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NONLINEAR THERMOELASTIC PLATE EQUATIONS – GLOBAL EXISTENCE AND DECAY RATES FOR THE CAUCHY PROBLEM

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Abstract. We consider the Cauchy problem in $\mathbb{R}^n$ for some fully nonlinear thermoelastic Kirchhoff type plate equations where heat conduction is modeled by either the Cattaneo law or by the Fourier law. Additionally, we take into account possible inertial effects. Considering nonlinearities which are of fourth-order in the space variable, we deal with a fully nonlinear system which triggers difficulties typical for nonlinear Schrödinger equations. The different models considered are systems of mixed type comparable to Schrödinger–parabolic or Schrödinger–hyperbolic systems. The main task consists in proving sophisticated a priori estimates with the achievement of obtaining the global existence of solutions for small data, neither known nor expected for the Cauchy problem in pure plate theory nor available before for the coupled system under investigation, where only special cases (bounded domains with analytic semigroup setting, or the Cauchy problem with semilinear nonlinearities) had been treated before.

Keywords: fully nonlinear thermoelastic plate, Fourier and Cattaneo law, global existence, Cauchy problem, inertial term

MSC 2010: 35B35, 35B40, 35M30, 35Q79, 74F05

1. Introduction

We consider the Cauchy problem for the following nonlinear thermoplastic plate equation, where heat conduction is modeled by Cattaneo's (Maxwell's, Vernotte's) law ($\tau > 0$) or by Fourier's law ($\tau = 0$), and where an inertial term may be present ($\mu > 0$) or not ($\mu = 0$):

$$\begin{aligned} u_{tt} + \Delta b(\Delta u) - \mu\Delta u_{tt} + \nu\Delta\theta &= 0, \\ \theta_t + \operatorname{div} q - \nu\Delta u_t &= 0, \\ \tau q_t + q + \nabla\theta &= 0. \end{aligned} \tag{1.1}$$

Here, u describes the elongation of a plate, while θ and q denote the temperature (difference to a fixed temperature) resp. the heat flux. For the Cattaneo law the relaxation parameter τ is a positive constant. The constant μ is a non-negative parameter in front of the inertial term. The function b is a given smooth function which satisfies $b'(0) > 0$. Without loss of generality, we assume $b(0) = 0$.

Not affecting the mathematical aspects, we have set most physical constants usually appearing in the equations equal to one, just keeping the constant τ, μ being relevant in particular for the type of the equations, and the positive ν for illustrating the effect in the estimates.

Taking $\tau = 0$, we obtain the standard nonlinear thermoelastic plate equation:

$$\begin{aligned} u_{tt} + \Delta b(\Delta u) - \mu \Delta u_{tt} + \nu \Delta \theta &= 0, \\ \theta_t - \Delta \theta - \nu \Delta u_t &= 0, \end{aligned} \tag{1.2}$$

where the Cattaneo law

$$\tau q_t + q + \nabla \theta = 0 \tag{1.3}$$

has turned into the Fourier law

$$q + \nabla \theta = 0, \tag{1.4}$$

leading to the classical parabolic heat equation appearing in (1.2). We start neglecting variations in time of the temperature, i.e. assuming

$$\theta_t \equiv 0$$

in (1.2), the system reduced to a standard damped nonlinear plate equation:

$$u_{tt} + \Delta b(\Delta u) - \mu \Delta u_{tt} - \nu^2 \Delta u_t = 0. \tag{1.5}$$

Our purpose of this paper is to construct global solutions in time for the Cauchy problem to the equations (1.5), then for (1.2) ($\tau = 0$, with both cases $\mu > 0$ or $\mu = 0$), and finally (1.1) with $\tau > 0$ and $\mu > 0$. Simultaneously we will describe the asymptotic behavior of the global solutions.

We recall the simple relation between the linear Schödinger equation for a complex-valued function v and the linear plate equation for a real-valued function u, saying that both the real part and the imaginary part of v satisfy a plate equation, and that $w := u_t + \mathrm{i}\,\Delta u$ satisfies the Schrödinger equation, we detect behind our fully nonlinear thermoelastic plate equation the nonlinear Schrödinger equation with all its difficulties. cp. [14], even for the *local* well-posedness. Treating fully nonlinear systems hence relies on some kind of damping requiring sophisticated a priori estimates which are the main task of our work.

For special situations we do have some contributions to nonlinear thermoelastic plate equations as there are: For *bounded* domains and Lasiecka and Wilke [13] obtaining global small existence for a fully nonlinear system, where $\tau = 0$ and $\mu = 0$. Extensions to more general nonlinearities, considering > 0, were given by Lasiecka, Pokojovy and Wan in [8], still in bounded domains. Semilinear problems for the *Cauchy problem* have been successfully treated by Fischer [6]. On the other hand, Liu and Kawashima [16] considered the fully nonlinear problem for the plate equation with the damping term u_t instead of $-\Delta u_t$ in (1.5) and obtained global existence in time.

We remark that the *linear* Cauchy problem has been extensively discussed in our paper [22] providing a detailed analysis of the time asymptotic behavior. The latter in turn provides the expectations on the nonlinear problems studied here. It is interesting to notice that for $\mu = 0$ there shows up a so-called regularity-loss when moving from $\tau = 0$ to $\tau > 0$. This kind of essentially changing the qualitative behavior can also be observed for bounded domains (instead of the Cauchy problem in $\mathbb{R}^n$), where the corresponding initial boundary value problem typically shows exponential stability for $\tau = 0$, while it looses this property for $\tau > 0$, see the papers of Quintanilla &

Racke [19, 20]. For bounded domains and $\tau = 0$, there are many results in particular on exponential stability, see for example [1, 7, 9, 10, 11, 12, 15, 17, 18]. For results for the Cauchy problem or in general exterior domains see for example [2, 3, 4, 17, 18]. For $\mu > 0$ the exponential stability is always given [5]. This different linear behavior is reflected in the different systems that are coupled and finally reflected in the different necessary a priori estimates and different regularity results.

Summarizing our new contributions we have

- The first treatment of fully nonlinear thermoelastic plate equations for the Cauchy problem.
- The treatment of a variety of models differing in the heat model or in the inertial term.
- The providing of a prior estimates and the description of the asymptotic behavior of the solutions as time tends to infinity.

The paper is organized as follows: We start in Section 2 with the discussion of the Cauchy problem for the damped plate equation (1.5). Section 3 discusses the nonlinear system (1.2), i.e. the system (1.1) with $\tau = 0$, both for the case with ($\mu > 0$) or without ($\mu = 0$) inertial term. In Section 4, we provide the global existence result for the system (1.1) for the case $\tau > 0$ and for $\mu > 0$; the case $\tau > 0$, $\mu = 0$ remains open. In the last Section 5 we collect some useful inequalities arising from the Gagliardo-Nirenberg inequality.

Throughout the paper, we use standard notation, in particular the Sobolev spaces $L^p = L^p(\mathbb{R}^n)$, $p \geq 1$, and $H^s = W^{s,2}(\mathbb{R}^n)$, $s \in \mathbb{N}_0$, with their associated norms $\|\cdot\|_{L^p}$ resp. $\|\cdot\|_{H^s}$. The symbol ∂_x^l stands for a typical derivative of order l, i.e. $\partial_x^l = \partial_{x_1}^{l_1} \dots \partial_{x_n}^{l_n}$, with $l_1 + \cdots + l_n = l$ and $\partial_{x_j} = \frac{\partial}{\partial x_j}$.

2. Damped plate equation ($\tau = 0$, $\mu \geq 0$, $\theta_t \equiv 0$)

2.1. **Global existence results.** We start in considering the Cauchy problem for the damped plate equation (1.5) arising from the general system (1.1) by taking the Fourier law, $\tau = 0$, and assuming θ_t to be negligible. Independently, it can be regarded as a plate equation with Kelvin-Voigt damping term:

$$
\begin{aligned}
&u_{tt} + \Delta b(\Delta u) - \mu \Delta u_{tt} - \nu^2 \Delta u_t = 0, \\
&u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x),
\end{aligned} \tag{2.1}
$$

where $\mu \geq 0$, $\nu > 0$ and b is a given nonlinear smooth function as introduced in Section 1.

We will prove the following global existence theorem for small data.

Theorem 2.1. [Global existence for $\mu = 0$] *Let $s \geq [n/2] + 1$ and $u_1, \Delta u_0 \in H^{s+2}(\mathbb{R}^n)$. There exists $\varepsilon_0 > 0$ such that if $\|(u_1, \Delta u_0)\|_{H^{s+2}} < \varepsilon_0$, then there is a unique solution u to the initial value problem (2.1), which satisfies $(u_t, \Delta u) \in C^0([0,\infty); H^{s+2}(\mathbb{R}^n))$ and $u_t \in C^1([0,\infty); H^s(\mathbb{R}^n))$ with the energy estimate:*

$$
\|(u_t, \Delta u)(t)\|_{H^{s+2}}^2 + \int_0^t \|\nabla (u_t, \Delta u)(\sigma)\|_{H^{s+2}}^2 d\sigma \leq C \|(u_1, \Delta u_0)\|_{H^{s+2}}^2
$$

for $t \geq 0$. Furthermore, we have the decay estimate:

$$\|\partial_x^\ell (u_t, \Delta u)(t)\|_{L^2} \leq C\|(u_1, \Delta u_0)\|_{H^{s+2}}(1+t)^{-\ell/2} \tag{2.2}$$

for $0 \leq \ell \leq s+2$, where C, here and in the sequel, denotes a positive constant not depending on t or on the data.

Theorem 2.2. [Global existence for $\mu > 0$] *Let $s \geq [n/2]+1$ and $u_1 \in H^{s+2}(\mathbb{R}^n)$, $\Delta u_0 \in H^{s+1}(\mathbb{R}^n)$. There exists $\varepsilon_0 > 0$ such that if $\|u_1\|_{H^{s+2}} + \|\Delta u_0\|_{H^{s+1}} < \varepsilon_0$, then there is a unique solution u to the initial value problem (2.1), which satisfies $u_t \in C^0([0,\infty); H^{s+2}(\mathbb{R}^n))$, $\Delta u \in C^0([0,\infty); H^{s+1}(\mathbb{R}^n))$ with the energy estimate:*

$$\begin{aligned}
&\|u_t(t)\|_{H^{s+2}}^2 + \|\Delta u(t)\|_{H^{s+1}}^2 + \int_0^t (\|\nabla u_t(\sigma)\|_{H^{s+1}}^2 + \|\nabla \Delta u(\sigma)\|_{H^s}^2) d\sigma \\
&\leq C_1(\|u_1\|_{H^{s+2}}^2 + \|\Delta u_0\|_{H^{s+1}}^2),
\end{aligned}$$

for $t \geq 0$. Furthermore, we have the decay estimate:

$$\|\partial_x^\ell u_t(t)\|_{H^2} + \|\Delta \partial_x^\ell u(t)\|_{H^1} \leq C(\|u_1\|_{H^{s+2}} + \|\Delta u_0\|_{H^{s+1}})(1+t)^{-\ell/2} \tag{2.3}$$

for $0 \leq \ell \leq s$.

Remark 2.3. *Comparing the two theorems above, the regularity of the initial data resp. the solutions are not same. This reflects that we essentially have two different types of differential equations for $\mu = 0$ and for $\mu > 0$, respectively.*

For the proof of the Theorems 2.1 and 2.2, we will combine a local existence result with a priori estimate. The final proof will be given in Subsection 2.4.

2.2. **Local existence.** In this subsection, we provide the local in time existence of solutions. These local solutions will finally be extended to global ones by employing a priori estimate. We introduce the following function space that will describe the regularity classes of the solutions:

$$\begin{aligned}
X^s[a,b] &:= \{u \mid (u_t, \Delta u) \in C([a,b]; H^s(\mathbb{R}^n)),\ \nabla u_t \in L^2(a,b; H^s(\mathbb{R}^n))\}, \\
X_\mu^s[a,b] &:= \{u \mid (u_t, \nabla u_t, \Delta u) \in C([a,b]; H^s(\mathbb{R}^n)),\ \nabla u_t \in L^2(a,b; H^s(\mathbb{R}^n))\}.
\end{aligned}$$

Then our local existence results are stated as follows.

Proposition 2.4. [Local existence for $\mu = 0$] *Let $s \geq [n/2]+1$, $t_0 \geq 0$ and $u_t(t_0), \Delta u(t_0) \in H^{s+2}(\mathbb{R}^n)$. There is $R_b > 0$ such that for $0 < R < R_b$ there are $R_0 = R_0(R)$ and $T_0 = T_0(R) > t_0$ such that for $\|(u_t, \Delta u)(t_0)\|_{H^{s+2}} \leq R_0$ there exists a unique solution u to the initial value problem (2.1), which satisfies $u \in X^{s+2}[t_0, T_0]$ and $u_t \in C^1([t_0, T_0]; H^s(\mathbb{R}^n))$ with*

$$\sup_{t \in [t_0, T_0]} \|(u_t, \Delta u)(t)\|_{H^{s+2}} \leq R.$$

Proposition 2.5. [Local existence for $\mu > 0$] *Let $s \geq [n/2]+1$, $t_0 \geq 0$ and $u_t(t_0) \in H^{s+2}(\mathbb{R}^n)$, $\Delta u(t_0) \in H^{s+1}(\mathbb{R}^n)$. There is $R_b > 0$ such that for $0 < R < R_b$ there are $R_0 = R_0(R)$ and $T_0 = T_0(R) > t_0$ such that for $\|u_t(t_0)\|_{H^{s+2}} + \|\Delta u(t_0)\|_{H^{s+1}} \leq R_0$*

there exists a unique solution u to the initial value problem (2.1)*, which satisfies* $u \in X_\mu^{s+1}[t_0, T_0]$ *and* $u_t \in C^1([t_0, T_0]; H^{s-1}(\mathbb{R}^n))$ *with*

$$\sup_{t\in[t_0 T_0]} (\|u_t(t)\|_{H^{s+2}} + \|\Delta u(t)\|_{H^{s+1}}) \le R.$$

Remark 2.6. *The choice of R_b will be determined by inequality* (2.14) *in the proof below.*

Proof of Propositions 2.4 and 2.5. We prove Propositions 2.4 and 2.5 simultaneously. and we can put $t_0 = 0$ without loss of generality. We first analyze the following problem defining iteratively a sequence $(u^k)_{k\in\mathbb{N}_0}$:

$$\begin{cases} u_{tt}^{k+1} + \operatorname{div}(b'(\Delta u^k)\nabla\Delta u^{k+1}) - \mu\Delta u_{tt}^{k+1} - \nu^2\Delta u_t^{k+1} = 0, \\ u^{k+1}(0,x) = u_0(x), \qquad u_t^{k+1}(0,x) = u_1(x) \end{cases} \tag{2.4}$$

where we start with $u^0 \equiv 0$. Here we note that (2.4) is, iteratively, a well-posed *linear* initial value problem for u^{k+1}.

Following the strategy for hyperbolic systems described in [21], we first claim that there exist $R_b > 0$ such that for any $R < R_b$ and for any $T > 0$, there is $R_0 = R_0(T, R) > 0$ such that for all $k \in \mathbb{N}_0$ we have

$$\sup_{0\le t\le T} \|(u_t^k, \Delta u^k)(t)\|_{H^{s+2}} \le R \tag{2.5}$$

for $\mu = 0$, resp.

$$\sup_{0\le t\le T} (\|u_t^k(t)\|_{H^{s+2}} + \|\Delta u^k(t)\|_{H^{s+1}}) \le R \tag{2.6}$$

for $\mu > 0$, provided the data satisfy

$$\|(u_1, \Delta u_0)\|_{H^{s+2}} \le R_0$$

for $\mu = 0$ resp.

$$\|u_1\|_{H^{s+2}} + \|\Delta u_0\|_{H^{s+1}} \le R_0$$

for $\mu > 0$. This claim is proved by induction. We first remark that

$$\sup_{0\le t\le T} (\|\Delta u_t^k(t)\|_{L^\infty} + \|\Delta u^k(t)\|_{W^{1,\infty}}) \le C_{s_0} R \tag{2.7}$$

for $\mu \ge 0$, which is obtained by (5.2) and (2.5).

For $k = 0$ (2.5) and (2.6) are satisfied since $u^0 \equiv 0$. Now we perform the induction step $k \to k+1$: We apply ∂_x^ℓ to (2.4) and obtain

$$\begin{aligned} &\partial_x^\ell u_{tt}^{k+1} + \operatorname{div}(b'(\Delta u^k)\nabla\Delta\partial_x^\ell u^{k+1}) - \mu\Delta\partial_x^\ell u_{tt}^{k+1} \\ &\qquad - \nu^2\Delta\partial_x^\ell u_t^{k+1} + \operatorname{div}([\partial_x^\ell, b'(\Delta u^k)]\nabla\Delta u^{k+1}) = 0 \end{aligned} \tag{2.8}$$

for $\ell \ge 0$. Here we remark that the last term of the left hand side in (2.8) is equal to zero if $\ell = 0$. We multiply (2.8) by $\partial_x^\ell u_t^{k+1}$, and then obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\mathcal{E}_{k+1}^\ell(t,x) + \operatorname{div}\mathcal{F}_{k+1}^\ell(t,x) + \nu^2|\nabla\partial_x^\ell u_t^{k+1}|^2 = \mathcal{R}_{k+1}^\ell(t,x) \tag{2.9}$$

for $\ell \geq 0$. Here we have defined

$$\mathcal{E}_{k+1}^{\ell}(t,x) := (\partial_x^\ell u_t^{k+1})^2 + b'(\Delta u^k)(\Delta \partial_x^\ell u^{k+1})^2 + \mu |\nabla \partial_x^\ell u_t^{k+1}|^2$$

$$\begin{aligned} \mathcal{F}_{k+1}^{\ell}(t,x) := {} & \partial_x^\ell u_t^{k+1} \partial_x^\ell (b'(\Delta u^k)\nabla \Delta u^{k+1}) - b'(\Delta u^k)\Delta \partial_x^\ell u^{k+1} \nabla \partial_x^\ell u_t^{k+1} \\ & - \mu \partial_x^\ell u_t^{k+1} \nabla \partial_x^\ell u_{tt}^{k+1} - \nu^2 \partial_x^\ell u_t^{k+1} \nabla \partial_x^\ell u_t^{k+1}, \end{aligned}$$

$$\begin{aligned} \mathcal{R}_{k+1}^{\ell}(t,x) := {} & -b''(\Delta u^k)\Delta \partial_x^\ell u^{k+1} \nabla \Delta u^k \cdot \nabla \partial_x^\ell u_t^{k+1} + \frac{1}{2} b''(\Delta u^k)\Delta u_t^k (\Delta \partial_x^\ell u^{k+1})^2, \\ & + \nabla \partial_x^\ell u_t^{k+1} \cdot [\partial_x^\ell, b'(\Delta u^k)] \nabla \Delta u^{k+1}. \end{aligned}$$

Then we integrate (2.9) and sum up the resulting equations with respect to $0 \leq \ell \leq s+2$ for $\mu = 0$, or $0 \leq \ell \leq s+1$ for $\mu > 0$, obtaining

$$E_{k+1}^{s+2}(t) + \nu^2 \int_0^t \|\nabla u_t^{k+1}\|_{H^{s+2}}^2 d\sigma = E_{k+1}^{s+2}(0) + \sum_{\ell=0}^{s+2} \int_0^t \int_{\mathbb{R}^n} \mathcal{R}_{k+1}^{\ell}(\sigma,x) dx d\sigma \tag{2.10}$$

for $\mu = 0$, or

$$E_{k+1}^{s+1}(t) + \nu^2 \int_0^t \|\nabla u_t^{k+1}\|_{H^{s+1}}^2 d\sigma = E_{k+1}^{s+1}(0) + \sum_{\ell=0}^{s+1} \int_0^t \int_{\mathbb{R}^n} \mathcal{R}_{k+1}^{\ell}(\sigma,x) dx d\sigma \tag{2.11}$$

for $\mu > 0$, where we have defined

$$E_{k+1}^{s}(t) := \|u_t^{k+1}(t)\|_{H^s}^2 + \mu \|\nabla u_t^{k+1}(t)\|_{H^s}^2 + \sum_{\ell=0}^{s} \int_{\mathbb{R}^n} b'(\Delta u^k)(\Delta \partial_x^\ell u^{k+1})^2 dx.$$

Now, we have $b'(v) = b'(0) + b''(\kappa v)v$ for some $0 < \kappa < 1$, and we obtain

$$\begin{aligned} E_{k+1}^{s}(t) \geq {} & \|u_t^{k+1}(t)\|_{H^s}^2 + b'(0)\|\Delta u^{k+1}(t)\|_{H^s}^2 + \mu \|\nabla u_t^{k+1}(t)\|_{H^s}^2 \\ & - C_{b,R}\|\Delta u^k(t)\|_{L^\infty}\|\Delta u^{k+1}(t)\|_{H^s}^2, \\ E_{k+1}^{s}(t) \leq {} & \|u_t^{k+1}(t)\|_{H^s}^2 + b'(0)\|\Delta u^{k+1}(t)\|_{H^s}^2 + \mu \|\nabla u_t^{k+1}(t)\|_{H^s}^2 \\ & + C_{b,R}\|\Delta u^k(t)\|_{L^\infty}\|\Delta u^{k+1}(t)\|_{H^s}^2, \end{aligned}$$

where we define

$$C_{b,R} := \sup_{|v| \leq C_{s_0} R} |b''(v)|, \tag{2.12}$$

where C_{s_0} is the Sobolev imbedding constant from Lemma 5.2. From these estimates we conclude that there exists C_0 such that

$$C_0^{-1} \tilde{E}_{k+1}^{s}(t) \leq E_{k+1}^{s}(t) \leq C_0 \tilde{E}_{k+1}^{s}(t) \tag{2.13}$$

if we fix R_b satisfying

$$b'(0) - C_{b,R} R_b > 0 \tag{2.14}$$

and choose $R < R_b$. Here we defined $\tilde{E}_{k+1}^{s}(t)$ that

$$\tilde{E}_{k+1}^{s}(t) := \|u_t^{k+1}(t)\|_{H^s}^2 + \|\Delta u^{k+1}(t)\|_{H^s}^2, \qquad \mu = 0,$$

$$\tilde{E}_{k+1}^{s}(t) := \|u_t^{k+1}(t)\|_{H^{s+1}}^2 + \|\Delta u^{k+1}(t)\|_{H^s}^2, \qquad \mu > 0.$$

We estimate the remainder terms as follows.

