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Reduced-Order Greedy Controllability of Finite Dimensional Linear Systems

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Abstract: Often a dynamical system is characterized by one or more parameters describing physical features of the problem or geometrical configurations of the computational domain. As a consequence, by assuming that the system is controllable, a range of optimal controls exists corresponding to different parameter values. The goal of the proposed approach is to avoid the computation of a control function for any instance of the parameters. The greedy controllability consists in the selection of the most representative values of the parameter set that allows a rapid approximation of the control function for any desired new parameter value, ensuring that the system is steered to the target within a certain accuracy. By proposing the reduced basis (RB) method in this framework, we are able to consider linear parametrized partial differential equations (PDEs) in our setting. The computational costs are drastically reduced and the efficiency of the greedy controllability approach is significantly improved. As a numerical example a heat equation with convection is studied to illustrate our proposed RB greedy controllability strategy.

Keywords: Optimal control, parametrized linear systems, controllability, greedy algorithm, reduced basis method, parametrized parabolic equations.

1. INTRODUCTION

In this paper we consider the parametrized, time-invariant, finite-dimensional system

$$\begin{aligned} \dot{x}_\nu(t) &= A_\nu x_\nu(t) + Bu(t) \quad \text{for } t \in (0, T], \\ x_\nu(0) &= x_o \end{aligned} \quad (1)$$

for a terminal time $T > 0$ and a parameter (vector) ν which belongs to a compact subset \mathcal{K} of \mathbb{R}^p . We call $x_\nu(t) \in \mathbb{R}^n$ the *state vector* for the given parameter ν . The parameter-dependent matrix $A_\nu \in \mathbb{R}^{n \times n}$ describes the free dynamics and $u(t)$ is the *control vector* in \mathbb{R}^m entering and acting on the system through the control operator $B \in \mathbb{R}^{n \times m}$. We suppose that the linear system (1) is controllable and that $m \leq n$ holds. Throughout the matrix A_ν is assumed to be Lipschitz-continuous with respect to the parameter ν .

In this paper we consider the following problem: suppose that an arbitrary initial state x_o , a terminal state $x_1 \in \mathbb{R}^n$, a parameter $\nu \in \mathcal{K}$ and a tolerance $\varepsilon > 0$ are given. Then, we are interested in a rapid computation of a control strategy $u_\nu(t)$ such that the solution x_ν of (1) for the control $u = u_\nu$ satisfies the inequality

$$\|x_\nu(T) - x_1\|_2 \leq \varepsilon$$

with the Euclidean norm $\|\cdot\|_2$. To attain this goal we apply a greedy approach in an offline phase in order to determine

appropriate parameter values $\{\nu_i\}_{i=1}^N \subset \mathcal{K}$ and associated control functions $\{u_{\nu_i}\}_{i=1}^N$ which ensure approximate controllability for an arbitrarily given parameter $\nu \in \mathcal{K}$. In this greedy approach we have to evaluate (1) and an associated dual system, which we specify later. To speed-up the computational time we apply a tailored reduced basis method.

This work is an extension of the method proposed in Lazar and Zuazua [2016]. We successfully combine their approach with the reduced basis method (see Patera and Rozza [2006] and Hesthaven et al. [2016], Quarteroni et al. [2016], for instance) to reduce the computational cost and to obtain a significant speed-up in the numerical performance. For the greedy algorithm we refer, e.g., to Cohen and DeVore [2016] for a general theory and to Buffa et al. [2012] for the reduced basis method. For parametrized model reduction we refer to Baur et al. [2011], for instance, and to the recent survey article Benner et al. [2015].

The paper is organized as follows. In Section 2 our problem is formulated mathematically. Moreover, we recall the classical approach and explain the greedy controllability approach utilized in Lazar and Zuazua [2016]. In Section 3 the reduced basis greedy controllability approach is explained. Further, we propose our numerical strategy, which is applied in Section 4 to a parametrized heat equation with convection. Finally, a conclusion is drawn in Section 5.

