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Luca Mechelli
Stefan Volkwein

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POD-Based Economic Model Predictive Control for Heat-Convection Phenomena

Luca Mechelli[1] and Stefan Volkwein[1]

University of Konstanz, Department of Mathematics and Statistics,
Universitätsstraße 10, D-78457 Konstanz, Germany
Luca.Mechelli@uni-konstanz.de*, Stefan.Volkwein@uni-konstanz.de

Abstract. In the setting of energy efficient building operation, an optimal boundary control problem governed by a linear parabolic advection-diffusion equation is considered together with bilateral control and state constraints. To keep the temperature in a prescribed range with the less possible heating cost, an economic model predictive control (MPC) strategy is applied. To speed-up the MPC method, a reduced-order approximation based on proper orthogonal decomposition (POD) is utilized. A-posteriori error analysis ensures the quality of the POD models. A numerical test illustrates the efficiency of the proposed strategy.

1 Introduction

We consider a linear parabolic advection-diffusion equation for describing the evolution of the temperature in a room. The heaters in the room are represented as boundary controls and on the rest of the boundary we parametrize the exchange of heat between the room and the outside. In order to treat the pointwise temperature constraints we perform a Lavrentiev regularization, see [12]. An economic model predictive control (MPC) strategy (see [5, Chapter 8] and references therein) is considered to treat the long-time horizon and possible parameter changes. Thus, the goal is to minimize only the controls while satisfying given state constraints. In each iteration of the MPC method the open-loop problem is solved by the primal dual active set strategy (PDASS); cf. [8]. To speed-up the numerical optimization we apply model-order reduction based on proper orthogonal decomposition (POD); see [9]. To ensure an accurate POD model in each iteration of the MPC algorithm we utilize the a-posteriori error estimate which is presented in [10] and follows from [6,13]. If the error is too large, we build a new POD basis utilizing the information from the current MPC iteration. Let us mention that POD is used for MPC methods in [1,4], but the authors do not consider economic MPC.

The paper is organized as follows: In Section 2 we introduce the optimal control problem and the Lavrentiev regularization. We describe the economic MPC algorithm in Section 3 and we propose two POD variants. In Section 4 we show numerical tests. Conclusions are drawn in Section 5.

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2 The optimal control problem

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded domain with Lipschitz-continuous boundary $\Gamma = \partial\Omega$. We suppose that Γ is split into the two disjoint subsets Γ_c and Γ_out, where at least Γ_c has nonzero (Lebesgue) measure. For large $T \gg 0$ we set $Q = (0,T) \times \Omega$, $\Sigma_\mathsf{c} = (0,T) \times \Gamma_\mathsf{c}$ and $\Sigma_\mathsf{out} = (0,T) \times \Gamma_\mathsf{out}$. By $L^2(0,T;V)$ we denote the space of measurable functions from $[0,T]$ to V, which are square integrable. We define the Hilbert spaces $\mathcal{H} = L^2(0,T;H)$ and $W(0,T) = \{\varphi \in L^2(0,T;V) \,|\, \varphi_t \in L^2(0,T;V')\}$, where V' denotes the dual space of V; cf. [3]. For $m \in \mathbb{N}$ let $b_i : \Gamma_\mathsf{c} \to \mathbb{R}$, $1 \le i \le m$, denote given control shape functions. For $\mathcal{U} = L^2(0,T;\mathbb{R}^m)$ the set of admissible controls $u = (u_i)_{1\le i\le m} \in \mathcal{U}$ is given as

$$\mathcal{U}_\mathsf{ad} = \big\{u \in \mathcal{U} \,\big|\, u_{\mathsf{a}i} \le u_i \le u_{\mathsf{b}i} \text{ for } i = 1,\dots,m \text{ and a.e. in } [0,T]\big\},$$

where $u_\mathsf{a} = (u_{\mathsf{a}i})_{1\le i\le m}$, $u_\mathsf{b} = (u_{\mathsf{b}i})_{1\le i\le m} \in \mathcal{U}$ are lower and upper bounds, respectively, and 'a.e.' stands for 'almost everywhere'. Then, for any control $u \in \mathcal{U}_\mathsf{ad}$ the state y is governed by the following *state equation*