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |\mathcal{R}^\ell_{k+1}(t,x)|dx \\
&\le \|b''(\Delta u^k)\nabla\Delta u^k\|_{L^\infty}\|\nabla\partial_x^\ell u_t^{k+1}\|_{L^2}\|\Delta\partial_x^\ell u^{k+1}\|_{L^2} + \frac{1}{2}\|b''(\Delta u^k)\Delta u_t^k\|_{L^\infty}\|\Delta\partial_x^\ell u^{k+1}\|_{L^2}^2 \\
&\quad + \|\nabla\partial_x^\ell u_t^{k+1}\|_{L^2}\|[\partial_x^\ell, b'(\Delta u^k)]\nabla\Delta u^{k+1}\|_{L^2} \\
&\le CC_{b,R}\|\nabla\Delta u^k\|_{L^\infty}\|\nabla\partial_x^\ell u_t^{k+1}\|_{L^2}\|\Delta\partial_x^\ell u^{k+1}\|_{L^2} + \frac{C_{b,R}}{2}\|\Delta u_t^k\|_{L^\infty}\|\Delta\partial_x^\ell u^{k+1}\|_{L^2}^2 \\
&\quad + C\|\partial_x^\ell b'(\Delta u^k)\|_{L^2}\|\nabla\Delta u^{k+1}\|_{L^\infty}\|\nabla\partial_x^\ell u_t^{k+1}\|_{L^2}.
\end{aligned}
$$

Here the last term of the last inequality can be neglected if $\ell = 0$. Furthermore, using (2.7) and (5.7) in Section 5, we can estimate $\|\partial_x^\ell b'(\Delta u^k)\|_{L^2} \le \tilde{C}_{b,R}\|\Delta\partial_x^\ell u^k\|_{L^2}$ for $\ell \ge 1$, where

$$
\tilde{C}_{b,R} := C\sum_{j=1}^{\ell}(C_{s_0}R)^{j-1}\sup_{|v|\le C_{s_0}R}|b^{j+1}(v)|.
$$

Thus, employing (2.5) and (2.7) again, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |\mathcal{R}^\ell_{k+1}(t,x)|dx \\
&\le C_R\|\nabla\partial_x^\ell u_t^{k+1}\|_{L^2}(\|\nabla\Delta u^{k+1}\|_{L^\infty} + \|\Delta\partial_x^\ell u^{k+1}\|_{L^2}) + C_R\|\Delta\partial_x^\ell u^{k+1}\|_{L^2}^2
\end{aligned}
$$

for $0 \le t \le T$, where C_R is a certain – generic – constant which depends on R. Using (5.2) and the Hölder inequality for (2.10), we get

$$
\begin{aligned}
E_{k+1}^{s+2}(t) + \frac{\nu^2}{2}\int_0^t \|\nabla u_t^{k+1}\|_{H^{s+2}}^2 d\sigma &\le E_{k+1}^{s+2}(0) + C_R\int_0^t \|\Delta u^{k+1}\|_{H^{s+2}}^2 d\sigma, \quad \mu = 0, \\
E_{k+1}^{s+1}(t) + \frac{\nu^2}{2}\int_0^t \|\nabla u_t^{k+1}\|_{H^{s+1}}^2 d\sigma &\le E_{k+1}^{s+1}(0) + C_R\int_0^t \|\Delta u^{k+1}\|_{H^{s+1}}^2 d\sigma, \quad \mu > 0.
\end{aligned}
$$

for $0 \le t \le T$, Therefore, employing Gronwall's inequality, we get

$$
\begin{aligned}
E_{k+1}^{s+2}(t) + C\int_0^t \|\nabla u_t^{k+1}\|_{H^{s+2}}^2 d\sigma &\le E_{k+1}^{s+2}(0)e^{C_R T}, \qquad \mu = 0, \\
E_{k+1}^{s+1}(t) + C\int_0^t \|\nabla u_t^{k+1}\|_{H^{s+1}}^2 d\sigma &\le E_{k+1}^{s+1}(0)e^{C_R T}, \qquad \mu > 0.
\end{aligned}
$$

Furthermore, using (2.13) and the fact that $\tilde{E}_{k+1}^{s+2}(0) \le R_0^2$ (resp. $\tilde{E}_{k+1}^{s+1}(0) \le R_0^2$), we arrive at

$$
\begin{aligned}
\tilde{E}_{k+1}^{s+2}(t) + C\int_0^t \|\nabla u_t^{k+1}\|_{H^{s+2}}^2 d\sigma &\le C_0^2 R_0^2 e^{C_R T}, \qquad \mu = 0, \\
\tilde{E}_{k+1}^{s+1}(t) + C\int_0^t \|\nabla u_t^{k+1}\|_{H^{s+1}}^2 d\sigma &\le C_0^2 R_0^2 e^{C_R T}, \qquad \mu > 0.
\end{aligned}
$$

Therefore, for a fixed arbitrary $T > 0$, taking $R_0 > 0$ such that $C_0^2 R_0^2 e^{C_R T} \le R$, we conclude the claimed estimates (2.5) resp. (2.6) for $k+1$ replacing k, thus finishing the proof by induction.

Next we demonstrate that $\{u^k\}_{k=0}^{\infty}$ is a Cauchy sequence in appropriate spaces. More precisely we show that there exists $T' = T'(R) > 0$ which satisfies $\bar{E}_{k+1}^{s+1}(T') < \bar{E}_{k}^{s+1}(T')$ for $\mu = 0$, or $\bar{E}_{k+1}^{s}(T') < \bar{E}_{k}^{s}(T')$ for $\mu > 0$, where

$$\bar{E}_k^s(t) := \|v_t^k(t)\|_{H^s}^2 + \mu\|\nabla v_t^k(t)\|_{H^s}^2 + \sum_{\ell=0}^{s}\int_{\mathbb{R}^n} b'(\Delta u^{k-1})(\Delta\partial_x^\ell v^k)^2 dx.$$

with $v^{k+1} := u^{k+1} - u^k$. The disturbance v^{k+1} satisfies

$$\begin{aligned} &v_{tt}^{k+1} - \mu\Delta v_{tt}^{k+1} - \nu^2\Delta v_t^{k+1} \\ &+ \mathrm{div}\{b'(\Delta u^k)\nabla\Delta v^{k+1} + (b'(\Delta u^k) - b'(\Delta u^{k-1}))\nabla\Delta u^k\} = 0. \end{aligned} \tag{2.15}$$

Here we remark that $v^{k+1}(0,x) = 0$ and $v_t^{k+1}(0,x) = 0$. We apply ∂_x^ℓ to (2.15), obtaining

$$\begin{aligned} &\partial_x^\ell v_{tt}^{k+1} + \mathrm{div}(b'(\Delta u^k)\nabla\Delta\partial_x^\ell v^{k+1}) - \mu\Delta\partial_x^\ell v_{tt}^{k+1} - \nu^2\Delta\partial_x^\ell v_t^{k+1} \\ &+ \mathrm{div}\partial_x^\ell\{(b'(\Delta u^k) - b'(\Delta u^{k-1}))\nabla\Delta u^k\} + \mathrm{div}([\partial_x^\ell, b'(\Delta u^k)]\nabla\Delta v^{k+1}) = 0 \end{aligned} \tag{2.16}$$

for $\ell \geq 0$. Here the last term of the left hand side can be neglect if $\ell = 0$. We multiply (2.16) by $\partial_x^\ell v_t^{k+1}$, and then obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\bar{\mathcal{E}}_{k+1}^\ell(t,x) + \mathrm{div}\bar{\mathcal{F}}_{k+1}^\ell(t,x) + \nu^2|\nabla\partial_x^\ell v_t^{k+1}|^2 = \bar{\mathcal{R}}_{k+1}^\ell(t,x) \tag{2.17}$$

for $\ell \geq 0$. where we define that

$$\begin{aligned} \bar{\mathcal{E}}_{k+1}^\ell(t,x) &:= (\partial_x^\ell v_t^{k+1})^2 + b'(\Delta u^k)(\Delta\partial_x^\ell v^{k+1})^2 + \mu|\nabla\partial_x^\ell v_t^{k+1}|^2 \\ \bar{\mathcal{F}}_{k+1}^\ell(t,x) &:= \partial_x^\ell v_t^{k+1}\partial_x^\ell\{b'(\Delta u^k)\nabla\Delta v^{k+1} + (b'(\Delta u^k) - b'(\Delta u^{k-1}))\nabla\Delta u^k\} \\ &\quad - b'(\Delta u^k)\Delta\partial_x^\ell v^{k+1}\nabla\partial_x^\ell v_t^{k+1} - \mu\partial_x^\ell v_t^{k+1}\nabla\partial_x^\ell v_{tt}^{k+1} - \nu^2\partial_x^\ell v_t^{k+1}\nabla\partial_x^\ell v_t^{k+1}, \\ \bar{\mathcal{R}}_{k+1}^\ell(t,x) &:= -b''(\Delta u^k)\Delta\partial_x^\ell v^{k+1}\nabla\Delta u^k\cdot\nabla\partial_x^\ell v_t^{k+1} + \frac{1}{2}b''(\Delta u^k)\Delta u_t^k(\Delta\partial_x^\ell v^{k+1})^2 \\ &\quad + \nabla\partial_x^\ell v_t^{k+1}\cdot\partial_x^\ell\{(b'(\Delta u^k) - b'(\Delta u^{k-1}))\nabla\Delta u^k\} \\ &\quad + \nabla\partial_x^\ell v_t^{k+1}\cdot[\partial_x^\ell, b'(\Delta u^k)]\nabla\Delta v^{k+1}. \end{aligned}$$

We integrate (2.17) and sum up the resulting equations with respect to $0 \leq \ell \leq s+1$ for $\mu = 0$, or $0 \leq \ell \leq s$ for $\mu > 0$. Then we have

$$\begin{aligned} &\bar{E}_{k+1}^{s+1}(T') + 2\nu^2\int_0^{T'}\|\nabla v_t^{k+1}\|_{H^{s+1}}^2 dt = 2\sum_{\ell=0}^{s+1}\int_0^{T'}\int_{\mathbb{R}^n}\bar{\mathcal{R}}_{k+1}^\ell(t,x)dxdt, \quad \mu = 0, \\ &\bar{E}_{k+1}^{s}(T') + 2\nu^2\int_0^{T'}\|\nabla v_t^{k+1}\|_{H^{s}}^2 dt = 2\sum_{\ell=0}^{s}\int_0^{T'}\int_{\mathbb{R}^n}\bar{\mathcal{R}}_{k+1}^\ell(t,x)dxdt, \quad \mu > 0. \end{aligned} \tag{2.18}$$

Here we note that $\bar{E}_{k+1}^{s+1}(0,x) = \bar{E}_{k+1}^{s}(0,x) = 0$. We also estimate the remainder terms. By Lemma 5.4 and

$$b'(\Delta u^k) - b'(\Delta u^{k-1}) = b''(\Delta u^{k-1} + \kappa\Delta v^k)\Delta v^k$$

for some $0 < \kappa < 1$, we calculate

$$\begin{aligned}
&\|\partial_x^\ell\{(b'(\Delta u^k) - b'(\Delta u^{k-1}))\nabla\Delta u^k\}\|_{L^2} \\
&\le C\|b'(\Delta u^k) - b'(\Delta u^{k-1})\|_{L^\infty}\|\nabla\Delta\partial_x^\ell u^k\|_{L^2} \\
&\qquad + C\|\nabla\Delta u^k\|_{L^\infty}\|\partial_x^\ell(b'(\Delta u^k) - b'(\Delta u^{k-1}))\|_{L^2} \\
&\le C\|b''(\Delta u^{k-1} + \kappa\Delta v^k)\|_{L^\infty}(\|\Delta v^k\|_{L^\infty}\|\nabla\Delta\partial_x^\ell u^k\|_{L^2} + \|\nabla\Delta u^k\|_{L^\infty}\|\Delta\partial_x^\ell v^k\|_{L^2}) \\
&\qquad + C\|\nabla\Delta u^k\|_{L^\infty}\|\Delta v^k\|_{L^\infty}\|\partial_x^\ell b''(\Delta u^{k-1} + \kappa\Delta v^k)\|_{L^2}.
\end{aligned}$$

Similarly as before, we can estimate

$$\|\partial_x^\ell b''(\Delta u^{k-1} + \kappa\Delta v^k)\|_{L^2} \le \tilde{C}_{b,2R}(\|\Delta\partial_x^\ell u^{k-1}\|_{L^2} + \|\Delta\partial_x^\ell v^k\|_{L^2})$$

for $\ell \ge 1$, by (2.7) and (5.7). Now we define

$$C_{b,2R} := \sup_{|v|\le 2C_{s_0}R}|b''(v)|, \qquad \tilde{C}_{b,2R} := C\sum_{j=1}^{\ell}(2C_{s_0}R)^{j-1}\sup_{|v|\le 2C_{s_0}R}|b^{j+2}(v)|.$$

Thus, we also obtain

$$\begin{aligned}
&\|\partial_x^\ell\{(b'(\Delta u^k) - b'(\Delta u^{k-1}))\nabla\Delta u^k\}\|_{L^2} \\
&\le CC_{b,2R}(\|\Delta v^k\|_{L^\infty}\|\nabla\Delta\partial_x^\ell u^k\|_{L^2} + \|\nabla\Delta u^k\|_{L^\infty}\|\Delta\partial_x^\ell v^k\|_{L^2}) \\
&\qquad + C\tilde{C}_{b,2R}\|\nabla\Delta u^k\|_{L^\infty}\|\Delta v^k\|_{L^\infty}(\|\Delta\partial_x^\ell u^{k-1}\|_{L^2} + \|\Delta\partial_x^\ell v^k\|_{L^2}).
\end{aligned}$$

and hence

$$\begin{aligned}
&\int_{\mathbb{R}^n}|\bar{\mathcal{R}}_{k+1}^\ell(t,x)|dx \\
&\le \|b''(\Delta u^k)\nabla\Delta u^k\|_{L^\infty}\|\nabla\partial_x^\ell v_t^{k+1}\|_{L^2}\|\Delta\partial_x^\ell v^{k+1}\|_{L^2} + \frac{1}{2}\|b''(\Delta u^k)\Delta u_t^k\|_{L^\infty}\|\Delta\partial_x^\ell v^{k+1}\|_{L^2}^2 \\
&\quad + \|\nabla\partial_x^\ell v_t^{k+1}\|_{L^2}(\|\partial_x^\ell\{(b'(\Delta u^k) - b'(\Delta u^{k-1}))\nabla\Delta u^k\}\|_{L^2} + \|[\partial_x^\ell, b'(\Delta u^k)]\nabla\Delta v^{k+1}\|_{L^2}) \\
&\le CC_{b,R}\|\nabla\Delta u^k\|_{L^\infty}\|\nabla\partial_x^\ell v_t^{k+1}\|_{L^2}\|\Delta\partial_x^\ell v^{k+1}\|_{L^2} + \frac{C_{b,R}}{2}\|\Delta u_t^k\|_{L^\infty}\|\Delta\partial_x^\ell v^{k+1}\|_{L^2}^2 \\
&\quad + C\|\partial_x^\ell b'(\Delta u^k)\|_{L^2}\|\nabla\Delta v^{k+1}\|_{L^\infty}\|\nabla\partial_x^\ell v_t^{k+1}\|_{L^2} \\
&\quad + \|\nabla\partial_x^\ell v_t^{k+1}\|_{L^2}\|\partial_x^\ell\{(b'(\Delta u^k) - b'(\Delta u^{k-1}))\nabla\Delta u^k\}\|_{L^2} \\
&\le C_R\|\nabla\partial_x^\ell v_t^{k+1}\|_{L^2}(\|\nabla\Delta v^{k+1}\|_{L^\infty} + \|\Delta\partial_x^\ell v^{k+1}\|_{L^2}) \\
&\quad + C_R\|\Delta\partial_x^\ell v^{k+1}\|_{L^2}^2 + C_R\|\nabla\partial_x^\ell v_t^{k+1}\|_{L^2}(\|\Delta v^k\|_{L^\infty} + \|\Delta\partial_x^\ell v^k\|_{L^2})
\end{aligned}$$

Applying this estimate and the Hölder inequality to (2.18), we obtain

$$
\begin{aligned}
&\bar{E}^{s+1}_{k+1}(T') + c\int_0^{T'} \|\nabla v_t^{k+1}\|^2_{H^{s+1}} dt \\
&\le C_R T' \sup_{0\le t\le T'} \|\Delta v^k(t)\|^2_{H^{s+1}} + C_R \int_0^{T'} \|\Delta v^{k+1}\|^2_{H^{s+1}} dt, \quad \mu = 0, \\
&\bar{E}^{s}_{k+1}(T') + c\int_0^{T} \|\nabla u_t^{k+1}\|^2_{H^{s}} dt \\
&\le C_R T' \sup_{0\le t\le T'} \|\Delta v^k(t)\|^2_{H^{s}} + C_R \int_0^{T'} \|\Delta v^{k+1}\|^2_{H^{s}} dt, \quad \mu > 0.
\end{aligned}
$$

Therefore, employing Gronwall's inequality, we arrive at

$$
\begin{aligned}
&\bar{E}^{s+1}_{k+1}(T') + c\int_0^{T'} \|\nabla v_t^{k+1}\|^2_{H^{s+1}} dt \le C_R T' e^{C_R T'} \sup_{0\le t\le T'} \|\Delta v^k(t)\|^2_{H^{s+1}}, \quad \mu = 0, \\
&\bar{E}^{s}_{k+1}(T') + c\int_0^{T'} \|\nabla v_t^{k+1}\|^2_{H^{s}} dt \le C_R T' e^{C_R T'} \sup_{0\le t\le T'} \|\Delta v^k(t)\|^2_{H^{s}}, \quad \mu > 0.
\end{aligned}
$$

Moreover, employing the same argument as (2.13), we get $\bar{E}^s_{k+1}(t) \ge C_0^{-1}\bar{\tilde{E}}^s_{k+1}(t)$, where

$$
\begin{aligned}
\bar{\tilde{E}}^s_{k+1}(t) &:= \|v_t^{k+1}(t)\|^2_{H^s} + \|\Delta v^{k+1}(t)\|^2_{H^s}, \qquad \mu = 0, \\
\bar{\tilde{E}}^s_{k+1}(t) &:= \|v_t^{k+1}(t)\|^2_{H^{s+1}} + \|\Delta v^{k+1}(t)\|^2_{H^s}, \qquad \mu > 0.
\end{aligned}
$$

Hence, this yields

$$
\begin{aligned}
&\bar{\tilde{E}}^{s+1}_{k+1}(T') + c\int_0^{T'} \|\nabla v_t^{k+1}\|^2_{H^{s+1}} dt \le C_0 C_R T' e^{C_R T'} \sup_{0\le t\le T'} \|\Delta v^k(t)\|^2_{H^{s+1}}, \quad \mu = 0, \\
&\bar{\tilde{E}}^{s}_{k+1}(T') + c\int_0^{T'} \|\nabla v_t^{k+1}\|^2_{H^{s}} dt \le C_0 C_R T' e^{C_R T'} \sup_{0\le t\le T'} \|\Delta v^k(t)\|^2_{H^{s}}, \quad \mu > 0.
\end{aligned}
$$

Consequently, choosing T' such that $C_0 C_R T' e^{C_R T'} < 1$, we conclude that $\{u^k\}_{k=0}^\infty$ is a Cauchy sequence in $X^{s+1}[0,T']$ for $\mu = 0$ (resp. $\{u^k\}_{k=0}^\infty$ is a Cauchy sequence $X^s_\mu[0,T']$ for $\mu > 0$).

