I. INTRODUCTION

Recent experiments investigated superconducting junctions containing atomic contacts or semiconductor nanowires with the objective to realize a novel type of versatile superconducting junction beyond the standard Josephson tunnel junctions. Superconducting atomic contacts (SAC) are the simplest example of a short junction hosting doublets of localized Andreev bound states (ABS) that carry the supercurrent in the junction. During the last years, a new class of experiments showed the possibility of driving transitions between these ABS formed in the SAC. Most importantly, such an Andreev spectroscopy allows for the detection of the occupation of the ABS. In semiconductor nanowires combining hybrid properties as strong spin-orbit interaction and superconducting proximity effect, Andreev bound states corresponding to Majorana modes are expected to emerge when the system is driven in a topological range of parameters. For a review about Majoranas we refer to [23–28]. Experiments reported so far confirmed several theoretical predictions, as the zero-bias conductance peak or the fractional ac-Josephson effect. Other experiments in topological junctions also confirmed characteristic features of highly transmitting conductance channels which are compatible with the theoretically predicted topological properties. For instance, the edge supercurrent associated to the helical edge states was observed in two-dimensional HgTe/HgCdTe quantum wells and evidence of the non-sinusoidal phase-supercurrent relation was also reported in other work.

SAC or nanowires are promising for realizing a new qubit architecture in which the information is encoded by microscopic degrees of freedom, i.e. the ABS, rather than the macroscopic BCS condensate, as in conventional superconducting qubits. Additionally, for nanowires, the ballistic regime is now within the reach of the experimental devices and the spectroscopic measurement has been now accomplished using the same method employed in SAC. Andreev spectroscopy, based on employing the microwave signal, represents a fundamental tool not only for spectroscopy and characterization but it represents a crucial issue towards the coherent control of the Andreev qubits. A first experiment has already reported coherent quantum manipulation of ABS in superconducting atomic contact. Moreover, junctions formed on ballistic nanowires have the adjoint value of being gate-tunable, an intriguing property that was used to devise a new superconducting qubit, i.e. the gate-mon. Finally, superconducting topological junctions based on nanowires enable topological protection against dissipation and decoherence that is based on the different Fermionic parity between two degenerate ground states.

An important and common problem in superconducting junctions is the nonequilibrium population of the long-lived continuum quasiparticles. It is well understood that, in superconducting junctions, the population of these quasiparticles, lying in the continuum above the gap, is not exponentially suppressed at low temperature as expected by assuming thermal equilibrium in the system. The underlying mechanism for their relaxation dynamics and their nonequilibrium properties are not fully understood. Quasiparticle excitations can compromise the performance of superconducting devices, causing high-frequency dissipation, decoherence in qubits, and braiding errors in proposed Majorana-based topological qubits. Previous experiments reported the observation of nonequilibrium Andreev populations and relaxation in atomic contact by measurements of switching currents. A nonequilibrium quasiparticle population was also reported in Aluminum nanobridges with submicron constrictions. However, in superconducting junctions with ballistic semiconductor nanowires, a parity lifetime (poisoning time) of the bound state ex-
the non-superconducting region to mimic the effect of finite transmission.

Both the short topological junction and the short conventional junction are described by the model Hamiltonian of the total system given by

$$H = H_{SSJ} + H_{res} + H_{int} + H_{mw}(t),$$  \hspace{1cm} (2)$$

with the Hamiltonian $H_{SSJ}$ of the SSJ, the Hamiltonian of the damped LC resonator $H_{res}$ and the interaction $H_{int}$ between the SSJ and the resonator. The Hamiltonian $H_{mw}(t)$ originates from the ac part of the magnetic flux driving the phase difference at microwave frequencies. Details of the derivation of this model are shown in appendix A.

We show that the steady state nonequilibrium occupation of the ABS is ruled by the microscopic, fundamental fermionic-parity changing processes involving the quasiparticles in the continuum and the emission/absorption of photons with the environment, see Fig. 2. We assume that the coupling strength between the resonator and the discrete states of SSJ is weak enough that the resonator’s damping overwhelms and we can disregard the coherent coupling between the resonator and the discrete states of SSJ. Finally, we treat the microwave source as an incoherent emission or absorption of photons at frequency $\Omega$ in the junction. This is valid if the energy $\hbar\Omega$, with the reduced Planck constant $\hbar$, is far away from the internal resonance $\Delta E = |E_{M, A}|$ and for strongly damped quasiparticles in the continuum.

In the rest of the paper, we refer to the Andreev bound state as an incoherent emission or absorption of photons.

Figure 1. Possible transitions in the short superconducting junction. For both the topological and the conventional junction, there are four transition rates (solid blue arrows) which change the fermionic parity of the Andreev bound states in an atomic contact (Andreev energy $E_A$) and a topological junction (Andreev energy $E_{M, A}$), respectively. The processes with out-rates $\Gamma_{\text{out}, i}$ empty the bound state, while the processes with in-rates $\Gamma_{\text{in}, i}$ fill the bound state. The index $i = 1, 2$ indicates how many quasiparticles are involved in this transition. For the conventional junction, there are additional parity-conserving processes with rates $\Gamma_{\text{in/out}, \text{AA}}$ between the ground state $E = 0$ and the Andreev bound state $E_A$ (dashed red arrows), which are absent for the topological junction since the Majorana bound state is nondegenerate.

In the limit in which the Josephson energy of the tunneling junction is larger than the superconducting coupling of the SSJ, the phase difference essentially drops on the SSJ and the external magnetic flux enables control of the phase difference. We analyze the dissipative effects on the short superconducting Josephson junction (SSJ) due to the damped LC resonator, together with the possibility of microwave absorption. We assume that the tunneling junction is formed by a second, smaller SQUID such that it behaves as a single Josephson tunneling junction of tunable Josephson energy $E_J$.

The Josephson junction is assumed to be in the Josephson regime and is therefore described as a damped LC resonator with capacitance $C_J$ and inductance $L_J = \hbar/2eI_c$ defined by the critical current $I_c$ of the junction. A resistance $R$ accounts for a finite damping on the resonator.

Motivated by Andreev spectroscopy experiments in short SAC, previous theoretical works tackle the problem of the nonequilibrium occupation of the ABS and a short topological junction. In this work, we discuss this problem by considering the experimental setup used in the experiments of the Saclay’s group and the recent experiment of the Delft’s group.

The dc-SQUID is formed by a Josephson junction and a short superconducting junction (SSJ), as sketched in Fig. 1(a). The phase difference between the two superconducting leads of each junction is described by $\chi$ for the Josephson junction and $\phi$ for the SSJ. We assume that the Josephson junction behaves as a damped LC resonator having a capacitance $C_J$ and an inductance $L_J$, as shown in Fig. 1(b). The whole dc-SQUID is penetrated by a dc magnetic flux with small ac part inducing the phase difference $\varphi + \delta \varphi(t)$. The phase differences in the superconducting ring are linked by the equation

$$\phi - \chi = \varphi + \delta \varphi(t).$$  \hspace{1cm} (1)$$

The Josephson energy is given by $E_J = \Phi_0/2\pi L_J I_c$. For the case of a conventional junction, we consider also a delta-like barrier in proceeding 10 ms was reported.\(^{52}\)

In the ballistic regime, we discuss a nanostructure characterized by one conducting transmission channel which gives rise to ABS when it is embedded between two superconducting leads.\(^{72,73}\) For the case of a conventional junction, we consider also a delta-like barrier in proceeding 10 ms was reported.\(^{52}\)
states formed in the short conventional junction as ABS whereas we refer to the Andreev bound states formed in the short topological junction as MBS (Majorana bound states).

II. MODEL HAMILTONIANS

A. \( H_{\text{SSJ}} \) for the short superconducting junction

The Hamiltonian of the SSJ is given by \( H_{\text{SSJ}} = \int dx \Psi_\beta^\dag(x) H_\beta(x) \Psi_\beta(x)/2 \) for the topological \((\beta = \text{tj})\) and the conventional \((\beta = \text{cj})\) junction, respectively, with the corresponding Bogoliubov-de Gennes (BdG) Hamiltonian \( H_\beta(x) \) and the Nambu spinor \( \Psi_\beta(x) \). The model of the SSJ is sketched in Fig. 3.

We diagonalize the Hamiltonian by solving the corresponding BdG equations to obtain the energy spectrum. The wave functions of the eigenstates are provided in appendix B for the short topological (appendix B1) and short conventional (appendix B2) junction. By introducing fermionic Bogoliubov quasiparticle annihilation (creation) operators \( \gamma_n^{(s)} \), we write the diagonalized Hamiltonian as

\[
H_{\text{SSJ}} = \sum_n E_n \gamma_n^{\dag} \gamma_n ,
\]

where we have a discrete spectrum \((n = \pm)\) for energies \(|E| < \Delta \) and a continuous spectrum of scattering states \((n = (E, s))\) at energies \(|E| > \Delta \). Here, \( s \) labels the four possible incident quasiparticles, with \( s = 1 \) \((s = 2)\) describing an electron-like (hole-like) quasiparticle impinging from the left and \( s = 3 \) \((s = 4)\) describing an electron-like (hole-like) quasiparticle impinging from the right lead.

![Figure 3. Sketch of the SSJ of length \( L \) formed by two superconductors (S) separated by a small normal (N) region. In S, there is an excitation gap of \( 2 \Delta \) around the Fermi energy \( E = 0 \) and there is a superconducting phase difference of \( \phi \) across the junction. For energies \( E > \Delta \), we have propagating quasiparticles whose wave functions are obtained by calculation of \( s \) scattering states \((s = 1, 2, 3, 4)\) labeling the incident quasiparticle \((e: \text{electron-like}, h: \text{hole-like})\) from the left or right. In general, each incident quasiparticle produces four outgoing quasiparticles due to normal or Andreev reflection.](image-url)

1. Short topological superconducting junction

We model the short topological junction by using the Fu-Kane-Model\(^3\) of superconductivity-proximized helical edge states in a two-dimensional topological insulator (TI), for which the BdG Hamiltonian is given by

\[
H_{\text{tj}}(x) = -i\hbar v_{\text{tj}} \sigma_3 \tau_3 \partial_x - \mu \tau_3 + \Delta(x) e^{i\phi(x)} \tau_1 , \tag{4}
\]

with \( \mu \) being the chemical potential and \( v_{\text{tj}} \) being the Fermi velocity of the edge states. This Hamiltonian can be related directly to the low-energy Hamiltonian of a spin-orbit coupled nanowire. As shown in the supplemental material of Ref. \[79\], the low-energy Hamiltonian of a clean nanowire junction in the limit of a strong magnetic field is the same as for a reflectionless S-TI-S junction, with different velocity \( v_{\text{tj}} \) and pairing gap \( \Delta \).

The matrices \( \tau_i \) and \( \sigma_i \) are Pauli matrices acting on particle-hole and right/left-movers sub-space (which corresponds to spin-space since spin is locked to momentum due to helicity), respectively, of the Nambu space defined by the spinor \( \Psi_\beta(x) = (\psi_\uparrow(x), \psi_\downarrow(x), \psi_e(x), -\psi_e^\dag(x))^T \), with the annihilation (creation) operator \( \psi_\sigma^{\dag}(x) \) of a quasiparticle with spin \( \sigma \). Matrices of different subspaces commute, i.e. \([\tau_i, \sigma_j] = 0\).

In the limit of short junctions, in which the superconducting coherence length fulfills \( \xi_{\text{tj}} = \hbar v_{\text{tj}}/\Delta \gg L \), we can consider \( L \to 0 \). Therefore, we write the inhomogeneous gap potential \( \Delta(x) \) and the phase \( \phi(x) \) as

\[
\Delta(x) = \begin{cases} \Delta, & x \neq 0 \\ 0, & x = 0 \end{cases} , \quad \phi(x) = \frac{\phi}{2} \text{sgn}(x) , \tag{5}
\]

with \( \phi \) being the total phase difference between the two superconducting leads and the sign function \( \text{sgn}(x) \).