$$\begin{aligned}
y_t(t,\boldsymbol{x}) - \lambda\Delta y(t,\boldsymbol{x}) + \boldsymbol{v}(t,\boldsymbol{x}) \cdot \nabla y(t,\boldsymbol{x}) &= f(t,\boldsymbol{x}), && \text{a.e. in } Q,\\
\lambda \frac{\partial y}{\partial \boldsymbol{n}}(t,\boldsymbol{s}) + \gamma_\mathsf{c} y(t,\boldsymbol{s}) &= \gamma_\mathsf{c} \sum_{i=1}^m u_i(t) b_i(\boldsymbol{s}), && \text{a.e. on } \Sigma_\mathsf{c},\\
\lambda \frac{\partial y}{\partial \boldsymbol{n}}(t,\boldsymbol{s}) + \gamma_\mathsf{out} y(t,\boldsymbol{s}) &= \gamma_\mathsf{out} y_\mathsf{out}(t), && \text{a.e. on } \Sigma_\mathsf{out},\\
y(0,\boldsymbol{x}) &= y_\circ(\boldsymbol{x}), && \text{a.e. in } \Omega.
\end{aligned} \tag{1}$$

We assume that $\lambda > 0$, $\gamma_\mathsf{c}, \gamma_\mathsf{out} \ge 0$, $\boldsymbol{v} \in L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^d))$, $y_\mathsf{out} \in L^2(0,T)$, $y_\circ \in H$, $b_1,\dots,b_m \in L^\infty(\Gamma_\mathsf{c})$ and $f \in \mathcal{H}$. By introducing the weak formulation we rewrite (1) as a parametrized dynamical system. For this purpose, we define the time-dependent mapping $\mathcal{M}(t;\cdot,\cdot) : V \times \mathbb{R}^m \to V'$ as

$$\mathcal{M}(t;\phi,\mathfrak{u}) = \mathcal{F}(t) + \gamma_\mathsf{c}\mathcal{B}\mathfrak{u} - \lambda\Delta\phi - \boldsymbol{v}(t)\cdot\nabla\phi,\ (\phi,\mathfrak{u}) \in V \times \mathbb{R}^m \text{ a.e. in } [0,T],$$

where the time-dependent linear functional $\mathcal{F}(t) : V \to V'$ is defined as

$$\langle \mathcal{F}(t), \varphi\rangle_{V',V} = \int_\Omega f(t)\varphi \,\mathrm{d}\boldsymbol{x} + \gamma_\mathsf{out} y_\mathsf{out}(t) \int_{\Gamma_\mathsf{out}} \varphi \,\mathrm{d}\boldsymbol{s} \quad \text{for } \varphi \in V$$

and $\langle\cdot\,,\cdot\rangle_{V',V}$ stands for the dual pairing between V and its dual space V'. The linear operator $\mathcal{B} : \mathbb{R}^m \to V'$ is defined as

$$\langle \mathcal{B}\mathfrak{u}, \varphi\rangle_{V',V} = \sum_{i=1}^m \mathfrak{u}_i \int_{\Gamma_\mathsf{c}} b_i\varphi \,\mathrm{d}\boldsymbol{s} \quad \text{for } \varphi \in V \text{ and } \mathfrak{u} = (\mathfrak{u}_i)_{1\le i\le m} \in \mathbb{R}^m.$$

Now, a weak solution to (1) satisfies the dynamical system

$$y_t(t) = \mathcal{M}(t;y(t),u(t)) \in V' \text{ a.e. in } (0,T], \quad y(0) = y_\circ \text{ in } H. \tag{2}$$

It is known that (2) admits a unique solution $y \in W(0,T)$; cf., e.g., [3]. To ensure that the state is in a given desired (temparature) range, we pose the pointwise state constraints

$$y_\mathsf{a}(t,\boldsymbol{x}) \le y(t,\boldsymbol{x}) \le y_\mathsf{b}(t,\boldsymbol{x}) \quad \text{a.e. in } Q, \tag{3}$$

where $y_\mathsf{a}, y_\mathsf{b} \in \mathcal{H}$ are given lower and upper bounds, respectively. To gain regular Lagrange multipliers we utilize a Lavrentiev regularization [12]. Let $\varepsilon > 0$ be a chosen regularization parameter and $w \in \mathcal{H}$ an additional (artificial) control. Then, (3) is replaced by the mixed control-state constraints

$$y_\mathsf{a}(t,\boldsymbol{x}) \le y(t,\boldsymbol{x}) + \varepsilon w(t,\boldsymbol{x}) \le y_\mathsf{b}(t,\boldsymbol{x}) \quad \text{a.e. in } Q.$$