Now we put $T_0 := \min\{T, T'\}$. From these arguments we conclude that there is $u \in X^{s+1}[0,T_0]$ such that u^k tends to u strongly in $X^{s+1}[0,T_0]$ as $k \to \infty$ for $\mu = 0$ (resp. $u \in X^s_\mu[0,T_0]$ for $\mu > 0$). Furthermore, from equation (2.4), we have $u_t \in C^1([0,T_0]; H^{s-1}(\mathbb{R}^n))$ for $\mu = 0$ (resp. $u_t \in C^1([0,T_0]; H^{s-2}(\mathbb{R}^n))$ for $\mu > 0$), and the differential equation (2.1) is satisfied.

Finally, we obtain by using the same arguments as in [21] the final regularity $u \in X^{s+2}[0,T_0]$ for $\mu = 0$ (resp. $u \in X^{s+1}_\mu[0,T_0]$ for $\mu > 0$), and moreover, $u_t \in C^1([0,T_0]; H^s(\mathbb{R}^n))$ for $\mu = 0$ (resp. $u_t \in C^1([0,T_0]; H^{s-1}(\mathbb{R}^n))$ for $\mu > 0$). Hence we completed the proofs for the local existence Propositions 2.4 and 2.5. □

2.3. **A priori estimates.** In this subsection, we prove a priori energy estimates that will allow us to extend the given local solution to a global one.

Proposition 2.7. [A priori estimate for $\mu = 0$] *Let $s \geq [n/2]+1$ and $u_1, \Delta u_0 \in H^{s+2}(\mathbb{R}^n)$. Let u be a solution to the initial value problem (2.1), which satisfies $(u_t, \Delta u) \in C^0([0,T]; H^{s+2}(\mathbb{R}^n))$ for some $T > 0$. There exists $\delta_0 > 0$ such that if the solution satisfies $\sup_{0\leq t\leq T} \|(u_t, \Delta u)(t)\|_{H^{s+2}} \leq \delta_0$, then the solution satisfies the following energy estimate:*

$$\|(u_t, \Delta u)(t)\|^2_{H^{s+2}} + \int_0^t \|\nabla(u_t, \Delta u)(\sigma)\|^2_{H^{s+2}} d\sigma \leq C_1^2 \|(u_1, \Delta u_0)\|^2_{H^{s+2}} \tag{2.19}$$

for $0 \leq t \leq T$, where $C_1 = C_1(\delta_0)$ is a positive constant depending essentially only on δ_0.

Proposition 2.8. [A priori estimate for $\mu > 0$] *Let $s \geq [n/2]+1$ and $u_1 \in H^{s+2}(\mathbb{R}^n)$, $\Delta u_0 \in H^{s+1}(\mathbb{R}^n)$. Let u be a solution u to the initial value problem (2.1), which satisfies $u_t \in C^0([0,T]; H^{s+2}(\mathbb{R}^n))$ and $\Delta u \in C^0([0,T]; H^{s+1}(\mathbb{R}^n))$ for some $T > 0$. There exists $\delta_0 > 0$ such that if the solution satisfies $\sup_{0\leq t\leq T}(\|u_t(t)\|_{H^{s+2}} + \|\Delta u(t)\|_{H^{s+1}}) \leq \delta_0$, then the solution satisfies the following energy estimate:*

$$\begin{aligned} &\|u_t(t)\|^2_{H^{s+2}} + \|\Delta u(t)\|^2_{H^{s+1}} + \int_0^t (\|\nabla u_t(\sigma)\|^2_{H^{s+1}} + \|\nabla \Delta u(\sigma)\|^2_{H^s}) d\sigma \\ &\leq C_1^2(\|u_1\|^2_{H^{s+2}} + \|\Delta u_0\|^2_{H^{s+1}}) \end{aligned} \tag{2.20}$$

for $0 \leq t \leq T$, where $C_1 = C_1 \delta_0)$ is a positive constant depending essentially only on δ_0.

Proof of Proposition 2.7. We note that

$$\sup_{0\leq t\leq T} (\|\Delta u_t(t)\|_{L^\infty} + \|\Delta u(t)\|_{W^{1,\infty}}) \leq C_{s_0}\delta_0 \tag{2.21}$$

for $\mu \geq 0$, which comes from (5.2) in Lemma 5.2 and the assumption in Theorem 2.7 or 2.8.

We first introduce the associated free energy φ,

$$\varphi(v) := \int_0^v b(\eta) d\eta. \tag{2.22}$$

We have

$$\varphi(v) = \frac{1}{2} b'(0) v^2 + \frac{1}{6} b''(\kappa v) v^3 \tag{2.23}$$

for some $0 < \kappa < 1$, and $\varphi \geq 0$ for small v. We obtain

$$\begin{aligned} &\Big\{\frac{1}{2}u_t^2 + \varphi(\Delta u) + \frac{\mu}{2}|\nabla u_t|^2\Big\}_t + \nu^2 |\nabla u_t|^2 \\ &+ \mathrm{div}\Big\{u_t \nabla b(\Delta u) - b(\Delta u)\nabla u_t - \mu u_t \nabla u_{tt} - \nu^2 u_t \nabla u_t\Big\} = 0. \end{aligned} \tag{2.24}$$

On the other hand, we have from (2.1) and the fact $\Delta b(\Delta u) = \mathrm{div}(b'(\Delta u)\nabla \Delta u)$, that

$$\partial_x^\ell u_{tt} + \mathrm{div} \partial_x^\ell (b'(\Delta u)\nabla\Delta u) - \mu \Delta \partial_x^\ell u_{tt} - \nu^2 \Delta \partial_x^\ell u_t = 0 \tag{2.25}$$

for $\ell \geq 0$. Multiplying (2.25) by $\partial_x^\ell u_t$, this yields

$$
\begin{aligned}
&\frac{1}{2}\{(\partial_x^\ell u_t)^2 + b'(\Delta u)(\Delta\partial_x^\ell u)^2 + \mu|\nabla\partial_x^\ell u_t|^2\}_t + \nu^2|\nabla\partial_x^\ell u_t|^2 \\
&- \nabla\partial_x^\ell u_t \cdot [\partial_x^\ell, b'(\Delta u)]\nabla\Delta u + b''(\Delta u)\Delta\partial_x^\ell u\nabla\Delta u \cdot \nabla\partial_x^\ell u_t \\
&- \frac{1}{2}b''(\Delta u)\Delta u_t(\Delta\partial_x^\ell u)^2 + \mathrm{div}\{\partial_x^\ell u_t\partial_x^\ell(b'(\Delta u)\nabla\Delta u)\} \\
&- \mathrm{div}\{b'(\Delta u)\Delta\partial_x^\ell u\nabla\partial_x^\ell u_t + \mu\partial_x^\ell u_t\nabla\partial_x^\ell u_{tt} + \nu^2\partial_x^\ell u_t\nabla\partial_x^\ell u_t\} = 0
\end{aligned}
\tag{2.26}
$$

for $\ell \geq 1$. Similarly, multiplying (2.25) by $-\Delta\partial_x^\ell u$, we have

$$
\begin{aligned}
&\Big\{\frac{\nu^2}{2}(\Delta\partial_x^\ell u)^2 - \partial_x^\ell u_t\Delta\partial_x^\ell u + \mu\Delta\partial_x^\ell u\Delta\partial_x^\ell u_t\Big\}_t \\
&+ b'(\Delta u)|\nabla\Delta\partial_x^\ell u|^2 - |\nabla\partial_x^\ell u_t|^2 - \mu|\Delta\partial_x^\ell u_t|^2 + \nabla\Delta\partial_x^\ell u \cdot [\partial_x^\ell, b'(\Delta u)]\nabla\Delta u \\
&+ \mathrm{div}\{\partial_x^\ell u_t\nabla\partial_x^\ell u_t - \Delta\partial_x^\ell u\partial_x^\ell(b'(\Delta u)\nabla\Delta u)\} = 0
\end{aligned}
\tag{2.27}
$$

for $\ell \geq 0$. We compute that $2\times(2.24)+\nu^2\times(2.27)$ with $\ell = 0$ or $2\times(2.26)+\nu^2\times(2.27)$. Then we obtain

$$
\frac{\partial}{\partial t}\mathcal{E}^\ell(t,x) + \mathcal{D}^\ell(t,x) + \mathrm{div}\mathcal{F}^\ell(t,x) = \mathcal{R}^\ell(t,x)
\tag{2.28}
$$

for $\ell \geq 0$. Here we define that

$$
\mathcal{E}^0(t,x) := u_t^2 + 2\varphi(\Delta u) + \mu|\nabla u_t|^2 + \frac{\nu^4}{2}(\Delta u)^2 - \nu^2 u_t\Delta u + \nu^2\mu\Delta u\Delta u_t, \quad \mathcal{R}^0(t,x) := 0,
$$

$$
\mathcal{F}^0(t,x) := 2u_t\nabla b(\Delta u) - 2b(\Delta u)\nabla u_t - 2\mu u_t\nabla u_{tt} - 2\nu^2 u_t\nabla u_t + \nu^2 u_t\nabla u_t - \nu^2\Delta u\nabla b(\Delta u),
$$

and

$$
\begin{aligned}
\mathcal{E}^\ell(t,x) &:= (\partial_x^\ell u_t)^2 + b'(\Delta u)(\Delta\partial_x^\ell u)^2 + \mu|\nabla\partial_x^\ell u_t|^2 \\
&\qquad + \frac{\nu^4}{2}(\Delta\partial_x^\ell u)^2 - \nu^2\partial_x^\ell u_t\Delta\partial_x^\ell u + \nu^2\mu\Delta\partial_x^\ell u\Delta\partial_x^\ell u_t, \\
\mathcal{F}^\ell(t,x) &:= 2\partial_x^\ell u_t\partial_x^\ell(b'(\Delta u)\nabla\Delta u) - 2b'(\Delta u)\Delta\partial_x^\ell u\nabla\partial_x^\ell u_t - 2\mu\partial_x^\ell u_t\nabla\partial_x^\ell u_{tt} \\
&\qquad - 2\nu^2\partial_x^\ell u_t\nabla\partial_x^\ell u_t + \nu^2\partial_x^\ell u_t\nabla\partial_x^\ell u_t - \nu^2\Delta\partial_x^\ell u\partial_x^\ell(b'(\Delta u)\nabla\Delta u) \\
\mathcal{R}^\ell(t,x) &:= 2\nabla\partial_x^\ell u_t \cdot [\partial_x^\ell, b'(\Delta u)]\nabla\Delta u - 2b''(\Delta u)\Delta\partial_x^\ell u\nabla\Delta u \cdot \nabla\partial_x^\ell u_t \\
&\qquad + b''(\Delta u)\Delta u_t(\Delta\partial_x^\ell u)^2 - \nu^2\nabla\Delta\partial_x^\ell u \cdot [\partial_x^\ell, b'(\Delta u)]\nabla\Delta u
\end{aligned}
$$

for $\ell > 0$, and

$$
\mathcal{D}^\ell(t,x) := \nu^2|\nabla\partial_x^\ell u_t|^2 + \nu^2 b'(\Delta u)|\nabla\Delta\partial_x^\ell u|^2 - \nu^2\mu|\Delta\partial_x^\ell u_t|^2
$$

for $\ell \geq 0$.

We integrate (2.28) and get

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\mathcal{E}^\ell(t,x)dx + \nu^2\int_0^t\|\nabla\partial_x^\ell u_t\|_{L^2}^2 d\sigma + \nu^2\int_0^t\int_{\mathbb{R}^n} b'(\Delta u)|\nabla\Delta\partial_x^\ell u|^2 dxd\sigma \\
&\qquad = \int_{\mathbb{R}^n}\mathcal{E}^\ell(0,x)dx + \int_0^t\int_{\mathbb{R}^n}\mathcal{R}^\ell(\sigma,x)dxd\sigma.
\end{aligned}
$$

Employing (2.21), (2.23) and the fact that,

$$\nu^2\int_{\mathbb{R}^n}|\partial_x^\ell u_t||\Delta\partial_x^\ell u|dx \le \frac{2\nu^4}{7}\|\Delta\partial_x^\ell u\|_{L^2}^2+\frac{7}{8}\|\partial_x^\ell u_t\|_{L^2}^2,$$
$$\nu^2\mu\int_{\mathbb{R}^n}|\Delta\partial_x^\ell u||\Delta\partial_x^\ell u_t|dx \le \frac{\nu^4}{7}\|\Delta\partial_x^\ell u\|_{L^2}^2+\frac{7\mu^2}{4}\|\Delta\partial_x^\ell u_t\|_{L^2}^2,$$

we calculate

$$\begin{aligned}\int_{\mathbb{R}^n}\mathcal{E}^\ell(t,x)dx &\ge \frac{1}{8}\|\partial_x^\ell u_t(t)\|_{L^2}^2+(b'(0)-C_{s_0}C_{b,\delta_0}\delta_0+\frac{\nu^4}{14})\|\Delta\partial_x^\ell u(t)\|_{L^2}^2\\ &\quad+\mu\|\nabla\partial_x^\ell u_t(t)\|_{L^2}^2-\frac{7\mu^2}{4}\|\Delta\partial_x^\ell u_t\|_{L^2}^2,\\ \int_{\mathbb{R}^n}\mathcal{E}^\ell(t,x)dx &\le \frac{15}{8}\|\partial_x^\ell u_t(t)\|_{L^2}^2+(b'(0)+C_{s_0}C_{b,\delta_0}\delta_0+\frac{13\nu^4}{14})\|\Delta\partial_x^\ell u(t)\|_{L^2}^2\\ &\quad+\mu\|\nabla\partial_x^\ell u_t(t)\|_{L^2}^2+\frac{7\mu^2}{4}\|\Delta\partial_x^\ell u_t\|_{L^2}^2,\end{aligned}\tag{2.29}$$

for $\ell\ge 0$, where we defined $C_{b,\delta_0}:=\sup_{|v|\le C_{s_0}\delta_0}|b''(v)|$ and fix δ_0 which satisfies

$$b'(0)-C_{s_0}C_{b,\delta_0}\delta_0>0.\tag{2.30}$$

Especially, in the case $\mu=0$ we are discussing at the moment, we can derive that there exists C such that

$$C^{-1}E^{s+2}(t)\le\sum_{\ell=0}^{s+2}\int_{\mathbb{R}^n}\mathcal{E}^\ell(t,x)dx\le CE^{s+2}(t),\tag{2.31}$$

where we defined $E^s(t):=\|(u_t,\Delta u)(t)\|_{H^s}^2$. We estimate the remainder terms.

$$\begin{aligned}&\int_{\mathbb{R}^n}|\mathcal{R}^\ell(t,x)|dx\\ &\le C\|\nabla\partial_x^\ell(u_t,\Delta u)\|_{L^2}\|[\partial_x^\ell,b'(\Delta u)]\nabla\Delta u\|_{L^2}+C\|b''(\Delta u)\Delta u_t\|_{L^\infty}\|\Delta\partial_x^\ell u\|_{L^2}^2\\ &\qquad+C\|b''(\Delta u)\nabla\Delta u\|_{L^\infty}\|\Delta\partial_x^\ell u\|_{L^2}\|\nabla\partial_x^\ell u_t\|_{L^2}\\ &\le C(C_{b,\delta_0}+\tilde{C}_{b,\delta_0})\|\nabla\Delta u\|_{L^\infty}\|\Delta\partial_x^\ell u\|_{L^2}\|\nabla\partial_x^\ell(u_t,\Delta u)\|_{L^2}\\ &\qquad+CC_{b,\delta_0}\|\Delta u_t\|_{L^\infty}\|\Delta\partial_x^\ell u\|_{L^2}^2\end{aligned}\tag{2.32}$$

for $\ell\ge 1$. Here we used that

$$\begin{aligned}\|[\partial_x^\ell,b'(\Delta u)]\nabla\Delta u\|_{L^2}&\le C(\|\partial_x b'(\Delta u)\|_{L^\infty}\|\nabla\Delta\partial_x^{\ell-1}u\|_{L^2}+\|\nabla\Delta u\|_{L^\infty}\|\partial_x^\ell b'(\Delta u)\|_{L^2})\\ &\le C(C_{b,\delta_0}+\tilde{C}_{b,\delta_0})\|\nabla\Delta u\|_{L^\infty}\|\Delta\partial_x^\ell u\|_{L^2}\end{aligned}$$

for $\ell\ge 1$, derived from Lemma 5.4 and the same argument as in the previous subsection, where

$$\tilde{C}_{b,\delta_0}:=C\sum_{j=1}^{\ell}(C_{s_0}\delta_0)^{j-1}\sup_{|v|\le C_{s_0}\delta_0}|b^{j+1}(v)|.$$

Thus, we derive

$$\begin{aligned}&\int_{\mathbb{R}^n}\mathcal{E}^\ell(t,x)dx+\nu^2\int_0^t\|\nabla\partial_x^\ell u_t\|_{L^2}^2d\sigma+(b'(0)-C_{s_0}C_{b,\delta_0}\delta_0)\int_0^t\|\nabla\Delta\partial_x^\ell u\|_{L^2}^2d\sigma\\&\le\int_{\mathbb{R}^n}\mathcal{E}^\ell(0,x)dx+CC_{b,\delta_0}\sup_{0\le\sigma\le t}\|\Delta u_t\|_{L^\infty}\int_0^t\|\Delta\partial_x^\ell u\|_{L^2}^2d\sigma\\&\quad+C(C_{b,\delta_0}+\tilde{C}_{b,\delta_0})\sup_{0\le\sigma\le t}\|\nabla\Delta u\|_{L^\infty}\int_0^t\|\Delta\partial_x^\ell u\|_{L^2}\|\nabla\partial_x^\ell(u_t,\Delta u)\|_{L^2}d\sigma\end{aligned}$$

for $\ell\ge 0$. Here the second and third terms in the right hand side can be neglected if $\ell=0$. Thus, using (2.21), the Hölder inequality and the fact that $b'(0)-C_{s_0}C_{b,\delta_0}\delta_0>0$, we can get

$$\begin{aligned}&\int_{\mathbb{R}^n}\mathcal{E}^\ell(t,x)dx+\int_0^t\|\nabla\partial_x^\ell(u_t,\Delta\partial_x^\ell u)\|_{L^2}^2d\sigma\\&\le C\int_{\mathbb{R}^n}\mathcal{E}^\ell(0,x)dx+C(1+\delta_0)\delta_0\int_0^t\|\Delta\partial_x^\ell u\|_{L^2}^2d\sigma\end{aligned}\tag{2.33}$$

for $\ell\ge 0$. Here we also remark that the last term in the right hand side can be neglect if $\ell=0$, and the certain constant C depends on δ_0.

