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GLOBAL WELL–POSEDNESS OF THE CAUCHY PROBLEM FOR THE JORDAN–MOORE–GIBSON–THOMPSON EQUATION

REINHARD RACKE[1] & BELKACEM SAID-HOUARI[2]

Abstract. In this paper, we consider the Cauchy problem of a third order in time nonlinear equation known as the Jordan–Moore–Gibson–Thompson (JMGT) equation arising in acoustics as an alternative model to the well-known Kuznetsov equation. First, using the contraction mapping theorem, we show a local existence result in appropriate function spaces. Second, by using the energy method together with a bootstrap argument, we prove a global existence result for small data. Third, polynomial decay rates in time for the solution will be obtained for space dimensions $N \geq 2$.

1. Introduction

In this paper, we consider the nonlinear Jordan–Moore–Gibson–Thompson equation:

$$\tau u_{ttt} + u_{tt} - c^2 \Delta u - \beta \Delta u_t = \frac{\partial}{\partial t} \left(\frac{1}{c^2} \frac{B}{2A} (u_t)^2 + |\nabla u|^2 \right), \tag{1.1a}$$

where $x \in \mathbb{R}^N$ (Cauchy problem), and $t > 0$. We consider the initial conditions:

$$u(t=0) = u_0, \qquad u_t(t=0) = u_1 \qquad u_{tt}(t=0) = u_2. \tag{1.1b}$$

1.1. **The model.** Equation (1.1a) appears as a generalization of the the Kuznetsov equation (see equation (1.8) below). Both equations are used as models in what is called nonlinear acoustics that deals with finite-amplitude wave propagation in fluids and solids and related phenomena, see the books of Beyer [1] or Rudenko and Soluyan [18]. In particular, the JMGT equation arises from modeling high-frequency ultra sound waves, see [13] for more details.

The derivation of equation (1.1a) (see [7] and [20]) can be obtained from the general equations of fluid mechanics by means of some asymptotic expansions in powers of small parameters. The motion of a viscous, heat-conducting fluid can be described by four equations: the conservation of mass (the continuity equation), the conservation of momentum (Newton's second law), conservation of energy (first law of thermodynamics or entropy balance) and an equation of state. Thus, the first three equations: conservation of mass, conservation of momentum and entropy balance in the model of thermo-viscous

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flow in compressible fluid, for the mass density ϱ, the acoustic particle velocity v and the absolute temperature θ, can be written as

$$\begin{cases} \varrho_t + \nabla \cdot (\varrho v) = 0, \\ \varrho(v_t + (v \cdot \nabla)v) = \nabla \cdot \mathbb{T}, \\ \varrho\theta(\eta_t + (v \cdot \nabla)\eta) = -\nabla q + \mathbb{T} : \mathbb{D}. \end{cases} \tag{1.2}$$

Here, η is the entropy and q is the heat flux vector. Moreover, $\mathbb{D}$ is the deformation tensor given by

$$\mathbb{D} = \frac{1}{2}(\nabla v + (\nabla v)^T),$$

anf $\mathbb{T}$ is the Cauchy–Poisson stress tensor given by

$$\begin{aligned} \mathbb{T} &= (-p + \lambda(\nabla \cdot v))\mathbb{I} + 2\mu\mathbb{D} \\ &= -p\mathbb{I} + 2\mu\Big(\mathbb{D} - \frac{1}{3}(\nabla \cdot v)\mathbb{I}\Big) + \zeta(\nabla \cdot v)\mathbb{I}, \end{aligned}$$

where p is the acoustic pressure $\mathbb{I}$ is the identity matrix, μ is the shear viscosity (the first coefficient of viscosity) $\lambda = \zeta - \frac{2}{3}\mu$, where ζ is the second coefficient of viscosity (the bulk viscosity), and the components of $\mathbb{T} : \mathbb{D}$ are $T_{ij}D_{ij}$ where T_{ij} are the components of the matrix $\mathbb{T}$ and D_{ij} are the components of the matrix $\mathbb{D}$. Hence, we can recast (1.2) as

$$\begin{cases} \varrho_t + \nabla \cdot (\varrho v) = 0, \\ \varrho(v_t + (v \cdot \nabla)v) = -\nabla p + (\lambda + \mu)\nabla(\nabla \cdot v) + \mu\Delta u, \\ \varrho\theta(\eta_t + (v \cdot \nabla)\eta) = -\nabla q + 2\mu\mathbb{D} : \mathbb{D} + \lambda(\nabla \cdot v)^2. \end{cases} \tag{1.3}$$

The equation of state (which describes the relationship between the pressure, the density and the entropy) is

$$p = p(\varrho, \eta). \tag{1.4}$$

First, we assume that the deviations of ϱ, p, η and θ from their equilibrium values ϱ_0, p_0, η_0 and θ_0 are small.

By taking the Taylor series expansion of (1.4) around values at rest ϱ_0 and η_0 and ignoring the higher-order terms, we get

$$p(\varrho, \eta) = p(\varrho_0, \eta_0) + \left(\frac{\partial p}{\partial \varrho}(\varrho_0, \eta_0)\right)(\varrho - \varrho_0) + \frac{1}{2}\left(\frac{\partial^2 p}{\partial \varrho^2}(\varrho_0, \eta_0)\right)(\varrho - \varrho_0)^2 + \left(\frac{\partial p}{\partial \eta}(\varrho_0, \eta_0)\right)(\eta - \eta_0).$$

We have

$$p_0 = p(\varrho_0, \eta_0), \quad A := \varrho_0\frac{\partial p}{\partial \varrho}(\varrho_0, \eta_0) = \varrho_0 c^2, \quad B := \varrho_0^2\frac{\partial^2 p}{\partial \varrho^2}(\varrho_0, \eta_0), \quad \varrho_0\frac{\gamma - 1}{\chi} = \frac{\partial p}{\partial \eta}(\varrho_0, \eta_0),$$

and the pressure p is given by

$$p(\varrho, \eta) = p_0 + \varrho_0 c^2\left[\frac{\varrho - \varrho_0}{\varrho_0} + \frac{B}{2A}\left(\frac{\varrho - \varrho_0}{\varrho_0}\right)^2 + \frac{\gamma - 1}{\chi c^2}(\eta - \eta_0)\right], \tag{1.5}$$

where $\nabla p_0 = 0$. In the above equations, c is the speed of sound, B/A the parameter of nonlinearity, χ the coefficient of volume expansion and $\gamma = c_p/c_v$ is the ratio of specific heat, where c_p and c_v are the specific heat capacities at constant pressure and constant volume. Assuming that the flow is rotation free, that is $\nabla \times v = 0$, by introducing the acoustic velocity potential $v = -\nabla u$, it has been shown in [6, 13] that equation (1.1a) can be derived from the above set of equations by assuming the Cattaneo law of heat conduction

$$\tau q_t + q = -K\nabla\theta, \tag{1.6}$$

where K is the thermal conductivity. The Cattaneo (or Maxwell–Cattaneo) law accounts for finite speed of propagation of the heat transfer and eliminates the paradox of infinite speed of propagation for pure heat conduction associated with the Fourier law (i.e., $\tau = 0$). Here τ is a small relaxation parameter. It is know that if we use in (1.2) the Fourier law

$$q = -K\nabla\theta, \tag{1.7}$$

then we can derive the Kuznetsov equation

$$u_{tt} - a^2\Delta u - b\Delta u_t = \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{B}{2A}(u_t)^2 + |\nabla u|^2\right), \tag{1.8}$$

which is a well-known model and widely used in nonlinear acoustics, see the derivation of (1.8) in [9] and [4]. Hence equation (1.1a) can be seen as a “hyperbolic" version of (1.8). Equation (1.8) is written in terms of the acoustic velocity potential $v = -\nabla u$. It can be also expressed in terms of the acoustic pressure fluctuation $\tilde{p}$ as

$$\frac{1}{c^2}\tilde{p}_{tt} - \Delta\tilde{p} - \frac{b}{c^2}\Delta\tilde{p}_t = \partial_{tt}\left(\frac{1}{\varrho_0 c^4}\frac{B}{2A}\tilde{p} + \frac{\varrho_0}{c^2}(v\cdot v)\right) \tag{1.9}$$

such that the identity

$$\varrho_0 v_t = -\Delta\tilde{p}$$

holds. Assuming that the local nonlinear effects can be neglected, that is making the replacement $v\cdot v = (\frac{1}{\varrho_0 c}\tilde{p})^2$ on the right-hand side of (1.9), we arrive at the so-called Westervelt equation:

$$\frac{1}{c^2}\tilde{p}_{tt} - \Delta\tilde{p} - \frac{b}{c^2}\Delta\tilde{p}_t = \partial_{tt}\left(\frac{1}{\varrho_0 c^4}\Big(1 + \frac{B}{2A}\Big)\tilde{p}\right) \tag{1.10}$$

or in terms of u through the relation $\varrho u_t = p$ as

$$u_{tt} - a^2\Delta u - b\Delta u_t = \frac{\partial}{\partial t}\left(\frac{1}{c^2}\Big(1 + \frac{B}{2A}\Big)(u_t)^2\right). \tag{1.11}$$

Analogously to the above reduction of the Kuznetsov equation to the Westervelt equation, we can reduce equation (1.1a) to

$$\tau u_{ttt} + u_{tt} - c^2\Delta u - \beta\Delta u_t = \frac{\partial}{\partial t}\left(\frac{1}{c^2}\Big(1 + \frac{B}{2A}\Big)(u_t)^2\right). \tag{1.12}$$

1.2. **Previous work.** The initial boundary value problem associated to (1.1a) has been studied recently in bounded domains. In [7] (see also [8]), the authors considered the linear equation

$$\tau u_{ttt} + \alpha u_{tt} + c^2 \mathcal{A} u + b \mathcal{A} u_t = 0 \tag{1.13}$$

where $\mathcal{A}$ is a positive self-adjoint operator, and showed that by neglecting diffusivity of the sound coefficient ($b = 0$) there arises a lack of existence of a semigroup associated with the linear dynamics. On the other hand, they proved that when the diffusivity of the sound is strictly positive ($b > 0$), the linear dynamics is described by a strongly continuous semigroup, which is exponentially stable provided the dissipativity condition $\gamma := \alpha - \tau c^2/b > 0$ is fulfilled, which is, for our equation (1.1a), equivalent to

$$\beta - \tau c^2 > 0. \tag{1.14}$$

This condition (1.14) will be assumed throughout the paper.

For $\gamma = 0$ the energy is conserved (the same type of results are obtained in [2] using energy methods, or in [12] using the analysis of the spectrum of the operator). The exponential decay rate results in [12] are completed in [16], where an explicit scalar product where the operator is normal allows the authors to obtain the optimal exponential decay rate of the solutions. Finally, in [3], the authors showed the chaotic behavior of the system when $\gamma < 0$, as we have mentioned above. Equation (1.13) with a viscoelastic damping of a memory type has been also considered in [11] and [10], where exponential stability results have been obtained.