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2. PROBLEM FORMULATION

2.1 Motivating PDE example

We consider the following parametrized heat equation governed by the multi-parameter $\nu = (\kappa_1, \kappa)$ with $\kappa = (\kappa_2, \kappa_3)$ living in a compact subset $\mathcal{K} \subset \mathbb{R}^3$:

$$\begin{aligned} y_t(t, \mathbf{x}) + \kappa \cdot \nabla y(t, \mathbf{x}) &= \kappa_1 \Delta y(t, \mathbf{x}) \quad \text{a.e. in } Q, \\ \kappa_1 \frac{\partial y}{\partial \mathbf{n}}(t, \mathbf{s}) &= \sum_{i=1}^m u_i(t) b_i(\mathbf{s}) \quad \text{a.e. on } \Sigma, \quad (2) \\ y(0, \mathbf{x}) &= y_o(\mathbf{x}) \quad \text{a.e. in } \Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$ is bounded domain with Lipschitz-continuous boundary $\Gamma = \partial\Omega$, $T > 0$, $\Sigma = (0, T) \times \Gamma$ and $b_i : \Gamma \rightarrow \mathbb{R}$, $1 \leq i \leq m$, denote given control shape functions (e.g., characteristic functions):

$$\Gamma = \bigcup_{i=1}^m \Gamma^i, \quad b_i(\mathbf{s}) = \chi_{\Gamma^i}(\mathbf{s}), \quad 1 \leq i \leq m, \quad \|b_i\|_{L^2(\Gamma)}^2 = |\Gamma^i|.$$

We are interested in weak solutions to (2). Let V stands for the Sobolev space $H^1(\Omega)$. For more details on Lebesgue and Sobolev spaces we refer to Evans [2008], for instance. For every $\phi, \tilde{\phi} \in V$ we define the parametrized bilinear form

$$a_\nu(\phi, \tilde{\phi}) = \int_{\Omega} \kappa_1 \nabla \phi(\mathbf{x}) \cdot \nabla \tilde{\phi}(\mathbf{x}) + (\kappa \cdot \nabla \phi(\mathbf{x})) \tilde{\phi}(\mathbf{x}) \, d\mathbf{x}$$

and the linear form

$$\ell_j(\phi) = \int_{\Gamma_j} b_j(\mathbf{x}) \phi(x) \, d\mathbf{x}, \quad j = 1, \dots, m.$$

We use a finite element (FE) space of piecewise linear basis functions $V^n = \text{span}\{\phi_i : i = 1, \dots, n\} \subset V$ for the spatial approximation of a weak solution to (2), we lead to the finite dimensional parametrized linear system:

$$\begin{cases} M \dot{\mathbf{x}}_\nu(t) = \tilde{A}_\nu \mathbf{x}_\nu(t) + \tilde{B} u(t) & \text{for } t \in (0, T], \\ \mathbf{x}_\nu(0) = \mathbf{x}_o \end{cases} \quad (3)$$

with

$$\begin{aligned} \tilde{A}_\nu &= ((a_\nu(\phi_j, \phi_i))_{1 \leq i, j \leq n}), \quad \tilde{B} = ((\ell_j(\phi_i))_{1 \leq i \leq n, 1 \leq j \leq m}), \\ \mathbf{x}_o &= (y_{oi}^n)_{1 \leq i \leq n}, \quad \mathbf{x}_\nu(t) = (y_{\nu i}^n(t))_{1 \leq i \leq n} \end{aligned}$$

for $t \in [0, T]$ and $\nu \in \mathcal{K}$. Suppose that \mathbf{x}_ν solves (3). Then, the FE solution of (2) is given in Galerkin form as

$$y_\nu^n(t, \mathbf{x}) = \sum_{i=1}^n y_{\nu i}^n(t) \phi_i(\mathbf{x}) \quad \text{for } t \in [0, T].$$

Notice that by setting $A_\nu = M^{-1} \tilde{A}_\nu$ and $B = M^{-1} \tilde{B}$, problem (3) can be expressed as (1). In particular, the matrix A_ν depends Lipschitz-continuously on the parameter $\nu \in \mathcal{K}$.

2.2 Approximate controllability

Recall that system (1) is said to be *controllable* when every initial state $x_o \in \mathbb{R}^n$ can be driven to any terminal state $x_1 \in \mathbb{R}^n$ in time $T > 0$ by choosing an appropriate control strategy u_ν . This controllability property can be characterized by a necessary and sufficient condition, the

so-called *Kalman condition*: System (1) is controllable if and only if:

$$\text{rank}[B, A_\nu B, \dots, A_\nu^{n-1} B] = n.$$

When this rank condition is fulfilled, the system is controllable for all $T > 0$. In the following we assume that (1) is controllable for all values of ν (cf. Zuazua [2014]).

In this work we want to compute the control u_ν for an arbitrarily given parameter $\nu \in \mathcal{K}$ very rapidly. Therefore, we are not interested in finding a control such that associated solution x_ν to (1) satisfies exactly $x_\nu(T) = x_1$. This is also motivated by our example presented in Section 2.1, where we only have an approximate FE solution for (2) at hand. Thus, we want to determine a control u_ν that ensures approximate controllability up to a given tolerance $\varepsilon > 0$.