We introduce the Hilbert space

$$\mathbb{X} = W(0,T) \times \mathcal{U} \times \mathcal{H}$$

endowed with the common product topology. The set of admissible solutions is given by

$$\mathbb{X}^\varepsilon_\mathsf{ad} = \big\{(y,u,w) \in \mathbb{X} \,\big|\, y \text{ solves (2)},\ u \in \mathcal{U}_\mathsf{ad},\ y_\mathsf{a} \le y + \varepsilon w \le y_\mathsf{b} \text{ in } \mathcal{H}\big\}.$$

The quadratic cost functional $J : \mathbb{X} \to \mathbb{R}$ is given by

$$J(x) = \sum_{i=1}^{m} \frac{\sigma_i}{2} \left\| u_i \right\|^2_{L^2(0,T)} + \frac{\sigma_w}{2} \left\| w \right\|^2_{\mathcal{H}} \quad \text{for } x = (y,u,w) \in \mathbb{X}$$

with weighting parameters $\sigma_1, \dots, \sigma_m, \sigma_w > 0$. The goal is to minimize the control costs and not to reach a prescribed target. The latter issue is taken into account by the state constraints. This approach is called economic optimal control; cf. [5]. Now, the optimal control problem is

$$\min J(x) \quad \text{subject to (s.t.)} \quad x \in \mathbb{X}^\varepsilon_\mathsf{ad}. \tag{$\mathbf{P}^\varepsilon$}$$

Problem ($\mathbf{P}^\varepsilon$), which can be formulated as pure control constrained problem, has a unique optimal solution $\bar{x}^\varepsilon = (\bar{y}^\varepsilon, \bar{u}^\varepsilon, \bar{w}^\varepsilon)$; see [10].

3 Model Predictive Control (MPC)

Since we want to control the temperature for a 'large' time horizon $[0,T]$ with $T \gg 0$, we use the features of MPC. The basic idea of MPC is to predict, stabilize and optimize a given dynamical system − like (2) − by reconstructing the optimal control $u(t) = \Phi(t, y(t))$ in a feedback form. In order to do that, we solve repetitively open-loop optimal control problems on smaller time horizons $N\Delta T \ll T$, $N \in \mathbb{N}$, with a primal-dual active set strategy (PDASS); cf. [10]. Then, the first part of the open-loop control is stored and implemented, before solving the next open-loop problem on a shifted time horizon. A general theory can be found in [5,11].

For chosen $0 \leq t_n < t_n^N \leq T$ with $t_n^N = t_n + N\Delta T$ and $y_n \in H$ we consider (2) on the time horizon $[t_n, t_n^N]$:

$$y_t(t) = \mathcal{M}(t; y(t), u(t)) \in V' \text{ a.e. in } (t_n, t_n^N], \quad y(t_n) = y_n \text{ in } H. \tag{4}$$

Next we define the function spaces related to $[t_n, t_n^N]$: $\mathcal{U}_n = L^2(t_n, t_n^N; \mathbb{R}^m)$, $\mathcal{H}_n = L^2(t_n, t_n^N; H)$ and $\mathbb{X}_n = W(t_n, t_n^N) \times \mathcal{U}_n \times \mathcal{H}_n$. Further, let

$$\mathcal{U}_{\mathsf{ad}}^n = \big\{ u \in \mathcal{U}_n \,\big|\, u_{\mathsf{a}i} \leq u_i \leq u_{\mathsf{b}i} \text{ for } i = 1, \ldots, m \text{ and a.e. in } [t_n, t_n^N] \big\},$$
$$\mathbb{X}_{\mathsf{ad}}^{\varepsilon,n} = \big\{ (y, u, w) \in \mathbb{X}_n \,\big|\, y \text{ solves (4)},\ u \in \mathcal{U}_{\mathsf{ad}}^n,\ y_{\mathsf{a}} \leq \mathcal{E}y + \varepsilon w \leq y_{\mathsf{b}} \text{ in } \mathcal{H}_n \big\}.$$

Now, the open-loop problem can be adapted by choosing the following cost:

$$J_n(x) = \sum_{i=1}^m \frac{\sigma_i}{2} \|u_i\|^2_{L^2(t_n, t_n^N)} + \frac{\sigma_w}{2} \|w\|^2_{\mathcal{H}_n} \quad \text{for } x = (y, u, w) \in \mathbb{X}_n.$$

The MPC method is summarized in Algorithm 1. If Φ^N is computed by the

Algorithm 1 (MPC method)