The dissipativity condition (1.14) can also be understood in looking at the zeros z_j, $j = 1, 2, 3$, of the characteristic polynomial associated to our equation (1.1a) after having applied the Fourier transform $\mathcal{F}_{x \to \xi}$ to the linarized part:

$$z^3 + z^2 + \beta |\xi|^2 z + c^2 |\xi|^2 = 0. \tag{1.15}$$

Computing the associated Hurwitz matrix

$$\mathbb{H}_3 := \begin{pmatrix} 1 & \tau & 0 \\ c^2|\xi|^2 & \beta|\xi|^2 & 1 \\ 0 & 0 & c^2|\xi|^2 \end{pmatrix},$$

and the determinants d_j of the minors $D_j = ((\mathbb{H}_3)_{km})_{k,m=1,\ldots,j}$, we have

$$d_1 = 1, \quad d_2 = |\xi|^2(\beta - \tau c^2), \quad d_3 = c^2 |\xi|^2 d_2.$$

Thus, $\Re z_j < 0$, $j = 1, 2, 3$, holds if and only if the dissipativity condition (1.14) holds.

Equation (1.12) (which is called Jordan-Moore-Gibson-Thompson-Westervelt) has been investigated in [8] and its linear form in [7]. The authors in [8] used the estimates of the higher-level energies obtained for the linear model in [7] to establish global well-posedness and decay rates of solutions to the initial and boundary value problem associated to (1.12). Of course (1.12) is simpler compared to (1.1a).

In this paper, we consider the Jordan–Moore–Gibson–Thompson equation in its full generality (i.e., (1.1a)) for the Cauchy problem $x \in \mathbb{R}^N$. Under the assumption $0 < \tau < \beta$: first, by using the contraction mapping theorem in appropriately chosen spaces, we show a local existence result in some appropriate functional spaces, second by using some energy-type estimates we prove a global existence result for small initial data by constructing an appropriate energy norm and show that this norm remains uniformly bounded with respect to time.

We rewrite the right-hand side of equation (1.1a) in the form

$$\frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{B}{2A}(u_t)^2 + |\nabla u|^2\right) = \frac{1}{c^2}\frac{B}{A}u_t u_{tt} + 2\nabla u \nabla u_t,$$

and introduce the new variables

$$v = u_t \qquad \text{and} \qquad w = u_{tt},$$

Without loss of generality, we assume from now on

$$c = 1.$$

Then equation (1.1a) can be rewritten as the following first order system

$$\begin{cases} u_t = v, \\ v_t = w, \\ \tau w_t = \Delta u + \beta \Delta v - w + \dfrac{B}{A} v w + 2\nabla u \nabla v, \end{cases} \tag{1.16}$$

with the initial data

$$u(t=0) = u_0, \qquad v(t=0) = v_0, \qquad w(t=0) = w_0. \tag{1.17}$$

Understanding the asymptotic behavior of the linearized problem is critical for proving the decay rate of the nonlinear problem. The first result (for $x \in \mathbb{R}^N$) in this direction has been presented in the recent paper [15], where the authors used the energy method in the Fourier space to show that under the assumption $\beta > \tau$ the energy norm of the solution $\|V(t)\|_{L^2} = \|(\tau u_{tt} + u_t, \nabla(\tau u_t + u), \nabla u_t)(t)\|_{L^2}$ decays with the rate $(1+t)^{-N/4}$. They also proved that this decay rate is optimal, by using the eigenvalues expansion method. Some other decay rates for $\|u(t)\|_{L^2}$ were also presented in [15] by using the explicit formula of the Fourier image of the solution.

The remaining part of this paper is organized as follows: In Section 2 we introduce some notations and some lemmas that we will use in the proof of the main results. In Section 3, we prove the global existence of solutions for small data. We employ the energy method together with some commutator estimates to prove a global existence result for small initial data in appropriate Sobolev spaces. We should mention that the method we used to prove the global existence does not depend on decay estimates for the linearized equation. As a result, the global existence is proved under the same regularity assumption required for the local existence which is proved in Section 4, where we apply the contraction mapping theorem to show the local well-posedness of (1.1). Finally, Section 5 is devoted to the decay estimate for the norm $\|(u_t + \tau u_{tt}, \nabla(u + \tau u_t), \nabla u_t)\|_{L^2}$. In fact, based on the decay estimates obtained in [15], for the linearized problem, we prove that the same decay result holds for the nonlinear problem for $N > 2$.

We introduce some notations that will be used throughout the paper. Let $\|.\|_q$ and $\|.\|_{H^\ell}$ stand for the $L^q(\mathbb{R}^N)$-norm ($2 \leq q \leq \infty$) and the $H^\ell(\mathbb{R}^N)$-norm. We define the weighted function space, $L^{1,1}(\mathbb{R}^N)$, $N \geq 1$ as follows: $u \in L^{1,1}(\mathbb{R}^N)$ iff $u \in L^1\left(\mathbb{R}^N\right)$ and

$$\|u\|_{1,1} = \int_{\mathbb{R}^N} (1 + |x|)|u(x)|dx < +\infty.$$

The symbol $[A, B] = AB - BA$ denotes the commutator. The constant C denotes a generic positive constant that appears in various inequalities and may change its value in different occurrences.

2. Preliminaries

The following lemma has been proved for instance in [5, Lemma 4.1].

Lemma 2.1. *Let $1 \leq p, q, r \leq \infty$ and $1/p = 1/q + 1/r$. Then, we have*

$$\|\nabla^k(uv)\|_{L^p} \leq C(\|u\|_{L^q}\|\nabla^k v\|_{L^r} + \|v\|_{L^q}\|\nabla^k u\|_{L^r}), \quad k \geq 0, \tag{2.1}$$

and the commutator estimate

$$\|[\nabla^k, f]g\|_{L^2} \leq C(\|f\|_{L^\infty}\|\nabla^k g\|_{L^2} + \|g\|_{L^\infty}\|\nabla^k f\|_{L^2}), \quad k \geq 1, \tag{2.2}$$

for some constant $C > 0$.

The next lemma has been proved in [19, Lemma 3.7].

Lemma 2.2. *Let $M = M(t)$ be a non-negative continuous function satisfying the inequality*

$$M(t) \leq c_1 + c_2 M(t)^\kappa,$$

in some interval containing 0, where c_1 and c_2 are positive constants and $\kappa > 1$. If $M(0) \leq c_1$ and

$$c_1 c_2^{1/(\kappa-1)} < (1 - 1/\kappa)\kappa^{-1/(\kappa-1)},$$

then in the same interval

$$M(t) < \frac{c_1}{1 - 1/\kappa}.$$

We will use the Gagliardo–Nirenberg interpolation inequality as follows.

Lemma 2.3. *([14]) Let $N \geq 1$. Let $1 \leq p, q, r \leq \infty$, and let m be a positive integer. Then for any integer j with $0 \leq j \leq m$, we have*

$$\left\|\nabla^j u\right\|_{L^p} \leq C \left\|\nabla^m u\right\|_{L^r}^{\alpha} \left\|u\right\|_{L^q}^{1-\alpha} \tag{2.3}$$

where

$$\frac{1}{p} = \frac{j}{N} + \alpha\left(\frac{1}{r} - \frac{m}{N}\right) + \frac{1-\alpha}{q}$$

for α satisfying $j/m \leq \alpha \leq 1$ and C is a positive constant.

We also recall the decay estimates of the linearized problem associated to (1.1a)

Proposition 2.4. *([15]) Let u be the solution of the linear problem*

$$\tau u_{ttt} + u_{tt} - c^2 \Delta u - \beta \Delta u_t = 0. \tag{2.4}$$

Assume that $0 < \tau < \beta$. Let $\mathbf{V} = (u_t + \tau u_{tt}, \nabla(u + \tau u_t), \nabla u_t)$ and assume in addition that $\mathbf{V}_0 \in L^1(\mathbb{R}^N) \cap H^s(\mathbb{R}^N)$. Then, for all $0 \leq j \leq s$, we have

$$\|\nabla^j \mathbf{V}(t)\|_{L^2(\mathbb{R}^N)} \leq C(1+t)^{-N/4-j/2} \|\mathbf{V}_0\|_{L^1(\mathbb{R}^N)} + Ce^{-ct} \|\nabla^j \mathbf{V}_0\|_{L^2(\mathbb{R}^N)}. \tag{2.5}$$

Also, differentiating (2.4) with respect t and following the same steps as in the proof of [15, Theorem 5.5], we have the following result.

Proposition 2.5. *Let $0 < \tau < \beta$ and let $v_0, v_1, v_2 \in L^1(\mathbb{R}^N) \cap H^s(\mathbb{R}^N)$. Also, let $(v_1, v_2) \in L^{1,1}(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} v_i(x)dx = 0$, $i = 1, 2$. Then, for $0 \leq j \leq s$, the following decay estimate holds:*

$$\begin{aligned} \left\|\nabla^j v\,(t)\right\|_{L^2(\mathbb{R}^N)} &\leq C(\|v_0\|_{L^1(\mathbb{R}^N)} + \|v_1\|_{L^{1,1}(\mathbb{R}^N)} + \|v_2\|_{L^{1,1}(\mathbb{R}^N)})(1+t)^{-N/4-j/2} \\ &+ C(\|\nabla^j v_0\|_{L^2(\mathbb{R}^N)} + \|\nabla^j v_1\|_{L^2(\mathbb{R}^N)} + \|\nabla^j v_2\|_{L^2(\mathbb{R}^N)})e^{-ct}. \end{aligned} \tag{2.6}$$

Here $v_1 = w_0$ and $v_2 = v_{tt}(t=0)$.

3. Global existence

In this section we prove the global existence for the nonlinear problem (1.1) resp. its first-order version (1.16). The required local existence theorem will be given in Theorem 4.1 below. We summarize the main result of this section in the following theorem.

Theorem 3.1. *Let $N \geq 1$ and $s > N/2 + 1$. Assume that $u_0, v_0, w_0 \in H^s(\mathbb{R}^N)$. Then there exists a small positive constant α, such that if*

$$\begin{aligned} \Upsilon_s(0) &= \|v_0 + \tau w_0\|_{H^s}^2 + \|\Delta v_0\|_{H^{s-1}}^2 + \|v_0\|_{H^{s-1}}^2 \\ &\quad + \|\Delta(u_0 + \tau v_0)\|_{H^{s-1}}^2 + \|\nabla(u_0 + \tau v_0)\|_{H^{s-1}}^2 \leq \alpha, \end{aligned}$$

then the local solution u to (1.1) given in Theorem 4.1 exists globally in time.

The proof of Theorem 3.1 will be given through several lemmas.