Particle-hole symmetry is expressed by the operator \( S_{\text{tj}} = \sigma_2 \tau_3 K \), \( K \) meaning complex conjugation, fulfilling \( \{S_{\text{tj}}, H_{\text{tj}}(x)\} = 0 \). Diagonalization of the Hamiltonian reveals a single pair of non-degenerate bound states \( E_{\pm}(\phi) = \pm E_{\text{M}}(\phi) \) with the 4\(\pi\)-periodic energy\(^{23\,24}\)

\[
E_{\text{M}}(\phi) = \Delta \cos \frac{\phi}{2} \tag{6}
\]

of the MBS. The current through the short topological junction can be expressed as

\[
I_{\text{M}}(\phi) = \frac{1}{\Phi_0} \frac{\partial E_{\text{M}}}{\partial \phi} \left( n_M - \frac{1}{2} \right) , \tag{7}
\]

with the flux quantum \( \Phi_0 = \hbar/2e \) and the occupation \( n_M \in \{0, 1\} \) of the MBS.

2. Short conventional superconducting junction

For the conventional junction, we start from a general second-quantized density Hamiltonian. Then, by replacing fermionic annihilation (creation) operators \( \psi_\sigma^{\dag}(x) \) of
electrons of spin $\sigma = \uparrow, \downarrow$ with
\[ \psi_{\uparrow}(x) = e^{\pm ik x} \psi_{\uparrow}(x) + e^{\mp ik x} \psi_{\uparrow}(x) \]
and linearizing around the Fermi surface, using the Nambu notation, we obtain that the BdG Hamiltonian of spin-up quasiparticles in the short conventional junction takes the form\(^2\)
\[ H_{\text{cj}}(x) = -i\hbar v_c \sigma_i \tau_3 \partial_x + \hbar v_c \sigma_3 \Delta(x) e^{i\phi(x)} \tau_1, \]
with $v_c = \hbar k_c/m$ being the Fermi velocity in the conventional junction, the mass $m$ of an electron, $k_c$ is the Fermi wave number and $\Delta(x)$ is the superconducting gap (phase difference).

The matrices $\tau_i$ and $\sigma_i$ are Pauli matrices acting on particle-hole and right/left-mover sub-space, respectively, of the Nambu space defined by the spinor $\Psi_{\text{cj}}(x) = (\psi_{\text{R} \uparrow}(x), \psi_{\text{L} \downarrow}(x), \psi_{\text{L} \uparrow}(x), \psi_{\text{R} \downarrow}(x))^T$, with the creation (annihilation) operator $\Psi_{\sigma}^{\dag}(x)$ of a quasiparticle with spin $\sigma$ moving in the direction $x$. Again, matrices of different sub-spaces commute, i.e. $[\tau_i, \sigma_j] = 0$. Again, in the short junction limit with a long coherence length $\xi_c = \hbar v_c / \Delta \gg L$, the superconducting gap $\Delta(x)$ and the phase bias $\phi(x)$ are given by Eq. (9). In Eq. (9), we model an arbitrary transmission $0 < T < 1$ through the conventional junction by including a finite $\delta$-barrier of strength $Z > 0$ at $x = 0$ leading to scattering at the interface, turning right- into left-movers and vice versa. We assume the transmission probability $T$ to be energy-indepedent and related to the barrier strength $Z$ by the relation $T = \cosh^{-2}(Z)$. For the conventional junction, particle-hole symmetry is described by the operator $\mathcal{S}_{\text{cj}} = i \sigma_1 \tau_2 \mathcal{K}$ which fulfills $\{\mathcal{S}_{\text{cj}}, H_{\text{cj}}(x)\} = 0$. Diagonalization of the Hamiltonian reveals a single pair of two-fold degenerate bound states $E_{\pm}(\phi, T) = \pm E_A(\phi, T)$ with the 2$\pi$-periodic ABS energy\(^2\)
\[ E_A(\phi, T) = \Delta \sqrt{1 - T \sin^2 \frac{\phi}{2}}. \]
In this case, the current can be expressed as
\[ I_A(\phi, T) = \frac{1}{\Phi_0} \frac{\partial E_A}{\partial \phi} (n_A - 1) \]
and it depends on the occupation $n_A \in \{0, 1, 2\}$ of the ABS. In contrast to the short topological junction, the conventional junction has a state of zero current corresponding to $n_A = 1$. Finally, the spin-down quasiparticles are described by the same Hamiltonian, given in Eq. (9), but with a different spinor given by $\mathcal{S}_{\text{cj}} \Psi_{\text{cj}}$.

B. Josephson junction as a dissipative resonator

We now specify the Hamiltonian $H_{\text{res}}$ of the damped resonator. We assume that the Josephson junction is in the Josephson regime in which the Josephson energy $E_J = \Phi_0^2 / L_1$ is large compared to the charging energy $E_C = (2e)^2 / 2C_1$, i.e. $E_J \gg E_C$, where $e$ is the elementary charge and $L_1 (C_1)$ is the inductance (capacitance) of the Josephson junction. Since fluctuations in the phase difference $\chi$ are small in this regime, the Josephson junction behaves like a LC resonator (cf. Fig. 4(b)) with an effective Hamiltonian
\[ H_{\text{res}} = \hbar \omega_0 b_0^\dagger b_0 + H_{\text{bath}}, \]
where we introduced bosonic creation and annihilation operators $b_0^\dagger$ and $b_0$, respectively, together with the Josephson plasma frequency $\omega_0 = \sqrt{2E_J E_C} / \hbar$. $H_{\text{bath}}$ describes the unavoidable dissipation of the LC resonator which is taken into account by assuming a resistor $R$ connected in parallel to the Josephson junction (cf. Fig. 4(b)). The bath can be formally described with the Caldeira-Leggett model\(^2\) i.e. coupling the resonator to an infinite set of independent harmonic oscillators producing an Ohmic damping $\gamma$. The resistor is assumed to be at environmental temperature $T_{\text{env}}$. The correlator of the damped LC resonator reads
\[ C(t) = \langle (b_0^\dagger(t) + b_0(t)) (b_0^\dagger + b_0) \rangle \]
\[ = \frac{1}{4} \int_0^\infty dE \chi(E) \left( n_B(E) e^{i\phi t/n_B(E)} + (1 + n_B(E)) e^{-i\phi t/n_B(E)} \right), \]
with the Bose-Einstein distribution $n_B(E) = 1 / (e^{E/k_B T_{\text{env}}} - 1)$, the Boltzmann constant $k_B$ and the spectral density of the LC resonator
\[ \chi(E) = \frac{8\hbar \omega_0 \gamma E / \pi}{E^2 - (\hbar \omega_0)^2 + 4\gamma^2 E^2}. \]

C. Interaction with the damped resonator and microwave irradiation

We discuss the interactions in the dc-SQUID between the SSJ and the damped LC resonator as well as the effect of a time dependent flux, via a small ac phase component $\delta \varphi(t)$ in the magnetic flux penetrating the SQUID given by $\delta \varphi(t) = \delta \varphi(\sin(\Omega t))$ with microwave frequency $\Omega$. The interaction Hamiltonian leading to dynamics in the SSJ is therefore given by
\[ H_{\text{int}} = \lambda (b_0^\dagger + b_0) \Phi_0 I_\beta, \]
\[ H_{\text{mw}}(t) = \delta \varphi(\sin(\Omega t)) \Phi_0 I_\beta, \]
with the coupling to the resonator $\lambda = \sqrt{E_C / \hbar \omega_0}$ and the current operator of the SSJ given by
\[ I_\beta = e v_{3j} \Phi_0^\dagger \Psi_\beta(0) \sigma_3 \Psi_\beta(0), \]
evaluated at the interface $x = 0$. Again, $\beta = tj (\beta = cj)$ labels the short topological (conventional) junction.
We note that a time-dependent ac phase bias induces a time-dependent voltage $V(t) = \theta_0 \partial \phi(t)$ which will be neglected since it only leads to a (time-dependent) renormalization of the energy levels and, thus, will not modify transition rates in our approach to the dynamics with a master equation.

III. RATE EQUATION FOR $n_M$ AND $n_A$

In this section, we describe the nonequilibrium dynamics of the SSJ by using a rate equation. In both cases, the short topological and the short conventional superconducting junction, there is a single pair of bound states at subgap energies $|E| < \Delta$, as described in Sec. II A. Depending on the type of the SSJ, there are several possible transitions between the ground state at $E = 0$, the MBS (ABS) $|E_M| < \Delta$ ($|E_A| < \Delta$) and the continuum at $|E| > \Delta$.

The rate equation for the occupation of the bound states can be formally derived by starting from the time-evolution of the density matrix of the total system and, finally, using a Born-Markov approximation. We neglect any coherence in the system described by off-diagonal elements in the density matrices. Tracing out the degrees of freedom of the damped resonator yields a reduced density matrix of the SSJ which is assumed to be a direct product of subgap parts $\rho$ and $\rho_c$. After tracing over the continuum, we obtain a density matrix of the bound states $\rho_\alpha$ for which we calculate rate equations for occupation probabilities $P_\alpha(t)$ of the bound state being occupied with $i$ quasiparticles. Since the SSJ obeys particle-hole symmetry as described in Sec. II A we can restrict the description to energies $E \geq 0$ because creating an excitation at energy $E > 0$ corresponds to destroying a quasi-particle at $-E$. Finally, we assume that continuum quasiparticles relax fast and that they are described by the Fermi-Dirac distribution $f(E) = 1/(e^{E/k_B T_{\text{qp}}} + 1)$ with a quasiparticle temperature $T_{\text{qp}}$. Notice that we assume $T_{\text{qp}} \neq T_{\text{env}}$ to mimic the effective, nonequilibrium distribution of the quasiparticles in the continuum.

A. Topological junction

For the topological junction, the first excited state corresponding to the MBS can only be empty ($i = 0$) or occupied with one quasiparticle ($i = 1$). Hence, the full rate equation for the probabilities $P_\alpha(t)$ reads

$$\frac{d}{dt} \begin{pmatrix} P_0(t) \\ P_1(t) \end{pmatrix} = \begin{pmatrix} -\Gamma_{\text{in}} & \Gamma_{\text{out}} \\ \Gamma_{\text{in}} & -\Gamma_{\text{out}} \end{pmatrix} \begin{pmatrix} P_0(t) \\ P_1(t) \end{pmatrix}$$

with the populating in-rate $\Gamma_{\text{in}} = \Gamma_{\text{in},1} + \Gamma_{\text{in},2}$ and the depopulating out-rate $\Gamma_{\text{out}} = \Gamma_{\text{out},1} + \Gamma_{\text{out},2}$, cf. Fig. 2. The rates are calculated using Fermi’s golden rule. The transition matrix elements obtained from the current operator in Eq. 10 in the case of a short topological junction are explicitly shown in appendix C1. Using the definition

$$\rho_{ij}^\pm(E) = \frac{\sqrt{\Delta^2 - E_M^2}}{\Delta} \frac{\sqrt{E^2 - \Delta^2}}{E \pm E_M}$$

for the topological junction, which has the meaning of an effective density of states (DOS) resulting from the product of the corresponding matrix element of the current operator and the DOS in a superconductor $D(E) = N_{ij} E / \sqrt{E^2 - \Delta^2}$, with $N_{ij} = L / \pi \hbar v_f$ being the DOS in the normal state, the rates for microwave radiation read

$$\Gamma_{\text{out},2/\text{in},1}^{\text{mw}} = \frac{(\delta \varphi)^2 \Delta^2}{16 \hbar} \rho_{ij}^\pm(h\Omega \mp E_M) \times f(h\Omega \mp E_M) \Theta(h\Omega - (\Delta \pm E_M))$$

for the photon emission and

$$\Gamma_{\text{in},2/\text{out},1}^{\text{mw}} = \frac{(\delta \varphi)^2 \Delta^2}{16 \hbar} \rho_{ij}^\pm(h\Omega \mp E_M) \times (1 - f(h\Omega \mp E_M)) \Theta(h\Omega - (\Delta \pm E_M))$$

for the photon absorption. The rates associated to the emission and absorption of photons in the damped LC resonator read

$$\Gamma_{\text{res},2/\text{in},1}^{\text{res}} = \frac{\lambda \Delta^2}{16 \hbar} \int_\Delta^\infty dE \rho_{ij}^\pm(E) f(E) \times \chi(E \pm E_M) (1 + n_B(E \pm E_M))$$

$$\Gamma_{\text{res},2/\text{out},1}^{\text{res}} = \frac{\lambda \Delta^2}{16 \hbar} \int_\Delta^\infty dE \rho_{ij}^\pm(E) (1 - f(E)) \times \chi(E \pm E_M) n_B(E \pm E_M)$$