Require: Initial state $y_\circ$, time horizon $N\Delta t$ and regularization parameter $\varepsilon > 0$;
1: Put $y_0 = y_\circ$ and $t_0 = 0$;
2: **for** $n = 0, 1, 2, \ldots$ **do**
3: Set $t_n^N = t_n + N\Delta t$;
4: Compute the solution $\bar{x}_n^\varepsilon = (\bar{y}_n^\varepsilon, \bar{u}_n^\varepsilon, \bar{w}_n^\varepsilon)$ to the linear-quadratic problem

$$\min J_n(x) \quad \text{s.t.} \quad x \in \mathbb{X}_{\mathsf{ad}}^{\varepsilon,n}; \tag{$\mathbf{P}_n^\varepsilon$}$$

5: Define the MPC feedback law $\Phi^N(t; \bar{y}_n^\varepsilon(t)) = \bar{u}_n^\varepsilon(t)$ for $t \in (t_n, t_n + \Delta t]$;
6: Set the associated MPC state $y^N(t) = \bar{y}_n^\varepsilon(t)$ for $t \in (t_n, t_n + \Delta t]$;
7: Put $y_{n+1} = y^N(t_n + \Delta t)$ and $t_{n+1} = t_n + \Delta t$.
8: **end for**

MPC algorithm, then state $\bar{y}^N$ solves (2) for the closed-loop control $\bar{u}^N = \Phi^N(\cdot\,; y_n^\varepsilon(\cdot))$ with a given initial condition $y_\circ$. Another advantage of MPC is that we can update the data during the for-loop. For example, suppose that we have a good forecast for the outside temperature until a certain time $\tilde{t} \in (0, T)$. Then, we can incorporate a new forecast at that time and update the data for the outside temperature in the next open-loop solves. The same can be done for the time-dependent velocity field $\boldsymbol{v}$ by solving Navier-Stokes equations with the new temperature and pressure informations obtained at time $\tilde{t}$. This updating strategy can not be done when we solve ($\mathbf{P}^\varepsilon$) by PDASS still ensuring convergence. Under appropriate conditions it is known that the larger is N the more $\bar{u}^N$ approximates the open-loop solution to ($\mathbf{P}^\varepsilon$).

For numerical realization of Algorithm 1, we have to discretize ($\mathbf{P}_n^\varepsilon$). For the temporal discretization we utilize the implicit Euler method. The spatial variable has to be approximated, too. Here, we utilize a standard Galerkin scheme based on piecewise linear finite elements (FE). To speed-up the numerical realization of the MPC method, we also implement a reduced-order Galerkin scheme for ($\mathbf{P}_n^\varepsilon$); cf. [2]. Here, we apply proper orthogonal decomposition (POD); cf. [7,9]. Moreover, the approach for ($\mathbf{P}^\varepsilon$) is described in [10]. However, there are several ways to do the reduced-order modeling, we concentrate on two methods in this paper:

- *Method 1:* In Algorithm 1 we solve ($\mathbf{P}_n^\varepsilon$) by using the FE Galerkin scheme for $n = 0$. Then, we take the state $\bar{y}_0^\varepsilon$ and the associated adjoint variable $\bar{p}_0^\varepsilon$ to build a POD basis of rank ℓ which is orthonormal in $V = H^1(\Omega)$; cf. [10]. Now, ($\mathbf{P}_n^\varepsilon$) is solved for all $n > 0$ applying its (from now on) fixed POD Galerkin approximation.
- *Method 2:* In Method 1 we add the a-posteriori error estimator from [10]. If the a-posteriori error is too big, we solve the current problem ($\mathbf{P}_n^\varepsilon$) by using the FE Galerkin scheme and update using the obtained optimal FE state and associated adjoint.

In our numerical experiments we do not change the POD rank ℓ (basis extension) in both methods.

