramping up the barrier between the condensates suddenly, in a non-adiabatic way. For this case an immediate damping of the Josephson oscillations has been predicted due to quasiparticle excitations [10], which is, however, not observed in the small traps of Ref. [7], revealing an incomplete understanding of the non-equilibrium dynamics.

In this Letter we present a detailed study of the temporal non-equilibrium dynamics of Josephson-coupled BECs after non-adiabatically switching-on the Josephson coupling, including interatomic interactions as well as quasiparticle (QP) excitations. As the main result we find that in small traps multiple, undamped Josephson oscillations are possibe up to a time scale τ_c. At this time

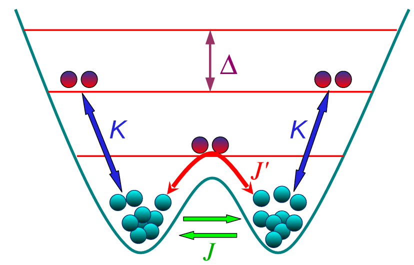

FIG. 1: (Color online). A BEC in a double-well potential after abrupt decrease of the barrier height. The definitions of the parameters are explained in the text, see Eqs. (3)–(5).

scale the dynamics switches abruptly but continuously from slow Josephson to fast Rabi oscillations between the discrete QP levels. We also predict that the self-trapped behavior is destroyed by the Rabi oscillations, i.e. for time $t > \tau_c$ the system switches to delocalized behavior, if it has previously been in a self-trapped state. This highly non-linear behavior results essentially from a separation of energy scales in small traps with discrete QP level spacing Δ, which can be chosen larger than J. Switching on J lowers the ground state energy by the amount $\Delta E = J\sqrt{N_1(0)N_2(0)}$, where $N_1(0)$, $N_2(0)$ are the occupation numbers of the two BECs in the initial state at time $T = 0$. Thus, after a sudden switching two initially separated BECs are in an excited state ΔE above the coupled ground state. Because of the large values of N_1, N_2 this energy is sufficient to excite QPs out of the BECs. The time-dependent BEC amplitude acts as a perturbation on the QP system. However, transitions to QP states are not allowed in perturbation theory, because the frequency of the oscillations is less than their excitation energy, $J < \Delta$. Our detailed calculations show that such transitions are only possible as a highly non-linear process after the characteristic time τ_c.

We consider a Bose-Einstein condensed atomic gas in a double-well trap as represented by Fig. 1. Such a system is most generally described by the Hamiltonian

$$H = \int d^3r \hat{\Psi}^\dagger(\mathbf{r},t) \left(-\frac{1}{2m}\Delta + V_{ext}(\mathbf{r},t) \right) \hat{\Psi}(\mathbf{r},t) + \frac{g}{2} \int d^3r \hat{\Psi}^\dagger(\mathbf{r},t)\hat{\Psi}^\dagger(\mathbf{r},t)\hat{\Psi}(\mathbf{r},t)\hat{\Psi}(\mathbf{r},t), \quad (1)$$

where $\Psi(\mathbf{r},t)$ is a bosonic field operator, and we assume a contact interaction between the bosons with $g = 4\pi a_s/m$ (a_s is the s-wave scattering length). V_{ext} is the external double-well trapping potential. Initially, the barrier between the two wells is assumed to be infinitely large, so that Josephson tunneling is absent. All bosons are condensed, and both condensates are in the equilibrium state. At time $t = 0$ the barrier is suddenly lowered so that a Josephson weak link is established between the wells. This nonadiabatic process drives the system out of thermodynamic equilibrium.

In order to develop the general non-equilibrium theory for this system and to analyze its dynamics, we wish to represent the Hamiltonian (1) in the complete basis of the exact single-particle eigenstates of the double-well potential $V_{ext}(\mathbf{r}, t > 0)$ after switching on the coupling J. In this basis the field operator reads,

$$\hat{\Psi}(\mathbf{r},t) = \phi_1(\mathbf{r})a_1(t) + \phi_2(\mathbf{r})a_2(t) + \sum_{n\neq 0} \varphi_n(\mathbf{r})\hat{b}_n(t), \quad (2)$$