Due to the nondegeneracy and different fermionic parity of the MBS, there is no direct transfer of a Cooper pair between the ground state and the first excited state. For $T_{\text{qp}} = 0$ implying $f(E) = 0$, our rates for microwave absorption, given in Eq. (20), coincide with the ones reported in Ref. [71] expressed in terms of the admittance $Y(\Omega)$ in a short topological junction. The notation of the transition rates $\Gamma_{j,k}^i$ can be understood by means of Fig. 2 with $j \in \{\text{in, out}\}$ referring to (out-) in-rates (de-) populating the MBS, $k \in \{1, 2\}$ referring to the number of QPs which are involved and $l \in \{\text{mw, res}\}$ labeling the source of perturbation (microwave and resonator).

Regarding the rates for microwave transitions, there are two sharp thresholds given by the function $\Theta(h\Omega - (\Delta \pm E_M))$ for absorption and emission of photons at microwave frequencies $\Omega > 0$. For instance, one QP from the continuum can decay to the MBS $\Gamma_{\text{res}}^{\text{in},1}$ or can be promoted from it to the continuum $\Gamma_{\text{res}}^{\text{out},1}$ for sufficient large energies $h\Omega > \Delta - E_M$. For processes involving the ground state, we need to transfer two QPs. Either
one continuum QP and the QP in the MBS combine to a Cooper pair by emission of a photon \((\Gamma_{\text{out}}^{\text{mw}})^2\) or a Cooper pair breaks up into two QPs, one is promoted to the MBS and one to the continuum, by photon absorption \((\Gamma_{\text{in}}^{\text{mw}})^2\). These transitions require energies \(h\Omega > \Delta + E_M\).

The transitions involving photons exchanged with the damped LC resonator in the dc-SQUID can be discussed in a similar fashion, although there is no sharp threshold anymore due to a finite broadening of the resonator as shown in Eq.\((14)\). A finite environmental temperature \(T_{\text{env}} > 0\) allows the same four processes shown in Fig.\(2\) involving the resonator. For transitions involving single QPs, the amount of energy being absorbed (emitted) by the SSJ is \(E - E_M\) which is described by the rate \(\Gamma_{\text{res}}^{\text{in}}(1/(\Gamma_{\text{res}}^{\text{in}}))\). Moreover, the transition of two QPs, one from the MBS and one from the continuum, is described by the rates \(\Gamma_{\text{res}}^{\text{out},2}(1/(\Gamma_{\text{res}}^{\text{in}}))\), in which an energy of \(E + E_M\) has to be emitted (absorbed).

From Eq.\((17)\), we calculate the stationary occupation \(n_M\) of the MBS for \(t \rightarrow \infty\). Using the property \(P_0(t) + P_1(t) = 1\) together with \(n_M(t) = \text{Tr}(\gamma_M^t\gamma_M^0P_M(t)) = P_1(t)\) for the topological junction, we find
\[
n_M = \frac{\Gamma_{\text{in}}}{\Gamma_{\text{in}} + \Gamma_{\text{out}}} ,
\]
with the total in-/out-rate \(\Gamma_{\text{in}}/\text{out}^\ast\) as defined in Eq.\((17)\).

**B. Conventional junction**

In contrast to the topological junction, the ABS in a conventional junction is twofold degenerate and can be occupied by two QPs. Therefore, the full set of equations for the probabilities \(P_{b(2)}(t)\) of zero (double) occupancy and \(P_{1(3)}^\ast\) of single occupancy reads
\[
\frac{d}{dt}\begin{pmatrix} P_0(t) \\ P_1(t) \end{pmatrix} = -\hat{R} \begin{pmatrix} P_0(t) \\ P_1(t) \end{pmatrix},
\]
with
\[
\hat{R} = \begin{pmatrix} \Gamma_{\text{in},AA} + 2\Gamma_{\text{in}} & -\Gamma_{\text{out}} & -\Gamma_{\text{out},AA} \\ -2\Gamma_{\text{in}} & \Gamma_{\text{in}} + \Gamma_{\text{out}} & -2\Gamma_{\text{out}} \\ -\Gamma_{\text{in},AA} & -\Gamma_{\text{in}} & \Gamma_{\text{out},AA} + 2\Gamma_{\text{out}} \end{pmatrix}.
\]

We have introduced the probability of single occupation as \(P_1(t) = P_{1(1)}^\ast(t) + P_{1(2)}^\ast(t)\), as we cannot distinguish which zero-current state is occupied since the two states are symmetric. The rates given in the matrix in Eq.\((23)\) are defined as
\[
\Gamma_{\text{in}} = \Gamma_{\text{in}}^{\text{mw}} + \Gamma_{\text{res}}^{\text{in},1} + \Gamma_{\text{in}}^{\text{mw},2} + \Gamma_{\text{res}}^{\text{in},2} ,
\]
\[
\Gamma_{\text{out}} = \Gamma_{\text{out}}^{\text{mw},1} + \Gamma_{\text{res}}^{\text{in},1} + \Gamma_{\text{out}}^{\text{mw},2} + \Gamma_{\text{res}}^{\text{in},2} ,
\]
\[
\Gamma_{\text{in},AA} = \Gamma_{\text{in}}^{\text{mw}} + \Gamma_{\text{in},AA}^{\text{mw},2} + \Gamma_{\text{res}}^{\text{in},AA} ,
\]
\[
\Gamma_{\text{out},AA} = \Gamma_{\text{out}}^{\text{mw},AA} + \Gamma_{\text{out}}^{\text{mw},2} + \Gamma_{\text{in},AA}^{\text{mw}}.
\]
The rates are calculated using Fermi's golden rule. The transition matrix elements obtained from the current operator in Eqn.\((16)\) in the case of a short conventional junction are explicitly shown in appendix\((C)\). Before, we define an effective density of states
\[
\rho_{\text{cj}}^\pm(E) = \frac{\sqrt{\Delta^2 - E_A^2} \sqrt{E^2 - \Delta^2}}{\Delta^2} \times \frac{E_A(E \pm E_A) + \Delta^2(\cos \varphi + 1)}{E_A}
\]
for the conventional junction. The individual rates entering Eq.\((25)\) due to the microwave read
\[
\Gamma_{\text{out},2/1,in}^{\text{mw}} = \frac{(\varphi)^2 \Delta^2}{32h} T \rho_{\text{cj}}^\pm(h\Omega \mp E_A) \times f(h\Omega \mp E_A) \Theta(h\Omega - (\Delta \pm E_A)) ,
\]
whereas we have
\[
\Gamma_{\text{res},2/1,out}^{\text{mw}} = \frac{(\varphi)^2 \Delta^2}{32h} T \rho_{\text{cj}}^\pm(h\Omega \mp E_A) \times (1 - f(h\Omega \mp E_A)) \Theta(h\Omega - (\Delta \pm E_A)) ,
\]
for the photons exchanged with the resonator. For the case of the conventional junction, there are additional parity-conserving transitions describing the excited state of two quasiparticles \(2E_A\) from the ground state
\[
\Gamma_{\text{b,AA}}^{\text{mw}} = \frac{(\varphi)^2 \Delta^3}{32h} (1 - T) \rho_{\text{cj}}^\pm(E_A) S_{\text{ph}}(h\Omega - 2E_A) ,
\]
where
\[
\Gamma_{\text{res},2/1,in}^{\text{mw}} = \frac{(\varphi)^2 \Delta^3}{32h} (1 - T) \rho_{\text{cj}}^\pm(E_A) \chi(2E_A) (\delta_{b,\text{out}} + n_B(2E_A)) ,
\]
with \(b \in \{\text{in,out}\}\) and an effective density of states
\[
\rho_{\text{cj}}^\pm(E_A) = \frac{\pi}{\Delta^3} \frac{(\Delta^2 - E_A^2)^2}{E_A^2} .
\]

These rates are associated to a transfer of a Cooper pair between the twofold degenerate Andreev level and the ground state. In passing by, we have introduced a phenomenological broadening \(S_{\text{ph}}(E) = (\gamma_\lambda/\pi)(E^2 + \gamma_\lambda^2)\) of the Andreev level with width \(\gamma_\lambda\) in the rates \(\Gamma_{\text{in}/\text{out},AA}^{\text{mw}}\) to resolve the transition of a Cooper pair between the ground state and the ABS.\(^{23}\)
We notice that the rates changing the parity for the conventional junction differ from the ones of the topological junction by the factor transmission $T$, beyond the obvious substitution $\rho_{ij} \rightarrow \rho_{ij}^c$. For $T = 1$, the two junctions are exactly equivalent and the occupations are related by $n_A = 2n_M$. For $T_{qp} = 0$, our rates in Eq. (27b) coincide with the ones calculated in Ref. [68] and with the calculation of the admittance $Y(\Omega)$ in a short conventional junction in Ref. [75]. Moreover, at $T_{env} = 0$, the parity-conserving rate $\Gamma_{\text{out,AA}}^{\text{res}}$, has the same form as the annihilation rate found in Ref. [68]. All rates due to the resonator (Eqs. (28) and (29b)) coincide with the rates found in Ref. [67].

The discussion of the in- and out-rates involving quasi-particles is analog to the case of a topological junction. In addition, there are new processes which directly switch the occupation of the ABS without changing the parity (cf. Eq. (29)). These rates appear in any short conventional junction as long as the transmission is $T < 1$. Since no QPs from the continuum are involved in these rates, they are completely independent of $T_{qp}$ and they occur at an energy of $2E_A$. In the case of microwave emission (absorption) $\Gamma_{\text{out(in,AA)}}^{\text{res}}$, the microwave energy has to be $\hbar \Omega \approx 2E_A$ for the process to be in resonance. In the same way the resonator has to provide energies of $2E_A$ in order to promote a Cooper pair to the ABS according to the rate $\Gamma_{\text{res,AA}}^{\text{in}}$.

From Eq. (23), we calculate the stationary occupation $n_A$ of the ABS for $t \rightarrow \infty$. Using the property $P_0(t) + P_1(t) + P_2(t) = 1$ together with $n_A(t) = \text{Tr}\{\gamma_A(t)\rho_M(t)\}$ for the conventional junction, we eliminate the probability $P_1(t)$ and find

$$n_A = 1 + P_2 - P_0,$$

with $P_2$ ($P_0$) being the stationary probability for the bound state of being occupied with two (zero) quasi-particles. For these probabilities, we find the relations

$$\begin{pmatrix} P_0 \\ P_2 \end{pmatrix} = \frac{\Gamma}{\text{det} \hat{\Gamma}} \begin{pmatrix} \Gamma_{\text{out}} \\ \Gamma_{\text{in}} \end{pmatrix},$$

with the matrix

$$\hat{\Gamma} = \begin{pmatrix} \Gamma_{\text{out,AA}} + \Gamma_{\text{in}} + 2\Gamma_{\text{out}} & \Gamma_{\text{out,AA}} - \Gamma_{\text{out}} \\ \Gamma_{\text{in,AA}} + \Gamma_{\text{in}} + 2\Gamma_{\text{in}} & \Gamma_{\text{in,AA}} - \Gamma_{\text{in}} \end{pmatrix}.$$

and the corresponding rates as defined in Eq. (25).