We apply the induction argument with respect to ℓ to (2.33), obtaining

$$\int_{\mathbb{R}^n}\mathcal{E}^\ell(t,x)dx+\int_0^t\|\nabla\partial_x^\ell(u_t,\Delta u)\|_{L^2}^2d\sigma\le C(1+\delta_0^2)^{2(s+2)}E^{s+2}(0)$$

for $0\le\ell\le s+2$. Consequently, employing (2.31), we arrive that

$$E^{s+2}(t)+\int_0^t\|\nabla(u_t,\Delta u)\|_{H^{s+2}}^2d\sigma\le C(1+\delta_0^2)^{2s+2}E^{s+2}(0),$$

and then we get the useful a priori estimate for $s\ge[n/2]+1$. □

Proof of Proposition 2.8. We start the proof from (2.28). In the case $\mu>0$, we must control the term $|\Delta\partial_x^\ell u_t|^2$ in (2.28). To this end, we combine (2.28) and $4\mu\times$ (2.26) which is replaced ℓ by $\ell+1$. Then we have

$$\frac{\partial}{\partial t}\tilde{\mathcal{E}}^\ell(t,x)+\tilde{\mathcal{D}}^\ell(t,x)+\mathrm{div}\tilde{\mathcal{F}}^\ell(t,x)=\tilde{\mathcal{R}}^\ell(t,x)\tag{2.34}$$

for $\ell\ge 0$, where

$$\begin{aligned}\tilde{\mathcal{E}}^\ell(t,x)&:=\mathcal{E}^\ell(t,x)+2\mu(\partial_x^{\ell+1}u_t)^2+2\mu b'(\Delta u)(\Delta\partial_x^{\ell+1}u)^2+2\mu^2|\nabla\partial_x^{\ell+1}u_t|^2,\\\tilde{\mathcal{D}}^\ell(t,x)&:=\nu^2|\nabla\partial_x^\ell u_t|^2+\nu^2b'(\Delta u)|\nabla\Delta\partial_x^\ell u|^2+3\nu^2\mu|\nabla\partial_x^{\ell+1}u_t|^2,\\\tilde{\mathcal{F}}^\ell(t,x)&:=\mathcal{F}^\ell(t,x)+4\mu\partial_x^{\ell+1}u_t\partial_x^{\ell+1}(b'(\Delta u)\nabla\Delta u)-4\mu b'(\Delta u)\Delta\partial_x^{\ell+1}u\nabla\partial_x^{\ell+1}u_t\\&\qquad-4\mu^2\partial_x^{\ell+1}u_t\nabla\partial_x^{\ell+1}u_{tt}-4\nu^2\mu\partial_x^{\ell+1}u_t\nabla\partial_x^{\ell+1}u_t,\\\tilde{\mathcal{R}}^\ell(t,x)&:=\mathcal{R}^\ell(t,x)+4\mu\nabla\partial_x^{\ell+1}u_t\cdot[\partial_x^{\ell+1},b'(\Delta u)]\nabla\Delta u\\&\qquad-4\mu b''(\Delta u)\Delta\partial_x^{\ell+1}u\nabla\Delta u\cdot\nabla\partial_x^{\ell+1}u_t+2\mu b''(\Delta u)\Delta u_t(\Delta\partial_x^{\ell+1}u)^2.\end{aligned}$$

Then integrating (2.34), we obtain

$$\int_{\mathbb{R}^n} \tilde{\mathcal{E}}^\ell(t,x)dx + \nu^2 \int_0^t (\|\nabla \partial_x^\ell u_t\|_{L^2}^2 + 3\mu \|\nabla \partial_x^{\ell+1} u_t\|_{L^2}^2) d\sigma$$
$$+ \nu^2 \int_0^t \int_{\mathbb{R}^n} b'(\Delta u)|\nabla \Delta \partial_x^\ell u|^2 dx d\sigma = \int_{\mathbb{R}^n} \tilde{\mathcal{E}}^\ell(0,x)dx + \int_0^t \int_{\mathbb{R}^n} \tilde{\mathcal{R}}^\ell(\sigma,x)dxd\sigma.$$

Here, employing (2.29), we compute that

$$\int_{\mathbb{R}^n} \tilde{\mathcal{E}}^\ell(t,x)dx \geq \frac{1}{8}\|\partial_x^\ell u_t(t)\|_{L^2}^2 + (b'(0) - C_{s_0} C_{b,\delta_0}\delta_0 + \frac{\nu^4}{14})\|\Delta \partial_x^\ell u(t)\|_{L^2}^2$$
$$+ 3\mu \|\nabla \partial_x^\ell u_t(t)\|_{L^2}^2 + \frac{\mu^2}{4}\|\Delta \partial_x^\ell u_t\|_{L^2}^2 + 2\mu(b'(0) - C_{s_0} C_{b,\delta_0}\delta_0)\|\Delta \partial_x^{\ell+1} u(t)\|_{L^2}^2,$$
$$\int_{\mathbb{R}^n} \tilde{\mathcal{E}}^\ell(t,x)dx \leq \frac{15}{8}\|\partial_x^\ell u_t(t)\|_{L^2}^2 + (b'(0) + C_{s_0} C_{b,\delta_0}\delta_0 + \frac{13\nu^4}{14})\|\Delta \partial_x^\ell u(t)\|_{L^2}^2$$
$$+ 3\mu \|\nabla \partial_x^\ell u_t(t)\|_{L^2}^2 + \frac{15\mu^2}{4}\|\Delta \partial_x^\ell u_t\|_{L^2}^2 + 2\mu(b'(0) + C_{s_0} C_{b,\delta_0}\delta_0)\|\Delta \partial_x^{\ell+1} u(t)\|_{L^2}^2.$$

Thus there exists C such that

$$C^{-1}\tilde{E}^s(t) \leq \sum_{\ell=0}^{s} \int_{\mathbb{R}^n} \tilde{\mathcal{E}}^\ell(t,x)dx \leq C\tilde{E}^s(t), \tag{2.35}$$

where we defined that $\tilde{E}^s(t) := \|u_t(t)\|_{H^{s+2}}^2 + \|\Delta u(t)\|_{H^{s+1}}^2$. For the remainder terms we use the same argument as before. Then we calculate

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\tilde{\mathcal{R}}^\ell(t,x)|dx \\
&\leq \int_{\mathbb{R}^n} |\mathcal{R}^\ell(t,x)|dx + C\|b''(\Delta u)\nabla \Delta u\|_{L^\infty}\|\Delta \partial_x^{\ell+1} u\|_{L^2}\|\nabla \partial_x^{\ell+1} u_t\|_{L^2} \\
&\quad + C\|\nabla \partial_x^{\ell+1} u_t\|_{L^2}\|[\partial_x^{\ell+1}, b'(\Delta u)]\nabla \Delta u\|_{L^2} + C\|b''(\Delta u)\Delta u_t\|_{L^\infty}\|\Delta \partial_x^{\ell+1} u\|_{L^2}^2 \\
&\leq \int_{\mathbb{R}^n} |\mathcal{R}^\ell(t,x)|dx + C(C_{b,\delta_0} + \tilde{C}_{b,\delta_0})\|\nabla \Delta u\|_{L^\infty}\|\Delta \partial_x^{\ell+1} u\|_{L^2}\|\nabla \partial_x^{\ell+1} u_t\|_{L^2} \\
&\quad + CC_{b,\delta_0}\|\Delta u_t\|_{L^\infty}\|\Delta \partial_x^{\ell+1} u\|_{L^2}^2.
\end{aligned} \tag{2.36}$$

Therefore we get

$$\begin{aligned}&\int_{\mathbb{R}^n}\tilde{\mathcal{E}}^\ell(t,x)dx+\nu^2\int_0^t(\|\nabla\partial_x^\ell u_t\|_{L^2}^2+3\mu\|\nabla\partial_x^{\ell+1}u_t\|_{L^2}^2)d\sigma\\&\qquad+\nu^2(b'(0)-C_{s_0}C_{b,\delta_0}\delta_0)\int_0^t\|\nabla\Delta\partial_x^\ell u\|_{L^2}^2d\sigma\\&\le\int_{\mathbb{R}^n}\tilde{\mathcal{E}}^\ell(0,x)dx+CC_{b,\delta_0}\sup_{0\le\sigma\le t}\|\Delta u_t\|_{L^\infty}\int_0^t\|\Delta\partial_x^\ell u\|_{L^2}^2d\sigma\\&\qquad+C(C_{b,\delta_0}+\tilde{C}_{b,\delta_0})\sup_{0\le\sigma\le t}\|\nabla\Delta u\|_{L^\infty}\int_0^t\|\Delta\partial_x^\ell u\|_{L^2}\|\nabla\partial_x^\ell(u_t,\Delta u)\|_{L^2}d\sigma\\&\qquad+CC_{b,\delta_0}\sup_{0\le\sigma\le t}\|\Delta u_t\|_{L^\infty}\int_0^t\|\Delta\partial_x^{\ell+1}u\|_{L^2}^2d\sigma\\&\qquad+C(C_{b,\delta_0}+\tilde{C}_{b,\delta_0})\sup_{0\le\sigma\le t}\|\nabla\Delta u\|_{L^\infty}\int_0^t\|\Delta\partial_x^{\ell+1}u\|_{L^2}\|\nabla\partial_x^{\ell+1}u_t\|_{L^2}d\sigma.\end{aligned}$$

Here we remark that the second and third terms in the right hand side can be neglect if $\ell=0$. Furthermore, using the Hölder inequality and the fact that $b'(0)-C_{s_0}C_{b,\delta_0}\delta_0>0$, this yields

$$\begin{aligned}&\int_{\mathbb{R}^n}\tilde{\mathcal{E}}^\ell(t,x)dx+\int_0^t(\|\nabla\partial_x^\ell u_t\|_{H^1}^2+\|\nabla\Delta\partial_x^\ell u\|_{L^2}^2)d\sigma\\&\qquad\le C\int_{\mathbb{R}^n}\tilde{\mathcal{E}}^\ell(0,x)dx+C\delta_0(1+\delta_0)\int_0^t\|\Delta\partial_x^{\ell+1}u\|_{L^2}^2d\sigma+C\delta_0\int_0^t\|\Delta\partial_x^\ell u\|_{L^2}^2d\sigma.\end{aligned}$$

Here we also remark that the last term in the right hand side can be neglect if $\ell=0$. We now choose δ_0 sufficiently small. Then we get

$$\begin{aligned}&\int_{\mathbb{R}^n}\tilde{\mathcal{E}}^\ell(t,x)dx+\int_0^t(\|\nabla\partial_x^\ell u_t\|_{H^1}^2+\|\nabla\Delta\partial_x^\ell u\|_{L^2}^2)d\sigma\\&\le C\int_{\mathbb{R}^n}\tilde{\mathcal{E}}^\ell(0,x)dx+C\delta_0\int_0^t\|\Delta\partial_x^\ell u\|_{L^2}^2d\sigma.\end{aligned}\tag{2.37}$$

Since the last term of (2.37) is neglected if $\ell=0$, we can employ the induction argument for $\ell\ge0$. Hence, utilizing (2.35), we arrive at the estimate

$$\int_{\mathbb{R}^n}\tilde{\mathcal{E}}^\ell(t,x)dx+\int_0^t(\|\nabla\partial_x^\ell u_t\|_{H^1}^2+\|\nabla\Delta\partial_x^\ell u\|_{L^2}^2)d\sigma\le C(1+\delta_0^2)^{2s}\tilde{E}^s(0).$$

for $0\le\ell\le s$, and hence

$$\tilde{E}^s(t)+\int_0^t(\|\nabla u_t\|_{H^{s+1}}^2+\|\nabla\Delta u\|_{H^s}^2)d\sigma\le C(1+\delta_0^2)^{2s}\tilde{E}^s(0).$$

Consequently we conclude the desired a priori estimate for $s\ge[n/2]+1$. □

2.4. **Proof of global existence and decay estimates.** In the last subsection, we construct the global solution and derive the corresponding decay estimates, that is we finish the proofs of Theorems 2.1 and 2.2.

Proof of Theorem 2.1. Let $R_b > 0$ be as in Proposition 2.4 satisfying (2.14). Then we choose $R > 0$ that $R := \min\{R_b/2, \delta_0\}$, where δ_0 is given in Proposition 2.7. According to Proposition 2.4 with $t_0 = 0$ there exists a unique solution u on some time interval $[0, T(R)]$ with

$$\sup_{0 \le t \le T} \|(u_t, \Delta u)(t)\|_{H^{s+2}} \le R \ \ (\le \delta_0). \tag{2.38}$$

Here we recall that there is $R > R_0(R) > 0$ for which the initial data must satisfy $\|(u_1, \Delta u_0)\|_{H^{s+2}} \le R_0$. Now, we choose the initial data which satisfy

$$\|(u_1, \Delta u_0)\|_{H^{s+2}} \le \varepsilon_0 \tag{2.39}$$

with the definition

$$\varepsilon_0 := \min\{R_0, \frac{R_0}{C_1(\delta_0)}\}.$$

Then, by the a priori estimate from Proposition 2.7 we conclude using (2.39)

$$\sup_{0 \le t \le T} \|(u_t, \Delta u)(t)\|_{H^{s+2}} \le C_1(\delta_0)\|(u_1, \Delta u_0)\|_{H^{s+2}} \le R_0. \tag{2.40}$$

This holds in particular for $t = T$, hence, with the same arguments as before, we have a solution u on some time interval $[0, 2T(R)]$, where u satisfies (2.40). Let

$$T^* := \sup\{\, S \mid u \text{ exists on } [0, S], \ \sup_{0 \le t \le S} \|(u_t, \Delta u)(t)\|_{H^{s+2}} \le R_0\}. \tag{2.41}$$

The assumption $T^* < \infty$ can be led to a contradiction by choosing $\tilde{T} < T^*$ such that $\tilde{T} + T(R) > T^*$ giving a solution existing beyond T^*. Indeed, from the assumption, we have $\|(u_t, \Delta u)(\tilde{T})\|_{H^{s+2}} \le R_0$, and employ Proposition 2.4 with $t_0 = \tilde{T}$ and Proposition 2.7 again. Thus, we can construct a unique solution u on the time interval $[0, \tilde{T} + T(R)]$ with

$$\sup_{0 \le t \le \tilde{T} + T} \|(u_t, \Delta u)(t)\|_{H^{s+2}} \le R_0, \tag{2.42}$$

contradicting the definition of T^*. Hence we have $T^* = \infty$, the solution hence exists globally in time. Additionally, we have that the solution remains uniformly bounded by R_0. Our purpose is to derive the following energy inequality.

$$\mathfrak{E}(t)^2 + \mathfrak{D}(t)^2 \le C\mathfrak{E}(0)^2 + C\mathfrak{E}(t)\mathfrak{D}(t)^2 \tag{2.43}$$

for $s \ge [n/2] + 1$, where

$$\mathfrak{E}(t)^2 := \sum_{\ell=0}^{s+2} \sup_{0 \le \sigma \le t} \Big\{(1+\sigma)^\ell \|\partial_x^\ell (u_t, \Delta u)(\sigma)\|_{L^2}^2\Big\},$$

$$\mathfrak{D}(t)^2 := \sum_{\ell=0}^{s+2} \int_0^t (1+\sigma)^\ell \|\nabla \partial_x^\ell (u_t, \Delta u)(\sigma)\|_{L^2}^2 d\sigma.$$

Once we derive (2.43), letting $\mathfrak{E}(0)$ be suitably small in (2.43), then we conclude (2.2). To do this, we start our proof from (2.28). Multiplying (2.28) by $(1+t)^\ell$, we have

$$\begin{aligned}&\frac{\partial}{\partial t}\{(1+t)^\ell\mathcal{E}^\ell(t,x)\}+\mathrm{div}\{(1+t)^\ell\mathcal{F}^\ell(t,x)\}+(1+t)^\ell\mathcal{D}^\ell(t,x)\\&\qquad=\ell(1+t)^{\ell-1}\mathcal{E}^\ell(t,x)+(1+t)^\ell\mathcal{R}^\ell(t,x)\end{aligned}$$

for $\ell\geq 0$. Noting that for $\ell\geq 1$ we have $\mathcal{E}^\ell(t,x)\leq C\mathcal{D}^{\ell-1}(t,x)$, we get

$$\begin{aligned}&\frac{\partial}{\partial t}\{(1+t)^\ell\mathcal{E}^\ell(t,x)\}+\mathrm{div}\{(1+t)^\ell\mathcal{F}^\ell(t,x)\}+(1+t)^\ell\mathcal{D}^\ell(t,x)\\&\qquad\leq C\sum_{j=0}^{\ell}(1+t)^j\mathcal{R}^j(t,x),\end{aligned}$$

and hence

$$\tag{2.44}\begin{aligned}&\int_{\mathbb{R}^n}(1+t)^\ell\mathcal{E}^\ell(t,x)dx+\int_0^t\int_{\mathbb{R}^n}(1+\sigma)^\ell\mathcal{D}^\ell(\sigma,x)dxd\sigma\\&\qquad\leq\int_{\mathbb{R}^n}\mathcal{E}^\ell(0,x)dx+C\sum_{j=0}^{\ell}\int_0^t(1+\sigma)^j\int_{\mathbb{R}^n}|\mathcal{R}^j(\sigma,x)|dxd\sigma\end{aligned}$$

for $\ell\geq 0$. Here the last term of the right hand side is neglected if $\ell=0$. To estimate the remainder term, we introduce

$$\tag{2.45}\mathfrak{N}(t):=\sup_{0\leq\sigma\leq t}\{(1+\sigma)^{1/2}\|\nabla\Delta u(\sigma)\|_{L^\infty}+(1+\sigma)\|\Delta u_t(\sigma)\|_{L^\infty}\}.$$

Then, it is easy to get $\mathfrak{N}(t)\leq C\mathfrak{E}(t)$ for $s\geq[n/2]+1$, which comes from the Gagliardo-Nirenberg inequality. We compute from (2.32)

$$\tag{2.46}\begin{aligned}&\int_0^t(1+\sigma)^\ell\int_{\mathbb{R}^n}|\mathcal{R}^\ell(\sigma,x)|dxd\sigma\\&\leq C(C_{b,\delta_0}+\tilde{C}_{b,\delta_0})\mathfrak{N}(t)\int_0^t(1+\sigma)^{\ell-1/2}\|\Delta\partial_x^\ell u\|_{L^2}\|\nabla\partial_x^\ell(u_t,\Delta u)\|_{L^2}d\sigma\\&\qquad+C\mathfrak{N}(t)\int_0^t(1+\sigma)^{\ell-1}\|\Delta\partial_x^\ell u\|_{L^2}^2d\sigma\\&\leq C\mathfrak{N}(t)\mathfrak{D}(t)^2\end{aligned}$$

for $0\leq\ell\leq s+2$. Therefore, combining the estimate (2.44), (2.46) with $\mathfrak{N}(t)\leq C\mathfrak{E}(t)$, we can get (2.43), thus finishing the proof of Theorem 2.1. □

Proof of Theorem 2.2. The arguments for the global existence of a small solution are the same as in the proof of Theorem 2.1. Thus we only derive the decay estimate. Our goal is also to derive the energy inequality for some $\tilde{T}>0$ that

$$\tag{2.47}\tilde{\mathfrak{E}}(t)^2+\tilde{\mathfrak{D}}(t)^2\leq C\tilde{\mathfrak{E}}(0)^2+C\tilde{\mathfrak{E}}(t)\tilde{\mathfrak{D}}(t)^2$$

for $t \ge \tilde{T}$ and $s \ge [n/2]+1$, where

$$\tilde{\mathfrak{E}}(t)^2 := \sum_{\ell=0}^{s} \sup_{0\le\sigma\le t} \big\{(1+\sigma)^\ell(\|\partial_x^\ell u_t\|_{H^2}^2 + \|\Delta\partial_x^\ell u\|_{H^1}^2)\big\},$$

$$\tilde{\mathfrak{D}}(t)^2 := \sum_{\ell=0}^{s} \int_0^t (1+\sigma)^\ell(\|\nabla\partial_x^\ell u_t\|_{H^1}^2 + \|\nabla\Delta\partial_x^\ell u\|_{L^2}^2)d\sigma.$$

Once we get (2.47), letting $\tilde{\mathfrak{E}}(0)$ suitably small in (2.47), we get (2.3) for $t \ge \tilde{T}$. Moreover, we had already obtained (2.20) for the solution, and we complete the proof of Theorem 2.2.