3.1. First order energy estimates.

Lemma 3.2. *The energy functional associated to system (1.16) is*

$$E_1(t) := \frac{1}{2}\int_{\mathbb{R}^N} \left(|v+\tau w|^2 + \tau(\beta-\tau)|\nabla v|^2 + |\nabla(u+\tau v)|^2\right) dx \tag{3.1}$$

and satisfies, for all $t \geq 0$, the identity

$$\frac{d}{dt}E_1(t) + (\beta-\tau)\,\|\nabla v\|^2_{L^2} = R_1, \tag{3.2}$$

where

$$R_1 := \int_{\mathbb{R}^N} \left(\frac{B}{A}vw + 2\nabla u\nabla v\right)(v+\tau w)\, dx.$$

Proof. Summing up the second and the third equation in (1.16), we get

$$(v+\tau w)_t = \Delta u + \beta\Delta v + \frac{B}{A}vw + 2\nabla u\nabla v. \tag{3.3}$$

Multiplying (3.3) by $v+\tau w$ and integrating by parts over $\mathbb{R}^N$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}|v+\tau w|^2 dx + \beta\int_{\mathbb{R}^N}|\nabla v|^2 dx\\
=\;& -\int_{\mathbb{R}^N}\nabla u\,(\nabla v+\tau\nabla w)\,dx - \beta\tau\int_{\mathbb{R}^N}\nabla v\nabla w dx\\
&+\int_{\mathbb{R}^N}\left(\frac{B}{A}vw + 2\nabla u\nabla v\right)(v+\tau w)\,dx.
\end{aligned} \tag{3.4}$$

We have

$$\frac{1}{2}\tau(\beta-\tau)\frac{d}{dt}\int_{\mathbb{R}^N}|\nabla v|^2 dx = \tau(\beta-\tau)\int_{\mathbb{R}^N}(\nabla w\nabla v)dx. \tag{3.5}$$

and

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}|\nabla(u+\tau v)|^2 dx\\
=\;& \tau\int_{\mathbb{R}^N}\nabla w\nabla u dx + \tau^2\int_{\mathbb{R}^N}\nabla w\nabla v dx + \int_{\mathbb{R}^N}\nabla v\nabla u dx + \tau\int_{\mathbb{R}^N}|\nabla v|^2 dx.
\end{aligned} \tag{3.6}$$

Summing up (3.4), (3.5) and (3.6), then (3.2) holds. This finishes the proof of Lemma 3.2. ☐

Next, we define the energy of second order

$$E_2(t) := \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla(v+\tau w)|^2 + \tau(\beta-\tau)|\Delta v|^2 + |\Delta(u+\tau v)|^2\right) dx. \tag{3.7}$$

The following lemma is proved analogously.

Lemma 3.3. *The energy functional $E_2(t)$ satisfies, for all $t \geq 0$, the identity*

$$\frac{d}{dt}E_2(t) + (\beta - \tau)\left\|\Delta v\right\|_{L^2}^2 = R_2, \tag{3.8}$$

where

$$R_2 := -\int_{\mathbb{R}^N}\left(\frac{B}{A}vw + 2\nabla u \nabla v\right)\Delta(v+\tau w)dx.$$

Now, we define the functional $F_1(t)$ as

$$F_1(t) := \int_{\mathbb{R}^N} \nabla(u+\tau v)\nabla(v+\tau w)dx. \tag{3.9}$$

Then, we have

Lemma 3.4. *For any $\epsilon_0 > 0$, we have*

$$\begin{aligned}
&\frac{d}{dt}F_1(t) + (1-\epsilon_0)\int_{\mathbb{R}^N}\left|\Delta\left(u+\tau v\right)\right|^2 dx \\
\leq\ &\int_{\mathbb{R}^N}|\nabla(v+\tau w)|^2 dx + C(\epsilon_0)\int_{\mathbb{R}^N}|\Delta v|^2 dx + |\tilde{R}_2|
\end{aligned} \tag{3.10}$$

with

$$\tilde{R}_2 = -\int_{\mathbb{R}^N}\left(\frac{B}{A}vw + 2\nabla u \nabla v\right)\Delta\left(u+\tau v\right)dx.$$

Proof. Multiplying equation (3.3) by $-\Delta\left(u+\tau v\right)$ and $(u_t+\tau v_t)$ by $-\Delta\left(v+\tau w\right)$ we get, respectively,

$$\begin{aligned}
-\int_{\mathbb{R}^N}(v+\tau w)_t\Delta(u+\tau v) &= -\int_{\mathbb{R}^N}(\Delta u + \beta\Delta v)(\Delta u + \tau \Delta v)dx \\
&\quad -\int_{\mathbb{R}^N}\left(\frac{B}{A}vw + 2\nabla u\nabla v\right)\Delta\left(u+\tau v\right)dx \\
&= -\int_{\mathbb{R}^N}(\Delta u + \beta\Delta v + \tau\Delta v - \tau\Delta v)(\Delta u + \tau\Delta v)dx \\
&\quad -\int_{\mathbb{R}^N}\left(\frac{B}{A}vw + 2\nabla u\nabla v\right)\Delta\left(u+\tau v\right)dx
\end{aligned}$$

and

$$-\int_{\mathbb{R}^N}(u+\tau v)_t\Delta(v+\tau w)dx = -\int_{\mathbb{R}^N}(\tau w + v)\Delta(v+\tau w)dx.$$

Integrating by parts and summing up the above two equations, we obtain

$$\begin{aligned}
&\frac{d}{dt}F_1(t) + \int_{\mathbb{R}^N}|\Delta(u+\tau v)|^2 dx - \int_{\mathbb{R}^N}|\nabla(v+\tau w)|^2 dx \\
=\ &(\tau-\beta)\int_{\mathbb{R}^N}(\Delta v(\Delta u + \tau\Delta v))dx - \int_{\mathbb{R}^N}\left(\frac{B}{A}vw + 2\nabla u\nabla v\right)\Delta\left(u+\tau v\right)dx.
\end{aligned}$$

Applying Young's inequality for any $\epsilon_0 > 0$, we obtain (3.10). This finishes the proof of Lemma 3.4. ∎

Next, we define the functional $F_2(t)$ as

$$F_2(t) := -\tau \int_{\mathbb{R}^N} \nabla v \nabla (v+\tau w) dx. \tag{3.11}$$

Lemma 3.5. *For any $\epsilon_1, \epsilon_2 > 0$, we have*

$$\begin{aligned} &\frac{d}{dt}F_2(t) + (1-\epsilon_1)\int_{\mathbb{R}^N} |\nabla(v+\tau w)|^2 dx \\ \leq\ & C(\epsilon_1,\epsilon_2)\int_{\mathbb{R}^N} (|\nabla v|^2 + |\Delta v|^2)dx + \epsilon_2 \int_{\mathbb{R}^N} |\Delta(u+\tau v)|^2 dx + R_3, \end{aligned} \tag{3.12}$$

where

$$R_3 = \tau \left| \int_{\mathbb{R}^N} \left(\frac{B}{A} vw + 2\nabla u \nabla v \right) \Delta v dx \right|.$$

Proof. Multiplying the second equation in (1.16) by $\tau\Delta(v+\tau w)$ and (3.3) by $\tau\Delta v$, and integrating over $\mathbb{R}^N$ we obtain, respectively,

$$\tau \int_{\mathbb{R}^N} v_t \Delta(v+\tau w) dx = \tau \int_{\mathbb{R}^N} w\Delta(v+\tau w)dx$$

and

$$\begin{aligned} &\tau \int_{\mathbb{R}^N} (v+\tau w)_t \Delta v dx \\ =\ & \tau \int_{\mathbb{R}^N} (\Delta u + \beta \Delta v)\Delta v dx + \tau \int_{\mathbb{R}^N} \left(\frac{B}{A} vw + 2\nabla u \nabla v\right) \Delta v dx \\ =\ & \int_{\mathbb{R}^N} \Big(\tau \Delta u + \tau\beta\Delta v + \tau^2 \Delta v - \tau^2\Delta v + (v+\tau w) - (v+\tau w)\Big)\Delta v dx \\ &+\tau \int_{\mathbb{R}^N} \left(\frac{B}{A} vw + 2\nabla u \nabla v\right) \Delta v dx. \end{aligned}$$

Using integration by parts, we obtain

$$\begin{aligned} &\frac{d}{dt}F_2(t) + \int_{\mathbb{R}^N} |\nabla(v+\tau w)|^2 dx - \tau(\beta-\tau)\int_{\mathbb{R}^N} |\Delta v|^2 dx \\ =\ & \tau \int_{\mathbb{R}^N} \Delta(u+\tau v)\Delta v dx + \int_{\mathbb{R}^N} \nabla(v+\tau w)\nabla v dx \\ &+\tau \int_{\mathbb{R}^N} \left(\frac{B}{A} vw + 2\nabla u \nabla v\right) \Delta v dx. \end{aligned}$$

Thus we obtain the estimate (3.12) for any $\epsilon_1, \epsilon_2 > 0$. □

We define the Lyapunov functional $L(t)$ as

$$L(t) := \gamma_0 \left(E_1(t) + E_2(t)\right) + F_1(t) + \gamma_1 F_2(t), \tag{3.13}$$

where γ_0 and γ_1 are positive numbers that will be fixed later on. Taking the derivative of (3.13) with respect to t and making use of (3.2), (3.10), (3.8) and (3.12), we obtain

$$\begin{aligned}
&\frac{d}{dt}L(t) + (\gamma_0(\beta-\tau) - \gamma_1 C(\epsilon_1,\epsilon_2))\,\|\nabla v\|_{L^2}^2 \\
&+(\gamma_0(\beta-\tau) - C(\epsilon_0) - \gamma_1 C(\epsilon_1,\epsilon_2))\,\|\Delta v\|_{L^2}^2 \\
&+(1-\epsilon_0-\gamma_1\epsilon_2)\,\|\Delta(u+\tau v)\|_{L^2}^2 \\
&+(\gamma_1(1-\epsilon_1)-1)\,\|\nabla(v+\tau w)\|_{L^2}^2 \\
\leq\;& \gamma_0(|R_1|+|R_2|)+|R_2|+|\tilde{R}_2|+\gamma_1|R_3|.
\end{aligned} \tag{3.14}$$

In the above estimate, we can fix our constants in such a way that the coefficients in front of the norm terms. This can be achieved as follows: we pick ϵ_0 and ϵ_1 small enough such that $\epsilon_0 < 1$ and $\epsilon_1 < 1$. After that, we take γ_1 large enough such that

$$\gamma_1 > \frac{1}{1-\epsilon_1}.$$

Once γ_1 and ϵ_0 are fixed, we select ϵ_2 small enough such that

$$\epsilon_2 < \frac{1-\epsilon_0}{\gamma_1}.$$

Finally, and recalling that $\tau < \beta$, we may choose γ_0 large enough such that

$$\gamma_0 > \frac{C(\epsilon_0)+\gamma_1 C(\epsilon_1,\epsilon_2)}{\beta-\tau}.$$

Consequently, we deduce that there exists a positive constant γ_2 such that for all $t \geq 0$,

$$\begin{aligned}
&\frac{d}{dt}L(t) + \gamma_2\left(\|\nabla v\|_{L^2}^2 + \|\Delta v\|_{L^2}^2 + \|\Delta(u+\tau v)\|_{L^2}^2 + \|\nabla(v+\tau w)\|_{L^2}^2\right) \\
&\leq C\sum_{i=1}^{3}|R_i| + C\tilde{R}_2.
\end{aligned} \tag{3.15}$$

In the following lemma, we show the equivalence between the functional $L(t)$ and $E(t) := E_1(t)+E_2(t)$.