4 Numerical Tests

All the tests in this section have been made on a Notebook Lenovo ThinkPad T450s with Intel Core i7-5600U CPU @ 2.60GHz and 12GB RAM. Let $T = 1$, $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$. Moreover, the b_i's are chosen to be (cf. Fig. 1)

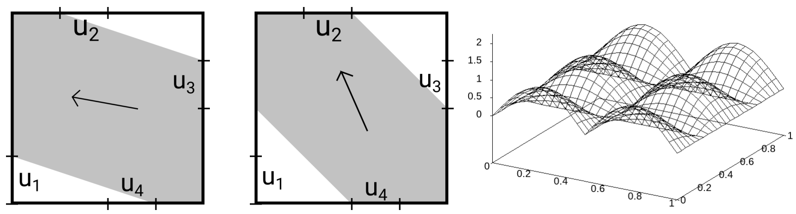
$$b_1(x) = \begin{cases} 1 \text{ if } x \in \{0\} \times [0.0, 0.25], \\ 0 \text{ otherwise}, \end{cases} \quad b_2(x) = \begin{cases} 1 \text{ if } x \in [0.25, 0.5] \times \{1\}, \\ 0 \text{ otherwise}, \end{cases}$$

$$b_3(x) = \begin{cases} 1 \text{ if } x \in \{1\} \times [0.5, 0.75], \\ 0 \text{ otherwise}, \end{cases} \quad b_4(x) = \begin{cases} 1 \text{ if } x \in [0.5, 0.75] \times \{0\}, \\ 0 \text{ otherwise}. \end{cases}$$

The FE discretization on a triangular mesh has 625 degrees of freedom. We choose $\Delta t = 0.01$ as time step. For the physical parameters we choose $\lambda = 1$, $\gamma_c = 1$, $\gamma_{out} = 0.03$. The initial condition is $y_\circ(x) = |\sin(2\pi x_1)\cos(2\pi x_2)|$ for $x = (x_1, x_2) \in \Omega$; cf. Fig. 1. In the first part of this tests we suppose to have as velocity field $\boldsymbol{v} = (v_1, v_2)$:

$$v_1(t,x) = \begin{cases} -1.6 & \text{if } x \in \mathcal{V}_\mathcal{F}, \\ 0 & \text{otherwise}, \end{cases} \quad v_2(t,x) = \begin{cases} 0.5 & \text{if } x \in \mathcal{V}_\mathcal{F}, \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

with $\mathcal{V}_\mathcal{F} = \big\{x = (x_1, x_2) \,\big|\, 12x_2 + 4x_1 \geq 3, 12x_2 + 4x_1 \leq 13\big\}$. The outside temperature is chosen to be $y_{\mathsf{out}}(t) = -1$ for all $t \in [0, T]$. As state constraints we take $y_\mathsf{a}(t) = 0.5 + \min(2t, 2)$, $y_\mathsf{b} = 3$ and $\varepsilon = 0.01$ and as control constraints



(a) $v^1(t,x)$ (b) $v^2(t,x)$ (c) $y_\circ(x)$

Fig. 1. Spatial domain Ω with the four boundary controls and the velocity fields (grey); initial condition $y_\circ$.

we take $u_{\mathsf{a}} = 0$ and $u_{\mathsf{b}} = 7$. Further, $\sigma_i = \sigma_w = 1$ for all $1 \le i \le m$. Now we solve the open-loop optimal control problem with PDASS. To illustrate that the MPC approach improves the control strategy if the data is changed during the optimization, suppose that at time $\tilde{t} = 0.5$ we measure again the outside temperature and the velocity field obtaining the new data be $\tilde{y}_{\mathsf{out}}(t) = 1$ and $\tilde{\boldsymbol{v}} = (\tilde{v}_1, \tilde{v}_2)$ with

$$\tilde{v}_1(t,x) = \begin{cases} -0.6 & \text{if } t \ge 0.5,\ x \in \tilde{\mathcal{V}}_{\mathcal{F}}, \\ 0 & \text{otherwise} \end{cases} \quad \tilde{v}_2(t,x) = \begin{cases} 1.5 & \text{if } t \ge 0.5,\ x \in \tilde{\mathcal{V}}_{\mathcal{F}}, \\ 0 & \text{otherwise} \end{cases}$$

with $\tilde{\mathcal{V}}_{\mathcal{F}} = \big\{x = (x_1, x_2)\,\big|\, x_1 + x_2 \ge 0.5, x_1 + x_2 \le 1.5\big\}$ (cf. (5)). Suppose that $\bar{x}^\varepsilon = (\bar{y}^\varepsilon, \bar{u}^\varepsilon, \bar{w}^\varepsilon)$ solves $(\mathbf{P}^\varepsilon)$ computed with the data y_{out} and $\boldsymbol{v}$. If we now solve the state equation utilizing the control $u = \bar{u}^\varepsilon$ and the new data $\tilde{y}_{\mathsf{out}}$, $\tilde{\boldsymbol{v}}$ for $t \in [\tilde{t}, T]$, the value of the cost J is 9.86. On the other hand, the value of the cost J utilizing the MPC control $\bar{u}^N$ is 8.797, i.e., significantly smaller. Moreover, if we solve the optimal control problem with PDASS taking in account these changes at time $\tilde{t}$, the optimal cost functional is 8.63. This illustrates that MPC takes care of changing data in the problem.