To derive (2.47), we start the proof from (2.34). We multiply (2.34) by $(1+t)^\ell$, and get

$$\begin{aligned}&\frac{\partial}{\partial t}\{(1+t)^\ell\tilde{\mathcal{E}}^\ell(t,x)\} + \mathrm{div}\{(1+t)^\ell\tilde{\mathcal{F}}^\ell(t,x)\} + (1+t)^\ell\tilde{\mathcal{D}}^\ell(t,x)\\&\qquad = \ell(1+t)^{\ell-1}\tilde{\mathcal{E}}^\ell(t,x) + (1+t)^\ell\tilde{\mathcal{R}}^\ell(t,x)\end{aligned}$$

for $\ell \ge 0$. Noting that $\tilde{\mathcal{E}}^\ell(t,x) \le \tilde{C}\tilde{\mathcal{D}}^\ell(t,x) + C\tilde{\mathcal{D}}^{\ell-1}(t,x)$, we get

$$\begin{aligned}&\frac{\partial}{\partial t}\{(1+t)^\ell\tilde{\mathcal{E}}^\ell(t,x)\} + \mathrm{div}\{(1+t)^\ell\tilde{\mathcal{F}}^\ell(t,x)\} + (1-\frac{\ell\tilde{C}}{1+t})(1+t)^\ell\tilde{\mathcal{D}}^\ell(t,x)\\&\quad \le C\sum_{j=0}^{\ell}(1+t)^j\tilde{\mathcal{R}}^j(t,x).\end{aligned}$$

Thus we can choose $\tilde{T} > 0$ such that $1 - s\tilde{C}/(1+\tilde{T}) \ge 1/2$, and obtain

$$\begin{aligned}&\int_{\mathbb{R}^n}(1+t)^\ell\tilde{\mathcal{E}}^\ell(t,x)dx + \frac{1}{2}\int_0^t\int_{\mathbb{R}^n}(1+\tau)^\ell\tilde{\mathcal{D}}^\ell(\sigma,x)dxd\sigma\\&\qquad \le \int_{\mathbb{R}^n}\tilde{\mathcal{E}}^\ell(0,x)dx + C\sum_{j=0}^{\ell}\int_0^t(1+\sigma)^j\int_{\mathbb{R}^n}|\tilde{\mathcal{R}}^j(\sigma,x)|dxd\sigma\end{aligned}$$

for $t \ge \tilde{T}$ and $0 \le \ell \le s$. Then the remainder term can be estimated from (2.32) and (2.36) that

$$\int_0^t(1+\sigma)^\ell\int_{\mathbb{R}^n}|\tilde{\mathcal{R}}^\ell(\sigma,x)|dxd\sigma \le C\mathfrak{N}(t)\tilde{\mathfrak{D}}(t)^2$$

for $0 \le \ell \le s$. Here $\mathfrak{N}(t)$ is defined in (2.45). Noting that $\mathfrak{N}(t) \le C\tilde{\mathfrak{E}}(t)$ for $s \ge [n/2]+1$, we arrive at (2.47). Hence we finish the proof of Theorem 2.2. □

3. Thermoelastic plate equation ($\tau = 0, \mu \ge 0$)

In this section, we consider the initial value problem for the thermoelastic system (1.2), i.e. for the general system (1.1) under the Fourier law ($\tau = 0$),

$$\begin{aligned}&u_{tt} + \Delta b(\Delta u) - \mu\Delta u_{tt} + \nu\Delta\theta = 0,\\&\theta_t - \Delta\theta - \nu\Delta u_t = 0, \qquad (3.1)\\&u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad \theta(0,x) = \theta_0(x),\end{aligned}$$

where $\mu \geq 0$, $\nu > 0$, and $b'(0) > 0$. The purpose is to derive the global existence of the solution for (3.1). We will obtain the following theorem.

Theorem 3.1. [Global existence for $\tau = 0, \mu = 0$] *Let $s \geq [n/2]+1$ and $(u_1, \Delta u_0, \theta_0) \in H^{s+2}(\mathbb{R}^n)$. There exists $\varepsilon_0 > 0$ such that if $\|(u_1, \Delta u_0, \theta_0)\|_{H^{s+2}} < \varepsilon_0$, then there is a unique solution u to the initial value problem* (3.1)*, which satisfies $(u_t, \Delta u, \theta) \in C^0([0,\infty); H^{s+2}(\mathbb{R}^n))$ with the energy estimate:*

$$\|(u_t, \Delta u, \theta)(t)\|^2_{H^{s+2}} + \int_0^t \|\nabla(u_t, \Delta u, \theta)(\tau)\|^2_{H^{s+2}} d\tau \leq C\|(u_1, \Delta u_0, \theta)\|^2_{H^{s+2}} \tag{3.2}$$

for $t \geq 0$. Moreover, we have the decay estimate:

$$\|\partial_x^\ell (u_t, \Delta u, \theta)(t)\|_{L^2} \leq C\|(u_1, \Delta u_0, \theta_0)\|_{H^{s+2}}(1+t)^{-\ell/2} \tag{3.3}$$

for $0 \leq \ell \leq s+2$.

Theorem 3.2. [Global existence for $\tau = 0, \mu > 0$] *Let $s \geq [n/2]+1$ and $u_1 \in H^{s+2}(\mathbb{R}^n)$, $(\Delta u_0, \theta_0) \in H^{s+1}(\mathbb{R}^n)$. There exists $\varepsilon_0 > 0$ such that if $\|u_1\|_{H^{s+2}} + \|(\Delta u_0, \theta_0)\|_{H^{s+1}} < \varepsilon_0$, then there is a unique solution u to the initial value problem* (3.1)*, which satisfies $u_t \in C^0([0,\infty); H^{s+2}(\mathbb{R}^n))$ and $(\Delta u, \theta) \in C^0([0,\infty); H^{s+1}(\mathbb{R}^n))$ with the energy estimate:*

$$\begin{aligned} &\|u_t(t)\|^2_{H^{s+2}} + \|(\Delta u, \theta)(t)\|^2_{H^{s+1}} + \int_0^t (\|\nabla(u_t, \theta)(\tau)\|^2_{H^{s+1}} + \|\nabla\Delta u(\tau)\|^2_{H^s}) d\tau \\ &\leq C(\|u_1\|^2_{H^{s+2}} + \|(\Delta u_0, \theta_0)\|^2_{H^{s+1}}), \end{aligned} \tag{3.4}$$

for $t \geq 0$. Moreover, we have the decay estimate:

$$\|\partial_x^\ell u_t(t)\|_{H^2} + \|\partial_x^\ell (\Delta u, \theta)(t)\|_{H^1} \leq C(\|u_1\|_{H^{s+2}} + \|(\Delta u_0, \theta_0)\|_{H^{s+1}})(1+t)^{-\ell/2} \tag{3.5}$$

for $0 \leq \ell \leq s$

In order to get the global existence result, we combine the local existence theory and a priori estimates. We introduce the function spaces

$$X^s[a,b] := \{(u,\theta) \mid (u_t, \Delta u, \theta) \in C([a,b]; H^s(\mathbb{R}^n)),\ \nabla\theta \in L^2(a,b; H^s(\mathbb{R}^n))\},$$

$$X^s_\mu[a,b] := \{(u,\theta) \mid (u_t, \nabla u_t, \Delta u, \theta) \in C([a,b]; H^s(\mathbb{R}^n)),\ \nabla\theta \in L^2(a,b; H^s(\mathbb{R}^n))\}.$$

Then our local existence results are stated as follows.

Proposition 3.3. [Local existence for $\tau = 0, \mu = 0$] *Let $s \geq [n/2]+1$, $t_0 \geq 0$ and $(u_t(t_0), \Delta u(t_0), \theta(t_0)) \in H^{s+2}(\mathbb{R}^n)$. There is $R_b > 0$ such that for $0 < R < R_b$ there are $R_0 = R_0(R)$ and $T_0 = T_0(R) > t_0$ such that for $\|(u_t, \Delta u, \theta)(t_0)\|_{H^{s+2}} \leq R_0$ there exists a unique solution u to the initial value problem* (3.1)*, which satisfies $(u,\theta) \in X^{s+2}[t_0, T_0]$ and $(u_t, \theta) \in C^1([t_0, T_0]; H^s(\mathbb{R}^n))$ with*

$$\sup_{t\in[t_0,T_0]} \|(u_t, \Delta u, \theta)(t)\|_{H^{s+2}} \leq R.$$

Proposition 3.4. [Local existence for $\tau = 0, \mu > 0$] *Let $s \geq [n/2]+1$, $t_0 \geq 0$ and $u_t(t_0) \in H^{s+2}(\mathbb{R}^n)$, $(\Delta u(t_0), \theta(t_0)) \in H^{s+1}(\mathbb{R}^n)$. There is $R_b > 0$ such that for $0 < R < R_b$ there are $R_0 = R_0(R)$ and $T_0 = T_0(R) > t_0$ such that for $\|u_t(t_0)\|_{H^{s+2}} +$*

$\|(\Delta u,\theta)(t_0)\|_{H^{s+1}} \le R_0$ *there exists a unique solution* u *to the initial value problem* (3.1), *which satisfies* $(u,\theta)\in X_\mu^{s+1}[t_0,T_0]$ *and* $(u_t,\theta)\in C^1([t_0,T_0];H^{s-1}(\mathbb{R}^n))$ *with*

$$\sup_{t\in[t_0,T_0]}(\|u_t(t)\|_{H^{s+2}}+\|(\Delta u,\theta)(t)\|_{H^{s+1}})\le R.$$

Crucial again are the following a priori estimates.

Proposition 3.5. [A priori estimate for $\tau=0,\mu=0$] *Let* $s\ge[n/2]+1$ *and* $(u_1,\Delta u_0,\theta_0)\in H^{s+2}(\mathbb{R}^n)$. *Let* u *be a solution to the initial value problem* (3.1), *which satisfies* $(u_t,\Delta u,\theta)\in C^0([0,T];H^{s+2}(\mathbb{R}^n))$ *for some* $T>0$. *There exists* $\delta_0>0$ *such that if the solution satisfies* $\sup_{0\le t\le T}\|(u_t,\Delta u,\theta)(t)\|_{H^{s+2}}\le\delta_0$, *then the solution satisfies* (3.2) *for* $0\le t\le T$.

Proposition 3.6. [A priori estimate for $\tau=0,\mu>0$] *Let* $s\ge[n/2]+1$ *and* $u_1\in H^{s+2}(\mathbb{R}^n)$, $(\Delta u_0,\theta_0)\in H^{s+1}(\mathbb{R}^n)$. *Let* u *be a solution to the initial value problem* (3.1), *which satisfies* $u_t\in C^0([0,T];H^{s+2}(\mathbb{R}^n))$ *and* $(\Delta u,\theta)\in C^0([0,T];H^{s+1}(\mathbb{R}^n))$ *for some* $T>0$. *There exists* $\delta_0>0$ *such that if the solution satisfies* $\sup_{0\le t\le T}(\|u_t(t)\|_{H^{s+2}}+\|(\Delta u,\theta)(t)\|_{H^{s+1}})\le\delta_0$, *then the solution satisfies* (3.4) *for* $0\le t\le T$.

The arguments for the local existence and for the construction of the global solution are analogous to those in Section 2. Therefore, we derive here the a priori estimates and decay estimates.

Proof of Proposition 3.5. Multiplying the first and second equation of (3.1) by u_t and θ, respectively, and combining the resulting equations, we obtain

$$\begin{aligned}&\Big\{\frac{1}{2}u_t^2+\varphi(\Delta u)+\frac{\mu}{2}|\nabla u_t|^2+\frac{1}{2}\theta^2\Big\}_t+|\nabla\theta|^2\\&+\operatorname{div}\big\{u_t\nabla b(\Delta u)-b(\Delta u)\nabla u_t-\mu u_t\nabla u_{tt}+\nu u_t\nabla\theta-\nu\theta\nabla u_t-\theta\nabla\theta\big\}=0.\end{aligned}\tag{3.6}$$

Here φ is defined by (2.22), which is described as a free energy. On the other hand, we have from (3.1) that

$$\begin{aligned}\partial_x^\ell u_{tt}+\operatorname{div}\partial_x^\ell(b'(\Delta u)\nabla\Delta u)-\mu\Delta\partial_x^\ell u_{tt}+\nu\Delta\partial_x^\ell\theta=0,\\ \partial_x^\ell\theta_t-\Delta\partial_x^\ell\theta-\nu\Delta\partial_x^\ell u_t=0\end{aligned}\tag{3.7}$$

for $\ell\ge0$. We multiply (3.7) by $\partial_x^\ell u_t$ and $\partial_x^\ell\theta$, respectively, and combine the resulting equations. This yields

$$\begin{aligned}&\frac{1}{2}\{(\partial_x^\ell u_t)^2+b'(\Delta u)(\Delta\partial_x^\ell u)^2+\mu|\nabla\partial_x^\ell u_t|^2+\nu(\partial_x^\ell\theta)^2\}_t+|\nabla\partial_x^\ell\theta|^2\\&-\nabla\partial_x^\ell u_t\cdot[\partial_x^\ell,b'(\Delta u)]\nabla\Delta u+b''(\Delta u)\Delta\partial_x^\ell u\nabla\Delta u\cdot\nabla\partial_x^\ell u_t-\frac{1}{2}b''(\Delta u)\Delta u_t(\Delta\partial_x^\ell u)^2\\&+\operatorname{div}\{\partial_x^\ell u_t\partial_x^\ell(b'(\Delta u)\nabla\Delta u)-b'(\Delta u)\Delta\partial_x^\ell u\nabla\partial_x^\ell u_t-\mu\partial_x^\ell u_t\nabla\partial_x^\ell u_{tt}\}\\&+\operatorname{div}\{\nu\partial_x^\ell u_t\nabla\partial_x^\ell\theta-\nu\partial_x^\ell\theta\nabla\partial_x^\ell u_t-\partial_x^\ell\theta\nabla\partial_x^\ell\theta\}=0\end{aligned}\tag{3.8}$$

for $\ell \geq 1$. Next we construct the dissipation terms. Multiplying (3.7) by $\partial_x^\ell \theta$ and $\partial_x^\ell u_t - \mu\Delta\partial_x^\ell u_t$, respectively, and combining the resulting equations, we get

$$
\begin{aligned}
&\{(\partial_x^\ell u_t - \mu\Delta\partial_x^\ell u_t)\partial_x^\ell\theta\}_t + \nu|\nabla\partial_x^\ell u_t|^2 + \mu\nu(\Delta\partial_x^\ell u_t)^2 - \nu|\nabla\partial_x^\ell\theta|^2 \\
&+ \nabla\partial_x^\ell u_t\cdot\nabla\partial_x^\ell\theta + \mu\Delta\partial_x^\ell u_t\Delta\partial_x^\ell\theta - \nabla\partial_x^\ell\theta\cdot\partial_x^\ell(b'(\Delta u)\nabla\Delta u) \\
&+ \mathrm{div}\{\partial_x^\ell\theta\partial_x^\ell(b'(\Delta u)\nabla\Delta u) + \nu\partial_x^\ell\theta\nabla\partial_x^\ell\theta - \nu\partial_x^\ell u_t\nabla\partial_x^\ell u_t - \partial_x^\ell u_t\nabla\partial_x^\ell\theta\} = 0
\end{aligned} \tag{3.9}
$$

for $\ell \geq 0$. Moreover, similarly as in the previous section, multiplying (3.7) by $-\Delta\partial_x^\ell u$, we have

$$
\begin{aligned}
&\big\{-\partial_x^\ell u_t\Delta\partial_x^\ell u + \mu\Delta\partial_x^\ell u\Delta\partial_x^\ell u_t\big\}_t + b'(\Delta u)|\nabla\Delta\partial_x^\ell u|^2 - |\nabla\partial_x^\ell u_t|^2 - \mu(\Delta\partial_x^\ell u_t)^2 \\
&\quad + \nu\nabla\Delta\partial_x^\ell u\nabla\partial_x^\ell\theta + \nabla\Delta\partial_x^\ell u\cdot[\partial_x^\ell, b'(\Delta u)]\nabla\Delta u \\
&\quad + \mathrm{div}\{\partial_x^\ell u_t\nabla\partial_x^\ell u_t - \nu\Delta\partial_x^\ell u\nabla\partial_x^\ell\theta - \Delta\partial_x^\ell u\partial_x^\ell(b'(\Delta u)\nabla\Delta u)\} = 0
\end{aligned} \tag{3.10}
$$

for $\ell \geq 0$. Then, calculating $2\times(3.9) + \nu\times(3.10)$, we get

$$
\begin{aligned}
&\{2\partial_x^\ell u_t\partial_x^\ell\theta - 2\mu\Delta\partial_x^\ell u_t\partial_x^\ell\theta - \nu\partial_x^\ell u_t\Delta\partial_x^\ell u + \mu\nu\Delta\partial_x^\ell u\Delta\partial_x^\ell u_t\}_t \\
&+ \nu|\nabla\partial_x^\ell u_t|^2 + \mu\nu(\Delta\partial_x^\ell u_t)^2 + \nu b'(\Delta u)|\nabla\Delta\partial_x^\ell u|^2 - 2\nu|\nabla\partial_x^\ell\theta|^2 \\
&+ 2\nabla\partial_x^\ell u_t\cdot\nabla\partial_x^\ell\theta + 2\mu\Delta\partial_x^\ell u_t\Delta\partial_x^\ell\theta + \nu^2\nabla\Delta\partial_x^\ell u\cdot\nabla\partial_x^\ell\theta \\
&- 2\nabla\partial_x^\ell\theta\cdot\partial_x^\ell(b'(\Delta u)\nabla\Delta u) + \nu\nabla\Delta\partial_x^\ell u\cdot[\partial_x^\ell, b'(\Delta u)]\nabla\Delta u \\
&- \mathrm{div}\{\nu\partial_x^\ell u_t\nabla\partial_x^\ell u_t + 2\partial_x^\ell u_t\nabla\partial_x^\ell\theta + \nu^2\Delta\partial_x^\ell u\nabla\partial_x^\ell\theta + \nu\Delta\partial_x^\ell u\partial_x^\ell(b'(\Delta u)\nabla\Delta u)\} \\
&+ \mathrm{div}\{2\partial_x^\ell\theta\partial_x^\ell(b'(\Delta u)\nabla\Delta u) + 2\nu\partial_x^\ell\theta\nabla\partial_x^\ell\theta\} = 0
\end{aligned} \tag{3.11}
$$

for $\ell \geq 0$. Here we also remark that the term which includes $[\partial_x^\ell, b'(\Delta u)]\nabla\Delta u$ is eliminated if $\ell = 0$. Now, for a sufficiently small α, we compute $2\times(3.6) + \alpha\nu\times(3.11)$ with $\ell = 0$ or $2\times(3.8) + \alpha\nu\times(3.11)$. Then we obtain

$$
\frac{\partial}{\partial t}\mathcal{E}^\ell(t,x) + \mathcal{D}^\ell(t,x) + \mathrm{div}\mathcal{F}^\ell(t,x) = \mathcal{R}^\ell(t,x) \tag{3.12}
$$

for $\ell \geq 0$. Here we define

$$
\begin{aligned}
\mathcal{E}^0(t,x) &:= u_t^2 + 2\varphi(\Delta u) + \mu|\nabla u_t|^2 + \theta^2 + \alpha\nu(2u_t\theta - 2\mu\Delta u_t\theta - u_t\Delta u + \mu\Delta u\Delta u_t), \\
\mathcal{F}^0(t,x) &:= 2u_t\nabla b(\Delta u) - 2b(\Delta u)\nabla u_t - 2\mu u_t\nabla u_{tt} + 2\nu u_t\nabla\theta - 2\nu\theta\nabla u_t - 2\theta\nabla\theta \\
&\qquad + \alpha\nu\{2\nu\theta\nabla\theta - \nu u_t\nabla u_t - 2u_t\nabla\theta - \nu^2\Delta u\nabla\theta + 2\theta\nabla b(\Delta u) - \nu\Delta u\nabla b(\Delta u)\}, \\
\mathcal{R}^0(t,x) &:= 2\alpha\nu\{b'(\Delta u) - b'(0)\}\nabla\Delta u\cdot\nabla\theta,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{E}^\ell(t,x) &:= (\partial_x^\ell u_t)^2 + b'(\Delta u)(\Delta \partial_x^\ell u)^2 + \mu|\nabla \partial_x^\ell u_t|^2 + (\partial_x^\ell \theta)^2 \\
&\quad + \alpha\nu\{2\partial_x^\ell u_t \partial_x^\ell \theta - 2\mu\Delta\partial_x^\ell u_t \partial_x^\ell \theta - \nu \partial_x^\ell u_t \Delta \partial_x^\ell u + \mu\nu \Delta\partial_x^\ell u \Delta \partial_x^\ell u_t\}, \\
\mathcal{F}^\ell(t,x) &:= 2\partial_x^\ell u_t \partial_x^\ell (b'(\Delta u)\nabla\Delta u) - 2b'(\Delta u)\Delta\partial_x^\ell u \nabla \partial_x^\ell u_t - 2\mu \partial_x^\ell u_t \nabla \partial_x^\ell u_{tt} \\
&\quad + 2\nu \partial_x^\ell u_t \nabla \partial_x^\ell \theta - 2\nu \partial_x^\ell \theta \nabla \partial_x^\ell u_t - 2\partial_x^\ell \theta \nabla \partial_x^\ell \theta \\
&\quad + \alpha\nu\{2\nu \partial_x^\ell \theta \nabla \partial_x^\ell \theta - \nu \partial_x^\ell u_t \nabla \partial_x^\ell u_t - 2\partial_x^\ell u_t \nabla \partial_x^\ell \theta - \nu^2 \Delta \partial_x^\ell u \nabla \partial_x^\ell \theta \\
&\quad - \nu \Delta \partial_x^\ell u \partial_x^\ell (b'(\Delta u)\nabla \Delta u) + 2\partial_x^\ell \theta \partial_x^\ell (b'(\Delta u)\nabla\Delta u)\}, \\
\mathcal{R}^\ell(t,x) &:= 2\nabla \partial_x^\ell u_t \cdot [\partial_x^\ell, b'(\Delta u)]\nabla\Delta u - 2b''(\Delta u)\Delta\partial_x^\ell u \nabla\Delta u \cdot \nabla \partial_x^\ell u_t + b''(\Delta u)\Delta u_t (\Delta \partial_x^\ell u)^2 \\
&\quad + \alpha\nu\{2\nabla\partial_x^\ell \theta \cdot \partial_x^\ell((b'(\Delta u) - b'(0))\nabla\Delta u) - \nu \nabla\Delta\partial_x^\ell u \cdot [\partial_x^\ell, b'(\Delta u)]\nabla\Delta u\}
\end{aligned}
$$

for $\ell \geq 1$, and

$$
\begin{aligned}
\mathcal{D}^\ell(t,x) &:= 2(1-\alpha\nu^2)|\nabla\partial_x^\ell \theta|^2 + \alpha\nu^2\{|\nabla \partial_x^\ell u_t|^2 + \mu(\Delta\partial_x^\ell u_t)^2 + b'(\Delta u)|\nabla\Delta\partial_x^\ell u|^2\} \\
&\quad + \alpha\nu\{2\nabla\partial_x^\ell u_t \cdot \nabla \partial_x^\ell \theta + 2\mu \Delta\partial_x^\ell u_t \Delta \partial_x^\ell \theta + (\nu^2 - 2b'(0))\nabla\Delta\partial_x^\ell u \cdot \nabla \partial_x^\ell \theta\}
\end{aligned}
$$

for $\ell \geq 0$.

