

# Firm Dynamics with Frictional Product and Labor Markets\*

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## Abstract

We examine empirically and theoretically the joint dynamics of prices, output, employment and wages across firms. We first analyze administrative firm data for the German manufacturing sector for which price and quantity information at the nine-digit product level, together with employment, working hours and wages are available. We then develop a dynamic model of heterogeneous firms who compete for workers and customers in frictional labor and product markets. Prices and wages are dispersed across firms, reflecting differences in firm productivity and demand. Productivity and demand shocks have distinct implications for the firms' employment, output and price adjustments. In a quantitative analysis, we evaluate the model predictions against the data.

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# 1 Introduction

The heterogeneity and dynamics of firms is important for the understanding of labor market flows and wage dispersion: Firms which differ in size, age or productivity create and destroy jobs at different rates and they pay different wages (e.g. Davis et al. (2006), Haltiwanger et al. (2013), Lentz and Mortensen (2010)). Much of the theoretical literature on firms in the labor market builds on the seminal contributions of Hopenhayn (1992) and Hopenhayn and Rogerson (1993), augmented by richer labor market features.<sup>1</sup> While the level and changes of firm productivity are crucial for both the size distribution of firms and for the firms' growth dynamics, productivity is generally based on the firms' revenue which reflects both a price and a quantity component. This is a reasonable theoretical shortcut, given that most datasets have no separate information on firm-level prices and output.

Recent empirical evidence points at a prominent role of firm-specific demand for firm growth. Using U.S. data on narrowly defined industries that permit a distinction between price and quantity, Foster et al. (2008) examine the separate contributions of demand and productivity for firm performance, finding that demand variation is the dominant driver of firm growth and firm survival.<sup>2</sup> While there are no significant productivity differences across firms of different ages, younger firms charge lower prices than incumbents which suggests that those firms build a customer base (relationship capital).<sup>3</sup>

This paper aims at better understanding the respective roles of firm-specific demand and supply for labor market dynamics. We first document the cross-sectional dynamics of prices, wages, labor productivity, and employment for German manufacturing firms. Second, we develop an equilibrium model of firm dynamics with (i) product and labor market frictions, (ii) wage and price dispersion, and (iii) separate roles for demand and technological shocks. In a quantitative analysis we examine how well this model captures several features of the data and we conduct a few counterfactual experiments.

We begin in Section 2 by analyzing administrative panel data for the manufacturing sector in Germany containing annual price and quantity information, together with employment,

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<sup>1</sup>Models based on the Hopenhayn-Rogerson framework can address a variety of issues, such as international trade (Melitz (2003)), industry dynamics (Ericson and Pakes (1995)), labor-market policy (Alvarez and Veracierto (2001)), and cross-country productivity differences (Restuccia and Rogerson (2008)). Search and matching frictions in the labor market have been introduced into such settings by, e.g., Smith (1999), Hawkins and Acemoglu (2014), Elsby and Michaels (2013) and Kaas and Kircher (2015).

<sup>2</sup>The quinquennial manufacturing census data they use does not permit them to study the dynamics of firms over time, or labor flows and wages, which are of particular interest in this paper.

<sup>3</sup>This idea motivates Foster et al. (2012) to build a structural econometric model of firm dynamics in which product demand stochastically adjusts slowly as firms actively expend resources to build a customer base. They find that demand gaps across firms are substantial and are bridged only slowly over time: in a given industry, a firm that is five years old or younger is expected to sell only 41 percent of the output of a firm that is more than fifteen years old. As they stress, the plight of young firms hinges on the coexistence of high technological efficiency and low demand, which is an indication of an inefficient allocation of customers across firms.

working hours and wages. On the one hand, we examine the respective roles of price and quantity adjustment and their relationships with employment growth. Although many firms do not change prices from one year to the next, the dispersion of annual price growth is still substantial. On average, price growth is negatively correlated with output growth, suggesting a dominant role of supply (cost) shocks. Nonetheless, price increases go together with employment expansions which suggests a prominent role of demand for firm growth. On the other hand, we are interested in the dispersion in prices, productivity and wages, and their respective correlations, and how these relate to firm characteristics, such as the size and growth of firms. For this purpose, we define for every firm-year observation a firm price index (analogous to the household price index of Kaplan and Menzio (2015)), defined as the ratio between the firm's actual sales value to the hypothetical sales of the firm if it would receive the average market prices for all its products. The firm's price index correlates negatively with quantity labor productivity which accounts for the observation that quantity labor productivity is more dispersed than revenue labor productivity.<sup>4</sup>