where $\phi_1(\mathbf{r})$, $\phi_2(\mathbf{r})$ are the respective ground state wavefunctions of the two wells after lowering the barrier, and the a_α are the corresponding, time-dependent condensate amplitudes (c-numbers), $a_\alpha(t) = \sqrt{N_\alpha}e^{i\theta_\alpha(t)}$, $\alpha = 1, 2$. This semiclassical treatment of the BECs neglects phase fluctuations. It is justified for the experiments [7], where the BECs are initially produced with fixed phase relation and the particle number is sufficiently large. The applicability of the semiclassical approximation has been discussed in detail in Refs. [10, 11, 12, 13] and has been tested experimentally in Ref. [14]. The quasiparticle dynamics will be treated fully quantum mechanically. The index $n \neq 0$ enumerates the exact single-particle excited states, with $\varphi_n(\mathbf{r})$ and $\hat{b}_n(t)$ the corresponding eigenfunctions and bosonic destruction operators, respectively. Note that by including excited states we go beyond the frequently used two-mode approximation [4, 9] for the BECs. For simplicity we assume that the ground state energies of the two wells before mixing are equal, $E_0 = 0$, and that the wavefunctions of the excited states extend over both wells. Inserting the field operator (2) into Eq. (1) and evaluating the overlap matrix elements in a straightforward way, the Hamiltonian takes for $t > 0$ the form, $H = H_{BEC} + H_{qp} + H_{mix}$. H_{BEC} describes condensate particles,

$$H_{BEC} = E_0 \sum_{\alpha=1}^{2} a^*_\alpha a_\alpha + \frac{U}{2} \sum_{\alpha=1}^{2} (a^*_\alpha a^*_\alpha a_\alpha a_\alpha) - J(a^*_1 a_2 + a^*_2 a_1), \quad (3)$$

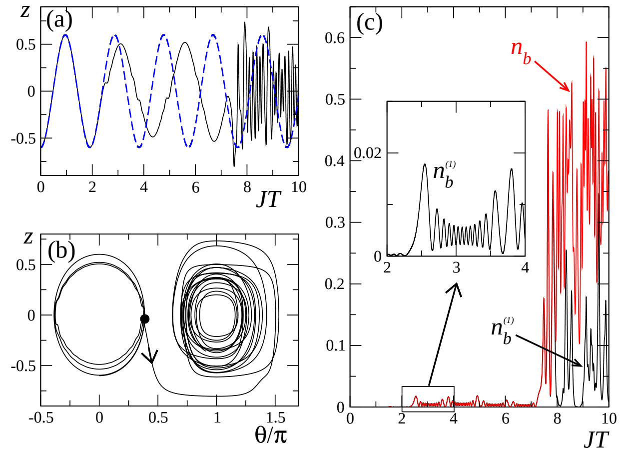

FIG. 2: (Color online). Time-evolution of condensed and non-condensed particles for the initial conditions $z(0) = -0.6$, $\theta(0) = 0$ and interaction parameters $u = u' = 5$, $j' = 60$, $k = 0$ (delocalized regime). 5 QP levels were included in the numerical evaluation. (a) The dynamics of the BEC population imbalance $z(T)$ is shown (solid black line). The dashed, blue line shows, for comparison, the behavior without QP coupling, $u' = j' = k = 0$, in agreement with Ref. [5]. (b) $z(T)$ vs. $\theta(T)$ map. The arrow indicates the direction of time evolution. The time $T = \tau_c$ is marked by the black dot. It is seen that at $T = \tau_c$ the system changes dynamically from a $\theta = 0$ to a $\theta = \pi$ junction with more erratic phase evolution. (c) Time evolution of the non-condensed particle population. The black curve (also in the inset) shows the particle occupation of the first level $n_b^{(1)}$, while the red curve is the sum of all five levels n_b. For $T < \tau_c$ the two curves practically coincide.

with the Josephson coupling J and the interaction between condensed particles, $U > 0$. H_{qp} corresponds to single-particle excitations,

$$H_{qp} = \sum_{n\neq 0} E_n \hat{b}^\dagger_n \hat{b}_n + \frac{U'}{2} \sum_{n,m} \hat{b}^\dagger_m \hat{b}^\dagger_n \hat{b}_n \hat{b}_m, \quad (4)$$

where E_n are the (bare) QP energies, and U' is the repulsive interaction between non-condensed particles. Mixing between the BECs and the QP system is described by

$$H_{mix} = J' \sum_n \left[(a^*_1 a_2 + a^*_2 a_1)\hat{b}^\dagger_n \hat{b}_n + \frac{1}{2}(a^*_1 a^*_2 \hat{b}_n \hat{b}_n + h.c.) \right] + K \sum_{n,\alpha=1}^{2} \left[(a^*_\alpha a_\alpha)\, \hat{b}^\dagger_n \hat{b}_n + \frac{1}{4}(a^*_\alpha a^*_\alpha \hat{b}_n \hat{b}_n + h.c.) \right]. \quad (5)$$