We integrate (3.12) and get

$$
\int_{\mathbb{R}^n} \mathcal{E}^\ell(t,x)dx + \int_0^t \int_{\mathbb{R}^n} \mathcal{D}^\ell(\sigma,x)dxd\sigma = \int_{\mathbb{R}^n} \mathcal{E}^\ell(0,x)dx + \int_0^t \int_{\mathbb{R}^n} \mathcal{R}^\ell(\sigma,x)dxd\sigma.
$$

Especially, in the case $\mu = 0$ we are discussing at the moment, we employ the same argument as in Section 2 and obtain

$$
C^{-1}\|\partial_x^\ell(u_t,\Delta u,\theta)(t)\|_{L^2}^2 \leq \int_{\mathbb{R}^n} \mathcal{E}^\ell(t,x)dx \leq C\|\partial_x^\ell(u_t,\Delta u,\theta)(t)\|_{L^2}^2
$$

and

$$
\int_{\mathbb{R}^n} \mathcal{D}^\ell(t,x)dx \geq c\|\nabla\partial_x^\ell(u_t,\Delta u,\theta)(t)\|_{L^2}^2
$$

for $\ell \geq 0$, where we choose α and δ_0 suitably small. Furthermore, we estimate the remainder terms

$$
\int_{\mathbb{R}^n} |\mathcal{R}^0(t,x)|dx \leq C\|\Delta u\|_{L^\infty}\|\nabla\Delta u\|_{L^2}\|\nabla\theta\|_{L^2},
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |\mathcal{R}^\ell(t,x)|dx \\
&\leq C\|\nabla\partial_x^\ell(u_t,\Delta u)\|_{L^2}\|[\partial_x^\ell, b'(\Delta u)]\nabla\Delta u\|_{L^2} + C\|\nabla\partial_x^\ell\theta\|_{L^2}\|\partial_x^\ell(b'(\Delta u)\nabla\Delta u)\|_{L^2} \\
&\qquad + C\|b''(\Delta u)\nabla\Delta u\|_{L^\infty}\|\Delta\partial_x^\ell u\|_{L^2}\|\nabla\partial_x^\ell u_t\|_{L^2} + C\|b''(\Delta u)\Delta u_t\|_{L^\infty}\|\Delta\partial_x^\ell u\|_{L^2}^2 \\
&\leq C\|\Delta u\|_{L^\infty}\|\Delta\partial_x^\ell u\|_{L^2}\|\nabla\partial_x^\ell(u_t,\Delta u)\|_{L^2} + C\|\Delta u_t\|_{L^\infty}\|\Delta\partial_x^\ell u\|_{L^2}^2 \\
&\qquad + C\|\nabla\partial_x^\ell\theta\|_{L^2}(\|\Delta u\|_{L^\infty}\|\nabla\Delta\partial_x^\ell u\|_{L^2} + \|\nabla\Delta u\|_{L^\infty}\|\Delta\partial_x^\ell u\|_{L^2})
\end{aligned}
$$

for $\ell \geq 1$, which comes from the same argument as in Section 2. Therefore we get

$$\int_{\mathbb{R}^n} \mathcal{E}^\ell(t,x)dx + \int_0^t \|\nabla\partial_x^\ell(u_t,\Delta u,\theta)\|_{L^2}^2 d\sigma$$
$$\leq C\int_{\mathbb{R}^n} \mathcal{E}^\ell(0,x)dx + C\delta_0(1+\delta_0)\int_0^t \|\Delta\partial_x^\ell u\|_{L^2}^2 d\sigma$$

for $\ell \geq 0$, where we take δ_0 suitably small, and the last term on the right-hand side can be neglected if $\ell = 0$. This estimate is equivalent to (2.33) in Section 2, and hence we can apply the same argument as before. Namely, we arrive at the desired a priori estimate and have proved Proposition 3.5. □

Proof of Proposition 3.6. We start the proof from (3.12). Now, in the case $\mu > 0$, we have to control the term $\Delta\partial_x^\ell u_t$ in $\mathcal{E}^0(t,x)$ and $\mathcal{E}^\ell(t,x)$. To this end, we combine (3.12) and $2\times$(3.8) where ℓ is replaced by $\ell+1$. Then we have

$$\frac{\partial}{\partial t}\tilde{\mathcal{E}}^\ell(t,x) + \tilde{\mathcal{D}}^\ell(t,x) + \mathrm{div}\tilde{\mathcal{F}}^\ell(t,x) = \tilde{\mathcal{R}}^\ell(t,x) \tag{3.13}$$

for $\ell \geq 0$, where

$$\tilde{\mathcal{E}}^\ell(t,x) := \mathcal{E}^\ell(t,x) + (\partial_x^{\ell+1}u_t)^2 + b'(\Delta u)(\Delta\partial_x^{\ell+1}u)^2 + \mu|\nabla\partial_x^{\ell+1}u_t|^2 + (\partial_x^{\ell+1}\theta)^2,$$
$$\tilde{\mathcal{D}}^\ell(t,x) := \mathcal{D}^\ell(t,x) + 2|\nabla\partial_x^{\ell+1}\theta|^2,$$
$$\tilde{\mathcal{F}}^\ell(t,x) := \mathcal{F}^\ell(t,x) + 2\partial_x^{\ell+1}u_t\partial_x^{\ell+1}(b'(\Delta u)\nabla\Delta u) - 2b'(\Delta u)\Delta\partial_x^{\ell+1}u\nabla\partial_x^{\ell+1}u_t$$
$$- 2\mu\partial_x^{\ell+1}u_t\nabla\partial_x^{\ell+1}u_{tt} + 2\partial_x^{\ell+1}u_t\nabla\partial_x^{\ell+1}\theta - 2\partial_x^{\ell+1}\theta\nabla\partial_x^{\ell+1}u_t,$$
$$\tilde{\mathcal{R}}^\ell(t,x) := \mathcal{R}^\ell(t,x) + 2\nabla\partial_x^{\ell+1}u_t\cdot[\partial_x^{\ell+1}, b'(\Delta u)]\nabla\Delta u$$
$$- 2b''(\Delta u)\Delta\partial_x^{\ell+1}u\nabla\Delta u\cdot\nabla\partial_x^{\ell+1}u_t + b''(\Delta u)\Delta u_t(\Delta\partial_x^{\ell+1}u)^2.$$

Then integrating (3.13) we get

$$\int_{\mathbb{R}^n}\tilde{\mathcal{E}}^\ell(t,x)dx + \int_0^t\int_{\mathbb{R}^n}\tilde{\mathcal{D}}^\ell(\sigma,x)dxd\sigma = \int_{\mathbb{R}^n}\tilde{\mathcal{E}}^\ell(0,x)dx + \int_0^t\int_{\mathbb{R}^n}\tilde{\mathcal{R}}^\ell(\sigma,x)dxd\sigma.$$

Moreover, we can derive

$$C^{-1}(\|\partial_x^\ell u_t\|_{H^2}^2 + \|\partial_x^\ell(\Delta u,\theta)\|_{H^1}^2) \leq \int_{\mathbb{R}^n}\tilde{\mathcal{E}}^\ell(t,x)dx \leq C(\|\partial_x^\ell u_t\|_{H^2}^2 + \|\partial_x^\ell(\Delta u,\theta)\|_{H^1}^2)$$

and

$$\int_{\mathbb{R}^n}\tilde{\mathcal{D}}^\ell(t,x)dx \geq c(\|\nabla\partial_x^\ell(u_t,\theta)\|_{H^1}^2 + \|\nabla\Delta\partial_x^\ell u\|_{L^2}^2).$$

For the remainder terms, we calculate

$$\int_{\mathbb{R}^n}|\tilde{\mathcal{R}}^\ell(t,x)|dx \leq \int_{\mathbb{R}^n}|\mathcal{R}^\ell(t,x)|dx + C\|\nabla\partial_x^{\ell+1}u_t\|_{L^2}\|[\partial_x^{\ell+1}, b'(\Delta u)]\nabla\Delta u\|_{L^2}$$
$$+ C\|b''(\Delta u)\nabla\Delta u\|_{L^\infty}\|\Delta\partial_x^{\ell+1}u\|_{L^2}\|\nabla\partial_x^{\ell+1}u_t\|_{L^2} + C\|b''(\Delta u)\Delta u_t\|_{L^\infty}\|\Delta\partial_x^{\ell+1}u\|_{L^2}^2$$
$$\leq \int_{\mathbb{R}^n}|\mathcal{R}^\ell(t,x)|dx + C\|\nabla\Delta u\|_{L^\infty}\|\Delta\partial_x^{\ell+1}u\|_{L^2}\|\nabla\partial_x^{\ell+1}u_t\|_{L^2} + C\|\Delta u_t\|_{L^\infty}\|\Delta\partial_x^{\ell+1}u\|_{L^2}^2.$$

Therefore we get for sufficiently small δ_0 that

$$\begin{aligned}(3.14)\qquad &\int_{\mathbb{R}^n} \tilde{\mathcal{E}}^\ell(t,x)dx + c\int_0^t (\|\nabla\partial_x^\ell(u_t,\theta)\|_{H^1}^2 + \|\nabla\Delta\partial_x^\ell u\|_{L^2}^2)d\sigma \\ &\le C\int_{\mathbb{R}^n} \tilde{\mathcal{E}}^\ell(0,x)dx + C\delta_0(1+\delta_0)\int_0^t \|\Delta\partial_x^\ell u\|_{L^2}^2 d\sigma,\end{aligned}$$

and thus obtain the a priori estimate, which proves Proposition 3.6. □

Proof of Theorem 3.1. Here we only prove the decay estimate (3.3). Namely our purpose is to derive the following energy inequality.

$$(3.15)\qquad \mathfrak{E}(t)^2 + \mathfrak{D}(t)^2 \le C\mathfrak{E}(0)^2 + C\mathfrak{E}(t)\mathfrak{D}(t)^2$$

for $s \ge [n/2]+1$, where

$$\begin{aligned}\mathfrak{E}(t)^2 &:= \sum_{\ell=0}^{s+2} \sup_{0\le\sigma\le t} \{(1+\sigma)^\ell \|\partial_x^\ell(u_t,\Delta u,\theta)(\sigma)\|_{L^2}^2\}, \\ \mathfrak{D}(t)^2 &:= \sum_{\ell=0}^{s+2} \int_0^t (1+\sigma)^\ell \|\nabla\partial_x^\ell(u_t,\Delta u,\theta)(\sigma)\|_{L^2}^2 d\sigma.\end{aligned}$$

To get (3.15), we start from (3.12). Multiplying (3.12) by $(1+t)^\ell$ and integrating the resulting equation, we obtain

$$\begin{aligned}(3.16)\qquad &\int_{\mathbb{R}^n} (1+t)^\ell \mathcal{E}^\ell(t,x)dx + \int_0^t (1+\sigma)^\ell \int_{\mathbb{R}^n} \mathcal{D}^\ell(\sigma,x)dxd\sigma \\ &\le \int_{\mathbb{R}^n} \mathcal{E}^\ell(0,x)dx + C\sum_{j=0}^{\ell}\int_0^t (1+\sigma)^j \int_{\mathbb{R}^n} |\mathcal{R}^j(\sigma,x)|dxd\sigma\end{aligned}$$

for $\ell \ge 0$, where we used $\mathcal{E}^\ell(t,x) \le C\mathcal{D}^{\ell-1}(t,x)$ for $\ell \ge 1$, and the last term of the right hand side is neglected if $\ell = 0$. Similarly as before, we also introduce

$$\mathfrak{N}(t) := \sup_{0\le\sigma\le t} \{(1+\sigma)^{1/2}\|\nabla\Delta u(\sigma)\|_{L^\infty} + (1+\sigma)\|\Delta u_t(\sigma)\|_{L^\infty}\},$$

and estimate

$$\begin{aligned}&\int_0^t (1+\sigma)^\ell \int_{\mathbb{R}^n} |\mathcal{R}^\ell(\sigma,x)|dxd\sigma \\ &\le C\delta_0 \int_0^t (1+\sigma)^\ell \int_{\mathbb{R}^n} \mathcal{D}^\ell(\sigma,x)dxd\sigma + C\mathfrak{N}(t)\int_0^t (1+\sigma)^{\ell-1}\|\Delta\partial_x^\ell u\|_{L^2}^2 d\sigma \\ &\qquad + C\mathfrak{N}(t)\int_0^t (1+\sigma)^{\ell-1/2}\|\Delta\partial_x^\ell u\|_{L^2}\|\nabla\partial_x^\ell(u_t,\Delta u,\theta)\|_{L^2}d\sigma \\ &\le C\delta_0 \int_0^t (1+\sigma)^\ell \int_{\mathbb{R}^n} \mathcal{D}^\ell(\sigma,x)dxd\sigma + C\mathfrak{N}(t)\mathfrak{D}(t)^2\end{aligned}$$

for $0 \le \ell \le s+2$. Therefore, combining the above estimate and (3.16) with small δ_0 and $\mathfrak{N}(t) \le C\mathfrak{E}(t)$ for $s \ge [n/2]+1$, we can conclude (3.15), thus finishing the proof of Theorem 3.1. □

Proof of Theorem 3.2. Here we only show the decay estimate. Our goal is also to derive the energy inequality for some $\tilde{T} > 0$ that

$$\tilde{\mathfrak{E}}(t)^2 + \tilde{\mathfrak{D}}(t)^2 \le C\tilde{\mathfrak{E}}(0)^2 + C\tilde{\mathfrak{E}}(t)\tilde{\mathfrak{D}}(t)^2 \tag{3.17}$$

for $t \ge \tilde{T}$ and $s \ge [n/2] + 1$, where

$$\tilde{\mathfrak{E}}(t)^2 := \sum_{\ell=0}^{s} \sup_{0\le\sigma\le t} \big\{(1+\sigma)^\ell(\|\partial_x^\ell u_t(\sigma)\|_{H^2}^2 + \|\partial_x^\ell(\Delta u, \theta)(\sigma)\|_{H^1}^2)\big\},$$
$$\tilde{\mathfrak{D}}(t)^2 := \sum_{\ell=0}^{s} \int_0^t (1+\sigma)^\ell(\|\nabla\partial_x^\ell(u_t,\theta)(\sigma)\|_{H^1}^2 + \|\nabla\Delta\partial_x^\ell u(\sigma)\|_{L^2}^2)d\sigma.$$

To derive (3.17), we start the proof from (3.13). We multiply (3.13) by $(1+t)^\ell$ and integrate the resulting equation. Then we get

$$\begin{aligned}(1+t)^\ell \int_{\mathbb{R}^n} \tilde{\mathcal{E}}^\ell(t,x)dx + \frac{1}{2}\int_0^t (1+\sigma)^\ell \int_{\mathbb{R}^n} \tilde{\mathcal{D}}^\ell(\sigma,x)dxd\sigma \\ \le \int_{\mathbb{R}^n} \tilde{\mathcal{E}}^\ell(0,x)dx + C\sum_{j=0}^{\ell}\int_0^t (1+\sigma)^j \int_{\mathbb{R}^n} |\tilde{\mathcal{R}}^j(\sigma,x)|dxd\sigma\end{aligned}$$

for $t \ge \tilde{T}$ and $0 \le \ell \le s$, where we used $\tilde{\mathcal{E}}^\ell(t,x) \le \tilde{C}\tilde{\mathcal{D}}^\ell(t,x) + C\tilde{\mathcal{D}}^{\ell-1}(t,x)$, and chose $\tilde{T} > 0$ such that $1 - s\tilde{C}/(1+\tilde{T}) \ge 1/2$. Furthermore the remainder term can be estimated

$$\int_0^t (1+\sigma)^\ell \int_{\mathbb{R}^n} |\tilde{\mathcal{R}}^\ell(\sigma,x)|dxd\sigma \le C\delta_0 \int_0^t (1+\sigma)^\ell \int_{\mathbb{R}^n} \mathcal{D}^\ell(\sigma,x)dxd\sigma + C\mathfrak{N}(t)\tilde{\mathfrak{D}}(t)^2$$

for $0 \le \ell \le s$. Thus, employing the smallness for δ_0 and $\mathfrak{N}(t) \le C\tilde{\mathfrak{E}}(t)$ for $s \ge [n/2] + 1$, we arrive at (3.17) and finish the proof of Theorem 3.2. □

4. Thermoelastic plate equation ($\tau > 0, \mu > 0$)

In this section, we consider the initial value problem for the thermoelastic system (1.1) under the Cattaneo law ($\tau > 0$), and with inertial term ($\mu > 0$),

$$\begin{aligned}&u_{tt} + \Delta b(\Delta u) - \mu\Delta u_{tt} + \nu\Delta\theta = 0,\\ &\theta_t + \mathrm{div} q - \nu\Delta u_t = 0,\\ &\tau q_t + q + \nabla\theta = 0,\\ &u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad \theta(0,x) = \theta_0(x), \quad q(0,x) = q_0(x),\end{aligned} \tag{4.1}$$

where $\nu > 0$ and $b'(0) > 0$. The purpose is again to derive the global existence of a small solution for (3.1) togehter with decay estimates.

Theorem 4.1. [Global existence for $\tau > 0, \mu > 0$] *Let $s \ge [n/2] + 1$ and $u_1 \in H^{s+2}(\mathbb{R}^n)$, $(\Delta u_0, \theta_0, q_0) \in H^{s+1}(\mathbb{R}^n)$. There exists $\varepsilon_0 > 0$ such that if $\|u_1\|_{H^{s+2}} + \|(\Delta u_0, \theta_0, q_0)\|_{H^{s+1}} < \varepsilon_0$, then there is a unique solution u to the initial value problem*

(4.1), *which satisfies* $u_t \in C^0([0,\infty); H^{s+2}(\mathbb{R}^n))$ *and* $(\Delta u, \theta, q) \in C^0([0,\infty); H^{s+1}(\mathbb{R}^n))$ *with the energy estimate:*

$$
\begin{aligned}
&\|u_t(t)\|^2_{H^{s+2}} + \|(\Delta u, \theta, q)(t)\|^2_{H^{s+1}} \\
&\quad + \int_0^t (\|\nabla u_t(\sigma)\|^2_{H^{s+1}} + \|\nabla(\Delta u, \theta)(\sigma)\|^2_{H^s} + \|q(\sigma)\|^2_{H^{s+1}}) d\sigma \\
&\le C(\|u_1\|^2_{H^{s+2}} + \|(\Delta u_0, \theta_0, q_0)\|^2_{H^{s+1}}),
\end{aligned} \tag{4.2}
$$

for $t \ge 0$. *Moreover, we have the decay estimate*

$$
\begin{aligned}
&\|\partial_x^\ell u_t(t)\|_{H^2} + \|\partial_x^\ell (\Delta u, \theta, q)(t)\|_{H^1} \\
&\qquad \le C(\|u_1\|_{H^{s+2}} + \|(\Delta u_0, \theta_0, q_0)\|_{H^{s+1}})(1+t)^{-\ell/2}
\end{aligned} \tag{4.3}
$$

for $0 \le \ell \le s$.