Lemma 3.6. *There exist two positive constants c_1 and c_2 such that for all $t \geq 0$*

$$c_1 E(t) \leq L(t) \leq c_2 E(t). \tag{3.16}$$

Proof. We have by using Hölder's inequality

$$\begin{aligned}
|F_1(t)+\gamma_1 F_2(t)| &\leq C\left(\|\nabla v\|_{L^2} + \|\nabla(u+\tau v)\|_{L^2}\right)\|\nabla(v+\tau w)\|_{L^2} \\
&\leq CE(t).
\end{aligned}$$

This gives

$$(\gamma_0 - C)\,E(t) \leq L(t) \leq (\gamma_0 + C)\,E(t).$$

We select γ_0 large enough, then (3.16) holds for $c_1 = \gamma_0 - C$ and $c_2 = \gamma_0 + C$. □

Integrating (3.15) form 0 to t and using (3.16), we get

$$F^{(0)}(t)+\gamma_2\int_0^t G^{(0)}(\sigma)d\sigma\leq F^{(0)}(0)+C\sum_{i=1}^{3}\int_0^t|R_i|d\sigma. \tag{3.17}$$

where

$$\begin{aligned} F^{(0)}(t) \;&:=\; \|(v+\tau w)(t)\|_{H^1}^2+\|\Delta v(t)\|_{L^2}^2+\|\nabla v(t)\|_{L^2}^2\\ &\quad+\|\Delta(u+\tau v)(t)\|_{L^2}^2+\|\nabla(u+\tau v)(t)\|_{L^2}^2 \end{aligned}$$

and

$$G^{(0)}(t)=\|\nabla v(t)\|_{L^2}^2+\|\Delta v(t)\|_{L^2}^2+\|\Delta\left(u+\tau v\right)(t)\|_{L^2}^2+\|\nabla(v+\tau w)(t)\|_{L^2}^2.$$

Our goal now is to estimate the remaining terms $|R_i|, 1\leq i\leq 3$ in (3.17). First, we have, by using Hölder's inequality,

$$\begin{aligned} |R_1| &= \left|\int_{\mathbb{R}^N}\left(\frac{B}{A}vw+2\nabla u\nabla v\right)(v+\tau w)\,dx\right|\\ &= \left|\int_{\mathbb{R}^N}\left(\frac{B}{\tau A}v\left(v+\tau w-v\right)+2\nabla\left(u+\tau v-\tau v\right)\nabla v\right)(v+\tau w)\,dx\right|\\ &\leq \int_{\mathbb{R}^N}\left|\left(\frac{B}{\tau A}v^2+(\tau w+v)\,v+2\nabla\left(u+\tau v\right)\nabla v-2\tau\left|\nabla v\right|^2\right)(v+\tau w)\right|dx\\ &\leq C\left(\|v\|_{L^\infty}\|v\|_{L^2}+\|v\|_{L^\infty}\|\tau w+v\|_{L^2}\right)\|v+\tau w\|_{L^2}\\ &\quad+C\left(\|\nabla v\|_{L^\infty}\|\nabla\left(u+\tau v\right)\|_{L^2}+\|\nabla v\|_{L^\infty}\|\nabla v\|_{L^2}\right)\|v+\tau w\|_{L^2}. \end{aligned}$$

Using integration by parts, we have

$$\begin{aligned} R_2 &= -\int_{\mathbb{R}^N}\left(\frac{B}{A}vw+2\nabla u\nabla v\right)\Delta(v+\tau w)dx\\ &= \int_{\mathbb{R}^N}\nabla\left(\frac{B}{\tau A}v\left(v+\tau w-v\right)+2\nabla\left(u+\tau v-\tau v\right)\nabla v\right)\nabla(v+\tau w)dx\\ &= \int_{\mathbb{R}^N}\left(\frac{B}{\tau A}v\nabla\left(v+\tau w\right)+\frac{B}{\tau A}\nabla v\left(v+\tau w\right)+\nabla\left|v\right|^2\right)\nabla(v+\tau w)dx,\\ &\quad+\int_{\mathbb{R}^N}\left(2H(u+\tau v)\nabla v+2H(v)\nabla\left(u+\tau v\right)-4\tau H(v)\nabla v\right)\nabla(v+\tau w)dx, \end{aligned}$$

where $H(f)=(\partial_{x_i}\partial_{x_j}f)_{,1\leq i,j\leq N}$ is the Hessian matrix of f. Using the fact that $\|H\left(f\right)\|_{L^2}=\|\Delta f\|_{L^2}$, together with Hölder's inequality, we get

$$\begin{aligned} |R_2| \;&\leq\; C\left(\|v\|_{L^\infty}\left(\|\nabla\left(v+\tau w\right)\|_{L^2}+\|\nabla v\|_{L^2}\right)+\|v+\tau w\|_{L^2}\|\nabla v\|_{L^\infty}\right)\|\nabla(v+\tau w)\|_{L^2}\\ &\quad+C\left(\|\nabla v\|_{L^\infty}\left(\|\Delta(u+\tau v)\|_{L^2}+\|\Delta v\|_{L^2}\right)+\|\nabla(u+\tau v)\|_{L^\infty}\|\Delta v\|_{L^2}\right)\|\nabla(v+\tau w)\|_{L^2}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} |\tilde{R}_2| \;&\leq\; C\left(\|v\|_{L^\infty}\left(\|\nabla\left(v+\tau w\right)\|_{L^2}+\|\nabla v\|_{L^2}\right)+\|v+\tau w\|_{L^2}\|\nabla v\|_{L^\infty}\right)\|\nabla(u+\tau v)\|_{L^2}\\ &\quad+C\left(\|\nabla v\|_{L^\infty}\left(\|\Delta(u+\tau v)\|_{L^2}+\|\Delta v\|_{L^2}\right)+\|\nabla(u+\tau v)\|_{L^\infty}\|\Delta v\|_{L^2}\right)\|\nabla(u+\tau v)\|_{L^2}. \end{aligned}$$

For R_3, we have, as in R_2,

$$\begin{aligned}|R_3| &\leq C\left(\|v\|_{L^\infty}\left(\|\nabla\left(v+\tau w\right)\|_{L^2}+\|\nabla v\|_{L^2}\right)+\|v+\tau w\|_{L^2}\|\nabla v\|_{L^\infty}\right)\|\nabla v\|_{L^2}\\ &\quad +C\left(\|\nabla v\|_{L^\infty}\left(\|\Delta(u+\tau v)\|_{L^2}+\|\Delta v\|_{L^2}\right)+\|\nabla(u+\tau v)\|_{L^\infty}\|\Delta v\|_{L^2}\right)\|\nabla v\|_{L^2}.\end{aligned}$$

Collecting the above estimates, we have

$$\begin{aligned}&F^{(0)}(t)+\gamma_2\int_0^t G^{(0)}(s)ds\\ \leq\ &F^{(0)}(0)+C\int_0^t \|v(s)\|_{L^\infty}\|v(s)\|_{L^2}\|(v+\tau w)(s)\|_{L^2}ds\\ &+C\int_0^t\left(\|\nabla v(s)\|_{L^\infty}+\|v(s)\|_{L^\infty}+\|\nabla(v+\tau w)(s)\|_{L^\infty}\right)F^{(0)}(s)ds.\end{aligned}\tag{3.18}$$

Now, multiplying the second equation in (1.16) by v and integrating over $\mathbb{R}^N$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}|v|^2dx &= \int_{\mathbb{R}^N} wvdx\\ &= \int_{\mathbb{R}^N}\frac{1}{\tau}v\left(\tau w+v-v\right)dx\\ &\leq C\|v(t)\|_{L^2}\|(\tau w+v)(t)\|_{L^2}.\end{aligned}\tag{3.19}$$

Integrating (3.19) with respect to t, we get

$$\|v(t)\|_{L^2}^2\leq\|v(0)\|_{L^2}^2+C\int_0^t\|v(s)\|_{L^2}\|(\tau w+v)(s)\|_{L^2}ds.\tag{3.20}$$

Collecting (3.18) and (3.20), we get

$$\begin{aligned}&K(t)+\gamma_2\int_0^t G^{(0)}(s)ds\\ \leq\ &K(0)+C\int_0^t\|v(s)\|_{L^\infty}\|v(s)\|_{L^2}\|(v+\tau w)(s)\|_{L^2}ds\\ &+C\int_0^t\left(\|\nabla v(s)\|_{L^\infty}+\|v(s)\|_{L^\infty}+\|\nabla(u+\tau v)(s)\|_{L^\infty}\right)F^{(0)}(s)ds.\\ &+C\int_0^t\|v(s)\|_{L^2}\|(\tau w+v)(s)\|_{L^2}ds,\end{aligned}\tag{3.21}$$

where

$$K(t):=F^{(0)}(t)+\|v(t)\|_{L^2}^2.$$

From (3.21), we get

$$K(t)+\gamma_2\int_0^t G^{(0)}(s)ds\leq K(0)+C\int_0^t\left(1+\|v(s)\|_{W^{1,\infty}}+\|\nabla u(s)\|_{L^\infty}\right)K(s)ds.\tag{3.22}$$

Applying Gronwall's inequality, we get

$$K(t)\leq K(0)e^{C\int_0^t\left(1+\|v(s)\|_{W^{1,\infty}}+\|\nabla u(s)\|_{L^\infty}\right)ds}.$$

Observing that the above inequality implies the boundedness of the norm $K(T)$ up to any finite time T provided that the quantity $\|v(s)\|_{W^{1,\infty}} + \|\nabla u(s)\|_{L^\infty}$ is bounded. One way to show this is to prove higher-order estimates and use a smallness assumption on the initial data.

3.2. **Higher-order energy estimates.** Applying the operator ∇^k, $k \geq 1$ to (1.16), we get for $U := \nabla^k u$, $V := \nabla^k v$ and $W := \nabla^k w$

$$\begin{cases} \partial_t U = V, \\ \partial_t V = W, \\ \tau \partial_t W = \Delta U + \beta \Delta V - W + \dfrac{B}{A}[\nabla^k, v]w + \dfrac{B}{A} vW + 2[\nabla^k, \nabla u]\nabla v + 2\nabla u \nabla V, \end{cases} \tag{3.23}$$

where $[A, B] = AB - BA$.

We define the first energy of order k as

$$\begin{aligned} E_1^{(k)}(t) \;&;= \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla^k v + \tau \nabla^k w|^2 + \tau(\beta - \tau)|\nabla^{k+1} v|^2 + |\nabla^{k+1} u + \tau \nabla^{k+1} v|^2 \right) dx \\ &= \frac{1}{2}\int_{\mathbb{R}^N} \left(|V + \tau W|^2 + \tau(\beta - \tau)|\nabla V|^2 + |\nabla(U + \tau V)|^2 \right) dx. \end{aligned} \tag{3.24}$$

Hence, we have the following estimate.

Lemma 3.7. *For all $t \geq 0$, it holds that*

$$\frac{d}{dt} E_1(t)^{(k)} + (\beta - \tau)\|\nabla V\|_{L^2}^2 = \int_{\mathbb{R}^N} R_1^{(k)}(t)\,(V + \tau W)\,dx, \tag{3.25}$$

where

$$R_1^{(k)}(t) = \frac{B}{A}[\nabla^k, v]w + \frac{B}{A} vW + 2[\nabla^k, \nabla u]\nabla v + 2\nabla u \nabla V. \tag{3.26}$$

We omit the proof of Lemma 3.7 since it can be done using the same steps as in Lemma 3.2.

As in the case $k = 0$, we define the second energy of order k as follows:

$$E_2^{(k)}(t) := \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla(V + \tau W)|^2 + \tau(\beta - \tau)|\Delta V|^2 + |\Delta(U + \tau V)|^2 \right) dx. \tag{3.27}$$

Then, we have the following lemma.