Goal: Suppose that an initial state $x_o \in \mathbb{R}^n$, a terminal state $x_1 \in \mathbb{R}^n$, a parameter $\nu \in \mathcal{K}$ and a tolerance $\varepsilon > 0$ are given. Then, we want to determine a control $u_\nu(t)$ such that the solution x_ν of (1) satisfies the *approximate controllability condition*:

$$|x_\nu(T) - x_1|_2 \leq \varepsilon.$$

2.3 Solution of the controllability problem

In this section we will present two different methods: the first is the classical approach (see Micu and Zuazua [2005] for further details), while the second is a greedy approach proposed by Lazar and Zuazua [2016].

Classical approach. For $\varphi^\circ \in \mathbb{R}$ we define the objective

$$J_\nu(\varphi^\circ) = \frac{1}{2} \int_0^T |B^\top \varphi_\nu(t)|_2^2 dt + x_1^\top \varphi^\circ + x_o^\top \varphi_\nu(0).$$

Then, we consider the quadratic problem

$$\min J_\nu(\varphi^\circ) \quad \text{subject to } \varphi^\circ \in \mathbb{R}^n \quad (4)$$

where

$$\begin{cases} -\dot{\varphi}_\nu(t) = A_\nu^\top \varphi_\nu(t) & \text{for } t \in [0, T), \\ \varphi_\nu(T) = \varphi^\circ. \end{cases} \quad (5)$$

Suppose that φ_ν° solves (4) for given $\nu \in \mathcal{K}$ and that φ_ν denotes the associated solution to (5) taking $\varphi^\circ = \varphi_\nu^\circ$. Then,

$$u_\nu(t) = B^\top \varphi_\nu(t), \quad t \in [0, T], \quad (6)$$

is the control of minimal $[L^2(0, T)]^m$ -norm that steers the solution of (1) to x_1 . Let Λ_ν be the quadratic form, known as Gramian, associated to the pair (A_ν, B) , *i.e.*

$$\langle \Lambda_\nu \varphi^\circ, \xi^\circ \rangle = \int_0^T \langle B^\top \varphi, B^\top \xi \rangle dt,$$

where φ, ξ are the solution of (5) with initial data φ° and ξ° , respectively. The matrix corresponding to the operator, which will be denoted the same, is given by the relation:

$$\Lambda_\nu = \int_0^T e^{(T-t)A_\nu} B B^\top e^{(T-t)A_\nu^\top} dt.$$

The minimiser φ_ν° can be equivalently determined as the solution to the system

$$\Lambda_\nu \varphi_\nu^\circ = x^1 - e^{TA_\nu} x_o$$

where Λ_ν is the Gramian associated to (A_ν, B) and the left hand side, up to the free dynamics component, represents the solution of the control system (1) with the control given by (6).

The numerical solution of (4) can be computed by using the conjugate gradient (CG) method. However, for each new parameter value ν we have to perform the minimization procedure in order to get the control which steers the solution to the target.

Greedy controllability. The main idea is to diminish the computational cost required by the classical approach by reducing the control space dimension. In particular, the most distinguished realizations of the parameter are identified and the corresponding set of control solutions are computed. This low dimensional set represents the best possible approximation of the whole space of controls. More precisely, given an error $\varepsilon > 0$ the goal is to find the minimum number of parameter values ν_i , $i = 1, \dots, N(\varepsilon)$, whose associated solutions $\varphi_i^\circ = \varphi_{\nu_i}^\circ$ to (4) and the controls $u_i = u_{\nu_i}$ given by (6), approximate all admissible controls with an error smaller than ε . This can be done in two phases: the *offline* and the *online* ones. In the offline phase the $N = N(\varepsilon)$ parameters ν_1, \dots, ν_N and the associated control solutions are computed with the classical approach, which is often computationally expensive. However, these calculations are performed only once. In the online phase, the computed control solutions can be utilized to easily construct approximations of the control solutions for any possible desired parameter.