The strategy to get the global existence result, we combine local existence with a priori estimates. We introduce appropriate function spaces

$$
X^s_\mu[a,b] := \{(u,\theta,q) \mid (u_t, \nabla u_t, \Delta u, \theta, q) \in C([a,b]; H^s(\mathbb{R}^n)),\ q \in L^2(a,b; H^s(\mathbb{R}^n))\}.
$$

Then our local existence results are stated as follows.

Proposition 4.2. [Local existence for $\tau > 0, \mu > 0$] *Let* $s \ge [n/2]+1$, $t_0 \ge 0$ *and* $u_t(t_0) \in H^{s+2}(\mathbb{R}^n)$, $(\Delta u(t_0), \theta(t_0), q(t_0)) \in H^{s+1}(\mathbb{R}^n)$. *There is* $R_b > 0$ *such that for* $0 < R < R_b$ *there are* $R_0 = R_0(R)$ *and* $T_0 = T_0(R) > t_0$ *such that for* $\|u_t(t_0)\|_{H^{s+2}} + \|(\Delta u, \theta, q)(t_0)\|_{H^{s+1}} \le R_0$ *there exists a unique solution* u *to the initial value problem* (4.1), *which satisfies* $(u,\theta,q) \in X^{s+1}_\mu[t_0, T_0]$ *and* $(u_t, \theta, q) \in C^1([t_0,T_0]; H^{s-1}(\mathbb{R}^n))$ *with*

$$
\sup_{t\in[t_0,T_0]} (\|u_t(t)\|_{H^{s+2}} + \|(\Delta u, \theta, q)(t)\|_{H^{s+1}}) \le R.
$$

A crucial tool are again the following a priori estimates.

Proposition 4.3. [A priori estimate for $\tau > 0, \mu > 0$] *Let* $s \ge [n/2]+1$ *and* $u_1 \in H^{s+2}(\mathbb{R}^n)$, $(\Delta u_0, \theta_0, q_0) \in H^{s+1}(\mathbb{R}^n)$. *Let* u *be a solution* (u,θ,q) *to the initial value problem* (4.1), *which satisfies* $u_t \in C^0([0,T]; H^{s+2}(\mathbb{R}^n))$ *and* $(\Delta u, \theta, q) \in C^0([0,T]; H^{s+1}(\mathbb{R}^n))$ *for some* $T > 0$. *Furthermore, assume that there exists* $\delta_0 > 0$ *such that the solution satisfies* $\sup_{0\le t\le T}(\|u_t(t)\|_{H^{s+2}} + \|(\Delta u, \theta, q)(t)\|_{H^{s+1}}) \le \delta_0$. *Then the solution satisfies* (4.2) *for* $0 \le t \le T$.

Similarly as in the previous section, we can construct the global solution in time and obtain the corresponding decay estimate. Thus we only prove Proposition 4.3 in this section.

Proof of Proposition 4.3. Multiplying the first, second and third equation of (4.1) by u_t, θ and q, respectively, and combining the resulting equations, we obtain

$$
\begin{aligned}
&\frac{1}{2}\Big\{u_t^2 + 2\varphi(\Delta u) + \mu|\nabla u_t|^2 + \theta^2 + \tau|q|^2\Big\}_t + |q|^2 \\
&+ \operatorname{div}\big\{u_t \nabla b(\Delta u) - b(\Delta u)\nabla u_t - \mu u_t \nabla u_{tt} + \nu u_t \nabla\theta - \nu\theta\nabla u_t + \theta q\big\} = 0.
\end{aligned} \tag{4.4}
$$

Here φ is defined by (2.22). On the other hand, we have from (4.1) that

$$
\begin{aligned}
\partial_x^\ell u_{tt} + \mathrm{div}\partial_x^\ell(b'(\Delta u)\nabla\Delta u) - \mu\Delta\partial_x^\ell u_{tt} + \nu\Delta\partial_x^\ell\theta &= 0,\\
\partial_x^\ell\theta_t + \mathrm{div}\partial_x^\ell q - \nu\Delta\partial_x^\ell u_t &= 0,\\
\tau\partial_x^\ell q_t + \partial_x^\ell q + \nabla\partial_x^\ell\theta &= 0
\end{aligned}
\tag{4.5}
$$

for $\ell \geq 0$. Here the last term on the left-hand side in the first equation is equal to zero if $\ell = 0$. We multiply (4.5) by $\partial_x^\ell u_t$, $\partial_x^\ell\theta$ and $\partial_x^\ell q$, respectively, and combine the resulting equations, yielding

$$
\begin{aligned}
&\frac{1}{2}\{(\partial_x^\ell u_t)^2 + b'(\Delta u)(\Delta\partial_x^\ell u)^2 + \mu|\nabla\partial_x^\ell u_t|^2 + (\partial_x^\ell\theta)^2 + \tau|\partial_x^\ell q|^2\}_t + |\partial_x^\ell q|^2\\
&- \nabla\partial_x^\ell u_t\cdot[\partial_x^\ell, b'(\Delta u)]\nabla\Delta u + b''(\Delta u)\Delta\partial_x^\ell u\nabla\Delta u\cdot\nabla\partial_x^\ell u_t - \frac{1}{2}b''(\Delta u)\Delta u_t(\Delta\partial_x^\ell u)^2\\
&+ \mathrm{div}\{\partial_x^\ell u_t\partial_x^\ell(b'(\Delta u)\nabla\Delta u) - b'(\Delta u)\Delta\partial_x^\ell u\nabla\partial_x^\ell u_t - \mu\partial_x^\ell u_t\nabla\partial_x^\ell u_{tt}\}\\
&+ \mathrm{div}\{\nu\partial_x^\ell u_t\nabla\partial_x^\ell\theta - \nu\partial_x^\ell\theta\nabla\partial_x^\ell u_t + \partial_x^\ell\theta\partial_x^\ell q\} = 0
\end{aligned}
\tag{4.6}
$$

for $\ell \geq 1$. Next we construct the dissipation terms. Multiplying the first and second equation of (4.5) by $\partial_x^\ell\theta$ and $\partial_x^\ell u_t - \mu\Delta\partial_x^\ell u_t$, respectively, and combining the resulting equations, we get

$$
\begin{aligned}
&\{(\partial_x^\ell u_t - \mu\Delta\partial_x^\ell u_t)\partial_x^\ell\theta\}_t + \nu|\nabla\partial_x^\ell u_t|^2 + \mu\nu(\Delta\partial_x^\ell u_t)^2 - \nu|\nabla\partial_x^\ell\theta|^2\\
&- \nabla\partial_x^\ell u_t\cdot\partial_x^\ell q - \mu\Delta\partial_x^\ell u_t\,\mathrm{div}\partial_x^\ell q - \nabla\partial_x^\ell\theta\cdot\partial_x^\ell(b'(\Delta u)\nabla\Delta u)\\
&+ \mathrm{div}\{\partial_x^\ell\theta\partial_x^\ell(b'(\Delta u)\nabla\Delta u) + \nu\partial_x^\ell\theta\nabla\partial_x^\ell\theta - \nu\partial_x^\ell u_t\nabla\partial_x^\ell u_t + \partial_x^\ell u_t\partial_x^\ell q\} = 0
\end{aligned}
\tag{4.7}
$$

for $\ell \geq 0$. Moreover, similarly as in the previous section, multiplying the first equation of (4.5) by $-\Delta\partial_x^\ell u$, we have

$$
\begin{aligned}
&- \{(\partial_x^\ell u_t - \mu\Delta\partial_x^\ell u_t)\Delta\partial_x^\ell u\}_t + b'(\Delta u)|\nabla\Delta\partial_x^\ell u|^2 - |\nabla\partial_x^\ell u_t|^2 - \mu(\Delta\partial_x^\ell u_t)^2\\
&+ \nu\nabla\Delta\partial_x^\ell u\nabla\partial_x^\ell\theta + \nabla\Delta\partial_x^\ell u\cdot[\partial_x^\ell, b'(\Delta u)]\nabla\Delta u\\
&+ \mathrm{div}\{\partial_x^\ell u_t\nabla\partial_x^\ell u_t - \nu\Delta\partial_x^\ell u\nabla\partial_x^\ell\theta - \Delta\partial_x^\ell u\partial_x^\ell(b'(\Delta u)\nabla\Delta u)\} = 0
\end{aligned}
\tag{4.8}
$$

for $\ell \geq 0$. Here we also note that the term which includes $[\partial_x^\ell, b'(\Delta u)]\nabla\Delta u$ in (4.8) can be neglected if $\ell = 0$. Then, calculating $2\times(4.7) + \nu\times(4.8)$, we get

$$
\begin{aligned}
&\{(\partial_x^\ell u_t - \mu\Delta\partial_x^\ell u_t)(2\partial_x^\ell\theta - \nu\Delta\partial_x^\ell u)\}_t + \nu|\nabla\partial_x^\ell u_t|^2 + \mu\nu(\Delta\partial_x^\ell u_t)^2\\
&+ \nu b'(\Delta u)|\nabla\Delta\partial_x^\ell u|^2 - 2\nu|\nabla\partial_x^\ell\theta|^2 - 2\nabla\partial_x^\ell u_t\cdot\partial_x^\ell q - 2\mu\Delta\partial_x^\ell u_t\,\mathrm{div}\partial_x^\ell q\\
&- 2\nabla\partial_x^\ell\theta\cdot\partial_x^\ell(b'(\Delta u)\nabla\Delta u) + \nu^2\nabla\Delta\partial_x^\ell u\cdot\nabla\partial_x^\ell\theta + \nu\nabla\Delta\partial_x^\ell u\cdot[\partial_x^\ell, b'(\Delta u)]\nabla\Delta u\\
&+ \mathrm{div}\{2\partial_x^\ell\theta\partial_x^\ell(b'(\Delta u)\nabla\Delta u) + 2\nu\partial_x^\ell\theta\nabla\partial_x^\ell\theta - \nu\partial_x^\ell u_t\nabla\partial_x^\ell u_t + 2\partial_x^\ell u_t\partial_x^\ell q\}\\
&- \mathrm{div}\{\nu^2\Delta\partial_x^\ell u\nabla\partial_x^\ell\theta + \nu\Delta\partial_x^\ell u\partial_x^\ell(b'(\Delta u)\nabla\Delta u)\} = 0
\end{aligned}
\tag{4.9}
$$

for $\ell \geq 0$. Here we also remark that the term which includes $[\partial_x^\ell, b'(\Delta u)]\nabla\Delta u$ is eliminated if $\ell = 0$. Furthermore, we multiply the second and third equation of (4.5) by $-\tau\mathrm{div}\partial_x^\ell q$ and $\nabla\partial_x^\ell\theta$, respectively, and combine the resulting equations. Then we get

$$
\begin{aligned}
&- \tau(\partial_x^\ell\theta\,\mathrm{div}\partial_x^\ell q)_t + \tau\mathrm{div}(\partial_x^\ell\theta\partial_x^\ell q_t) + |\nabla\partial_x^\ell\theta|^2 - \tau(\mathrm{div}\partial_x^\ell q)^2 \\
&+ \nabla\partial_x^\ell\theta\cdot\partial_x^\ell q + \nu\tau\Delta\partial_x^\ell u_t\,\mathrm{div}\partial_x^\ell q = 0
\end{aligned} \tag{4.10}
$$

for $\ell \geq 0$. Now, for sufficiently small α_1, we compute $\alpha_1\nu\times(4.9)+\times(4.10)$, obtaining

$$
\begin{aligned}
&\{-\tau\partial_x^\ell\theta\,\mathrm{div}\partial_x^\ell q + \alpha_1\nu(\partial_x^\ell u_t - \mu\Delta\partial_x^\ell u_t)(2\partial_x^\ell\theta - \nu\Delta\partial_x^\ell u)\}_t \\
&+ \alpha_1\nu^2(|\nabla\partial_x^\ell u_t|^2 + \mu(\Delta\partial_x^\ell u_t)^2 + b'(\Delta u)|\nabla\Delta\partial_x^\ell u|^2) + (1-2\alpha_1\nu^2)|\nabla\partial_x^\ell\theta|^2 \\
&- \tau(\mathrm{div}\partial_x^\ell q)^2 + \nabla\partial_x^\ell\theta\cdot\partial_x^\ell q + \nu\tau\Delta\partial_x^\ell u_t\,\mathrm{div}\partial_x^\ell q + \tau\mathrm{div}(\partial_x^\ell\theta\partial_x^\ell q_t) \\
&+ \alpha_1\nu\{\nu^2\nabla\Delta\partial_x^\ell u\cdot\nabla\partial_x^\ell\theta - 2\nabla\partial_x^\ell u_t\cdot\partial_x^\ell q - 2\mu\Delta\partial_x^\ell u_t\,\mathrm{div}\partial_x^\ell q \\
&- 2\nabla\partial_x^\ell\theta\cdot\partial_x^\ell(b'(\Delta u)\nabla\Delta u) + \nu\nabla\Delta\partial_x^\ell u\cdot[\partial_x^\ell, b'(\Delta u)]\nabla\Delta u\} \\
&+ \alpha_1\nu\mathrm{div}\{2\partial_x^\ell\theta\partial_x^\ell(b'(\Delta u)\nabla\Delta u) + 2\nu\partial_x^\ell\theta\nabla\partial_x^\ell\theta + 2\partial_x^\ell u_t\partial_x^\ell q\} \\
&- \alpha_1\nu\mathrm{div}\{\nu\partial_x^\ell u_t\nabla\partial_x^\ell u_t + \nu^2\Delta\partial_x^\ell u\nabla\partial_x^\ell\theta + \nu\Delta\partial_x^\ell u\partial_x^\ell(b'(\Delta u)\nabla\Delta u)\} = 0.
\end{aligned} \tag{4.11}
$$

Moreover, combining (4.4), (4.6) and (4.11), we obtain

$$
\frac{\partial}{\partial t}\tilde{\mathcal{E}}^\ell(t,x) + \tilde{\mathcal{D}}^\ell(t,x) + \mathrm{div}\tilde{\mathcal{F}}^\ell(t,x) = \tilde{\mathcal{R}}^\ell(t,x) \tag{4.12}
$$

for $\ell \geq 0$. Here we define

$$
\begin{aligned}
\tilde{\mathcal{E}}^0(t,x) &:= \sum_{j=0}^{1}\{(\partial_x^j u_t)^2 + \mu|\nabla\partial_x^j u_t|^2 + (\partial_x^j\theta)^2 + \tau|\partial_x^j q|^2\} + 2\varphi(\Delta u) \\
&\quad + b'(\Delta u)(\Delta\partial_x u)^2 - \alpha_0\tau\theta\,\mathrm{div}q + \alpha_0\alpha_1\nu(u_t - \mu\Delta u_t)(2\theta - \nu\Delta u), \\
\tilde{\mathcal{F}}^0(t,x) &:= 2\sum_{j=0}^{1}\{\partial_x^j u_t\partial_x^j(b'(\Delta u)\nabla\Delta u) - \mu\partial_x^j u_t\nabla\partial_x^j u_{tt} + \nu\partial_x^j u_t\nabla\partial_x^j\theta \\
&\quad - \nu\partial_x^j\theta\nabla\partial_x^j u_t + \partial_x^j\theta\partial_x^j q\} - 2b(\Delta u)\nabla u_t - 2b'(\Delta u)\Delta\partial_x u\nabla\partial_x u_t + \alpha_0\tau\theta q_t \\
&\quad + \alpha_0\alpha_1\nu\{2\nu\theta\nabla\theta - \nu u_t\nabla u_t + 2u_t q - \nu^2\Delta u\nabla\theta \\
&\quad + 2\theta b'(\Delta u)\nabla\Delta u - \nu\Delta u(b'(\Delta u)\nabla\Delta u)\}, \\
\tilde{\mathcal{R}}^0(t,x) &:= 2\nabla\partial_x u_t\cdot[\partial_x, b'(\Delta u)]\nabla\Delta u - 2b''(\Delta u)\Delta\partial_x u\nabla\Delta u\cdot\nabla\partial_x u_t \\
&\quad + b''(\Delta u)\Delta u_t(\Delta\partial_x u)^2 + 2\alpha_0\alpha_1\nu(b'(\Delta u) - b'(0))\nabla\Delta u\cdot\nabla\theta,
\end{aligned}
$$

and

$$\tilde{\mathcal{E}}^\ell(t,x) := \sum_{j=0}^{1}\{(\partial_x^{\ell+j}u_t)^2 + b'(\Delta u)(\Delta\partial_x^{\ell+j}u)^2 + \mu|\nabla\partial_x^{\ell+j}u_t|^2 + (\partial_x^{\ell+j}\theta)^2 + \tau|\partial_x^{\ell+j}q|^2\}$$
$$- \alpha_0\tau\partial_x^\ell\theta\,\mathrm{div}\partial_x^\ell q + \alpha_0\alpha_1\nu(\partial_x^\ell u_t - \mu\Delta\partial_x^\ell u_t)(2\partial_x^\ell\theta - \nu\Delta\partial_x^\ell u),$$

$$\tilde{\mathcal{F}}^\ell(t,x) := 2\sum_{j=0}^{1}\{\partial_x^{\ell+j}u_t\partial_x^{\ell+j}(b'(\Delta u)\nabla\Delta u) - b'(\Delta u)\Delta\partial_x^{\ell+j}u\nabla\partial_x^{\ell+j}u_t - \mu\partial_x^{\ell+j}u_t\nabla\partial_x^{\ell+j}u_{tt}$$
$$+ \nu\partial_x^{\ell+j}u_t\nabla\partial_x^{\ell+j}\theta - \nu\partial_x^{\ell+j}\theta\nabla\partial_x^{\ell+j}u_t + \partial_x^{\ell+j}\theta\partial_x^{\ell+j}q\} + \alpha_0\tau\partial_x^\ell\theta\partial_x^\ell q_t$$
$$+ \alpha_0\alpha_1\nu\{2\nu\partial_x^\ell\theta\nabla\partial_x^\ell\theta - \nu\partial_x^\ell u_t\nabla\partial_x^\ell u_t + 2\partial_x^\ell u_t\partial_x^\ell q - \nu^2\Delta\partial_x^\ell u\nabla\partial_x^\ell\theta$$
$$+ 2\partial_x^\ell\theta\partial_x^\ell(b'(\Delta u)\nabla\Delta u) - \nu\Delta\partial_x^\ell u\partial_x^\ell(b'(\Delta u)\nabla\Delta u)\},$$

$$\tilde{\mathcal{R}}^\ell(t,x) := \sum_{j=0}^{1}\{2\nabla\partial_x^{\ell+j}u_t\cdot[\partial_x^{\ell+j}, b'(\Delta u)]\nabla\Delta u - 2b''(\Delta u)\Delta\partial_x^{\ell+j}u\nabla\Delta u\cdot\nabla\partial_x^{\ell+j}u_t$$
$$+ b''(\Delta u)\Delta u_t(\Delta\partial_x^{\ell+j}u)^2\}$$
$$+ \alpha_0\alpha_1\nu\{2\nabla\partial_x^\ell\theta\cdot\partial_x^\ell((b'(\Delta u) - b'(0))\nabla\Delta u) - \nu\nabla\Delta\partial_x^\ell u\cdot[\partial_x^\ell, b'(\Delta u)]\nabla\Delta u\}$$

for $\ell > 1$, and

$$\tilde{\mathcal{D}}^\ell(t,x) := \alpha_0\alpha_1\nu^2\{|\nabla\partial_x^\ell u_t|^2 + \mu(\Delta\partial_x^\ell u_t)^2 + b'(\Delta u)|\nabla\Delta\partial_x^\ell u|^2\} + \alpha_0(1-2\alpha_1\nu^2)|\nabla\partial_x^\ell\theta|^2$$
$$+ 2|\partial_x^\ell q|^2 + 2|\partial_x^{\ell+1}q|^2 - \alpha_0\tau(\mathrm{div}\partial_x^\ell q)^2 + \alpha_0\nabla\partial_x^\ell\theta\cdot\partial_x^\ell q + \alpha_0\nu\tau\Delta\partial_x^\ell u_t\,\mathrm{div}\partial_x^\ell q$$
$$+ \alpha_0\alpha_1\nu\{(\nu^2 - 2b'(0))\nabla\Delta\partial_x^\ell u\cdot\nabla\partial_x^\ell\theta - 2\nabla\partial_x^\ell u_t\cdot\partial_x^\ell q - 2\mu\Delta\partial_x^\ell u_t\,\mathrm{div}\partial_x^\ell q\}$$

for $\ell \geq 0$. We integrate (4.12) and get

(4.13)
$$\int_{\mathbb{R}^n}\tilde{\mathcal{E}}^\ell(t,x)dx + \int_0^t\int_{\mathbb{R}^n}\tilde{\mathcal{D}}^\ell(\sigma,x)dxd\sigma = \int_{\mathbb{R}^n}\tilde{\mathcal{E}}^\ell(0,x)dx + \int_0^t\int_{\mathbb{R}^n}\tilde{\mathcal{R}}^\ell(\sigma,x)dxd\sigma.$$