Lemma 3.8. *The energy functional $E_2(t)$ satisfies, for all $t \geq 0$, the identity*

$$\frac{d}{dt} E_2^{(k)}(t) + (\beta - \tau)\|\Delta v\|_{L^2}^2 = -\int_{\mathbb{R}^N} R_1^{(k)} \Delta(V + \tau W) dx. \tag{3.28}$$

The proof of the above lemma is similar to that of Lemma 3.3. We omit the details.

We define now the functional $F_1^{(k)}(t)$ as

$$F_1^{(k)}(t) := \int_{\mathbb{R}^N} \nabla(U + \tau V)\nabla(V + \tau W) dx. \tag{3.29}$$

Then, we have the following estimate, where it's proof can be done following the same strategy as in Lemma 3.4.

Lemma 3.9. *For any* $\epsilon_0' > 0$, *we have*

$$\begin{aligned} &\frac{d}{dt}F_1^{(k)}(t) + (1-\epsilon_0')\int_{\mathbb{R}^N} |\Delta\left(U+\tau V\right)|^2 dx \\ \leq\ & \int_{\mathbb{R}^N} |\nabla(V+\tau W)|^2 dx + C(\epsilon_0')\int_{\mathbb{R}^N} |\Delta V|^2 dx \\ &+ \int_{\mathbb{R}^N} |R_1^{(k)}||\Delta\left(U+\tau V\right)| dx \end{aligned} \tag{3.30}$$

As in the case $k=0$, we define the functional $F_2^{(k)}(t)$ as

$$F_2^{(k)}(t) := -\tau \int_{\mathbb{R}^N} \nabla V \nabla(V+\tau W) dx. \tag{3.31}$$

Hence, we have the following estimate.

Lemma 3.10. *For any* $\epsilon_1', \epsilon_2' > 0$, *we have*

$$\begin{aligned} &\frac{d}{dt}F_2^{(k)}(t) + (1-\epsilon_1')\int_{\mathbb{R}^N} |\nabla(V+\tau W)|^2 dx \\ \leq\ & C(\epsilon_1',\epsilon_2')\int_{\mathbb{R}^N} (|\nabla V|^2 + |\Delta V|^2) dx \\ &+\epsilon_2'\int_{\mathbb{R}^N} |\Delta(U+\tau V)|^2 dx + \tau\int_{\mathbb{R}^N} |R_1^{(k)}||\Delta V| dx. \end{aligned} \tag{3.32}$$

We can prove Lemma 3.10 following the same steps as in the proof of Lemma 3.5, we omit the details.

As in the case $k=0$, if we define the functional

$$L^{(k)}(t) := \gamma_0'\left(E_1^{(k)}(t) + E_2^{(k)}(t)\right) + F_1^{(k)}(t) + \gamma_1' F_2^{(k)}(t), \tag{3.33}$$

and we proceed exactly as in the case $k=0$ to get the following estimate, which is similar to (3.17),

$$\begin{aligned} &\|(V+\tau W)(t)\|_{H^1}^2 + \|\Delta V(t)\|_{L^2}^2 + \|\nabla V(t)\|_{L^2}^2 \\ &+\|\Delta(U+\tau V)(t)\|_{L^2}^2 + \|\nabla(U+\tau V)(t)\|_{L^2}^2 \\ &+\gamma_2\int_0^t \Big\{ \|\nabla V(s)\|_{L^2}^2 + \|\Delta V(s)\|_{L^2}^2 + \|\Delta\left(U+\tau V\right)(s)\|_{L^2}^2 \\ &+\|\nabla(V+\tau W)(s)\|_{L^2}^2 \Big\} ds \\ &\leq \|\nabla(V+\tau W)(0)\|_{H^1}^2 + \|\Delta V(0)\|_{L^2}^2 + \|\nabla V(0)\|_{L^2}^2 \\ &+\|\Delta(U+\tau V)(0)\|_{L^2}^2 + \|\nabla(U+\tau V)(0)\|_{L^2}^2 \\ &+C\sum_{i=1}^3 \int_0^t I_i^{(k)}(s) ds. \end{aligned} \tag{3.34}$$

where

$$\begin{aligned} I_1^{(k)} &:= \left| \int_{\mathbb{R}^N} R_1^{(k)}(t)\,(V+\tau W)\,dx \right|, \\ I_2^{(k)} &:= \left| \int_{\mathbb{R}^N} R_1^{(k)}(t)\Delta(U+\tau V)dx \right|, \\ I_3^{(k)} &:= \left| \tau \int_{\mathbb{R}^N} |R_1^{(k)}\Delta V\,dx \right|. \end{aligned}$$

Our goal now is to estimate the above terms $I_i^{(k)}$, $1 \leq i \leq 3$. For $I_1^{(k)}$, we use Hölder's inequality to write

$$I_1^{(k)} \leq \|R_1^{(k)}\|_{L^2}\|\,(V+\tau W)\,\|_{L^2}.$$

Similarly, we have for $I_2^{(k)}$,

$$I_2^{(k)} \leq \|\nabla R_1^{(k)}\|_{L^2}\|\nabla\,(U+\tau V)\,\|_{L^2}$$

and

$$I_3^{(k)} \leq \tau\|\nabla R_1^{(k)}\|_{L^2}\|\nabla V\|_{L^2}.$$

In the following lemma, we estimate $\|R_1^{(k)}\|_{L^2}$ and $\|\nabla R_1^{(k)}\|_{L^2}$.

Lemma 3.11. *For $k \geq 1$, it holds that*

$$\|R_1^{(k)}\|_{L^2} \leq C\Lambda(t)\left(\|W\|_{L^2} + \|V\|_{H^1} + \|\nabla\,(U+\tau V)\,\|_{L^2}\right). \tag{3.35}$$

and

$$\|\nabla R_1^{(k)}\|_{L^2} \leq C\Lambda(t)\left(\|\nabla V\|_{L^2} + \|\nabla W\|_{L^2} + \|\Delta V\|_{L^2} + \|\Delta\,(U+\tau V)\,\|_{L^2}\right) \tag{3.36}$$

where

$$\Lambda(t) = \|v\|_{W^{1,\infty}} + \|w\|_{L^\infty} + \|\nabla u\|_{L^\infty}.$$

Proof. We have from (3.26),

$$\begin{aligned} \|R_1^{(k)}\|_{L^2} \quad &\leq \quad C\left(\|[\nabla^k, v]w\|_{L^2} + C\|v\|_{L^\infty}\|W\|_{L^2}\right) \\ &\quad\quad +C\left(\|[\nabla^k, \nabla u]\nabla v\|_{L^2} + \|\nabla u\|_{L^\infty}\|\nabla V\|_{L^2}\right). \end{aligned} \tag{3.37}$$

Now applying the commutator estimate in Lemma 2.1, we get

$$\|[\nabla^k, v]w\|_{L^2} \leq C(\|v\|_{L^\infty}\|\nabla^k w\|_{L^2} + \|w\|_{L^\infty}\|\nabla^k v\|_{L^2}).$$

Similarly,

$$\begin{aligned} \|[\nabla^k, \nabla u]\nabla v\|_{L^2} \quad &\leq \quad C(\|\nabla v\|_{L^\infty}\|\nabla^{k+1}u\|_{L^2} + \|\nabla u\|_{L^\infty}\|\nabla^{k+1}v\|_{L^2}) \\ &\leq \quad C(\|\nabla v\|_{L^\infty}\left(\|\nabla^{k+1}\,(u+\tau v)\,\|_{L^2} + \|\nabla^{k+1}v\|_{L^2}\right) + \|\nabla u\|_{L^\infty}\|\nabla^{k+1}v\|_{L^2}). \end{aligned}$$

Plugging the last two estimates into (3.37), then (3.35) holds.

Now, to get (3.36), we have

$$\nabla R_1^{(k)} = \nabla^{k+1}\left(\frac{B}{A}vw + 2\nabla u\nabla v\right)$$

Thus, applying (2.1), we get

$$\begin{aligned}\|\nabla R_1^{(k)}\|_{L^2} \quad &\leq \quad C\left(\|w\|_{L^\infty}\|\nabla^{k+1}v\|_{L^2} + \|v\|_{L^\infty}\|\nabla^{k+1}w\|_{L^2}\right) \\ &\quad +C\left(\|\nabla u\|_{L^\infty}\|\nabla^{k+2}v\|_{L^2} + \|\nabla v\|_{L^\infty}\|\nabla^{k+2}u\|_{L^2}\right).\end{aligned} \tag{3.38}$$

Now, we estimate the terms in the second line of the above formula as

$$\begin{aligned}&\|\nabla u\|_{L^\infty}\|\nabla^{k+2}v\|_{L^2} + \|\nabla v\|_{L^\infty}\|\nabla^{k+2}u\|_{L^2} \\ \leq \quad & C\left(\|\nabla u\|_{L^\infty}\|\Delta\nabla^k v\|_{L^2} + \|\nabla v\|_{L^\infty}\|\Delta\nabla^k u\|_{L^2}\right) \\ \leq \quad & C\left(\|\nabla u\|_{L^\infty}\|\Delta\nabla^k v\|_{L^2} + \|\nabla v\|_{L^\infty}\left(\|\Delta\nabla^k(u+\tau v)\|_{L^2} + \|\Delta\nabla^k v\|_{L^2}\right)\right).\end{aligned}$$

Inserting the above estimates into (3.38), we get (3.36). □

Now, taking (3.35) and (3.36) into account, then (3.34) yields

$$F^{(k)}(t) + \int_0^t G^{(k)}(s)ds \leq CF^{(k)}(0) + C\int_0^t \Lambda(s)\Big(\Phi_1^{(k)}(s) + \Phi_2^{(k)}(s)\Big)ds \tag{3.39}$$

where

$$\begin{cases} F^{(k)}(t) & := \|(V+\tau W)(t)\|_{H^1}^2 + \|\Delta V(t)\|_{L^2}^2 + \|\nabla V(t)\|_{L^2}^2 \\ & \quad + \|\Delta(U+\tau V)(t)\|_{L^2}^2 + \|\nabla(U+\tau V)(t)\|_{L^2}^2, \\ G^{(k)}(t) & := \|\nabla V(t)\|_{L^2}^2 + \|\Delta V(t)\|_{L^2}^2 + \|\Delta(U+\tau V)(t)\|_{L^2}^2 + \|\nabla(V+\tau W)(t)\|_{L^2}^2, \\ \Phi_1^{(k)}(t) & := \left(\|W(t)\|_{L^2} + \|V(t)\|_{H^1} + \|\nabla(U+\tau V)(t)\|_{L^2}\right)\|\nabla(V+\tau W)(t)\|_{L^2} \\ \Phi_2^{(k)}(t) & := \left(\left(\|\nabla V(t)\|_{L^2} + \|\nabla W(t)\|_{L^2} + \|\Delta V(t)\|_{L^2} + \|\Delta(U+\tau V)(t)\|_{L^2}\right)\right. \\ & \quad \left.\times\left(\|\nabla(U+\tau V)(t)\|_{L^2} + \|\nabla V(t)\|_{L^2}\right)\right). \end{cases}$$