IV. RESULTS

We study the case in which we have two different temperatures in the system. The environmental temperature $T_{env}$, which is considered to be the temperature at which the experiment is performed, i.e. the resonator, and a quasiparticle temperature $T_{qp}$. This is motivated by experiments on superconducting low temperature circuits where the continuum quasiparticle population does not correspond to thermal equilibrium.

A. Effect of the damped LC-resonator

First, we investigate the effect of the damped resonator on the occupation of the bound state in absence of any microwave radiation. As a first observation, we note that if both temperatures are equal, i.e. $T_{env} = T_{qp} = T$, it can be shown that $\Gamma_{\text{out}}^{\text{res}}/\Gamma_{\text{in}}^{\text{res}} = e^{E_M/T}$ (detailed balance) for both the topological ($\alpha = M$) and the conventional ($\alpha = A$) junction. In this case, the stationary solutions of the bound state occupations reduce to $n_M = f(E_M)$ and $n_A = 2f(E_A)$ for the topological and the conventional junction, respectively, with the Fermi function $f(E)$ at the equilibrium temperature $T$.

From now on, we consider different temperatures for continuum QPs and the environment. We plot the occupation as a function of the phase bias $\varphi$ and the Josephson plasma frequency $\omega_0$ at zero environmental temperature $T_{env} = 0$, (namely $k_B T_{env} \ll \hbar \omega_0$). In this limit, $n_M(E) = 0$ and the resonator is only able to absorb energy emitted by transitions of quasi-particles (QPs) in the SSJ. The expressions for the occupation of the MBS and the ABS, Eqs. (22) and (31), respectively, reduce to

$$n_M = \frac{\Gamma_{\text{res,in,2}}^{\text{res}}}{\Gamma_{\text{res,in,1}}^{\text{res}} + \Gamma_{\text{res,out,2}}^{\text{res}}}$$

and

$$n_A = \frac{2\Gamma_{\text{res,in,1}}^{\text{res}}(2\Gamma_{\text{res,in,1}}^{\text{res}} + \Gamma_{\text{res,out,2}}^{\text{res}}) + \Gamma_{\text{res,AA}}^{\text{res}}}{\Gamma_{\text{res,in,1}}^{\text{res}}(3\Gamma_{\text{res,in,1}}^{\text{res}} + \Gamma_{\text{res,out,2}}^{\text{res}}) + 2\Gamma_{\text{res,in,1}}^{\text{res}} + \Gamma_{\text{res,out,2}}^{\text{res}}},$$

respectively. As expected, in the limit $\Gamma_{\text{res,out,AA}}^{\text{res}} \ll \Gamma_{\text{res,in,1}}^{\text{res}}, \Gamma_{\text{res,out,2}}^{\text{res}}$, the two junctions behave in a similar way and the occupation is simply rescaled, $n_A \approx 2n_M$. Hence, the different behaviour relies on the presence of the process $\Gamma_{\text{res,out,AA}}^{\text{res}}$ (red dotted arrows in Fig. 2). The results are shown in Fig. 4 at $k_B T_{qp} = 0.1 \Delta$.

Next, we discuss the occupation in case of a topological SSJ shown in Fig. 2(a). The rates entering $n_M$ in Eq. (34) are given by

$$\Gamma_{\text{res,out,2}}^{\text{res}} \sim \int_{-\infty}^{\infty} \text{d}E f_{\text{eff}}(E) \chi_{\text{eff}}(E \pm E_M),$$

where we defined the effective occupation function $f_{\text{eff}}(E) = f(E)\sqrt{E^2 - \Delta^2}/\Delta$ and the effective spectral density $\chi_{\text{eff}}(E) = \chi(E)\Delta/E$. For $\hbar \omega_0 > \Delta$, we distinguish between two regions: region (I), with $\Delta + E_M < \hbar \omega_0 < 2\Delta + E_M$, showing an empty MBS, and region (II), with $\Delta < \hbar \omega_0 < \Delta + E_M$, in which the occupation of the MBS varies strongly. This can be understood by means of Fig. 3. The absolute value of the rates is determined by the convolution of the effective functions $f_{\text{eff}}(E)$ and $\chi_{\text{eff}}(E \pm E_M)$.

For region (I), there is a strong overlap between $f_{\text{eff}}(E)$ and $\chi_{\text{eff}}(E + E_M)$ in the emission rate $\Gamma_{\text{res,out,2}}^{\text{res}}$ due to the location of the majority of the continuum QPs closely above the gap leading to $\Gamma_{\text{res,out,2}}^{\text{res}} \gg \Gamma_{\text{res,in,1}}^{\text{res}}$ (cf. Fig. 3(b)). Therefore, the interaction with the resonator leads to a
strong depopulation of the MBS with \( n_M \ll 1 \). In region (II), the occupation strongly depends on the value of \( \omega_0 \) and the phase \( \varphi \). By looking at Fig. 5(a), we start to decrease the value of \( \omega_0 \) from \( \hbar \omega_0 = \Delta + E_M \) to \( \hbar \omega_0 = \Delta \). A value of \( \hbar \omega_0 \lesssim \Delta + E_M \) shows that there is a large overlap for \( f_{\text{eff}}(E) \) and \( \chi_{\text{eff}}(E + E_M) \), while the overlap between \( f_{\text{eff}}(E) \) and \( \chi_{\text{eff}}(E - E_M) \) is negligible. By decreasing \( \omega_0 \), we shift the peaks of \( \chi_{\text{eff}}(E \pm E_M) \) to lower energies and, in particular, \( \chi_{\text{eff}}(E - E_M) \) below the gap \( \Delta \) which drastically reduces the previously large overlap, eventually leading to \( \Gamma_{\text{res},\text{AA}} \gg \Gamma_{\text{out},\text{AA}} \) and a highly populated MBS.

Decreasing the Josephson plasma frequency further, i.e. \( \hbar \omega_0 < \Delta \) (regions (III) and (IV)), the depopulating rate \( \Gamma_{\text{out},\text{AA}} \) becomes negligible everywhere except for phase differences \( \varphi \approx \pi \) in which \( \Gamma_{\text{res},\text{AA}} \approx \Gamma_{\text{res},\text{in}} \) (region (IV)). Therefore, coupling to the resonator leads to a high occupation \( n_M \lesssim 1 \) of the MBS.

In the case of the conventional junction, the discussion of the occupation of the ABS, shown in Fig. 4(b), is analogous to the case of the topological junction, i.e. albeit the presence of the process \( \Gamma_{\text{out},\text{AA}} \), the behaviour of the occupation can be understood in a similar way compared to the topological case.

For region (I), with \( \Delta + E_A < \hbar \omega_0 < 2\Delta + E_A \), the populating rate \( \Gamma_{\text{res},\text{in}} \) is again strongly suppressed and the occupation of the ABS becomes \( n_A \ll 1 \), similar to the case of the topological junction. The other regions (II), (III) and (IV) can be explained by the competition between the relaxation process associated with the rate \( \Gamma_{\text{out},\text{AA}} \) and the refilling process involving continuum.

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**Figure 4.** Stationary nonequilibrium bound state occupation in SSJ as a function of phase difference \( \varphi \) and Josephson plasma frequency \( \omega_0 \). (a) MBS occupation \( n_M \) in topological junctions. The black solid lines separating regions (I) and (II) is given by \( \hbar \omega_0 = \Delta + E_M \) and by \( \hbar \omega_0 = \Delta - E_M \) for the region (III) and (IV). The black dashed line separating regions (II) and (III) is \( \hbar \omega_0 = \Delta \). (b) ABS occupation \( n_A \) for finite transmission \( T = 0.99 \) in conventional junctions. The black solid line separating regions (I) and (II) is given by \( \hbar \omega_0 = \Delta + E_A \) and by \( \hbar \omega_0 = \Delta - E_A \) for the region (II)/ (III) and (IV). The black dashed line separating regions (II) and (III) is given by \( \Gamma_{\text{out},\text{AA}} \approx 2\Gamma_{\text{in},1} \) in the limit \( \Gamma_{\text{out},\text{AA}}, \Gamma_{\text{res},1} \gg \Gamma_{\text{in},1} \). Common parameters: \( \gamma = 0.001\Delta, k_B T_{\text{qp}} = 0.1\Delta, T_{\text{env}} = 0 \).

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**Figure 5.** Sketch of the contributions to the transition rates \( \Gamma_{\text{res},\text{in}/\text{out}} \) at \( T_{\text{env}} = 0 \) which determine the absolute value of the rates depending on the value of the Josephson plasma frequency \( \omega_0 \). Quasiparticles in the continuum with energies \( E > \Delta \) are occupied by the effective function \( f_{\text{eff}}(E) = f(E)\sqrt{E^2 - \Delta^2}/\Delta \), which is a combination of matrix elements, density of states in a superconductor and the Fermi-Dirac distribution. Most of the continuum quasiparticles are located closely above the gap in an energy range approximately proportional to \( T_{\text{qp}} \). The effective spectral function \( \chi_{\text{eff}}(E) = \chi(E)\Delta/E \) is related to the absorption peak of the resonator shifted to values \( \omega_0 \pm E_M \) according to \( \chi_{\text{eff}}(E \pm E_M) \) for the rate \( \Gamma_{\text{res},\text{in}/\text{out}} \). The maximal shifts \( \omega_0 \pm \Delta \) can be achieved at a phase difference \( \varphi = 0 \), while the minimal shifts are at \( \varphi = \pi \). The cases (a), (b), (c) and (d) correspond to the regions (II), (I), (III) and (IV), respectively, defined in Fig. 4(a).
umb QPs with rate $\Gamma^\text{res}_{\text{out,AA}}$. For $\Delta < \hbar\omega_0 < \Delta + E_A$, which is mainly region (II), transitions associated to the rate $\Gamma^\text{res}_{\text{out,AA}}$ dominate the behaviour of the occupation, i.e. $\Gamma^\text{res}_{\text{out,AA}} \gg \Gamma^\text{res}_{\text{in,1}}, \Gamma^\text{res}_{\text{out,2}}$. Therefore, the occupation reduces to

$$n_A \approx \frac{2\Gamma^\text{res}_{\text{in,1}}}{3\Gamma^\text{res}_{\text{in,1}} + \Gamma^\text{res}_{\text{out,2}}}.$$  \hfill (37)

Due to the presence of the process $\Gamma^\text{res}_{\text{out,AA}}$, there is a strong reduction of the occupation of the ABS compared to the topological case leading to an occupation of $n_A \approx 2/3$ for $\Gamma^\text{res}_{\text{in,1}} \gg \Gamma^\text{res}_{\text{out,2}}$.