For each term we compute

$$C^{-1}(\|\partial_x^\ell u_t\|_{H^2}^2 + \|\partial_x^\ell(\Delta u,\theta,q)\|_{H^1}^2) \leq \int_{\mathbb{R}^n}\tilde{\mathcal{E}}^\ell(t,x)dx \leq C(\|\partial_x^\ell u_t\|_{H^2}^2 + \|\partial_x^\ell(\Delta u,\theta,q)\|_{H^1}^2),$$

and

$$\int_{\mathbb{R}^n}\tilde{\mathcal{D}}^\ell(t,x)dx \geq c(\|\nabla\partial_x^\ell u_t\|_{H^1}^2 + \|\nabla\partial_x^\ell(\Delta u,\theta)\|_{L^2}^2 + \|\partial_x^\ell q\|_{H^1}^2) \tag{4.14}$$

for $\ell \geq 0$. For the remainder terms, we also compute

$$\int_{\mathbb{R}^n}|\tilde{\mathcal{R}}^0(t,x)|dx \leq C\|\nabla\partial_x u_t\|_{L^2}\|[\partial_x, b'(\Delta u)]\nabla\Delta u\|_{L^2} + C\|\nabla\Delta u\|_{L^\infty}\|\Delta\partial_x u\|_{L^2}\|\nabla\partial_x u_t\|_{L^2}$$
$$+ C\|\Delta u_t\|_{L^\infty}\|\Delta\partial_x u\|_{L^2}^2 + C\|\Delta u\|_{L^\infty}\|\nabla\Delta u\|_{L^2}\|\nabla\theta\|_{L^2}$$
$$\leq C\|\nabla\Delta u\|_{L^\infty}\|\Delta\partial_x u\|_{L^2}\|\nabla\partial_x u_t\|_{L^2}$$
$$+ C\|\Delta u_t\|_{L^\infty}\|\Delta\partial_x u\|_{L^2}^2 + C\|\Delta u\|_{L^\infty}\|\nabla\Delta u\|_{L^2}\|\nabla\theta\|_{L^2}$$
$$\leq C\delta_0\int_{\mathbb{R}^n}\tilde{\mathcal{D}}^0(t,x)dx$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |\tilde{\mathcal{R}}^\ell(t,x)|dx \\
&\le C\sum_{j=0}^{1}\{\|\nabla\partial_x^{\ell+j}u_t\|_{L^2}\|[\partial_x^{\ell+j}, b'(\Delta u)]\nabla\Delta u\|_{L^2} \\
&\quad + \|b''(\Delta u)\nabla\Delta u\|_{L^\infty}\|\Delta\partial_x^{\ell+j}u\|_{L^2}\|\nabla\partial_x^{\ell+j}u_t\|_{L^2} + \|b''(\Delta u)\Delta u_t\|_{L^\infty}\|\Delta\partial_x^{\ell+j}u\|_{L^2}^2\} \\
&\quad + C\|\nabla\partial_x^\ell\theta\|_{L^2}\|\partial_x^\ell\{(b'(\Delta u)-b'(0))\nabla\Delta u\}\|_{L^2} + C\|\nabla\Delta\partial_x^\ell u\|_{L^2}\|[\partial_x^\ell, b'(\Delta u)]\nabla\Delta u\|_{L^2} \\
&\le C\sum_{j=0}^{1}\{\|\nabla\Delta u\|_{L^\infty}\|\nabla\partial_x^{\ell+j}u_t\|_{L^2}\|\Delta\partial_x^{\ell+j}u\|_{L^2} + \|\Delta u_t\|_{L^\infty}\|\Delta\partial_x^{\ell+j}u\|_{L^2}^2\} \\
&\quad + C\|\nabla\partial_x^\ell\theta\|_{L^2}(\|\nabla\Delta u\|_{L^\infty}\|\Delta\partial_x^\ell u\|_{L^2} + \|\Delta u\|_{L^\infty}\|\nabla\Delta\partial_x^\ell u\|_{L^2}) \\
&\quad + C\|\nabla\Delta u\|_{L^\infty}\|\nabla\Delta\partial_x^\ell u\|_{L^2}\|\Delta\partial_x^\ell u\|_{L^2} \\
&\le C\delta_0\Big\{\int_{\mathbb{R}^n}\tilde{\mathcal{D}}^\ell(t,x)dx + \int_{\mathbb{R}^n}\tilde{\mathcal{D}}^{\ell-1}(t,x)dx\Big\}
\end{aligned}
$$

for $\ell \ge 1$. Therefore, substituting the above estimates into (4.13) and letting δ_0 sufficiently small, we obtain

$$
\int_{\mathbb{R}^n}\tilde{\mathcal{E}}^\ell(t,x)dx + \int_0^t\int_{\mathbb{R}^n}\tilde{\mathcal{D}}^\ell(\sigma,x)dxd\sigma \le C\int_{\mathbb{R}^n}\tilde{\mathcal{E}}^\ell(0,x)dx + C\delta_0\int_0^t\int_{\mathbb{R}^n}\tilde{\mathcal{D}}^{\ell-1}(\sigma,x)dxd\sigma.
$$

Here the last term on the right-hand side can be also neglected if $\ell = 0$. Consequently, using the same induction arguments as before, we arrive at the desired a priori estimate, thus proving Proposition 4.3. □

Remark 4.4. *In the case $\tau > 0$ and $\mu = 0$, we do not have the term $(\Delta\partial_x^\ell u_t)^2$ in the dissipation $\tilde{\mathcal{D}}^\ell(t,x)$. Therefore we can not control the term $\Delta\partial_x^\ell u_t\,\mathrm{div}\partial_x^\ell q$ in $\tilde{\mathcal{D}}^\ell(t,x)$. That is, this case remains open and asks for even more sophisticated estimates in the future. Semilinear nonlinearities have been dealt with successfully in* [6].

5. Useful Inequalities

In this section, we collect some useful estimates, mainly based on the Gagliardo-Nirenberg inequality which is stated as follows.

Lemma 5.1. *Let q, r be real numbers such that $1 \le q, r \le \infty$, and n, j, k be integers such that $n \ge 1$ and $0 \le j < k$. Then, if p and α satisfy the conditions:*

$$
\frac{1}{p} = \frac{j}{n} + \alpha\Big(\frac{1}{r} - \frac{k}{n}\Big) + (1-\alpha)\frac{1}{q}, \qquad \frac{j}{k} \le \alpha < 1, \tag{5.1}
$$

then the following estimate holds:

$$
\|\partial_x^j u\|_{L^p(\mathbb{R}^n)} \le C\|u\|_{L^q(\mathbb{R}^n)}^{1-\alpha}\|\partial_x^k u\|_{L^r(\mathbb{R}^n)}^{\alpha},
$$

where C is a certain constant which depends on n, j, k, q, r, α.

We often use the well-known Sobolev inequlaty given by

Lemma 5.2. *Let $n \geq 1$ and $s_0 = [n/2] + 1$. Then the following estimates hold:*

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C_{s_0} \|u\|_{H^{s_0}(\mathbb{R}^n)}, \tag{5.2}$$

where C_{s_0} is a certain constant which depends on n.

Next we show

Lemma 5.3. *Let $n \geq 1$, $k \geq 2$ and $1 \leq p, q, r \leq \infty$ with $1/p = 1/q + 1/r$. Then the following estimate holds:*

$$\|\prod_{j=1}^k \partial_x^{\ell_j} u_j\|_{L^p(\mathbb{R}^n)} \leq C \sum_{j=1}^k (\sum_{h \neq j} \|u_h\|_{L^\infty(\mathbb{R}^n)})^{k-2} (\sum_{h \neq j} \|u_h\|_{L^q(\mathbb{R}^n)}) \|\partial_x^\ell u_j\|_{L^r(\mathbb{R}^n)}. \tag{5.3}$$

for $\ell_j \geq 0$, where $\ell = \sum_{j=1}^k \ell_j$.

Proof. We use the Hölder inequality, obtaining

$$\|\prod_{j=1}^k \partial_x^{\ell_j} u_j\|_{L^p(\mathbb{R}^n)} \leq \prod_{j=1}^k \|\partial_x^{\ell_j} u_j\|_{L^{p_j}(\mathbb{R}^n)}$$

with $1/p = \sum_{j=1}^k 1/p_j$. For each terms of the right hand side, we apply the Gagliardo-Nirenberg inequality. Precisely, we have

$$\|\partial_x^{\ell_j} u_j\|_{L^{p_j}(\mathbb{R}^n)} \leq C \|u_j\|_{L^{q_j}(\mathbb{R}^n)}^{1-\alpha_j} \|\partial_x^\ell u_j\|_{L^{r_j}(\mathbb{R}^n)}^{\alpha_j},$$

where these parameters must satisfy

$$\frac{1}{p_j} = \frac{\ell_j}{n} + \alpha_j (\frac{1}{r_j} - \frac{\ell}{n}) + (1 - \alpha_j) \frac{1}{q_j}, \qquad \frac{\ell_j}{\ell} \leq \alpha_j < 1, \tag{5.4}$$

From (5.4), we can choose that $\alpha_j = \ell_j/\ell$, and then this yields $\sum_{j=1}^k \alpha_j = 1$ and

$$\frac{1}{p} = \sum_{j=1}^k (\alpha_j \frac{1}{r_j} + (1 - \alpha_j) \frac{1}{q_j}),$$

Thus we can take $q_j = (k-1)q$ and $r_j = r$ for $1 \leq j \leq k$, and hence $1/p = 1/q + 1/r$. Furthermore, we arrive at the following estimate from the above inequalities and the

Young inequality

$$\begin{aligned}
\|\prod_{j=1}^{k}\partial_x^{\ell_j}u_j\|_{L^p(\mathbb{R}^n)} &\le C\prod_{j=1}^{k}\|u_j\|_{L^{(k-1)q}(\mathbb{R}^n)}^{1-\ell_j/\ell}\|\partial_x^{\ell}u_j\|_{L^r(\mathbb{R}^n)}^{\ell_j/\ell}\\
&\le C\prod_{j=1}^{k}(\|u_j\|_{L^\infty(\mathbb{R}^n)}^{1-1/(k-1)}\|u_j\|_{L^q(\mathbb{R}^n)}^{1/(k-1)})^{1-\ell_j/\ell}\|\partial_x^{\ell}u_j\|_{L^r(\mathbb{R}^n)}^{\ell_j/\ell}\\
&= C\prod_{j=1}^{k}((\prod_{h\neq j}\|u_h\|_{L^\infty(\mathbb{R}^n)}^{1-1/(k-1)}\|u_h\|_{L^q(\mathbb{R}^n)}^{1/(k-1)})\|\partial_x^{\ell}u_j\|_{L^r(\mathbb{R}^n)})^{\ell_j/\ell}\\
&\le C\sum_{j=1}^{k}(\prod_{h\neq j}\|u_h\|_{L^\infty(\mathbb{R}^n)}^{1-1/(k-1)}\|u_h\|_{L^q(\mathbb{R}^n)}^{1/(k-1)})\|\partial_x^{\ell}u_j\|_{L^r(\mathbb{R}^n)}\\
&= C\sum_{j=1}^{k}(\prod_{h\neq j}\|u_h\|_{L^\infty(\mathbb{R}^n)})^{1-1/(k-1)}(\prod_{h\neq j}\|u_h\|_{L^q(\mathbb{R}^n)})^{1/(k-1)}\|\partial_x^{\ell}u_j\|_{L^r(\mathbb{R}^n)}\\
&\le C\sum_{j=1}^{k}(\sum_{h\neq j}\|u_h\|_{L^\infty(\mathbb{R}^n)})^{k-2}(\sum_{h\neq j}\|u_h\|_{L^q(\mathbb{R}^n)})\|\partial_x^{\ell}u_j\|_{L^r(\mathbb{R}^n)}.
\end{aligned}$$

To get this estimate, we used that

$$\begin{aligned}
\prod_{j=1}^{k}A_j^{1-\alpha_j}B_j^{\alpha_j} &= (A_1^{\alpha_2}A_1^{\alpha_3}\cdots A_1^{\alpha_k}B_1^{\alpha_1})(A_2^{\alpha_1}A_2^{\alpha_3}\cdots A_2^{\alpha_k}B_2^{\alpha_2})\cdots(A_k^{\alpha_1}A_k^{\alpha_2}\cdots A_k^{\alpha_{k-1}}B_k^{\alpha_k})\\
&= (A_2^{\alpha_1}A_3^{\alpha_1}\cdots A_k^{\alpha_1}B_1^{\alpha_1})(A_1^{\alpha_2}A_3^{\alpha_2}\cdots A_k^{\alpha_2}B_2^{\alpha_2})\cdots(A_1^{\alpha_k}A_2^{\alpha_k}\cdots A_{k-1}^{\alpha_k}B_k^{\alpha_k})\\
&= \prod_{j=1}^{k}((\prod_{h\neq j}A_h)B_j)^{\alpha_j},
\end{aligned}$$

where $\sum_{j=1}^{k}\alpha_j = 1$. Consequently, we can conclude (5.3) and complete the proof. $\square$

We present in the next lemma some useful estimates of standard Gagliardo-Nirenberg type, which we employ to control the nonlinear terms.

Lemma 5.4. *Let $n \ge 1$ and $1 \le p, q, r \le \infty$ with $1/p = 1/q + 1/r$. Then the following estimates hold:*

$$\|\partial_x^{\ell}(uv)\|_{L^p(\mathbb{R}^n)} \le C(\|u\|_{L^q(\mathbb{R}^n)}\|\partial_x^{\ell}v\|_{L^r(\mathbb{R}^n)} + \|v\|_{L^q(\mathbb{R}^n)}\|\partial_x^{\ell}u\|_{L^r(\mathbb{R}^n)}) \tag{5.5}$$

for $\ell \ge 0$, and

$$\|[\partial_x^{\ell}, u]\partial_x v\|_{L^p(\mathbb{R}^n)} \le C(\|\partial_x u\|_{L^q(\mathbb{R}^n)}\|\partial_x^{\ell}v\|_{L^r(\mathbb{R}^n)} + \|\partial_x v\|_{L^q(\mathbb{R}^n)}\|\partial_x^{\ell}u\|_{L^r(\mathbb{R}^n)}) \tag{5.6}$$

for $\ell \ge 1$, where $[\cdot,\cdot]$ denotes a commutator defined by $[A, B] = AB - BA$.

Proof. It is easy to prove (5.5) by Leipniz formula and Lemma 5.3. We only prove (5.6). We have from the definition of a commutator

$$
\begin{aligned}
[\partial_x^\ell, u]\partial_x v &= \partial_x^\ell(u\partial_x v) - u\partial_x^{\ell+1} v \\
&= \sum_{j=0}^{\ell} \binom{\ell}{j} \partial_x^j u \partial_x^{\ell-j+1} v - u\partial_x^{\ell+1} v \\
&= \sum_{j=1}^{\ell} \binom{\ell}{j} \partial_x^j u \partial_x^{\ell-j+1} v = \sum_{k=0}^{\ell-1} \binom{\ell}{k+1} \partial_x^k(\partial_x u)\partial_x^{\ell-k-1}(\partial_x v).
\end{aligned}
$$

On the other hand, we can obtain from Lemma 5.3.

$$
\|\partial_x^k(\partial_x u)\partial_x^{\ell-k-1}(\partial_x v)\|_{L^p(\mathbb{R}^n)} \le C(\|\partial_x u\|_{L^q(\mathbb{R}^n)}\|\partial_x^\ell v\|_{L^r(\mathbb{R}^n)} + \|\partial_x v\|_{L^q(\mathbb{R}^n)}\|\partial_x^\ell u\|_{L^r(\mathbb{R}^n)}),
$$

and hence we arrive at (5.6). □

In the last of this section, we state the small corollary derived by Lemma 5.3.

Corollary 5.5. *Let $n \ge 1$ and $1 \le p \le \infty$. Suppose that f is a smooth function of u. Then the following estimates hold:*

$$
\|\partial_x^\ell f(u)\|_{L^p} \le C\|\partial_x^\ell u\|_{L^p} \sum_{j=1}^{\ell} \|u\|_{L^\infty}^{j-1} \|f^{(j)}(u)\|_{L^\infty} \tag{5.7}
$$

for $\ell \ge 1$.

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References

[1] Avalos, G., Lasiecka, I.: Exponential stability of a thermoelastic system without mechanical dissipation. *Rend. Instit. Mat. Univ. Trieste Suppl.* **28** (1997), 1–28.

[2] Denk, R., Racke, R.: L^p resolvent estimates and time decay for generalized thermoelastic plate equations, *Electronic J. Differential Equations* **48** (2006), 1–16.

[3] Denk, R., Racke, R., Shibata, Y.: L_p theory for the linear thermoelastic plate equations in bounded and exterior domains. *Advances Differential Equations* **14** (2009), 685–715.

[4] Denk, R., Racke, R., Shibata, Y.: Local energy decay estimate of solutions to the thermoelastic plate equations in two- and three-dimensional exterior domains. *J. Analysis Appl.* **29** (2010), 21–62.

[5] Fernández Sare, H.D., Muñoz Rivera, J.E.: Optimal rates of decay in 2-d thermoelasticity with second sound. *J. Math. Phys.* **53** (2012), 073509-1–13.

[6] Fischer, L.: *Globale Lösungen zu Cauchyproblemen bei nichtlinearen thermoelastischen Plattengleichungen.* Master thesis, University of Konstanz (2016).

[7] Kim, J.U.: On the energy decay of a linear thermoelastic bar and plate. *SIAM J. Math. Anal.* **23** (1992), 889–899.

[8] Lasiecka, I., Pokojovy, M., Wan, X.: Global existence and exponential stability for a nonlinear thermoelastic Kirchhoff-Love plate. arXiv:1612.00798v1 (2016).

[9] Lasiecka, I., Triggiani, R.: Two direct proofs on the analyticity of the S.C. semigroup arising in abstract thermoelastic equations. *Adv. Differential Equations* **3** (1998), 387–416.

[10] Lasiecka, I., Triggiani, R.: Analyticity, and lack thereof, of thermo-elastic semigroups. *ESAIM, Proc.* **4** (1998),199–222.
[11] Lasiecka, I., Triggiani, R.: Analyticity of thermo-elastic semigroups with coupled hinged/Neumann boundary conditions. *Abstract Appl. Anal.* **3** (1998), 153–169.
[12] Lasiecka, I., Triggiani, R.: Analyticity of thermo-elastic semigroups with free boundary conditions. *Annali Scuola Norm. Sup. Pisa* **27** (1998), 457–482.
[13] Lasiecka, I., Wilke, M.: Maximal regularity and global existence of solutions to a quasilinear thermoelastic plate system. *Discr. Cont. Dyn. Sys.* **33** (2013), 5189–5202.
[14] Lineares, F., Ponce, G.: *Introduction to nonlinear dispersive equations.* Springer-Verlag, New York (2009).
[15] Liu, K., Liu, Z.: Exponential stability and analyticity of abstract linear thermoelastic systems. *Z. angew. Math. Phys.* **48** (1997), 885–904.
[16] Liu, Y., Kawashima, S.: Global existence and asymptotic behavior of solutions for quasi-linear dissipative plate equation. *Discrete Contin. Dyn. Syst.* **29** (2011), no.3, 1113–1139.
[17] Muñoz Rivera, J. E., Racke, R.: Smoothing properties, decay, and global existence of solutions to nonlinear coupled systems of thermoelastic type. *SIAM J. Math. Anal.* **26** (1995), 1547–1563.
[18] Muñoz Rivera, J.E., Racke, R.: Large solutions and smoothing properties for nonlinear thermoelastic systems. *J. Differential Equations* **127** (1996), 454–483.
[19] Quintanilla, R., Racke, R.: Qualitative aspects of solutions in resonators. *Arch. Mech.* **60** (2008), 345–360.
[20] Quintanilla, R., Racke, R.: Addendum to: Qualitative aspects of solutions in resonators. *Arch. Mech.* **63** (2011), 429–435.
[21] Racke, R.: *Lectures on nonlinear evolution equations. Initial value problems.* Second edition. Birkhäuser, Basel (2015).
[22] Racke, R., Ueda, Y.: Dissipative structures for thermoelastic plate equations in $\mathbb{R}^n$. *Adv. Dfferential Equations* **21** (2016), 601–630 .

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