2.4 Offline phase

During the offline phase we explore the parameter domain \mathcal{K} by considering a fine discrete (training) set $\mathcal{K}_{\text{tr}} \subset \mathcal{K}$ and we select with a greedy approach a set of the most relevant parameter values $\mathcal{K}_{\text{gr}} = \{\nu_1, \dots, \nu_N\} \in \mathcal{K}_{\text{tr}}$, $N = N(\varepsilon)$. Then, we compute the corresponding minimizers $\varphi_i^\circ = \varphi_{\nu_i}^\circ$ (solution of (4)), the adjoint solutions $\varphi_i = \varphi_{\nu_i}$ (solutions of (5)) and the controls $u_i = u_{\nu_i}$ by (6) for $i = 1, \dots, N$. For a subset $\Phi^\circ \subset \mathbb{R}^n$ and for $\varphi^\circ \in \mathbb{R}^n$ let us define the distance

$$\text{dist}(\varphi^\circ, \Phi^\circ) = \inf \{|\tilde{\varphi}^\circ - \varphi^\circ|_2 : \tilde{\varphi}^\circ \in \Phi^\circ\}.$$

Algorithm 1 (Greedy method)

- 1: **Require:** training set \mathcal{K}_{tr} ; tolerance $\varepsilon > 0$
 - 2: Select $\nu_1 \in \mathcal{K}_{\text{tr}}$;
 - 3: Find $\varphi_1^\circ, \varphi_1, u_1$ from (4), (5), (6), respectively;
 - 4: Set $\Phi_1^\circ = \text{span}\{\varphi_1^\circ\} \subset \mathbb{R}^n$ and $i = 2$;
 - 5: **while** $\max_{\nu \in \mathcal{K}_{\text{tr}}} \text{dist}(\varphi_\nu^\circ, \Phi_i^\circ) > \varepsilon$;
 - 6: Compute $\nu_i = \text{argmax}_{\nu \in \mathcal{K}_{\text{tr}}} \text{dist}(\varphi_\nu^\circ, \Phi_i^\circ)$;
 - 7: Find $\varphi_i^\circ, \varphi_i, u_i$ from (4), (5), (6), respectively;
 - 8: Set $\Phi_i^\circ = \text{span}\{\varphi_1^\circ, \dots, \varphi_i^\circ\} \subset \mathbb{R}^n$ and $i = i + 1$;
 - 9: **end**;
-

We observe that in Steps 5 and 6 of Algorithm 1 the computation of $\text{dist}(\varphi_\nu^\circ, \Phi_i^\circ)$ requires φ_i° (which have been computed in Steps 3 and 7) and the computation of the minimizers φ_ν° for all $\nu \in \mathcal{K}_{\text{tr}}$ which is computationally unfeasible. Hence, a surrogate is utilized to approximate $\text{dist}(\varphi_\nu^\circ, \Phi_i^\circ)$; cf. [Lazar and Zuazua, 2016]. Let $\nu_1 \in \mathcal{K}_{\text{tr}}$ be chosen arbitrarily. We compute the associated $\varphi_1^\circ, \varphi_1, u_1$ from (4), (5), (6), respectively. Then, we choose any $\nu \in \mathcal{K}_{\text{tr}} \setminus \{\nu_1\}$. The surrogate distance is given by $|\bar{x}_\nu(T) - x_1(T)|_2$, where

- $x_1(T)$ is the solution of (1) at time T choosing the control $u = u_1$ and the parameter $\nu = \nu_1$,

- $\bar{x}_\nu(T)$ is the solution of (1) at time T choosing the control $u = u_1$ and the parameter $\nu \neq \nu_1$.

Since the control u_1 is optimal for $\nu = \nu_1$, we have $x_1(T) = x_1$. On the other hand, u_1 is not optimal for $\nu \neq \nu_1$. Thus, $\bar{x}_\nu(T)$ is a perturbation of x_1 (see Fig. 1). However, the value of $\bar{x}_\nu(T)$ is ob-

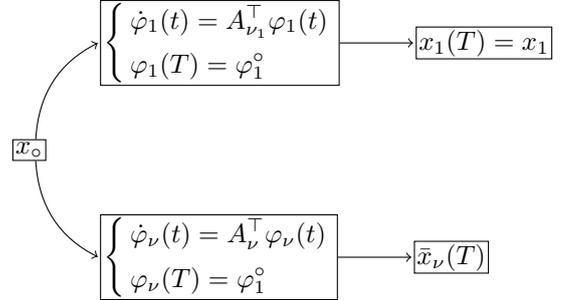


Fig. 1. The surrogate of the distance $\text{dist}(\varphi_\nu^\circ, \varphi_1^\circ)$.