Now, summing up (3.40) for k with $1 \leq k \leq s-1$ and adding the result to (3.22), we get

$$\begin{aligned}\Upsilon_s(t) + \int_0^t G^{(s)}(\sigma)d\sigma \quad &\leq \quad C\Upsilon_s(0) + C\int_0^t \Lambda(\sigma)\Big(\Phi_1^{(s)}(\sigma) + \Phi_2^{(s)}(\sigma)\Big)d\sigma \\ &\quad +C\int_0^t (1+\Lambda(\sigma))\Upsilon_s(\sigma)d\sigma\end{aligned} \tag{3.40}$$

with

$$\begin{aligned}\Upsilon_s(t) \quad &:= \quad \|(v+\tau w)(t)\|_{H^s}^2 + \|\Delta v(t)\|_{H^{s-1}}^2 + \|v(t)\|_{H^{s-1}}^2 \\ &\quad + \|\Delta(u+\tau v)(t)\|_{H^{s-1}}^2 + \|\nabla(u+\tau v)(t)\|_{H^{s-1}}^2\end{aligned} \tag{3.41}$$

and

(3.42)

$$\begin{cases} G^{(s)}(t) &= \|\nabla v(t)\|_{H^{s-1}}^2 + \|\Delta v(t)\|_{H^{s-1}}^2 + \|\Delta\left(u+\tau v\right)(t)\|_{H^{s-1}}^2 + \|\nabla (v+\tau w)(t)\|_{H^{s-1}}^2 , \\ \Phi_1^{(s)}(t) &= \left(\|w(t)\|_{H^{s-1}} + \|v(t)\|_{H^s} + \|\nabla\left(u+\tau v\right)(t)\|_{H^{s-1}}\right)\|\nabla\left(v+\tau w\right)(t)\|_{H^{s-1}} \\ \Phi_2^{(s)}(t) &= \Big(\left(\|\nabla v(t)\|_{H^{s-1}} + \|\nabla w(t)\|_{H^{s-1}} + \|\Delta v(t)\|_{H^{s-1}} + \|\Delta\left(u+\tau v\right)(t)\|_{H^{s-1}}\right) \\ & \quad \times \left(\|\nabla\left(u+\tau v\right)(t)\|_{H^{s-1}} + \|\nabla v(t)\|_{H^{s-1}}\right)\Big). \end{cases}$$

It is clear that

$$\Phi_i^{(s)}(t) \le C\Upsilon_s(t), \qquad i=1,2.$$

Hence, (3.40) takes the form

$$\Upsilon_s(t) + \int_0^t G^{(s)}(\sigma)d\sigma \le C\Upsilon_s(0) + C\int_0^t (1+\Lambda(\sigma))\Upsilon_s(\sigma)d\sigma. \tag{3.43}$$

Applying Gronwall's inequality, to (3.43), we get

$$\Upsilon_s(t) \le C_1\Upsilon_s(0)e^{(1+\sup_{0\le t\le T}\Lambda(t))T}, \tag{3.44}$$

where $T = T(\Upsilon_s(0))$ is the maximal time of the local existence.

Hence, if we had that

$$\sup_{0\le t\le T}\Lambda(t)$$

was bounded then our result holds. One way to show that the above quantity is bounded is by assuming smallness assumption on the initial data in some regular spaces and using the bootstrap argument. Indeed, we assume the following a *priori* assumption

$$\sup_{0\le t\le T}\Upsilon_s(t) \le \delta, \tag{3.45}$$

for a constant $\delta > 0$, small enough. Due to the embedding $H^s(\mathbb{R}^N) \hookrightarrow W^{m,\infty}(\mathbb{R}^N)$, for $s > m + N/2$, we have

$$\sup_{0\le t\le T}\|v(t)\|_{W^{1,\infty}} \le C\sup_{0\le t\le T}\|v(t)\|_{H^{s_2}} \le C\delta \tag{3.46}$$

for $s_2 > N/2 + 1$. Similarly, we have

$$\sup_{0\le t\le T}\|\nabla u(t)\|_{L^\infty} \le C\sup_{0\le t\le T}\|\nabla u(t)\|_{H^{s_1}} \le C\delta \tag{3.47}$$

for $s_1 > N/2$ and

$$\sup_{0\le t\le T}\|w(t)\|_{L^\infty} \le C\sup_{0\le t\le T}\|w(t)\|_{H^{s_3}} \le C\delta \tag{3.48}$$

for $s_3 > N/2$.

Hence, from (3.44), (3.46), (3.47) and (3.48), we deduce that if (3.45) holds, (3.44) yields

$$\Upsilon_s(t) \le C_1\Upsilon_s(0)e^{(1+C_2\delta)T}, \tag{3.49}$$

with C_1 and C_2 being positive constants not depending on T and δ.

Now, we fix $\delta > 0$ and choose α small enough such that

$$\alpha \leq \alpha_0(\delta) := \frac{\delta}{2C_1 e^{(1+C_2\delta)}} < \frac{\delta}{2},$$

and

$$\Upsilon_s(0) \leq \alpha.$$

Then, there is $T = T(\Upsilon_s(0)) = T(\alpha) = T(\delta)$ where (3.45) holds and hence by (3.49), we obtain

$$\Upsilon_s(t) \leq \frac{\delta}{2}, \tag{3.50}$$

for all $0 \leq t \leq T$. In particular

$$\Upsilon_s(T) \leq \frac{\delta}{2} < \delta, \tag{3.51}$$

Consequently, the bootstrap argument together with the local existence implies that we can continue the local solution to $T(\delta) + T(\delta) = 2T(\delta)$ and we get analogously, (3.49), now for $0 \leq t \leq 2T = 2T(\delta)$, then $\Upsilon_s(2T) < \delta$ and so on, we can continue the solution to $T = \infty$.

4. A local existence theorem

In this section, we use the contraction mapping theorem to show the following local existence theorem.

Theorem 4.1. *Let $s > \frac{N}{2} + 1$. Let $\mathbf{U}_0 = (u_0, u_1, u_2)^T = (u_0, v_0, w_0)^T$ be such that*

$$\begin{aligned} \Upsilon_s(0) \;=\; & \|(v_0 + \tau w_0)\|_{H^s}^2 + \|\Delta v_0\|_{H^{s-1}}^2 + \|v_0\|_{H^{s-1}}^2 \\ & + \|\Delta(u_0 + \tau v_0)\|_{H^{s-1}}^2 + \|\nabla(u_0 + \tau v_0)\|_{H^{s-1}}^2 \leq \tilde{\delta}_0 \end{aligned} \tag{4.1}$$

for some $\tilde{\delta}_0 > 0$. Then, there exists a small time $T = T(\Upsilon_s(0)) > 0$ such that problem (1.1) has a unique solution u on $[0, T) \times \mathbb{R}^N$ satisfying

$$\sup_{0 \leq t \leq T} \Upsilon_s(t) + \int_0^T G^{(s)}(\sigma) d\sigma \leq C_{\tilde{\delta}_0},$$

where $\Upsilon_s(t)$ and $G^{(s)}(t)$ are given in (3.41) and (3.42), respectively, determining the regularity of u, and $C_{\tilde{\delta}_0}$ is a positive constant depending on $\tilde{\delta}_0$.

Proof. First, we write problem (1.1) as a first-order evolution equation of the form

$$\begin{cases} \dfrac{d}{dt}\mathbf{U}(t) = \mathcal{A}\mathbf{U}(t) + \mathcal{F}(\mathbf{U}, \nabla\mathbf{U}), \quad t > 0 \\ \mathbf{U}(0) = \mathbf{U}_0, \end{cases} \tag{4.2}$$

where $\mathbf{U}(t) = (u, v, w)^T = (u, u_t, u_{tt})^T$, $\mathbf{U}_0 = (u_0, u_1, u_2)^T$ and $\mathcal{A}$ is the following linear operator which generates a semigroup (see [15]):

$$\mathcal{A}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} v \\ w \\ \dfrac{1}{\tau}\Delta(u+\beta v) - \dfrac{1}{\tau}w \end{pmatrix}$$

and $\mathcal{F}$ is the nonlinear term

$$\mathcal{F}(\mathbf{U}, \nabla \mathbf{U}) = \begin{pmatrix} 0 \\ 0 \\ \dfrac{1}{\tau}\dfrac{B}{A}vw + \dfrac{2}{\tau}\nabla u \nabla v \end{pmatrix}.$$

If $\mathbf{U}$ is a smooth solution of (1.1), then

$$\mathbf{U}(t) = \Phi(\mathbf{U})(t) = e^{t\mathcal{A}}\mathbf{U}_0 + \int_0^t e^{(t-r)\mathcal{A}}\mathcal{F}(\mathbf{U}, \nabla \mathbf{U})(r)dr. \tag{4.3}$$

We define for $s > s_0 = 1 + N/2$,

$$\begin{aligned} X \ := \ & \Big\{ \mathbf{U} = (u, v, w) \mid \nabla u, \Delta u \in C^0([0,T], H^{s-1}), u \in W, \ v \in C^0([0,T], H^{s+1}), \\ & w \in C^0(0,T], H^s), \ \|\mathbf{U}\|_X^2 \equiv \sup_{0\le t\le T} \Upsilon_s(t) < \infty \Big\}, \end{aligned}$$

where W is the completion of C_0^∞ under the seminorm $\|\nabla \cdot\|_{L^2}$.

We also define

$$Y := \left\{ \mathbf{U} \mid \|\mathbf{U}\|_Y^2 \equiv \int_0^T G^{(s)}(\sigma)d\sigma < \infty \right\}$$

where Υ_s and G^s are defined in (3.41) and (3.42), respectively. It is clear that

$$\|\mathbf{U}\|_Y \le CT\|\mathbf{U}\|_X,$$

for some $C > 0$. Define

$$Z := \{\mathbf{U} \in X \cap Y; \mathbf{U}(x, 0) = \mathbf{U}_0(x)\}.$$

Hence, from the above computation, we deduce that $\Phi(\mathbf{U})$ is well defined and it maps Z into $X \cap Y$. We define the ball D_R as:

$$D_R = \big\{\mathbf{U} \in Z : \|\mathbf{U}\|_{X\cap Y} \le R\big\}.$$

It is clear that D_R is closed subset of the space Z and non-empty for all $R > R_0$ with $R_0^2 = \Upsilon_s(0)$.

Our goal is to show that:

(1) Φ maps the ball D_R into itself,
(2) Φ is a contraction in D_R.

As we will see properties (1) and (2) are valid for R large enough, depending on the initial data, and for T sufficiently small and its choice is given later. Once the properties (1)-(2) are verified, the application of the contraction mapping theorem gives the existence of a uniques solution of (4.2).