In Section 3 we build an equilibrium model in which heterogeneous firms compete for workers in a frictional labor market and simultaneously compete for buyers in a frictional product market. Productivity and demand are firm-specific state variables, and idiosyncratic shocks to both variables have distinct implications for the employment, wage, output and price policies of firms. Frictions in labor and product markets imply that growing firms need to build a workforce as well as a customer base over time. Sluggish adjustment gives rise to dispersion of wages and prices across firms: firms with a higher desire to grow offer higher wages to recruit more workers and they offer lower discount prices to attract new customers. In this way our model combines the competitive-search settings of Gourio and Rudanko (2014) and of Kaas and Kircher (2015), and it introduces firm-specific demand shocks besides the usual productivity shocks into this combined framework.<sup>5</sup> In Section 4 we calibrate the model to account for the joint dynamics of output and prices observed in German firm-level data. We show that the calibrated model generates several cross-sectional observations, such as the positive relationship between firm size and productivity and the negative relationship between firm size and prices, although it features too little overall dispersion of prices.

Different from the standard Diamond-Mortensen-Pissarides framework, firms in our model employ multiple workers and operate under decreasing returns. Next to labor market frictions, we also introduce a search-and-matching process in the product market which is meant to capture the observation that firms spend substantial time and resources for sales and marketing activities in order to attract customers. As in Gourio and Rudanko (2014), firms adjust sluggishly to productivity shocks since customer acquisition is costly and time-consuming. Different from

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<sup>4</sup>These findings are in line with those of Foster et al. (2008) about TFP dispersion. Lacking information on the cost structure of firms in our panel data, we cannot examine TFP dispersion in this paper.

<sup>5</sup>See also Felbermayr et al. (2014) who examine patterns of wage dispersion in Germany using a two-country version of Kaas and Kircher (2015).

their contribution, we address the role of demand for firm growth and we examine the labor market spillovers. Several recent contributions also introduce product market search frictions into macroeconomic models. In the presence of such frictions, Bai et al. (2012) and Michaillat and Saez (2015) argue that aggregate demand shocks play more prominent roles than technology shocks. Kaplan and Menzio (forthcoming), Petrosky-Nadeau and Wasmer (2015) and Den Haan (2013) combine frictions in product and labor market, introducing new amplification mechanisms for business-cycle dynamics. Unlike our paper, none of these contributions addresses firm heterogeneity and the role of firm-specific (idiosyncratic) demand.

Our work relates to an empirical literature examining the dispersion of firm-level prices and productivity. While Abbott (1991) and Foster et al. (2008) document dispersion of producer prices in specific industries, Carlsson and Skans (2012) and Carlsson et al. (2014) use Swedish firm-level data for the manufacturing sector showing that unit labor costs are transmitted less than one-to-one to output prices and that much of the variation in output prices remains unexplained by productivity shocks. Furthermore, they find that employment responds negligibly to productivity shocks, while permanent demand shocks are the main driving force of employment adjustment. Smeets and Warzynski (2013) and De Loecker (2011) examine the price effects of international trade on the basis of Danish and Belgian (resp.) manufacturing firms. Kugler and Verhoogen (2011) use Colombian firm data to document a positive relationship between plant size and prices, which contrasts with a negative correlation between firm size and prices observed in Foster et al. (2008) and Roberts and Supina (1996). Our findings, based on rather homogeneous products, are consistent with the latter results.

## 2 Empirical Findings

This section describes the data, how we treat them, and the results. Further details are contained in Appendix A.

### 2.1 Data

We use the German administrative firm-level data *Amtliche Firmendaten für Deutschland* (AFiD) which are provided by the Research Data Centers of the Federal Statistical Office and the statistical offices of the *Länder* (federal states). Officially collected by the statistical offices based in each of the federal states, the data are a result of several combined statistics whereby all firms or establishments with operations in the country are required by law to report them. Depending on the underlying statistics, the data are provided in different panels. We work with the *AFiD-Panel Industriebetriebe* covering establishments in manufacturing, mining and quarrying in the years 2005–2007.<sup>6</sup> For establishments with at least 20 employees, the panel

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<sup>6</sup>The actual panel covers the much longer horizon 1995–2014, and we plan to extend this analysis for the

includes total annual sales value, total employment, wages and hours worked in the reporting year.<sup>7</sup> We merge this panel with the *AFiD-Module Produkte* that has recordings on quantities and sales values for nine-digit products that the establishment produced in a given year. For around 20 percent of the establishments in the year 2006, we can further match the panel with the *AFiD-Module Verdienste* which is a matched employer–employee dataset containing detailed information for a subsample of the establishment’s workers, such as wage, working hours, age, education, gender, and job tenure.<sup>8</sup> For our analysis of wage dispersion, we consider this subsample of establishments.