Here the coupling constant J' arises as a QP-assisted Josephson tunneling as well as a pairwise QP creation/destruction out of both BECs *simultaneously.* K represents the density-density interaction of condensed and non-condensed particles and the pairwise QP creation/destruction out of each of the BECs *separately.* In deriving Eq. (5) we neglected the off-diagonal in n and

m elements because of different spatial dependence of the wavefunctions.

To treat the non-equilibrium quantum dynamics of the system, we use the Keldysh Green's function $\mathbf{G}+\mathbf{C}$, generalized to Bose-condensed systems. The QP part reads,

$$\begin{aligned}\mathbf{G}_{nm}(t,t') &= -i\begin{pmatrix}\langle T_C\hat{b}_n(t)\hat{b}^\dagger_m(t')\rangle & \langle T_C\hat{b}_n(t)\hat{b}_m(t')\rangle\\ \langle T_C\hat{b}^\dagger_n(t)\hat{b}^\dagger_m(t')\rangle & \langle T_C\hat{b}^\dagger_n(t)\hat{b}_m(t')\rangle\end{pmatrix}\\ &= \begin{pmatrix} G_{nm}(t,t') & F_{nm}(t,t')\\ \overline{F}_{nm}(t,t') & \overline{G}_{nm}(t,t'),\end{pmatrix}, \qquad (6)\end{aligned}$$

where T_C implies time ordering along the Keldysh contour, i.e. each of the normal and anomalous bosonic Green's functions G and F is a 2×2 matrix in Keldysh space. The condensate part is classical with trivial time ordering,

$$\mathbf{C}_{\alpha\beta}(t,t') = -i\begin{pmatrix} a_\alpha(t)a^*_\beta(t') & a_\alpha(t)a_\beta(t')\\ a^*_\alpha(t)a^*_\beta(t') & a^*_\alpha(t)a_\beta(t')\end{pmatrix}. \qquad (7)$$

The equations of motion for $\mathbf{G}+\mathbf{C}$ are derived in a standard way [15]. Transforming the time variables to center-of mass and relative coordinates, $T=(t+t')/2$ and $\tau=(t-t')$, observing that the Josephson dynamics depending on T is slow compared to the inverse QP energies ($J<\Delta$), the relative coordinate can be set $\tau=0$ in all propagators and self-energies. We treat the QP interaction in Eq. (4) within the self-consistent Bogoliubov-Hartree-Fock approximation [16]. This will be sufficient for the present purpose, since QP collisions, neglected here, will play a role only for sufficiently high population of QP levels (see below). The normal and anomalous QP self-energies, $\Sigma_n(\tau=0,T)$, $\Omega_n(\tau=0,T)$, are then diagonal in the QP level index n and read,

$$\begin{aligned}\Sigma_n(T) &= K(N_1+N_2)+J'(a_1^*a_2+a_2^*a_1)\\ &+ 2iU'\sum_m G^<_{mm}(T), \qquad (8)\end{aligned}$$

$$\Omega_n(T) = \frac{K}{2}\sum_{\alpha=1}^{2}a_\alpha a_\alpha + J'a_1a_2 + iU'\sum_m F^<_{mm}(T), \quad (9)$$

with $\overline{\Sigma}=\Sigma$ and $\overline{\Omega}=\Omega^*$. After lengthy but straightforward calculations one arrives at the coupled set of equations for the non-condensate propagators $G^<(\tau=0,T)$, $F^<(\tau=0,T)$ and the complex condensate amplitudes $a_1(T)$, $a_2(T)$,