tained only by solving of (5) and (1), whereas we avoid the solution of the minimization problem (4). We note that $\text{dist}(\varphi_\nu^\circ, \Phi_N^\circ) \sim \text{dist}(\Lambda_\nu \varphi_\nu^\circ, \Lambda_\nu \Phi_N^\circ) = \text{dist}(x^1 - e^{T A_\nu} x_0, \Lambda_\nu \Phi_N^\circ)$, where $\Lambda_\nu \Phi_N^\circ = \text{span}\{\Lambda_\nu \varphi_1^\circ, \dots, \Lambda_\nu \varphi_N^\circ\}$. So the surrogate $\text{dist}(x^1 - e^{T A_\nu} x_0, \Lambda_\nu \Phi_N^\circ)$ can be easily computed projecting the vector $x^1 - e^{T A_\nu} x_0$ to a linear space $\Lambda_\nu \Phi_N^\circ$, whose basis is obtained by solving (1) and (5) N times.

2.5 Online phase

Once the approximate space Φ_N° is defined offline, we can compute rapidly for an arbitrary given value $\nu \in \mathcal{K}$, the approximate control $u_\nu^*(t) = B^\top \varphi_\nu^*(t)$, where

$$\varphi_\nu^{\circ,*} = \sum_{i=1}^N c_i \varphi_i^\circ \in \text{span}\{\varphi_1^\circ, \dots, \varphi_N^\circ\}$$

such that the solution of (1) with the control u_ν^* satisfies $|x_\nu^*(T) - x_1| \leq \varepsilon$. We note that the coefficients c_i are obtained solving the system

$$P_n \Lambda_\nu \varphi_\nu^\circ = c_i \Lambda_\nu \varphi_i^\circ, \quad i = 1, \dots, N$$

where $P_n \Lambda_\nu \varphi_\nu^\circ$ denotes the projection of $\Lambda_\nu \varphi_\nu^\circ$ to $\Lambda_\nu \Phi_N^\circ$

3. REDUCED BASIS GREEDY CONTROLLABILITY

The exploration of the parameter domain in the offline phase requires repetitive evaluations of the state and adjoint systems (1) and (5) which can be very costly. In order to speed up the offline (and consequently the online phase), we propose to reduce the complexity of the state and the adjoint equations. The Reduced Basis (RB) method (Hesthaven et al. [2016]) allows to replace the high dimensional problems (1) and (5) with new small dimensional ones. The first step consists in building a good set of basis functions for the state and for the adjoint equations.

Reduced basis scheme for the state equation: Find a basis $\{\psi_i\}_{i=1}^\ell$ of rank $\ell \ll n$ and define $V^\ell = \text{span}\{\psi_1, \dots, \psi_\ell\} \subset V^n$. Then, set $\Psi = [\psi_1, \dots, \psi_\ell] \in$

$\mathbb{R}^{n \times \ell}$, $A_\nu^\ell = \Psi^\top A_\nu \Psi \in \mathbb{R}^{\ell \times \ell}$, $B^\ell = \Psi^\top B \in \mathbb{R}^{\ell \times m}$ and $x_\circ^\ell = \Psi^\top x_\circ \in \mathbb{R}^\ell$. Now, the reduced-order approximation $x^\ell(t) \in V^\ell$ for $t \in [0, T]$ is governed by

$$\begin{aligned} \dot{x}_\nu^\ell(t) &= A_\nu^\ell x^\ell(t) + B^\ell u(t) \quad \text{for } t \in (0, T], \\ x_\nu^\ell(0) &= x_\circ^\ell. \end{aligned} \quad (7)$$

The space V^ℓ for being a good approximation of V^n has to be large enough to approximate the solution of (1) for any possible choice of the parameter ν and any possible control needed. In order to build a space V^ℓ which takes into account both variations (controls and parameters), we first define a distributed set of parameter values $\{\nu_1, \nu_2, \dots, \nu_L\}$ and an index set $\mathcal{J} = \{1, \dots, L\}$. Then, we compute the corresponding minimizers φ_i° , $i \in \mathcal{J}$, by solving (4) for the parameter value $\nu = \nu_i$. Then, we compute $\varphi_i = \varphi_{\nu_i}$, $i \in \mathcal{J}$, from (5) by choosing $\varphi^\circ = \varphi_i^\circ$. Finally, we obtain the controls $u_i = u_{\nu_i}$, $i \in \mathcal{J}$, from (6) by taking $\varphi_\nu = \varphi_i$. In this way we have collected a certain number of possible optimal controls.