The original merged dataset is unbalanced and has 375,016 establishment-product-years.<sup>9</sup> We lose 87,912 observations by deleting products without quantity information, e.g. services such as cleaning or repair which are measured in Euros. Further cleaning involves dropping all products whose sales value is zero (17,751), products which are processed in contract work (tenth digit “2”; 16,518), establishments with missing employment (1,043), and those with zero or missing wage information (176). Because only quantities of products measured in the same unit can be compared, we drop one product which is measured in different units across establishments or years. The remainder of the observations have positive sales values, quantities, employment and wage information. Reported working hours are missing for some establishments; we still keep those observations for the analysis with the exception of those statistics regarding hourly wages or hourly productivity.

Since pricing decisions are likely made at the firm level, and in order to eliminate product sold to establishments that belong to the same firm, we remove all establishments that belong to multi-establishment firms. We refer to these single-establishment firms as “firms” in the following terminology, keeping in mind that these firms represent the portion of smaller and medium-sized firms in German manufacturing.

We measure the price of a product by its sales value divided by the quantity. This price can reflect a quality component of the product which may differ across firms producing the same nine-digit product in the same year, or across time in the same firm. For our analysis of firm dynamics, i.e. year-to-year changes in price and quantity, we presume that these quality differences within the same firm are not substantial. This leads us to consider the *full sample* defined below for our analysis of firm dynamics. For cross-sectional price and productivity variation, however, quality differences between firms should be a bigger concern. Therefore we also consider a *homogeneous sample* which is based on those products which are measured in

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complete time period.

<sup>7</sup>The annual sales value is the sum of sales values reported in quarterly production surveys. Employment, wages and hours are obtained as averages from monthly surveys. We only consider establishments which are active throughout the full year.

<sup>8</sup>For the longer period 1995–2014, matched employer-employee modules are available in the years 1996, 2001, 2006, 2010 and 2014.

<sup>9</sup>This corresponds to 143,780 establishment-years (hence, an establishment produces 2.6 products on average) and 14,363 product-years (hence, a product is produced by 26.1 establishments on average).

physical units of weight, length, area, and volume, whereas we remove all products which are measured in other units such as “items” or “pairs”. The underlying hypothesis is that products measured in physical units have a lower degree of processing, so that quality differences are less important.<sup>10</sup> For further details about the two samples and the removal of outliers, see Appendix A.

## 2.2 Firm Dynamics

We now describe our empirical findings about the dynamics of firms, i.e. year-to-year changes of prices, output, employment, wages and productivity. This analysis is based on the *full sample* as described above. Formal definitions of all variables and further details are in Appendix A. We track a firm’s product portfolio between any two years, and if the product portfolio changes we replace the missing product price with the quantity-weighted average price of this product obtained from other firms in the sample. Based on price and quantity information in any two consecutive years, we calculate a firm-level Paasche price index (referred to as the firm’s *price*). The firm’s *quantity* (output) equals the total sales value divided by the price index. We consider two measures of labor productivity. *Quantity labor productivity* (QLP) is obtained by dividing quantity by employment, and *revenue labor productivity* (RLP) and is obtained by dividing total sales by employment. Alternatively, when we divide by working hours instead of employment, we refer to (quantity or revenue) hourly labor productivity (QHP and RHP). Growth rates are expressed as log annual changes of the respective variables. A firm’s sales growth is naturally decomposed as the sum of its price growth and its quantity growth. It also follows that quantity labor productivity is quantity growth minus employment (or hours) growth; and revenue labor productivity growth is quantity growth plus price growth minus employment (or hours) growth. Figure 4 shows kernel density plots of the various log growth rates.

In what follows we report statistics of residuals obtained by fitting a random-effects model with controls for year, industry and region. Absent enough years of observation per firm (currently at most 2 years) an arguably more appropriate fixed-effects model is not suitable. We are currently fitting an OLS model with industry-year and region-year controls. Unweighted and employment-weighted statistics are reported in Appendix A.3.