$$\begin{aligned} i\frac{\partial}{\partial T}G^<_{nn}(T) &= \Omega_n(T)\overline{F}^<_{nn}(T)-\overline{\Omega}_n(T)F^<_{nn}(T),\\ \Big(i\frac{\partial}{\partial T}-2E_n &- 2\Sigma_n(T)\Big)F^<_{nn}(T)=\Omega_n(T)\overline{G}^<_{nn}(T)\\ &+ \Omega_n(T)G^<_{nn}(T), \qquad (10)\end{aligned}$$

$$\begin{aligned} i\frac{\partial}{\partial T}a_1(T) &= \left[U|a_1(T)|^2+KN_b(T)\right]a_1(T)-Ja_2(T)\\ +J'N_b(T)a_2(T) &+ i\left[\frac{K}{2}a_1^*(T)+\frac{J'}{2}a_2^*(T)\right]\sum_n F^<_{nn}(T) \qquad (11)\end{aligned}$$

The equation for $a_2(T)$ is obtained from Eq. (11) by $a_1 \rightleftarrows a_2$. From Eqs. (10), (11) we compute the occupation numbers for bosons out of condensate, $N_b^{(n)}(T)=\langle\hat{b}^\dagger_n(T^+)\hat{b}_n(T)\rangle$, $N_b(T)=\sum_n N_b^{(n)}(T)$, the condensate population imbalance $z(T)=[N_1(T)-N_2(T)]/N$, normalized by the total particle number $N=N_1(0)+N_2(0)+N_b(0)$, and the time evolution of the phase difference $\theta(T)=\theta_2(T)-\theta_1(T)$. To absorb the large factors of particle numbers appearing in Eqs. (9)–(11) it is useful to define the dimensionless parameters $u=NU/J$, $u'=NU'/J$, $j'=NJ'/J$, $k=NK/J$, and $n_b^{(n)}(T)=N_b^{(n)}(T)/N$.

Without coupling to the QP excitations ($j'=k=0$), Eq. (11) reduces to the two-mode model of Smerzi et al. [5], exhibiting the self-trapped and delocalized regimes, with a Josephson oscillation frequency of

$$\omega_J^{(0)} = 2|J|\sqrt{1+u/2} \qquad (12)$$

in the linear regime (Eq. (10) in Ref. [5]). When, however, $j'\neq 0$, $k\neq 0$ and QPs are excited, $N_b(T)>0$, the QP-assisted Josephson tunneling term in H_{mix} becomes active (J'-term in Eq. (11)). One then expects an enhanced Josephson frequency, with roughly J replaced by $J[1-j'n_b(T)]$ in Eq. (12). At the same time, Rabi oscillations of the $N_b^{(n)}(T)$, i.e. of QP pairs between the BEC and the excited levels, with frequencies $\omega_R\approx 2E_n$ set in, c.f. Eq. (10). As a result, in this QP-dominated regime one expects complex, high-frequency anharmonic oscillatory behavior.

The complete numerical solutions of Eqs. (9)–(11) for a finite-size trap with $N=5\cdot 10^5$ particles (level spacing $\Delta=E_{n+1}-E_n=10J$, taking 5 QP levels into account [17]) are shown for typical parameter values in Fig. 2 for the delocalized regime and in Fig. 3 for the initially self-trapped regime.

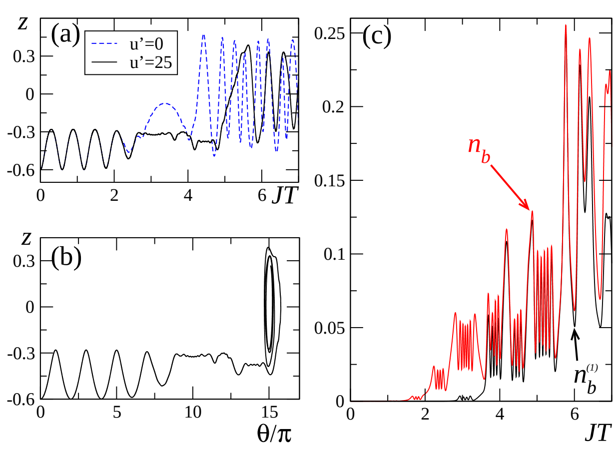

FIG. 3: (Color online). Same as in Fig. 2, but for $u=u'=25$ (self-trapped regime) and $j'=30$. In (a) the dashed, blue line shows the behavior without QP interactions, $u'=0$.