We write

$$\Phi(\mathbf{U}) = \mathbf{U}^0 + \mathcal{G}(\mathbf{U}),$$

where $\mathbf{U}^0$ and $\mathcal{G}(\mathbf{U})$ are given by

$$\mathbf{U}^0 = e^{t\mathcal{A}}\mathbf{U}_0 \qquad \text{and} \qquad \mathcal{G}(\mathbf{U}) = \int_0^t e^{(t-r)\mathcal{A}}\mathcal{F}(\mathbf{U},\nabla\mathbf{U})(r)dr.$$

It is clear that $\mathbf{U}^0$ satisfies the linear equation

$$\mathbf{U}_t^0 - \mathcal{A}\mathbf{U}^0 = 0, \qquad \mathbf{U}^0(x) = \mathbf{U}_0(x), \tag{4.4}$$

and $\mathcal{G}(\mathbf{U})$ satisfies the nonlinear equation with zero initial data, that is

$$\partial_t\mathcal{G}(\mathbf{U}) - \mathcal{A}\mathcal{G}(\mathbf{U}) = \mathcal{F}(\mathbf{U},\nabla\mathbf{U}), \qquad \mathcal{G}(\mathbf{U})(x,0) = 0. \tag{4.5}$$

As in the proof of Theorem 3.1, we have for all $t \in [0,T)$,

$$\|\mathbf{U}^0(t)\|_X^2 + \|\mathbf{U}^0(t)\|_Y^2 \le C\Upsilon_s(0). \tag{4.6}$$

Now, to bound $\mathcal{G}$ in D_R, we have by applying once again the proof of Theorem 3.1, (especially the estimate (3.43)) for all $0 \le t \le T$,

$$\|\mathcal{G}(\mathbf{U}(t))\|_X^2 + \|\mathcal{G}(\mathbf{U}(t))\|_Y^2 \le C\int_0^t (1+\Lambda(\sigma))\Upsilon_s(\sigma)d\sigma. \tag{4.7}$$

Now, using the Sobolev embedding $H^s(\mathbb{R}^N) \hookrightarrow W^{1,\infty}(\mathbb{R}^N)$, for $s > 1 + N/2$, we get

$$\Lambda(t) \le C\sqrt{\Upsilon_s(t)} \le C\|\mathbf{U}(t)\|_X. \tag{4.8}$$

Hence, (4.7) yields

$$\begin{aligned} \|\mathcal{G}(\mathbf{U}(t))\|_X^2 + \|\mathcal{G}(\mathbf{U}(t))\|_Y^2 &\le CT\|\mathbf{U}(t)\|_X^2 + CT\|\mathbf{U}(t)\|_X^{3/2} \\ &\le TC(R^2 + R^{3/2}). \end{aligned} \tag{4.9}$$

Collecting (4.6) and (4.9), we obtain

$$\|\Phi(\mathbf{U})\|_{X\cap Y}^2 \le C\Upsilon_s(0) + TC(R^2 + R^{3/2})$$

Choosing R sufficiently large and T sufficiently small such that

$$C\Upsilon_s(0) + TC(R^2 + R^{3/2}) \le R^2,$$

That is

$$T \le \frac{R^2 - C\Upsilon_s(0)}{C(R^2 + R^{3/2})},$$

provided that $R^2 > C\Upsilon_s(0)$, then, we obtain

$$\|\Phi(\mathbf{U})\|_{X\cap Y} \le R.$$

Hence, we have prove that $\Phi\left(D_R\right) \subset D_R$.

Now, we need to prove that Φ is contractive. We have, as above for $\mathbf{U}$ and $\mathbf{V}$ in D_R, $\mathcal{G}(\mathbf{U})$ and $\mathcal{G}(\mathbf{V})$ solve the equation, hence, we obtain (4.5)

$$\begin{aligned} &\partial_t\Big(\mathcal{G}(\mathbf{U})-\mathcal{G}(\mathbf{V})\Big)-\mathcal{A}\big(\mathcal{G}(\mathbf{U})-\mathcal{G}(\mathbf{V})\big)=\mathcal{F}(\mathbf{U},\nabla\mathbf{U})-\mathcal{F}(\mathbf{V},\nabla\mathbf{V}),\\ &\mathcal{G}(\mathbf{U})\,(t=0)=\mathcal{G}(\mathbf{V})\,(t=0)=0. \end{aligned} \tag{4.10}$$

We put $\mathbf{W}(t)=\mathcal{G}(\mathbf{U}(t))-\mathcal{G}(\mathbf{V}(t))$. Then, we obtain from above

$$\begin{cases} \partial_t\mathbf{W}-\mathcal{A}(\mathbf{W})=\mathcal{F}(\mathbf{U},\nabla\mathbf{U})-\mathcal{F}(\mathbf{V},\nabla\mathbf{V}),\\ \mathbf{W}(t=0)=0. \end{cases}$$

Let $\mathbf{U}=(u,v,w)^T$ and $\mathbf{V}=(\bar{u},\bar{v},\bar{w})^T$, then we have

$$\begin{aligned} &\mathcal{F}(\mathbf{U},\nabla\mathbf{U})-\mathcal{F}(\mathbf{V},\nabla\mathbf{V})\\ &=\begin{pmatrix} 0\\ 0\\ \frac{1}{\tau}\frac{B}{A}(vw-\bar{v}\,\bar{w})+\frac{2}{\tau}(\nabla u\nabla v-\nabla\bar{u}\nabla\bar{v}) \end{pmatrix}\\ &=\begin{pmatrix} 0\\ 0\\ \frac{1}{\tau}\frac{B}{A}\big((v-\bar{v})\bar{w}+v(w-\bar{w})\big)+\frac{2}{\tau}\big((\nabla u-\nabla\bar{u})\nabla\bar{v}+\nabla u(\nabla v-\nabla\bar{v})\big) \end{pmatrix} \end{aligned}$$

Following the same steps as before, we obtain (instead of (3.40)), for all $0\le t\le T$,

$$\begin{aligned} &\Upsilon_s(\mathbf{W})(t)+\int_0^t G^{(s)}(\mathbf{W})(\sigma)d\sigma\\ \le\;& C\int_0^t\left[\Lambda(\mathbf{U}(\sigma))+\Lambda(\mathbf{V}(\sigma))\right]\sqrt{\Upsilon(\mathbf{W}(\sigma))}\sqrt{\Upsilon_s\Big(\mathbf{U}(\sigma)-\mathbf{V}(\sigma)\Big)}d\sigma\\ &+C\int_0^t(1+\Lambda(\mathbf{U}(\sigma))+\Lambda(\mathbf{V}(\sigma)))\sqrt{\Upsilon(\mathbf{W}(\sigma))}\sqrt{\Upsilon_s\Big(\mathbf{U}(\sigma)-\mathbf{V}(\sigma)\Big)}d\sigma. \end{aligned}$$

Applying (4.8), we deduce that

$$\|\mathbf{W}\|_{X\cap Y}\le CT\|\mathbf{U}-\mathbf{V}\|_X\left(1+\|\mathbf{U}\|_X+\|\mathbf{V}\|_X\right)$$

This implies

$$\|\mathcal{G}(\mathbf{U}-\mathbf{V})\|_{X\cap Y}\le CT\|\mathbf{U}-\mathbf{V}\|_{X\cap Y}\big(1+2R\big).$$

Now, we fix T small enough such that $CT\big(1+2R\big)=\kappa<1$. Hence, we deduce that

$$\|\Phi(\mathbf{U})-\Phi(\mathbf{V})\|_{X\cap Y}\le\kappa\|\mathbf{U}-\mathbf{V}\|_{X\cap Y}.$$

Thus, we conclude that Φ is a contraction in D_R. The application of the contraction mapping principle shows that there exists a unique solution $\mathbf{U}\in Z$ of (1.1). This finishes the proof of Theorem 4.1. □

5. Decay rates

In this section, we prove decay rates for solutions to (1.1). Let

$$\mathbf{V} := (v + \tau w, \nabla(u + \tau v), \nabla v)$$

be the global solution according to Theorem 3.1, with $v = u_t$ and $w = u_{tt}$. Let

$$\|(u, v, w)\|_{\mathcal{H}}^2 \ = \tau(\beta - \tau)\|\nabla v\|_{L^2(\mathbb{R}^N)}^2 + \|\nabla(u + \tau v)\|_{L^2(\mathbb{R}^N)}^2 + \|v + \tau w\|_{L^2(\mathbb{R}^N)}^2. \tag{5.1}$$

defining a norm in $\mathcal{H} = H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$. It is clear that the norm above is equivalent to the norm $\|\mathbf{V}(t)\|_{L^2}^2$.

Theorem 5.1. *Assume that* $0 < \tau < \beta$ *and* $s > \frac{N}{2} + 1$. *Let* u *be the global solution of* (1.1). *Let* $v_0 = u_t(t = 0)$, $v_1 = u_{tt}(t = 0)$ *and* $v_2 = u_{ttt}(t = 0)$ *satisfying* $v_0, v_1, v_2 \in L^1(\mathbb{R}^N) \cap H^s(\mathbb{R}^N)$ *and* $(v_1, v_2) \in L^{1,1}(\mathbb{R}^N)$ *with* $\int_{\mathbb{R}^N} v_i(x)dx = 0$, $i = 1, 2$. *Assume that* $\|\mathbf{V}_0\|_{H^s \cap L^1}$ *is small enough. Then, the following decay estimates hold:*

$$\|\nabla^j \mathbf{V}(t)\|_{L^2} + \|\nabla^j u_t(t)\|_{L^2} \leq \begin{cases} C(1+t)^{-N/4-j/2}, & \text{if} \quad N > 2 \\ C(1+t)^{-1/2-j/2}\log(t+1), & \text{if} \quad N = 2, \end{cases}$$

for all $0 \leq j \leq s$.

Proof. First, inspired by the decay estimates of the linear problem (see Propositions 2.4 and 2.5), we define

$$\mathcal{M}(t) := \sup_{0 \leq \sigma \leq t} \sum_{j=0}^{s} (1+\tau)^{N/4+j/2} (\|\nabla^j \mathbf{V}(\sigma)\|_{L^2} + \|\nabla^j v(\sigma)\|_{L^2}).$$

We also define the quantities

$$\begin{aligned} M_0(t) \ &:= \ \sup_{0 \leq \sigma \leq t} (1+\sigma)^{\frac{N}{2}} \left(\|\mathbf{V}(\sigma)\|_{L^\infty} + \|v(\sigma)\|_{L^\infty} \right), \\ M_1(t) \ &:= \ \sup_{0 \leq \sigma \leq t} (1+\sigma)^{\frac{N}{2}+\frac{1}{2}} \|\nabla \mathbf{V}(\sigma)\|_{L^\infty}. \end{aligned}$$

So, our goal is to show that $\mathcal{M}(t)$ is bounded uniformly in t if $\|\mathbf{V}_0\|_{H^s \cap L^1} = \|\mathbf{V}_0\|_{H^s} + \|\mathbf{V}_0\|_{L^1}$ is small enough. From (4.3), we have for $\mathbf{U} = (u, v, w)$, and for $0 \leq j \leq s$,

$$\begin{aligned} \|\nabla^j \mathbf{U}(t)\|_{\mathcal{H}} \ &\leq \ \|\nabla^j e^{t\mathcal{A}} \mathbf{U}_0\|_{\mathcal{H}} + \int_0^t \left\|\nabla^j e^{(t-r)\mathcal{A}} \mathcal{F}(\mathbf{U}, \nabla \mathbf{U})(r)\right\|_{\mathcal{H}} dr \\ &= \ \|\nabla^j e^{t\mathcal{A}} \mathbf{U}_0\|_{\mathcal{H}} + \int_0^{t/2} \left\|\nabla^j e^{(t-r)\mathcal{A}} \mathcal{F}(\mathbf{U}, \nabla \mathbf{U})(r)\right\|_{\mathcal{H}} dr \\ &\quad + \int_{t/2}^t \left\|\nabla^j e^{(t-r)\mathcal{A}} \mathcal{F}(\mathbf{U}, \nabla \mathbf{U})(r)\right\|_{\mathcal{H}} dr \\ &\equiv \ \|\nabla^j e^{t\mathcal{A}} \mathbf{U}_0\|_{\mathcal{H}} + J_1 + J_2. \end{aligned}$$

This gives, by using the estimate (2.5),

$$\|\nabla^j e^{t\mathcal{A}} \mathbf{U}_0\|_{\mathcal{H}} \leq C(1+t)^{-N/4-j/2} \left(\|\mathbf{V}_0\|_{L^1(\mathbb{R}^N)} + \|\nabla^j \mathbf{V}_0\|_{L^2(\mathbb{R}^N)} \right).$$

Now, for J_1, we have (also using the estimate (2.5))

$$\begin{aligned} J_1 &\leq C\int_0^{t/2}(1+t-r)^{-N/4-j/2}\|\tilde{\mathcal{F}}(\mathbf{U},\nabla\mathbf{U})(r)\|_{L^1(\mathbb{R}^N)}dr \\ &\quad +C\int_0^{t/2}e^{-(t-r)}\|\nabla^j\tilde{\mathcal{F}}(\mathbf{U},\nabla\mathbf{U})(r)\|_{L^2(\mathbb{R}^N)}dr \\ &\equiv J_{11}+J_{12}, \end{aligned} \tag{5.2}$$

where $\tilde{\mathcal{F}}(\mathbf{U},\nabla\mathbf{U})(t)=\Big(\dfrac{B}{A}vw+2\nabla u\nabla v,0,0\Big)$.