Table 1 shows that at a standard deviation of 0.099, price growth is less dispersed than sales growth, quantity growth and labor productivity growth (however measured) whose standard deviations are all between 0.15 and 0.17. Price growth is more dispersed than employment

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<sup>10</sup>To give examples, the *homogeneous sample* includes products “1720 32 144: Fabric of synthetic fibers (with more than 85% synthetic) for curtains (measured in  $m^2$ )” and “2112 30 200: Cigarette paper, not in the form of booklets, husks, or rolls less than 5 cm broad (measured in  $t$ )”, whereas it does not include “1740 24 300: Sleeping bags (measured in ‘items’)” and “2513 60 550: Gloves made of vulcanized rubber for housework usage (measured in ‘pairs’)”.

growth (0.062) or wage (per employee or per hour) growth (0.079 and 0.088 resp.). The standard deviation of quantity labor productivity growth is of similar magnitude as the one of revenue labor productivity, regardless of whether they are measured per employees or per hour.

Table 1: Summary statistics for growth dynamics

	$\hat{S}$	$\hat{P}$	$\hat{Q}$	$\hat{E}$	$\hat{W}$	$\hat{H}$	$\widehat{W/E}$	$\widehat{W/H}$	$\widehat{QLP}$	$\widehat{QHP}$	$\widehat{RLP}$	$\widehat{RHP}$
$\hat{S}$	1.000											
$\hat{P}$	0.281	1.000										
$\hat{Q}$	0.792	-0.364	1.000									
$\hat{E}$	0.247	0.021	0.227	1.000								
$\hat{W}$	0.297	0.036	0.265	0.615	1.000							
$\hat{H}$	0.265	0.031	0.238	0.533	0.502	1.000						
$\widehat{W/E}$	0.106	0.023	0.088	-0.247	0.580	0.064	1.000					
$\widehat{W/H}$	-0.016	0.001	-0.016	-0.033	0.336	-0.631	0.457	1.000				
$\widehat{QLP}$	0.672	-0.381	0.895	-0.214	-0.019	-0.012	0.200	0.000	1.000			
$\widehat{QHP}$	0.582	-0.367	0.798	-0.120	-0.065	-0.386	0.044	0.371	0.859	1.000		
$\widehat{RLP}$	0.887	0.274	0.686	-0.207	0.005	0.008	0.224	0.002	0.784	0.648	1.000	
$\widehat{RHP}$	0.783	0.250	0.600	-0.111	-0.044	-0.382	0.060	0.387	0.653	0.809	0.847	1.000
Std. dev.	0.154	0.099	0.160	0.062	0.076	0.084	0.079	0.088	0.169	0.158	0.165	0.152

Notes: Each variable is expressed in logs.  $\hat{S}$  is sales growth,  $\hat{P}$  is price growth,  $\hat{Q}$  is quantity growth,  $\hat{E}$  is employment growth,  $\hat{W}$  wage bill growth,  $\hat{H}$  is hours growth,  $\widehat{W/E}$  is wage per employee growth,  $\widehat{W/H}$  is wage per hour growth,  $\widehat{QHP}$   $\widehat{QLP}$  is (hourly) quantity labor productivity growth and  $\widehat{RHP}$   $\widehat{RLP}$  is (hourly) revenue labor productivity growth. All statistics control for year, industry and region.

The table further shows that price growth shows no strong correlations with the growth rates of employment and hours. In contrast, it correlates strongly negatively with quantity growth which itself correlates positively with employment and hours. Both price growth and quantity growth are weakly correlated with the growth of wages per employee or per hour.

Based on these statistics, we conduct a variance decomposition of the sales growth into a component due to price growth, and a component due to quantity growth. Likewise, we can decompose the variance of revenue labor productivity growth into a component due to price growth and one due to quantity labor productivity growth. In both cases, the price variation accounts for around 21% of the changes in sales growth and in revenue productivity growth. When it comes to changes in quantity labor productivity growth, output changes account for 91% (78%), whereas employment (hours) adjustment account for the remaining 9% (22%). We conclude that quantity changes are the most important source of variation in a firm's sales, quantity or revenue labor productivity. Table 6 in the Appendix shows the details for these

decompositions.

To explore further the observation that employment growth correlates more strongly with quantity growth than with price growth, we sorted firms into bins of log quantity growth and of log price growth. For each bin, we calculate the average log employment growth rate. The resulting relationships are shown in Figure 1 which are smoothed plots using locally weighted regressions. The figure shows that for firms that record positive growth in either price or quantity, they also expand employment, although the responsiveness to quantity growth is much stronger (an elasticity around one third) which is little surprising given that firms need to adjust their factor inputs to produce more output. On the other hand, employment growth responds much weaker to negative quantity growth. Indeed, on average firms cut employment only when quantity growth is smaller than 11 percent. In contrast, when firms cut prices, they do not reduce employment on average which may be an indication that positive supply (cost) shocks led many firms to expand employment together with price cuts.