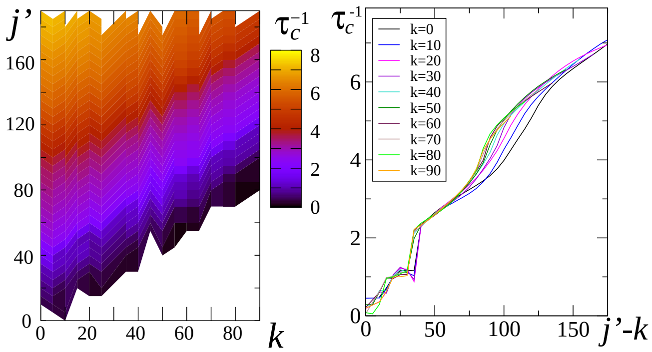

FIG. 4: (Color online). Dependence of the inverse time-scale τ_c^{-1} on two main parameters j' and k, for the initial conditions as in Fig. 3. In the left panel the white area corresponds to j' and k values for which τ_c was not found; we assume $\tau_c \to \infty$ in this case. The right panel shows a collapse of all τ_c^{-1} curves onto a single one with the simple law $\tau_c^{-1} \sim j' - k$. The scatter is due to the ambiguity in the numerical definition of τ_c.

The results reproduce the expected behavior discussed above in the regime with finite QP population. The parameters were chosen such that the energy ΔE stored in the two BECs by the initial, non-adiabatic switching-on of J is much greater than the QP energies, $\Delta E \approx J\sqrt{N_1(0)N_2(0)} \gg \Delta$. The most striking and most important feature seen in both figures is that nevertheless multiple undamped Josephson oscillations occur for an extended period of time without QPs being excited. The reason for this behavior is that in the initial state the QP population $n_b(t)$ is vanishing and, therefore, the QP-assisted Josephson tunneling term J' in Eq. (5) does not contribute. Hence, the Josephson oscillations have the bare frequency $\omega_J \approx \omega_J^{(0)} < 2\Delta$ which is not sufficient to excite a QP pair perturbatively. Only for times greater than a characteristic time τ_c the highly non-linear dynamics of the system makes QP excitations possible. In this long-time regime the finite QP population $n_b(t)$ and fast oscillations of the BEC population imbalance $z(t)$ stabilize each other mutually: $n_b(t) > 0$ implies a QP-enhanced Josephson frequency, and the resulting fast oscillations ($\omega_J > 2\Delta$) of $z(t)$ can efficiently excite QPs via the mixing Hamiltonian (5).

The fast, QP-induced dynamics implies two further features. (1) As seen from Fig. 3 (a) an initially self-trapped state is destroyed and the system changes to a delocalized state at the same time when $n_b(T)$ becomes sizeable. (2) At the onset of the fast dynamics the system changes from a $\theta = 0$ to $\theta = \pi$ Josephson junction, see Figs. 2 (b), 3 (b). This can be understood qualitatively, in that the large phase difference $\theta(T) \approx \pi$ is required to sustain the large Josephson current in the state with fast dynamics.

Since the transition to the QP-dominated regime is not described by a Fermi golden rule, it is hard to analyse the time scale τ_c analytically. We defined τ_c numerically as the scale where $n_b(t)$ first exceeds 0.05 and extracted it from our solutions. As seen in Fig. 3 (a), τ_c is essentially independent of the QP interaction u'. The dependence of $1/\tau_c$ on the parameters j' and k (Fig. 4) is remarkably linear, and for $j' < k$ no transition to the QP-dominated regime is found.

To conclude, we have presented a detailed quantum dynamical study of the non-linear Josephson dynamics of BECs confined in a finite-size double-well potential, including coupling to quasiparticle states. Remarkably, the system can sustain multile, undamped Josephson oscillations for an extended time period before quasiparticles get excited and the behavior changes abruptly to a regime of fast Josephson and Rabi oscillations. Only in this quasiparticle-dominated regime we expect strong damping of the oscillations due to inelastic quasiparticle collisions, equilibrating the system at a finite temperature. This will be a subject of further research. The sharp but continuous transition from the Josephson- to the quasiparticle-dominated regime should be experimentally observable.

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[17] We have performed similar calculations for 2, 3 and 4 QP levels with fixed Δ. It was verified that the generic behavior described in the text, in particular the values of τ_c, are essentially independent of the number of QP levels taken into account.