Then, we consider a discretization of the space \mathcal{K} defining a set $\mathcal{K}_{\text{tr}} \subset \mathcal{K}$ with $\dim(\mathcal{K}_{\text{tr}}) \gg L$ and we define an artificial parameter $\mu = (\nu, i)$ with $\nu \in \mathcal{K}_{\text{tr}}$, $i \in \mathcal{J}$ which refers to different combinations of parameter values and controls, $\mu \in \mathcal{D} = \mathcal{K}_{\text{tr}} \times \mathcal{J}$. As usually done in the offline phase of the RB method, through a POD-greedy algorithm (Haasdonk and Ohlberger [2008]), a small set of parameter values $\mu_j \in \mathcal{D}$ is determined and only for these values the problem (1) is solved. These solutions are utilized as snapshots in order to get basis functions $\{\psi_i\}_{i=1}^\ell$ in \mathbb{R}^n by applying the POD method.

Reduced basis scheme for the adjoint equation: Find a basis $\{\xi_i\}_{i=1}^\varphi$ of rank $\varphi \ll n$ and define $V^\varphi = \text{span}\{\xi_1, \dots, \xi_\varphi\} \subset V^n$. Then, set $\Xi = [\xi_1, \dots, \xi_\varphi] \in \mathbb{R}^{n \times \varphi}$, $A_\nu^{\varphi, \top} = \Xi^\top A_\nu^\top \Xi \in \mathbb{R}^{\varphi \times \varphi}$, $B^\varphi = \Xi^\top B \in \mathbb{R}^{\varphi \times m}$ and $\varphi^{\circ, \varphi} = \Xi^\top \varphi^\circ \in \mathbb{R}^\varphi$. Now, the reduced-order approximation $\varphi^\varphi(t) \in V^\varphi$ for $t \in [0, T]$ solves

$$\begin{aligned} -\dot{\varphi}_\nu^\varphi(t) &= A_\nu^{\varphi, \top} \varphi_\nu^\varphi(t) \quad \text{for } t \in [0, T], \\ \varphi_\nu^\varphi(T) &= \varphi^{\circ, \varphi}. \end{aligned} \quad (8)$$

The reduced basis computation for the adjoint problem is the same described for the state equation. In this case the parameter $\mu = (\nu, i)$, $\nu \in \mathcal{K}_{\text{tr}}$, $i \in \mathcal{J}$, refers to different combinations of parameter values and minimizers φ_i° . Again applying the classical RB POD-greedy algorithm we select φ parameter values that define the reduced basis solutions of (5) for $\mu_1, \dots, \mu_\varphi$.

Solution strategy In our algorithm we perform two offline phases. During the first one we build the basis for the state and for the adjoint equation, while in the second one we perform the same steps described in Section 2. The advantage here is that in the greedy algorithm instead of using (1) and (5), we use the reduced systems (7) and (8) which are already computed in the first offline phase. In our numerical example we observe that we gain a huge speed-up. Moreover, we propose to use a *reduced-order surrogate* of the form $|\Psi x_\nu^{\ell, \star}(T) - x_1|$, where again the reduced-order systems are utilized to get $\Psi x_\nu^{\ell, \star}(T) \approx x_\nu^\star(T)$.

As already mentioned the advantage is that the exploration of the parameter domain requires now repetitive

evaluations of reduced adjoint and reduced state systems. During the online phase, which is the same described in Section 2, we use again the reduced system obtaining a speed up also in this phase.

4. NUMERICAL EXAMPLE

We consider (2), where we fix $\kappa = 0.1 \cdot (1, 1)$ and consider $\nu = \kappa_1$ as the varying parameter of the system. The set \mathcal{K} is given as the bounded interval $[2, 4]$. We choose $T = 1$ and $\Omega = (0, 2) \times (0, 1) \subset \mathbb{R}^2$. The desired state is $y_1(\mathbf{x}) = 10$ and the initial condition is $y_\circ(\mathbf{x}) = -(x_1 - 1)^2$, where x_1 is the first component of \mathbf{x} . Furthermore for $m = 1$ the shape function $b_1(x)$ has the support in the whole boundary domain $\partial\Omega$. The high fidelity spatial approximations of the problem solution, used for the basis computations and the error comparisons is computed by a finite element (FE) model that uses piecewise linear elements with $n = 685$ degrees of freedom. For the temporal discretization the Crank-Nicolson method is applied with equidistant step size $\Delta t = 0.01$ steps. To solve (4) we use the CG method. To highlight the advantage of the proposed algorithm we compare the results in terms of the error (difference between the final state and the desired state y_1) and the CPU time applying three different approaches: (1) the classical approach (see Section 2.1), (2) the greedy controllability method (Lazar and Zuazua [2016]) denoted here as FE greedy controllability and the proposed RB greedy controllability method. Table 4 shows the results with the classical optimization approach for some selected parameter values.