In Table 1, the results for the MPC algorithm in this scenario are shown. First of all, we have to mention that $\varepsilon w = (y_{\mathsf{a}} - y)\,\chi_{\mathcal{A}_{\mathsf{a}}^{\mathcal{H}}(x)} + (y_{\mathsf{b}} - y)\,\chi_{\mathcal{A}_{\mathsf{b}}^{\mathcal{H}}(x)}$,

Table 1. Results of the MPC algorithm with FE and POD.

Spatial Discretization	Rank ℓ	$J(x)$	$\Vert\varepsilon w\Vert_{\mathcal{H}}$	Speed-up
MPC-FE	–	8.797	0.0182	–
MPC-POD, Method 1	8	8.888	0.0180	3.59
MPC-POD, Method 1	12	8.801	0.0183	3.63
MPC-POD, Method 1	16	8.799	0.0182	3.32
MPC-POD, Method 2	8	8.928	0.0192	2.49
MPC-POD, Method 2	12	8.800	0.0182	2.69
MPC-POD, Method 2	16	8.798	0.0182	2.62

where $\mathcal{A}_{\mathsf{a}}^{\mathcal{H}}(x)$ and $\mathcal{A}_{\mathsf{b}}^{\mathcal{H}}(x)$ are the active sets for state constraints; [10]. Thus, $\|\varepsilon w\|_{\mathcal{H}}$ can be used to measure how much the state constraints are violated. As one can see, $\|\varepsilon w\|_{\mathcal{H}}$ has on order of magnitude of 10^{-2}, which coincides with the error Δt occuring from the temporal discretization. Regarding the MPC-POD, Methods 1 and 2 produce a good approximation of the MPC-FE system, with a reasonable speed-up. Since we evalute the a-posteriori error estimate in each iteration of the MPC algorithm, Method 2 is slower than Method 1. Recall that the computation of the error estimate requires FE state and adjoint solves [10,13]. For the a-posteriori error estimator we choose the

Table 2. Relative errors between the MPC FE and its POD approximation.

Spatial Discretization	Rank ℓ	rel-err(y)	rel-err(u)	Basis updates
MPC-POD, Method 1	8	0.02304	0.07415	–
MPC-POD, Method 1	12	0.00252	0.01815	–
MPC-POD, Method 1	16	0.00248	0.01771	–
MPC-POD, Method 2	8	0.00290	0.02264	6
MPC-POD, Method 2	12	0.00208	0.01622	3
MPC-POD, Method 2	16	0.00193	0.01540	1

tolerance $0.3\|u^{\mathrm{POD}}\|$. Let us define the quantities

$$\begin{aligned}
\|u^{\mathrm{POD}}\|^2 &= \sum_{i=1}^{m} \|u_i^{\mathrm{POD}}\|^2_{L^2(0,T)}.\\
\text{rel-err}(y) &= \|y^{\mathrm{FE}} - y^{\mathrm{POD}}\|_{\mathcal{H}} / \|y^{\mathrm{FE}}\|_{\mathcal{H}}.\\
\text{rel-err}(u)^2 &= \sum_{i=1}^{m} \|u_i^{\mathrm{FE}} - u_i^{\mathrm{POD}}\|^2_{L^2(0,T)} \Big/ \sum_{i=1}^{m} \|u_i^{\mathrm{FE}}\|^2_{L^2(0,T)}.
\end{aligned}$$

Method 2 does what we expected: improving the POD Method 1 approximation of the FE optimal state and control.

5 Conclusion

To conclude, we have compared a MPC algorithm based on the PDASS and the PDASS strategy itself: the PDASS alone can not take in account parameters' changes, unless we are able to predict them in advance, and it is well known that can result costly for a long-time horizon problem. On the other hand, we can improve the PDASS combining it with MPC, that can react on these changes in a reasonable time and produce sub-optimal results closed to the optimal ones. It can also treat long-time horizon problems, due to the structure of the algorithm itself that splits the problem in several problems on a shorter time horizon, having still convergence to the optimal solution, see [5]. We have successfully applied a POD approximation and we were able to improve the results refreshing the POD basis, according to a-posteriori error estimator presented in [10].

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