To estimate J_1 it is convenient to divide the integral into two parts J_{11} and J_{12} corresponding to $[0,t/2]$ and $[t/2,t]$ and then estimate each term separately, cp. Lemma 7.4 in [17]. First, we have by using Hölder's inequality,

$$\|\tilde{\mathcal{F}}(\mathbf{U},\nabla\mathbf{U})(t)\|_{L^1(\mathbb{R}^N)}\leq\|\mathbf{V}(t)\|_{L^2}^2+\|v(t)\|_{L^2}^2.$$

Hence, J_{11} can be estimated as follow:

$$\begin{aligned} J_{11} &= \int_0^{t/2}(1+t-r)^{-N/4-j/2}\|\tilde{\mathcal{F}}(\mathbf{U},\nabla\mathbf{U})(r)\|_{L^1(\mathbb{R}^N)}dr \\ &\leq C\mathcal{M}^2(t)\int_0^{t/2}(1+t-r)^{-N/4-j/2}(1+r)^{-N/2}dr \\ &\leq C\mathcal{M}^2(t)\int_0^{t/2}(1+t)^{-N/4-j/2}(1+r)^{-N/2}dr \\ &\leq C\mathcal{M}^2(t)(1+t)^{-N/4-j/2}\int_0^{t/2}(1+r)^{-N/2}dr \\ &\leq \begin{cases} C\mathcal{M}^2(t)(1+t)^{-N/4-j/2}, & \text{if} \quad N>2, \\ C\mathcal{M}^2(t)(1+t)^{-N/4-j/2}\log(t+1), & \text{if} \quad N=2. \end{cases} \end{aligned}$$

On the other hand,

$$\begin{aligned} \|\nabla^j\tilde{\mathcal{F}}(\mathbf{U},\nabla\mathbf{U})(r)\|_{L^2(\mathbb{R}^N)} &\leq C\left(\|\nabla^j(vw)\|_{L^2}+\|\nabla^j(\nabla u\nabla v)\|_{L^2}\right) \\ &\leq C\left(\|\nabla^j(v(v+\tau w))\|_{L^2}+\|\nabla^j(v^2)\|_{L^2}+\|\nabla^j(\nabla u\nabla v)\|_{L^2}\right). \end{aligned}$$

This gives, by applying (2.1),

$$\begin{aligned} &\|\nabla^j\tilde{\mathcal{F}}(\mathbf{U},\nabla\mathbf{U})(t)\|_{L^2(\mathbb{R}^N)} \\ \leq\ & C\|v\|_{L^\infty}\left(\|\nabla^j(v+\tau w)\|_{L^2}+\|\nabla^j v\|_{L^2}\right) \\ & +C\|v+\tau w\|_{L^\infty}\|\nabla^j v\|_{L^2}+C\|\nabla v\|_{L^\infty}\|\nabla^j\nabla v\|_{L^2} \\ & +C\|\nabla v\|_{L^\infty}\|\nabla^j\nabla(u+\tau v)\|_{L^2}+C\|\nabla(u+\tau v)\|_{L^\infty}\|\nabla^j\nabla v\|_{L^2} \\ \leq\ & C(1+t)^{-N/2}(1+t)^{-N/4-j/2}M_0(t)\mathcal{M}(t) \\ & +C(1+t)^{-N/2-1/2}(1+t)^{-N/4-j/2}M_1(t)\mathcal{M}(t). \end{aligned}$$

Consequently, using these estimates, we deduce that

$$J_{12} \leq C(1+t)^{-3N/2-j/2}(M_0(t)+M_1(t))\mathcal{M}(t).$$

Next, J_2 is estimated by applying (2.5) with $j=1$ and using $\nabla^{j-1}\tilde{\mathcal{F}}(\mathbf{U},\nabla\mathbf{U})(t)$ instead of $\mathbf{V}_0$, to obtain, for $j\geq 1$,

$$\begin{aligned} J_2 &= \int_{t/2}^t \left\|\nabla e^{(t-r)\mathcal{A}}\nabla^{j-1}\tilde{\mathcal{F}}(\mathbf{U},\nabla\mathbf{U})(r)\right\|_{L^2} dr \\ &\leq C\int_{t/2}^t (1+t-r)^{-\frac{N}{4}-\frac{1}{2}}\left\|\nabla^{j-1}\tilde{\mathcal{F}}(\mathbf{U},\nabla\mathbf{U})(r)\right\|_{L^1} dr \\ &\quad +C\int_{t/2}^t e^{-c(t-r)}\left\|\nabla^{j}\tilde{\mathcal{F}}(\mathbf{U},\nabla\mathbf{U})(r)\right\|_{L^2} dr \\ &\equiv J_{21}+J_{22}. \end{aligned}$$

On the other hand, we have by applying (2.1),

$$\begin{aligned} \|\nabla^{j-1}\tilde{\mathcal{F}}(\mathbf{U},\nabla\mathbf{U})(t)\|_{L^1(\mathbb{R}^N)} &\leq C(\|\mathbf{V}(t)\|_{L^2}+\|v(t)\|_{L^2})(\|\nabla^{j-1}\mathbf{V}(t)\|_{L^2}+\|\nabla^{j-1}v(t)\|_{L^2}) \\ &\leq C\mathcal{M}^2(t)(1+t)^{-\frac{N}{2}-\frac{j-1}{2}}. \end{aligned}$$

Thus,

$$\begin{aligned} J_{21} &\leq C\mathcal{M}^2(t)\int_{t/2}^t (1+t-r)^{-\frac{N}{4}-\frac{1}{2}}(1+r)^{-\frac{N}{2}-\frac{j-1}{2}}dr \\ &\leq C\mathcal{M}^2(t)(1+t/2)^{-\frac{N}{2}-\frac{j-1}{2}}\int_{t/2}^t (1+t-r)^{-\frac{N}{4}-\frac{1}{2}}dr \\ &\leq (1+t/2)^{-\frac{N}{2}-\frac{j-1}{2}}\int_0^{t/2}(1+r)^{-\frac{N}{4}-\frac{1}{2}}dr \\ &\leq C(1+t)^{-\frac{N}{2}-\frac{j-1}{2}}\begin{cases}(1+t)^{-N/4-1/2+1}+1, & \text{if } N\neq 2,\\ \log(t+1), & \text{if } N=2.\end{cases} \\ &\leq C\begin{cases}(1+t)^{-\frac{N}{4}-\frac{j}{2}}, & \text{if } N>2,\\ (1+t)^{-\frac{1}{2}-\frac{j}{2}}\log(t+1), & \text{if } N=2.\end{cases} \qquad (5.3) \end{aligned}$$

For J_{22}, we have as in the estimate of J_{12},

$$J_{22} \leq C(1+t)^{-3N/2-j/2}(M_0(t)+M_1(t))\mathcal{M}(t).$$

Therefore, collecting the above estimates, we have

$$\begin{aligned} \|\nabla^j\mathbf{U}(t)\|_{\mathcal{H}} &\leq C(1+t)^{-N/4-j/2}\left(\|\mathbf{V}_0\|_{L^1(\mathbb{R}^N)}+\|\nabla^j\mathbf{V}_0\|_{L^2(\mathbb{R}^N)}\right) \\ &\quad +C\mathcal{M}^2(t)(1+t)^{-N/4-j/2}+C(1+t)^{-N/4-j/2}(M_0(t)+M_1(t))\mathcal{M}(t) \qquad (5.4) \end{aligned}$$

This yields

$$\begin{aligned} \mathcal{M}(t) &\leq C\left(\|\mathbf{V}_0\|_{L^1(\mathbb{R}^N)}+\|\nabla^j\mathbf{V}_0\|_{L^2(\mathbb{R}^N)}\right) \\ &\quad +C\mathcal{M}^2(t)+C(M_0(t)+M_1(t))\mathcal{M}(t). \qquad (5.5) \end{aligned}$$

Applying Lemma 2.3 with $\alpha = \frac{N}{2m}$, $q = r = 2$, $j = 0$ and $p = \infty$, we get for $m > \frac{N}{2}$

$$\|\mathbf{V}\|_{L^\infty} \leq C \|\nabla^m \mathbf{V}\|_{L^2}^{\frac{N}{2m}} \|\mathbf{V}\|_{L^2}^{1-\frac{N}{2m}},$$

and similar estimates can be used for $\|v\|_{L^\infty}$. This yields

$$M_0(t) \leq C\mathcal{M}(t),$$

provided that $s \geq m > \frac{N}{2}$.

Next, to estimate $M_1(t)$, we apply Lemma 2.3 with $\alpha = \frac{N+2}{2m}$, $q = r = 2$, $j = 1$ and $p = \infty$, we get for $m > \frac{N+2}{2}$,

$$\|\nabla \mathbf{V}\|_{L^\infty} \leq C \|\nabla^m \mathbf{V}\|_{L^2}^{\frac{N+2}{2m}} \|\mathbf{V}\|_{L^2}^{1-\frac{N+2}{2m}}.$$

This leads to

$$M_1(t) \leq C\mathcal{M}(t),$$

provided that $s \geq m > \frac{N}{2} + 1$. Hence, since $M_0(t) + M_1(t) \leq C\mathcal{M}(t)$, then (5.5) implies that

$$\mathcal{M}(t) \leq C \left(\|\mathbf{V}_0\|_{L^1(\mathbb{R}^N)} + \|\nabla^j \mathbf{V}_0\|_{L^2(\mathbb{R}^N)} \right) + C\mathcal{M}^2(t).$$

Consequently, applying Lemma 2.2 gives the desired result, provided that $\|\mathbf{V}_0\|_{L^1(\mathbb{R}^N)} + \|\nabla^j \mathbf{V}_0\|_{L^2(\mathbb{R}^N)}$ is small enough for all $0 \leq j \leq s$. This finishes the proof of Theorem 5.1. □

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