At energies $\hbar\omega_0 < \Delta$, mainly region (III), (IV) and partially region (II), transitions due to the process $\Gamma^\text{res}_{\text{out,2}}$ are negligible and, therefore, the two competing rates are $\Gamma^\text{res}_{\text{out,AA}}$ and $\Gamma^\text{res}_{\text{in,1}}$. The occupation reduces to

$$n_A \approx \frac{4\Gamma^\text{res}_{\text{in,1}} + 2\Gamma^\text{res}_{\text{out,AA}}}{2\Gamma^\text{res}_{\text{in,1}} + 3\Gamma^\text{res}_{\text{out,AA}}}.$$ \hfill (38)

The regions (II) and (III), for which $\Delta - E_A < \hbar\omega_0 < \Delta$, are separated by the dashed line $n_A \approx 1$ for $\Gamma^\text{res}_{\text{out,AA}} \approx 2\Gamma^\text{res}_{\text{in,1}}$ with either $n_A \lesssim 2$ for $\Gamma^\text{res}_{\text{out,AA}} \ll \Gamma^\text{res}_{\text{in,1}}$ (region (III)) or $n_A \approx 2/3$ for $\Gamma^\text{res}_{\text{out,AA}} \gg \Gamma^\text{res}_{\text{in,1}}$ (region (II)). For $\hbar\omega_0 < \Delta - E_A$, we find again $\Gamma^\text{res}_{\text{out,AA}} \gg \Gamma^\text{res}_{\text{in,1}}$ leading to an occupation of $n_A \approx 2/3$. This is a consequence of the lack of continuum QPs at energies which can be absorbed by the resonator in order to refill the bound state. The reason that $\Gamma^\text{res}_{\text{out,AA}} \gg \Gamma^\text{res}_{\text{in,1}}, \Gamma^\text{res}_{\text{out,2}}$ almost everywhere except the red region is due to the fact that the absolute value of $\Gamma^\text{res}_{\text{out,AA}}$ is independent the continuum. In contrast, $\Gamma^\text{res}_{\text{in,1}}$ and $\Gamma^\text{res}_{\text{out,2}}$ consist of a convolution of continuum functions and the absorption of the resonator.

As described in Sec. \[11\], both MBS and ABS carry a supercurrent $I_M$ and $I_A$, respectively, which depends on the occupation of the bound state (cf. Eqns. \[7\] and \[11\], respectively). In the case of a topological junction, the occupation $n_M$ shows a non-trivial behaviour for $\Delta \leq \hbar\omega_0 \leq 2\Delta$ (cf. Fig. \[9\](a)). The corresponding current $I_M$ as a function of the phase difference $\varphi$ for fixed $\omega_0$ in this range is shown in Fig. \[9\]. For $\varphi = 0$ and $\varphi = \pi$, the corresponding current is always zero which is independent of the value of $\omega_0$ since either $\partial E_M/\partial \varphi \big|_{\varphi = 0} = 0$ and $n_M(\pi) = 1/2$ at $\varphi = \pi$. Moreover, for $\varphi \neq 0, \pi$, there exists another zero crossing for $\Delta < \hbar\omega_0 \leq 1.5\Delta$ since the occupation crosses $n_M(\varphi) = 1/2$, leading to a change of the direction of the current across this junction. For $\hbar\omega_0 \geq 1.5\Delta$ and $\hbar\omega_0 \leq \Delta$, the occupation is always $n_M < 1/2$ and $n_M \approx 1/2$, respectively, such that the additional zero current crossing disappears.

In the case of a conventional junction, the occupation $n_A$ shows a non-trivial behaviour for $\hbar\omega_0 < 2\Delta$. Then, in this regime, we plot the corresponding current $I_A$ as a function of the phase difference $\varphi$ for fixed $\omega_0$, in Fig. \[11\](a) for $\hbar\omega_0 \leq 1.1\Delta$ and in Fig. \[11\](b) for $\hbar\omega_0 \geq 1.3\Delta$. For $\varphi = 0$ and $\varphi = \pi$, the corresponding current is always zero which is independent of the value of $\omega_0$, similar to the topological case, yet both zero crossings originate from $\partial E_A/\partial \varphi \big|_{\varphi = \pi} = 0$. For phase differences $\varphi \neq 0, \pi$, there exists an additional zero crossing if the occupation of the ABS crosses $n_A = 1$. We observe a such zero current crossing for $\hbar\omega_0 \leq \Delta$ due to the presence of the rate $\Gamma^\text{res}_{\text{out,AA}}$ in a highly transmitting contact, at finite value of the phase. Eventually, the zero current crossing point.
shifts to very small phase differences $\varphi \ll \pi$ at $\Delta < \hbar \omega_0 \lesssim 1.5\Delta$. For values $\hbar \omega_0 \gtrsim 1.5\Delta$, this additional zero current crossing disappears since the occupation is always $n_A < 1$.

**B. LC resonator and microwave**

Now, we discuss the occupation of the bound states in the presence of microwave radiation. We plot the occupation as a function of the phase difference $\varphi$ and the microwave frequency $\Omega$. Considering a setup with a resonator at a fixed energy $\hbar \omega_0 \leq \Delta$, i.e. we focus on the cases $\hbar \omega_0 = 0.2\Delta$ and $\hbar \omega_0 = \Delta$ at $T_{\text{env}} = 0$, we now allow for transitions due to absorption or emission of energy $\hbar \Omega$ in the presence of a microwave.

We now regard the occupation of the bound states in the presence of microwave radiation as shown in Fig.8 for the previously discussed values of $\hbar \omega_0$. We define three regions labeled (I), (II) and (III). In region (I), which is given by $h\Omega < \Delta - E_{M,A}$, the energy of the microwave is too low for possible transitions of QPs (cf. rates in Eqs. (29a), (29b) and (29c)). Therefore, the contribution to the rates in this regime is purely due to the interaction of the SSJ with the damped resonator. To understand the behavior of the SSJ in region (I), we recall the case without microwave irradiation, shown in Fig.4 at $\hbar \omega_0 = 0.2\Delta$ and $\hbar \omega_0 = \Delta$, respectively. On the one hand, at $\hbar \omega_0 = \Delta$ for the case of the topological junction, we have $n_M \approx 1$ for all phases $\varphi$ whereas, for the conventional case at $\hbar \omega_0 = \Delta$, the occupation saturates at $2/3$ (actually it switches from 2 to $2/3$ at very small phase, see previous discussion and Fig.4). On the other hand, both types of junctions show a high occupation at $\hbar \omega_0 = 0.2\Delta$ and phases $\varphi < \varphi_c$, where the critical phase $\varphi_c$ is defined by $E_{M,A}(\varphi_c) \approx \Delta - \hbar \omega_0$. For phases $\varphi > \varphi_c$, the behavior differs drastically. In the case of a topological junction, the occupation starts to decrease smoothly from $n_M(\varphi_c) \approx 1$ to $n_M(\pi) = 1/2$, while for the conventional junction the occupation immediately drops from $n_A \approx 2$ to $n_A \approx 2/3$ in a short range around $\varphi_c$. For $h\Omega > \Delta - E_{M,A}$, microwave photon absorption and emission become possible.

The behaviour of the bound state occupations in region (II) is dominated by transitions due to the microwave, i.e. $\Gamma_{\text{mw}}^p \ll \Gamma_{\text{mw}}^\Omega$, where $p$ labels all possible rates defined in Sec. II. The crossover behavior in region (II) is set by the condition $h\Omega = \Delta + E_{M,A}$ and it can be explained in a similar way to the discussion for Fig.4 with the difference that below such threshold the absorption processes dominate. For $\Delta - E_{M,A} < h\Omega < \Delta + E_{M,A}$, the bound states are almost empty, i.e. $n_{M,A} \ll 1$. This is because $\Gamma_{\text{mw}}^\Omega_1 \gg \Gamma_{\text{mw}}^\Omega_2$ due to the small number of continuum QPs at the chosen $T_{\text{qp}}$. Moreover, the energy of the microwave radiation is still too low for transitions from or into the ground state, i.e. $\Gamma_{\text{mw}}^\text{out}_{1,2} \approx \Gamma_{\text{mw}}^\text{in}_{1,2} = 0$. Comparing the topological with the conventional case, there exists a line $h\Omega \approx 2E_A$ of non-zero occupation in the case of the conventional junction. This is due to a resonant process associated with the rates $\Gamma_{\text{mw}}^\text{in/out}_{\text{in/out}}$, i.e. $\Gamma_{\text{mw}}^\text{in/out}_{\text{in/out}} \gg \Gamma_{\text{mw}}^\Omega_1$, which cannot appear in a topological junction. Away from this resonance, the occupation in the dark blue area is given by $n_M \approx f(h\Omega + E_A)$ and $n_A \approx 2f(h\Omega + E_A)$, respectively, with $f(E)$ being the Fermi-Dirac distribution at the temperature $T_{\text{qp}}$. At $h\Omega > \Delta + E_{M,A}$ for the topological junction, the occupation shows almost no phase dependence in region (II) and it is possible to demonstrate that it approaches $n_M(h\Omega \gg 2\Delta) \approx 0.5$. At $h\Omega > \Delta + E_{M,A}$ for the conventional junction, there is a weak phase dependence visible due to the different effective density of states. However, the occupation also approaches $n_A \approx 0.5$ in the limit $h\Omega \gg 2\Delta$. Finally, in Fig.8(c) and Fig.8(d), there is a sharp vertical line separating regions (II) and (III) which is defined by $\hbar \omega_0 = \Delta - E_{M,A}$ with $\hbar \omega_0 = 0.2\Delta$. This line is moved to $\varphi \approx \pi$ since $\hbar \omega_0 = \Delta$ for Fig.8(a) and Fig.8(b). In region (III), we have a strong competition between the microwave radiation and the damped resonator. At $\hbar \omega_0 = 0.2\Delta$, coupling to the resonator leads to a refilling of the bound state while the microwave contribution leads to a depopulation. This results in occu-
Figure 9. Subgap currents as a function of phase difference $\varphi$ for different microwave frequencies $\Omega$ in the presence of the resonator with plasma frequency $\omega_0 = 0.2\Delta / h$. (a) Current $I_M$ carried by Majorana bound states in short topological superconducting junctions. (b) Current $I_A$ carried by Andreev bound states in short conventional superconducting junctions. Parameters are the same as for Fig.8(c) and (d), respectively. Current is plotted in units of $I_0 = e^2 / h$.

V. SUMMARY AND CONCLUSIONS

We calculated the nonequilibrium bound state occupations for short topological and short conventional superconducting junctions being part of a dc-SQUID in the presence of an applied ac microwave field and of phase fluctuations due to a damped resonator of frequency $\omega_0$. We used a simple rate equation to obtain the stationary state of the occupations. We assumed that the continuum quasiparticles above the gap relax much faster than the dynamics of the discrete ABS and MBS level. We discussed the case when quasiparticles are still described by a Fermi distribution but with higher effective temperature $T_{ap} > T_{env}$ with respect to the environmental temperature in the system. This mimics, in a phenomenological way, an effective nonequilibrium distribution.

In the absence of a microwave field, at low environmental temperature $(k_B T_{env} \ll h \omega_0)$, we have shown that if the resonator’s Josephson plasma energy is $\hbar \omega_0 < 2\Delta$, the resulting occupations of the short superconducting junctions differ drastically for the topological and conventional case (cf. Fig.4) despite the fact that the transmission is $T \approx 1$ in the case of the conventional junction. This result is due to the process of transfer a Cooper pair between the ground state and the excited bound state — occurring with energy $2E_A$ — in the conventional junction that leads to a decreased occupation. In contrast, the lack of this process in the topological case leads to a high occupation in a wide region (see Fig.4). In the presence of photon pumping due to microwaves, the occupations can be generally determined by the competition of microwaves and the photon emitted in the damped resonator. When the microwave dominates, we found that the steady occupation of the bound states is in correspondence of the quasiparticle temperature (blue regions in Fig.5). Hence, in this regime of strong pumping, measuring the equilibrium occupation can give a priori information about the nonequilibrium quasiparticle population. Finally, we show that the theoretically regions of different occupation and their switching and crossover appear in a one to one correspondence in the behavior of the supercurrent as a function of the phase difference.