Table 1. Results with the classical optimization approach with eight chosen parameters

$\nu = \kappa_1$	2.00	2.44	2.67	2.89
CG iterations	32	34	28	29
CPU time	37 s	38 s	32 s	32 s
$ x_\nu(T) - x_1 _2$	8.6e-3	2.8e-3	9.6e-3	7.4e-3
$\nu = \kappa_1$	3.11	3.56	3.78	4.00
CG iterations	28	24	20	20
CPU time	32 s	26 s	22 s	22 s
$ x_\nu(T) - x_1 _2$	8.1e-3	6.9e-3	7.7e-3	6.5e-3

For the greedy approach we select a set \mathcal{K}_{gr} of dimension $N = 17$ over a set \mathcal{K}_{tr} of dimension 1000. The needed CPU time for the offline part is about 80 minutes. However, in Figure 2 we can observe a huge speed up in the online phase. The speed-up factor is around 4 if we compare the classical approach and the FE greedy controllability. Figure 3 shows the decay of the error between the final state and the target x_1 when vary N . We observe that we have accurate results for $N = 17$ (the average error is $\simeq 10^{-3}$). Figure 4 compares the decay of the error when we increase the dimension of \mathcal{K}_{gr} ($N = 1, \dots, 17$) applying the FE and the RB greedy controllability method during the offline part over a range of 1000 parameters values. For both methods we observe that for $N = 17$ we have a good approximation of the desired state y_1 . The error is $\simeq 10^{-3}$. Regarding the CPU time for the offline part we have a big speed up using the RB greedy approach: we can observe a reduction of 75% (80 minutes for the FE greedy and 20 minutes for the RB greedy).

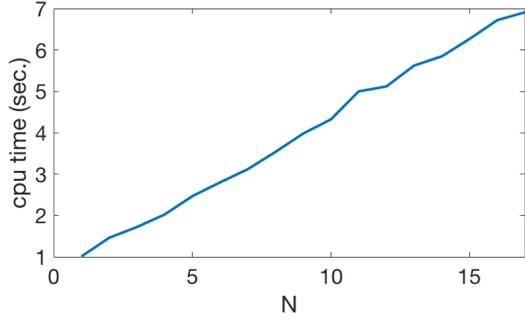


Fig. 2. Online phase of greedy controllability approximation: CPU time increasing the dimension of \mathcal{K}_{gr} .

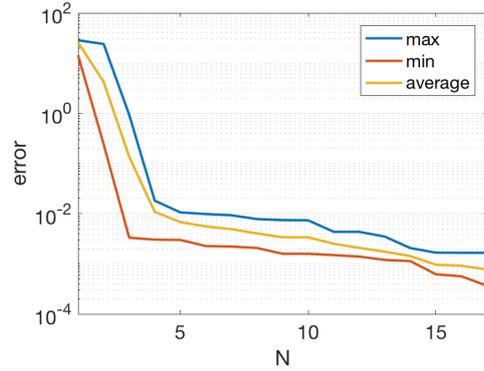


Fig. 3. Online part: Error between the final state and the desired state y_1 using the FE greedy controllability approach by varying the dimension of \mathcal{K}_{gr} : minimum, maximum and average over a set of 20 random parameters values.

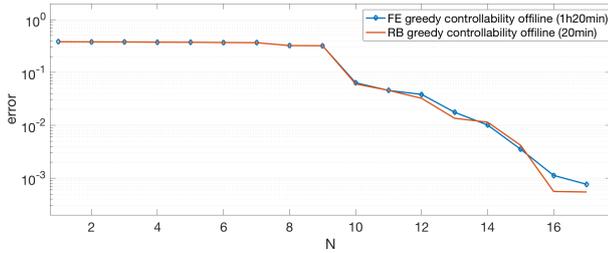


Fig. 4. Offline part: Error between the final state and the desired state y_1 using the FE greedy controllability approach and the RB greedy method during the definition of \mathcal{K}_{gr} among a set of 1000 parameters values.

Figure 5 shows a similar comparison in the online part, here we computed the solution of our problem by selecting 100 random parameter values online and plotting the corresponding averaged error for both RB and FE greedy approach. We observe that for $N = 17$ we have a satisfying approximation of the desired state x_1 comparable with the ones obtained with the classical approach (Table 4). Regarding the CPU time, as mentioned before, using the proposed algorithm we have also an online speed up of factor 3, *cf.* Figure 6, which means that we further speed up the classical approach by a factor of around 8.