Before to conclude, we further discuss possible methods to measure the nonequilibrium population. In experiments for the SAC[S12] as well as in nanowires[H1], the SQUID system sketched in Fig.1 was capacitively coupled to a voltage biased Josephson junction acting as (incoherent) emitter of photons directed towards the SQUID as well as a spectrometer of the system. The external Josephson junction of the spectrometer was designed with a typical characteristic impedance $\text{Re}[Z_0] \ll R_Q$, with $R_Q = \hbar / 4 e^2$, such that the current observed at finite voltage bias $V_{\text{spec}}$ can be expressed, using $P(E)$...
theory\textsuperscript{38} as a function of the impedance seen by the junctions itself\textsuperscript{37,39}

\[ I_{\text{spec}} = \frac{I_{\text{spec}}^2}{2} \frac{\text{Re}[Z(\omega)]}{V_{\text{spec}}} \]  

(39)

in which \( Z(\omega) \) corresponds to the impedance associated to the SQUID and \( I_{\text{spec}} \) is the critical current of the spectrometer. Far away of the frequency range associated to the plasma mode \( \omega_0 \), one can reasonably assume that the impedance of the SQUID is essentially given by the impedance of the SSJ, i.e. \( Z(\omega) \approx Z_{\text{SSJ}}(\omega) \). The latter quantity is the result of the processes of emission and absorption occuring in the SSJ (described in this work) and it is directly related to the occupation numbers of the ABS in the conventional SSJ\textsuperscript{20} as well as in the topological one\textsuperscript{21,22}.

Another way to measure the ABS occupation can be via switching current measurements\textsuperscript{23}. By measuring the switching current, these occupations should be experimentally accessible since the subgap current carried by the bound states is proportional to the occupation, as described in the theoretical work of Ref.\textsuperscript{71}.

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Appendix A: Derivation of the total Hamiltonian

In this appendix, we provide the derivation of the final Hamiltonian of the combined system, given in Eqn.\textsuperscript{2}, consisting of the short superconducting junction (SSJ), the damped LC resonator and the microwave radiation due to a small ac part of the magnetic flux penetrating the SQUID.

1. Hamiltonian of the LC resonator

First, we start with the Hamiltonian of the damped LC resonator which is formed by a conventional Josephson junction in the Josephson regime, together with dissipation described within the Caldeira-Legett model via coupling to a bath. The general Hamiltonian of a Josephson junction is given by

\[ H_{\text{res}} = E_C \left( N - \frac{Q_e}{2e} \right)^2 - E_J \cos \chi, \]  

(A1)

with the residual offset charge \( Q_e = eN_e \) carried by \( N_e \) single charge carriers, the number of charge carrying Cooper pairs \( N \) and the phase difference \( \chi \) across the junction which satisfy the commutation relation \( [\chi, N] = i \). In the Josephson regime for \( E_J \gg E_C \), phase fluctuations are small, i.e. \( \langle \chi^2 \rangle \ll 1 \). Moreover, we write the accumulated charge in a symmetrized way, \( Q_t = e(N_c - N_a) / 2 \), in terms of the quasiparticle operators of the SSJ, with

\[ N_c - N_a = \left\{ \begin{array}{ll} \sum_{\sigma} \int \text{d}x \text{sgn}(x) \psi^\dagger_\sigma \psi_\sigma & , \text{topological} \\ \sum_{\sigma} \int \text{d}x \text{sgn}(x) \psi^\dagger_\sigma \psi_{\sigma \alpha} & , \text{conventional} \end{array} \right. \]  

(A2)

spin \( \sigma = \uparrow, \downarrow \) and \( \alpha = R, L \) referring to right/left-movers.

Expanding the resulting Hamiltonian to lowest order in \( \chi \), we obtain

\[ H_{\text{res}} = E_C \left( N - \frac{N_c - N_a}{4} \right)^2 + \frac{E_J}{2} \chi^2. \]  

(A3)

By means of the unitary transformation \( U = \exp(-i\chi(N_c - N_a)/4) \), we achieve the shift \( N \rightarrow N + (N_c - N_a)/4 \) yielding the final result presented in Eqn.\textsuperscript{12}, where we have introduced the Josephson plasma frequency \( \omega_0 = \sqrt{2E_CE_f}/h \) together with a bosonic creation and annihilation operator \( b^\dagger_0 \) and \( b_0 \), respectively, satisfying \( \chi = \sqrt{E_C/\hbar \omega_0} (b^\dagger_0 - b_0) \) and \( N = (i/2) \sqrt{\hbar \omega_0}/E_C (b^\dagger_0 + b_0) \).

2. Hamiltonian of the short superconducting junction

Now, we describe the Hamiltonian of the SSJ, see Sec.\textsuperscript{14} which can be written as

\[ H_{\text{SSJ}} = \frac{1}{2} \int \text{d}x \Psi^\dagger_\beta(x) H_\beta(x) \Psi_\beta(x), \]  

(A4)

with the Hamiltonian for the short topological junction (\( \beta = tj \)) in the Fu-Kan\textsuperscript{3} model

\[ H_{bj}(x) = -i\hbar v_{bj} \sigma_3 \tau_3 \partial_x - \mu \tau_3 + \Delta(x) e^{i\phi(x) \tau_3} \tau_1 \]  

(A5)

and in the short conventional junction (\( \beta = cj \))

\[ H_{cj}(x) = -i\hbar v_{cj} \sigma_3 \tau_3 \partial_x + i\tau_3 + \Delta(x) e^{i\phi(x) \tau_3} \tau_1, \]  

(A6)

both with \( \phi(x) \) and \( \Delta(x) \) as defined in Eqn.\textsuperscript{5}. Since the SSJ is placed in the dc-SQUID, all phases are related according to Eqn.\textsuperscript{6}, i.e. we replace the phase \( \phi \rightarrow \varphi + \chi + \delta \varphi(t) \) in \( \phi(x) \). In a first step, we remove the time-dependent phase \( \delta \varphi(t) \) by means of the unitary transformation \( U(x, t) = \exp(-i\text{sgn}(x) \delta \varphi(t) \tau_3/4) \) which does not affect the Hamiltonian of the resonator. Transforming the time-dependent BdG equation \( [H_\beta(x) - i\hbar \partial_t] \) such that

\[ U(x, t) [H_\beta(x) - i\hbar \partial_t] U^\dagger(x, t) = -i\hbar \partial_t + \tilde{H}_\beta(x) \]  

\[ + \frac{eV(t)}{2} \text{sgn}(x) \tau_3 + \frac{eV(t)}{2} \sigma_3 \delta(x) \delta \varphi(t), \]  

(A7)
where $\mathcal{H}_\beta(x)$ is given by the corresponding Hamiltonians in Eqsns. (A5) and (A6), respectively, with the replacement $\phi \to \phi + \chi$ in $\phi(x)$. Here, the time-dependent voltage $V(t)$ is given by the Josephson relation $V(t) = \Phi_0 \dot{\phi}(t)$, with the flux quantum $\Phi_0 = \hbar/2e$, whose contribution will be neglected in the following since it does not contribute to transition rates entering a master equation. The last term in Eqn. (A7) gives our first contribution to the interaction Hamiltonian as

$$H_{\text{int}}(t) = \delta \varphi(t) \Phi_0 I_\beta$$

(A8)

describing the coupling to the classical microwave field $\delta \varphi(t)$, with the current operator as defined in Eqn. (16). Now, we still have to perform the unitary transformation $U = \exp(-i\chi(N_+ - N_-)/4)$ as described in appendix A1 with $(N_+ - N_-)$ defined in Eqn. (A2). Therefore, we expand $H_\beta(x)$ to linear order in $\chi$ such that

$$H_\beta(x) \approx H_\beta(x) \bigg|_{\chi=0} + \chi J_0 ,$$

(A9)

with $J_0 = \frac{\partial \mathcal{H}_\beta(x)}{\partial \chi} |_{\chi=0}$. Applying the expanded transformation $U \approx 1 - i\chi(N_+ - N_-)/4$ for small $\chi$ on the Hamiltonian $\mathcal{H}_\beta = \int dx \Psi_\beta^d(x) \mathcal{H}_\beta(x) \Psi_\beta^d/2$, we create a counter-term $-\chi J_0$ and obtain

$$U H_\beta U^\dagger \approx \tilde{H}_\beta + \frac{i}{4} \left[ H_\beta, (N_+ - N_-) \right]$$

$$= \tilde{H}_\beta \bigg|_{\chi=0} + \chi \Phi_0 I_\beta ,$$

(A10)

with, again, the current operator of the SSJ as defined in Eqn. (16). From $\tilde{H}_\beta|_{\chi=0}$, we can read off the corresponding Hamiltonians of the topological and the conventional junction, as presented in Eqsns. (4) and (9), respectively, while

$$H_{\text{int}} = \chi \Phi_0 I_\beta$$

(A11)

is our second contribution to the interaction Hamiltonian describing the coupling to the resonator. The total interaction $H_{\text{int}} + H_{\text{int}}(t)$ is the one presented in Eqn. (15).

### Appendix B: Solutions of the Bogoliubov-de Gennes equations

In this appendix, we provide the solutions of the Bogoliubov-de Gennes (BdG) equation for the short topological (appendix B1) and the short conventional (appendix B2) superconducting junction.

#### 1. Solutions for the topological junction

The BdG Hamiltonian for the topological junction in the short junction limit is given in Eqn. (4). Applying the local unitary transformation $U(x) = e^{-i\phi(x)\gamma_3/2}$ on this Hamiltonian, $H_\beta(x) \to H_\beta'(x) = U(x)H_\beta(x)U^\dagger(x)$, we obtain

$$H_\beta'(x) = -i\hbar v_{ij}\sigma_3 \partial_x - \mu \tau_3$$

$$+ (\Delta(x) \tau_1 + i\hbar v_{ij} \frac{\phi}{2} \sigma_3 \delta(x)),$$

(B1)

from which we find the solutions to the BdG equation $H_\beta'(x)\Phi_E(x) = \mathcal{E}\Phi_E(x)$. Since this is a first-order differential equation, we write the solutions as $\Phi_E(x) = B(x, x_0)\Phi_E(x_0)$ with the transfer matrix

$$B(x, x_0) = \text{P} \exp \left( i\frac{v_{ij}\sigma_3}{\hbar} \int_{x_0}^x dx' \left[ \mathcal{E} - \Delta(x') \right] \right)$$

(B2)

with the ordered exponential for some reference point $x_0$. Hence, the boundary exponential at the interface $x = 0$ is given by

$$B(0^+, 0^-) = e^{-i\phi\tau_3/2}$$

(B3)

which links the solutions on the left and right side of the interface. Due to particle-hole symmetry described in Sec. III A1 we can restrict the calculation to positive energies $\mathcal{E} \geq 0$.

**a. Continuum wave functions**

For $\mathcal{E} > \Delta$, we calculate scattering states $\Phi_{E,s}(x)$, $s \in \{1, 2, 3, 4\}$, which correspond to four possible incident quasiparticles. First, we define the four possible outgoing quasiparticles

$$\Phi^h_-(x) = \sqrt{\frac{N E}{L}} (e^{-\alpha/2}, e^{\alpha/2}, 0, 0)^T e^{i k_{h} x} \Theta(-x),$$

(B4a)

$$\Phi^v_-(x) = \sqrt{\frac{N E}{L}} (0, 0, e^{\alpha/2}, e^{-\alpha/2})^T e^{-i k_{v} x} \Theta(-x),$$

(B4b)

$$\Phi^h_+(x) = \sqrt{\frac{N E}{L}} (e^{\alpha/2}, e^{-\alpha/2}, 0, 0)^T e^{i k_{v} x} \Theta(x),$$

(B4c)

$$\Phi^v_+(x) = \sqrt{\frac{N E}{L}} (0, e^{-\alpha/2}, e^{\alpha/2})^T e^{-i k_{h} x} \Theta(x),$$

(B4d)

where $\Phi^q_\alpha$ is a $q$-like ($q$: electron, $h$: hole) quasiparticle moving to the $q$ ($\leftarrow$: left, $\rightarrow$: right) lead. Here, we defined the energy dependent scattering phase $\alpha(\mathcal{E})$ via

$$e^{\pm \alpha(\mathcal{E})} = \frac{\mathcal{E}}{\Delta} \pm \sqrt{\frac{\mathcal{E}^2 - \Delta^2}{\Delta^2}} - 1 ,$$

(B5)

and the wave numbers $k_{e,h}(\mathcal{E}) = k_{ij} \pm \kappa(\mathcal{E})$ for electron- and hole-like quasiparticles, with the definitions $k_{ij} = \mu/\hbar v_{ij}$ and

$$\kappa(\mathcal{E}) = \frac{\sqrt{\mathcal{E}^2 - \Delta^2}}{\hbar v_{ij}}.$$  

(B6)
Moreover, $N_E = \Delta/2E$ is a normalization constant.

In the same way, we define the four possible incident quasiparticles as

$$\Phi_1^i(x) = \frac{N_E}{L} (e^{\alpha/2}, e^{-\alpha/2}, 0, 0)^T e^{i k_0 x} \Theta(-x), \tag{B7a}$$

$$\Phi_2^i(x) = \frac{N_E}{L} (0, 0, e^{-\alpha/2}, e^{\alpha/2})^T e^{-i k_0 x} \Theta(-x), \tag{B7b}$$

$$\Phi_3^i(x) = \frac{N_E}{L} (0, 0, e^{\alpha/2}, e^{-\alpha/2})^T e^{-i k_0 x} \Theta(x), \tag{B7c}$$

$$\Phi_4^i(x) = \frac{N_E}{L} (e^{-\alpha/2}, e^{\alpha/2}, 0, 0)^T e^{i k_0 x} \Theta(x). \tag{B7d}$$

With the incident and outgoing quasiparticles defined in Eqs. (B7) and (B4), respectively, we write the four scattering states as

$$\Phi_{E,s}(x) = \Phi_1^i(x) + A_0 \Phi_2^o(x) + B_0 \Phi_3^o(x) + C_0 \Phi_4^o(x) \tag{B8}$$

The corresponding scattering coefficients $A_0$, $B_0$, $C_0$, and $D_0$ are obtained by solving the boundary condition \textbf{B3} and the equation

$$\Phi_{E,s}(0^+) = B(0^+, 0^-) \Phi_{E,s}(0^-) \tag{B9}$$

for each of the four scattering states. The full solution reads

$$(A_1, B_1, C_1, D_1) = (A, 0, B, 0), \tag{B10a}$$

$$(A_2, B_2, C_2, D_2) = (0, A^*, 0, B^*), \tag{B10b}$$

$$(A_3, B_3, C_3, D_3) = (0, B^*, 0, A^*), \tag{B10c}$$

$$(A_4, B_4, C_4, D_4) = (B, 0, A, 0), \tag{B10d}$$

with the scattering coefficients

$$A = \frac{\Delta^2}{E^2 - E_M^2} (-i) \sin \phi \left( \alpha - i \frac{\phi}{2} \right), \tag{B11a}$$

$$B = \frac{\Delta^2}{E^2 - E_M^2} \sin \alpha \left( \alpha - i \frac{\phi}{2} \right), \tag{B11b}$$

satisfying $|A|^2 + |B|^2 = 1$. Here, we already introduced $E_M = \Delta \cos(\phi/2)$ which is the energy of the Majorana bound state inside the gap (see appendix B11b).

\textit{b. Majorana bound state wave function}

For energies $E < \Delta$, there is no incident quasiparticle and $\alpha \rightarrow i \alpha$ and $\kappa \rightarrow i \kappa$ become complex. Therefore, the scattering state for the Majorana bound state following from Eqn. (B8) reads

$$\Phi_M(x) = \left( \begin{array}{c} e^{-i \alpha/2} A_0 \exp(i k_0 x) \\ e^{i \alpha/2} B_0 \exp(-i k_0 x) \\ e^{-i \alpha/2} B_0 \exp(-i k_0 x) \\ e^{i \alpha/2} D_0 \exp(i k_0 x) \end{array} \right) e^{\kappa x} \Theta(-x), \tag{B12}$$

with $\alpha(E)$ defined via

$$e^{\pm i \alpha(E)} = \frac{E}{\Delta} \pm i \sqrt{1 - \frac{E^2}{\Delta^2}}. \tag{B13}$$

and the wave numbers $k_{ij} = \mu/\hbar v_{ij}$ and

$$\kappa(E) = \frac{\sqrt{\Delta^2 - E^2}}{\hbar v_{ij}}. \tag{B14}$$

Applying the boundary condition in Eqn. (B3), i.e.

$$\Phi_M(0^+) = B(0^+, 0^-) \Phi_M(0^-), \tag{B15}$$

we find the energy has to fulfill $E = E_M$ with the Majorana bound state energy

$$E_M(\phi) = \Delta \cos \left( \frac{\phi}{2} \right), \tag{B16}$$

in order to find non-trivial solutions for the scattering coefficients $A_0$, $B_0$, $C_0$ and $D_0$. Under this condition, Eqn. (B15) reveals

$$A_0 = C_0, \tag{B17a}$$

$$B_0 = D_0. \tag{B17b}$$

To normalize the subgap wave function, we use

$$1 = \int dx \Phi_M^* \Phi_M(x) = \frac{2}{k_M} \left( |A_0|^2 + |B_0|^2 \right), \tag{B18}$$

with $k_M = \kappa(E_M)$. In the absence of a magnetic field in the non-superconducting part of the topological junction, we lack of one more condition for the scattering amplitudes since left- and right-moving quasiparticles experience no normal scattering without a magnetic field due to the helicity of the conducting states. To match the solution found in Ref. 11 in this limit, we take $A_0 = 0$. We could also choose $B_0 = 0$ in order to achieve maximum current, this would simply lead to current flowing in the other direction. Finally, we obtain the solution

$$A_0 = 0, \tag{B19a}$$

$$B_0 = \sqrt{ \frac{\Delta}{2 \hbar v_{ij}} \sin \left( \frac{\phi}{2} \right) }. \tag{B19b}$$
2. Solutions for the conventional junction

The BdG Hamiltonian for the conventional junction in the short junction limit is given in Eqn. (10). Applying the local unitary transformation $U(x) = e^{i\phi(x)\tau_z/2}$ on this Hamiltonian, $H_{ij}(x) \rightarrow H_{ij}'(x) = U(x)H_{ij}(x)U^\dagger(x)$, we obtain

$$H_{ij}'(x) = -i\hbar v_{cl}\sigma_3\tau_3 + \Delta(x)\tau_1$$
$$+ \hbar v_{cl}\left[\frac{\phi}{2}\sigma_3 + Z\sigma_1\tau_3\right]\delta(x) \quad (B20)$$

from which we find the solutions to the BdG equation $H_{ij}'(x)\Phi_E(x) = E\Phi_E(x)$. Since this is a first-order differential equation, we write the solutions as $\Phi_E(x) = B(x, x_0)\Phi_E(x_0)$ with the transfer matrix

$$B(x, x_0) = \text{Pexp}\left(\frac{1}{\hbar v_{cl}}\int_{x_0}^{x} \left[ E - \Delta(x')\tau_1 \right] \right.$$
$$\left. - \hbar v_{cl}\left[\frac{\phi}{2}\sigma_3 + Z\sigma_1\tau_3\right]\delta(x')\right) \quad (B21)$$

with the ordered exponential for some reference point $x_0$. Hence, the boundary condition at the interface $x = 0$ is given by

$$B(0^+, 0^-) = e^{-i\phi/2} \frac{1}{\sqrt{T}} \left(1 + \sigma_2\sqrt{1 - T}\right) \quad (B22)$$

which links the solutions on the left and right side of the interface. In Eqn. (B22), we have defined the transmission $T = 1/cosh^2 z$. Due to particle-hole symmetry described in Sec. II A 2, we can restrict the calculation to positive energies $E \geq 0$.

a. Continuum wave functions

For $E > \Delta$, we calculate scattering states $\Phi_{E,s}(x)$, $s \in \{1, 2, 3, 4\}$, which correspond to four possible incident quasiparticles. First, we define the four possible outgoing quasiparticles

$$\Phi_{L}^s(x) = \sqrt{\frac{N_E}{L}} (e^{-\alpha/2}, e^{\alpha/2}, 0, 0)^T e^{ikx} \Theta(-x), \quad (B23a)$$
$$\Phi_{R}^s(x) = \sqrt{\frac{N_E}{L}} (0, 0, e^{\alpha/2}, e^{-\alpha/2})^T e^{ikx} \Theta(-x), \quad (B23b)$$
$$\Phi_{L}^s(x) = \sqrt{\frac{N_E}{L}} (e^{-\alpha/2}, e^{\alpha/2}, 0, 0)^T e^{ikx} \Theta(x), \quad (B23c)$$
$$\Phi_{R}^s(x) = \sqrt{\frac{N_E}{L}} (0, 0, e^{-\alpha/2}, e^{\alpha/2})^T e^{ikx} \Theta(x), \quad (B23d)$$

where $\Phi_{L}^s$ is a $r$-like (c: electron, h: hole) quasiparticle moving to the $q$ ($\leftarrow$: left, $\rightarrow$: right) lead. Here, we defined the energy dependent scattering phase $\alpha(E)$ via

$$e^{\pm\alpha(E)} = \frac{E}{\Delta} \pm \sqrt{\frac{E^2}{\Delta^2} - 1}, \quad (B24)$$

and the wave number

$$k(E) = \frac{\sqrt{E^2 - \Delta^2}}{\hbar v_{cl}}. \quad (B25)$$

Moreover, $N_E = \Delta/2E$ is a normalization constant.

In the same way, we define the four possible incident quasiparticles as

$$\Phi_1^s(x) = \sqrt{\frac{N_E}{L}} (e^{\alpha/2}, e^{-\alpha/2}, 0, 0)^T e^{ikx} \Theta(-x), \quad (B26a)$$
$$\Phi_2^s(x) = \sqrt{\frac{N_E}{L}} (0, 0, e^{\alpha/2}, e^{-\alpha/2})^T e^{ikx} \Theta(-x), \quad (B26b)$$
$$\Phi_3^s(x) = \sqrt{\frac{N_E}{L}} (e^{-\alpha/2}, e^{\alpha/2}, 0, 0)^T e^{-ikx} \Theta(x), \quad (B26c)$$
$$\Phi_4^s(x) = \sqrt{\frac{N_E}{L}} (0, 0, e^{-\alpha/2}, e^{\alpha/2})^T e^{-ikx} \Theta(x). \quad (B26d)$$

With the incident and outgoing quasiparticles defined in Eqns. (B26) and (B23), respectively, we write the four scattering states as

$$\Phi_{E,s}(x) = \Phi_{L}^s(x) + A_s \Phi_{R}^s(x) + B_s \Phi_{L}^s(x) + C_s \Phi_{R}^s(x) \quad (B27)$$