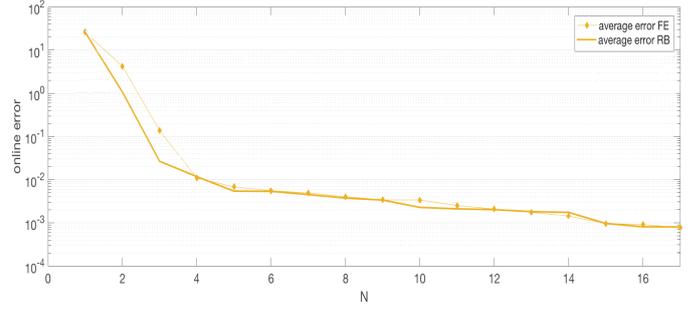


Fig. 5. Online phase: Error between the final state and the desired state y_1 using the FE greedy controllability approach and the RB greedy method by varying the dimension of \mathcal{K}_{gr} : average values over a set of 20 random parameters values.

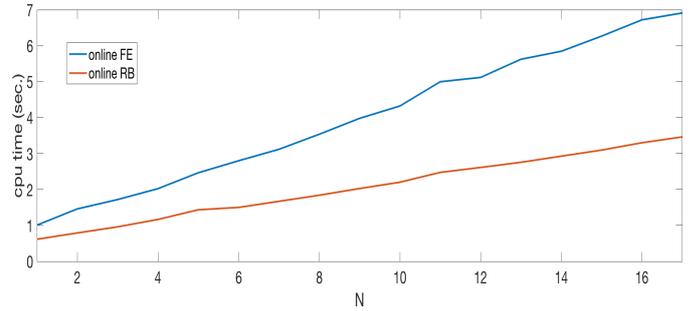


Fig. 6. Online part. CPU time increasing the dimension of \mathcal{K}_{gr} and using the FE greedy controllability approach and the RB greedy method.

5. CONCLUSION

The proposed Reduced Greedy controllability approach allows to solve controllability problems for parameter dependent dynamical systems, the problem is not solved entirely by the RB method, but the latter is used only locally into a greedy controllability technique. Compared with the already existing greedy controllability approach the new reduced one further reduces the computational time by maintaining the same solution accuracy. Future developments will involve the implementation of rigorous error bounds for the RB offline computations involving the state and adjoint system. Moreover we want to consider test cases with more parameters (*e.g.*, varying the parameter κ in front of the advection term), larger number of controls acting in the system and more complex configurations.

REFERENCES

- Interpolatory projection methods for parameterized model reduction. *SIAM J. Sci. Comput.*, 33(5), 2489–2518.
- Benner, P., Gugercin, S. and Willcox, K. (2015). A survey of projection-based model reduction methods for parametric dynamical systems. *SIAM Review*, 57(4), 483–531.
- Buffa, A., Maday, Y., Patera, A.T., Prud'homme, C. and Turinici, G. (2012). A priori convergence of the greedy algorithm for the parameterized reduced basis. *Math. Model. Numer. Anal.*, 46, 595–603.
- Cohen, A. and DeVore, R. (2016). Kolmogorov widths under holomorphic mappings. *IMA J. Numer. Anal.* 36, 1.

- Evans, L.C. (2008). *Partial Differential Equations*. American Mathematical Society, Providence, Rhode Island.
- Haasdonk, B. and Ohlberger, M. (2008). Reduced basis method for finite volume approximations of parametrized linear evolution equations. *M2AN Math. Model. Numer. Anal.*, 42(2), 277–302.
- Hesthaven, J.S., Rozza, G. and Stamm, B. (2016). *Certified Reduced Basis Methods for Parametrized Partial Differential Equations*. Springer Briefs in Mathematics.
- Lazar, M. and Zuazua, E. (2016). Greedy controllability of finite dimensional linear systems. *Automatica*.
- Lions, J.L. (1971). *Optimal Control of Systems Governed by Partial Differential Equations*. Springer.
- A. T. Patera, A.T. and G. Rozza, G. (2006). *Reduced Basis Approximation and A Posteriori Error Estimation for Parametrized Partial Differential Equations*. ©MIT.
- Quarteroni, A., Manoni, A., and Negri, F. (2016) *Reduced Basis Methods for Partial Differential Equations*. Springer.
- Micu, S. and Zuazua, E. (2005). An introduction to the controllability of linear PDE. *Contrôle non linéaire et applications, Collection Travaux en Cours Hermann*, 67–150.
- Zuazua, E. (2014). Averaged control. *Automatica*.