The corresponding scattering coefficients $A_s, B_s, C_s$ and $D_s$ are obtained by using the boundary condition in Eqn. (B22), i.e. solving the equation

$$\Phi_{E,s}(0^+) = B(0^+, 0^-) \Phi_{E,s}(0^-) \quad (B28)$$

for each of the four scattering states. The full solution reads

$$(A_1, B_1, C_1, D_1) = (A, B, C, D), \quad (B29a)$$
$$(A_2, B_2, C_2, D_2) = (B^*, A^*, D^*, C^*), \quad (B29b)$$
$$(A_3, B_3, C_3, D_3) = (-D^*, C^*, -B^*, A^*), \quad (B29c)$$
$$(A_4, B_4, C_4, D_4) = (C, -D, A, -B) \quad (B29d)$$

with the scattering coefficients

$$A = \frac{\Delta^2}{E^2 - E_A^2} (-i\hat{T}) \sinh\left(\alpha - \frac{\phi}{2}\right), \quad (B30a)$$
$$B = \frac{\Delta^2}{E^2 - E_A^2} (-i\sqrt{1 - \hat{T}}) \sin^2 \alpha \quad (B30b)$$
$$C = \frac{\Delta^2}{E^2 - E_A^2} \sqrt{T} \sin \alpha \sinh\left(\alpha - \frac{i\phi}{2}\right), \quad (B30c)$$
$$D = \frac{\Delta^2}{E^2 - E_A^2} \sqrt{1 - \hat{T}} \sqrt{T} \sin \frac{\phi}{2} \sinh \alpha \quad (B30d)$$

satisfying $AB + CD = 0$ and $|A|^2 + |B|^2 + |C|^2 + |D|^2 = 1$. Here, we already introduced $E_A = \Delta \sqrt{1 - \hat{T}} \sin^2(\phi/2)$. This is the energy of the Andreev bound state inside the gap (see appendix B2b).
b. Andreev bound state wave function

For energies \( E < \Delta \), there is no incident quasiparticle and \( \alpha \to i \alpha \) and \( k \to ik \) become complex. Therefore, the scattering state for the Andreev bound state following from Eqn. (B27) reads

\[
\Phi_A(x) = \begin{pmatrix} e^{-i\alpha/2}A_0 \\ e^{i\alpha/2}A_0 \\ e^{-i\alpha/2}B_0 \\ e^{i\alpha/2}B_0 \end{pmatrix} e^{kx} \Theta(-x) + \begin{pmatrix} e^{i\alpha/2}C_0 \\ e^{-i\alpha/2}C_0 \\ e^{i\alpha/2}D_0 \\ e^{-i\alpha/2}D_0 \end{pmatrix} e^{-kx} \Theta(x), \tag{B31}
\]

with \( \alpha(E) \) defined via

\[
e^{\pm i\alpha(E)} = \frac{E}{\Delta} \pm i \sqrt{1 - \frac{E^2}{\Delta^2}}. \tag{B32}
\]

and the wave number

\[
k(E) = \frac{\sqrt{\Delta^2 - E^2}}{\hbar v_c}. \tag{B33}
\]

Applying the boundary condition in Eqn. (B22), i.e.

\[
\Phi_A(0^+) = B(0^+, 0^-) \Phi_A(0^-), \tag{B34}
\]

we find the energy has to fulfill \( E = E_A \) with the Andreev bound state energy

\[
E_A(\phi, T) = \Delta \sqrt{1 - T \sin^2 \frac{\phi}{2}}. \tag{B35}
\]

in order to find non-trivial solutions for the scattering coefficients \( A_0, B_0, C_0 \) and \( D_0 \). Under this condition, Eqn. (B34) reveals

\[
\begin{align*}
A_0 &= -C_0, \tag{B36a} \\
B_0 &= D_0, \tag{B36b} \\
B_0 &= -\frac{i}{\sqrt{1 - T}} \left( \sqrt{T} \cos \frac{\phi}{2} + \frac{E_A}{\Delta} \right) A_0. \tag{B36c}
\end{align*}
\]

To normalize the sub-gap wave function, we use

\[
1 = \int dx \left( \Phi_A^*(x) \right)^T \Phi_A(x) = \frac{2}{k_A} \left( |A_0|^2 + |B_0|^2 \right), \tag{B37}
\]

with \( k_A = k(E_A) \), and finally obtain the solution

\[
\begin{align*}
A_0 &= \sqrt{N_0} \sqrt{1 - T} \sin \frac{\phi}{2}, \tag{B38a} \\
B_0 &= -i \sqrt{N_0} \left( \sqrt{T} \cos \frac{\phi}{2} + \frac{E_A}{\Delta} \right) \sin \frac{\phi}{2}, \tag{B38b} \\
N_0 &= \frac{\Delta^2 \sqrt{T}}{4\hbar v_c E_A \sin \frac{\phi}{2} \left( \sqrt{T} \cos \frac{\phi}{2} + \frac{E_A}{\Delta} \right)}, \tag{B38c}
\end{align*}
\]

satisfying \( A_0B_0 + C_0D_0 = 0 \).

Appendix C: Current operator matrix elements

In this appendix, we provide the matrix elements of the current operator for the short topological (appendix C1) and the short conventional (appendix C2) superconducting junction.

First, we are going to write the current operator of Eqn. (16) in terms of the solutions of the Bogoliubov-de Gennes equation for the topological (conventional) junction obtained in appendix B1 (appendix B2). Since the Hamiltonian \( \mathcal{H}_{ij(\sigma)}(x) \) in Eqn. (4) (Eqn. (9)) obeys particle-hole symmetry, described in Sec. II A, we can write the spinors for each junction as

\[
\Psi_{ij}^{(0^+)} = \sum_{n > 0} \left\{ \Phi_n(0^-) \gamma_n + (S_{ij} \Phi_n(0^-))^\dagger \right\}, \tag{C1a}
\]

\[
\Psi_{ij}^{(0^-)} = \sum_{n > 0} \left\{ \Phi_n(0^-) \gamma_{1,n} + (S_{ij} \Phi_n(0^-)) \gamma_{2,n}^\dagger \right\}, \tag{C1b}
\]

by using only the solutions at positive energy, with \( n = (E, s) \) for continuum states \( s \in \{1, 2, 3, 4\} \) with energy \( E > \Delta \) and \( n = E_{M(A)} \) for the Majorana (Andreev) bound state in the topological (conventional) junction. We note that quasiparticle states in the conventional junction are spin-degenerate. Therefore, particle-hole symmetry \( S_{ij} = i \sigma_1 \tau_2 K \) yields \( \gamma_{1,-n} = \gamma_{2,n}^\dagger \), with the definition of the spinor of spin-down quasiparticles \( S_{ij} \Psi_{ij}(x) = (\psi_{R1}(x), -\psi_{L1}(x), \psi_{L1}(x), -\psi_{R1}(x))^T \) denoted by the index \( \nu/2 \).

1. Matrix elements for the topological junction

Using the definition of the spinor in Eqn. (C1a), we calculate the current operator in Eqn. (16) yielding

\[
I_{ij} = \sum_{m,n > 0} \int dx \left( \Phi_{mn}^{(1)} \gamma_n - I_{mn} \gamma_{1,n}^\dagger + \text{h.c.} \right), \tag{C2}
\]

with the matrix elements

\[
\begin{align*}
I_{mn}^{(1)} &= \frac{e v_f}{2} \Phi_m(0^-) \sigma_3 \Phi_n(0^-), \tag{C3a} \\
I_{mn}^{(2)} &= \frac{e v_f}{2} \Phi_m(0^-) \sigma_3 S_{ij} \Phi_n(0^-), \tag{C3b}
\end{align*}
\]

describing transitions of one or two quasiparticles, respectively. As already introduced in Sec. II A, \( S_{ij} = \sigma_2 \tau_2 K \) describes particle-hole symmetry in the topological junction. Choosing \( m = n = E_M \), we obtain the subgap current \( I_M \) carried by the Majorana bound state (MBS) as

\[
I_M = e \Delta \sin \frac{\phi}{2} \left( n_M - \frac{1}{2} \right) = 2e \left. \frac{\partial E_M}{\partial \phi} \right|_{n_M = \frac{1}{2}} \left( n_M - \frac{1}{2} \right), \tag{C4}
\]
with \( n_M \) being the occupation of the MBS. This is the subgap current presented in Eqn. (7) in the main text.

We note that the transfer of a Cooper pair from the ground state to the MBS is not possible because the corresponding matrix element \( t_{\text{MM}}^{\text{two}} = 0 \). Moreover, the matrix elements for transitions involving both the MBS and continuum states are given by

\[
|I_{\text{ME}}^{\text{one/two}}|^2 = \frac{e^2}{4\hbar^2N_{ij}} \frac{E^2 - \Delta^2}{E \mp E_M}, \tag{C5}
\]

where we defined \( |I_{\text{ME}}^{\text{one/two}}|^2 = \sum_{i=1}^4 |I_{\text{ME}(E,s)}^{\text{one/two}}|^2 \) and introduced the density of states \( N_{ij} = L/\pi \hbar v_{ij} \) in one dimension in the normal state of the topological junction.

2. Matrix elements for the conventional junction

Using the definition of the spinor in Eqn. (C1b), we calculate the current operator in Eqn. (16) yielding

\[
I_{\text{cj}} = \sum_{m,n>0} I_{\text{mn}}^{\text{one}}(\gamma_{1,m}^\dagger \gamma_{1,n} + \gamma_{2,m}^\dagger \gamma_{2,n} - 1) + \sum_{m,n>0} (I_{\text{mn}}^{\text{two}} \gamma_{1,m}^\dagger \gamma_{2,n} + \text{h.c.}) \triangleq \sum_{m,n>0} (I_{\text{mn}}^{\text{one/two}}), \tag{C6}
\]

with the matrix elements

\[
I_{\text{mn}}^{\text{one}} = \frac{eV_{ij}}{2} \Phi_m^* T(0^-) \sigma_3 \Phi_n(0^-), \tag{C7a}
\]

\[
I_{\text{mn}}^{\text{two}} = \frac{eV_{ij}}{2} \Phi_m^* T(0^-) \sigma_3 \mathcal{S}_{\text{cj}} \Phi_n(0^-), \tag{C7b}
\]

describing transitions of one or two quasiparticles, respectively. As already introduced in Sec. II A 2, \( \mathcal{S}_{\text{cj}} = \text{sgn}(E - E_M) \) and \( \mathcal{I}_{\text{cj}} \) describes particle-hole symmetry in the conventional junction. Choosing \( m = n = E_A \), we obtain the subgap current \( I_A \) carried by the Andreev bound state (ABS) as

\[
I_A = -\frac{e\Delta^2}{4\hbar} T \sin \phi \left( n_A - 1 \right) - \frac{e^2}{2\hbar} \frac{\partial^2}{\partial \phi^2} \left( n_A - 1 \right), \tag{C8}
\]

with \( n_A = n_1 + n_2 \) being the occupation of the twofold degenerate ABS. The factor 1/2 is a result of the fact that the Bogoliubov-de Gennes Hamiltonian of the conventional junction describes only spin-up particles, while spin-down particles give the same contribution. The subgap current presented in Eqn. (11) in the main text is the full current \( 2I_A \) taking both spins into account.

Moreover, the matrix elements for transitions involving both the ABS and continuum states are given by

\[
|I_{\text{AE}}^{\text{one/two}}|^2 = \frac{e^2 T}{4\pi\hbar^2 N_{ij}} \sqrt{\frac{E_A^2 - \Delta^2}{E_A}} \frac{E^2 - E_A^2}{E_A} \left( E_A(E \mp E_A) \pm \Delta^2 (\cos \phi + 1) \right), \tag{C10}
\]

where we defined \( |I_{\text{AE}}^{\text{one/two}}|^2 = \sum_{i=1}^4 |I_{\text{AE}(E,s)}^{\text{one/two}}|^2 \) and introduced the density of states \( N_{ij} = L/\pi \hbar v_{ij} \) in one dimension in the normal state of the conventional junction.

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This coherent coupling leads to an anticrossing in the spectrum and it occurs for strong coupling between the ABS in a SAC in high-quality superconducting microwave resonators, as observed. Such a coupling was also proposed as new circuit architecture for the circuit QED. The exact treatment of the time-dependent term leading to a coherent evolution was studied in previous works (but neglecting the quasiparticles in the continuum) and is beyond the goal of this work.