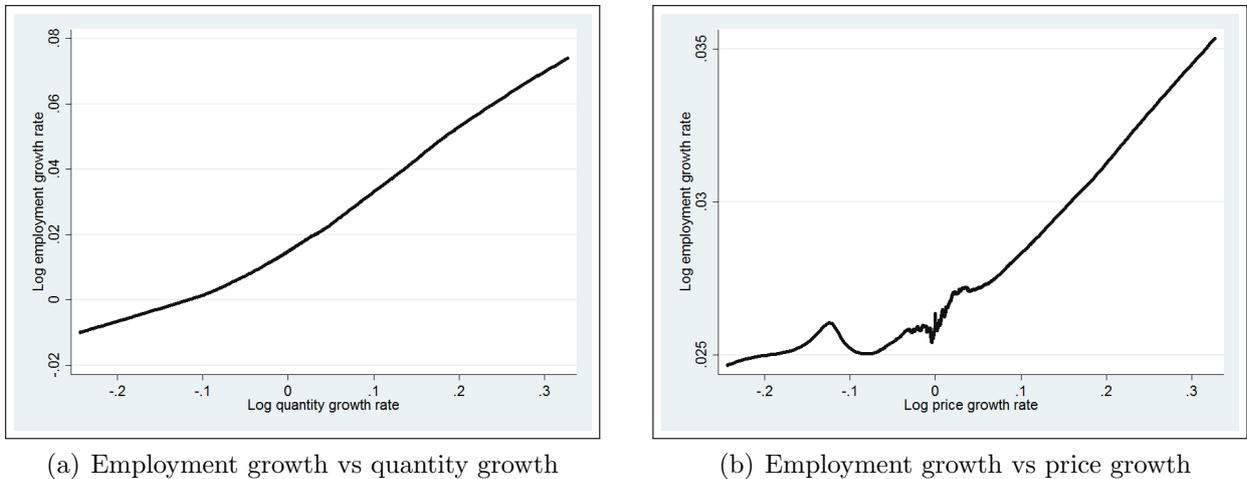


Figure 1: Nonlinear relations between employment growth and price growth, and between employment growth and quantity growth.

### 2.3 Price, Wage and Productivity Dispersion

This section discusses the cross-sectional dispersion in prices, productivity as well as wages. The cross-firm dispersion of average hourly wages can be measured in any year. Only in the year 2006 can we decompose this into an observable (worker-specific) and a firm-specific (unobservable) part.<sup>11</sup>

<sup>11</sup>We describe the procedure in Appendix A and Figure 6 for kernel density plots. The next version of this paper will contain more elaborate results.

We define a firm-level relative price index, termed as the firm-specific price, as the firm’s total sales value relative to that which it would make if each of its products were sold at the (quantity-weighted) average market price.<sup>12</sup> As before, we define revenue labor productivity (RLP) as the total sales value divided by employment. Quantity labor productivity is total sales value evaluated at average market prices divided by employment, denoted QLP.<sup>13</sup> Hence revenue productivity is the product between quantity productivity and the firm-specific price (see Appendix A). All these variables are expressed in logs. Lastly, quantity or revenue labor productivity is alternatively measured per hour (RHP and QHP respectively). Figure 5 shows kernel density plots of firm-specific prices, revenue and quantity productivity, both unweighted and weighted by employer size.

Table 2 summarizes statistics for the residuals of price, productivity, and wages obtained as in the preceding section by fitting a random-effects model for each variable with controls for year, industry and region. The table shows that the firm-specific price is more dispersed than wage per hour and wage per employee. For either measure, quantity labor productivity is more dispersed than revenue labor productivity, and revenue labor productivity is less dispersed than firm-specific price. The table also shows that the firm-specific price is weakly positively correlated with wage per hour, wage per employee and with both measures of revenue labor productivity.<sup>14</sup> Furthermore, the firm-specific price is strongly and negatively correlated with both measures of quantity labor productivity. This is the main reason why quantity labor productivity is much more dispersed than revenue labor productivity, similar as in Foster et al. (2008) for TFP dispersion.

### 3 The Model

In this section we build a canonical model that describes the dynamics of firms in the presence of frictions in product and labor markets. In the product market, firms compete for buyers via costly sales activities and by offering discounts on their products, which helps to build a customer base. In the labor market, firms build up a workforce by spending resources on recruitment and by offering long-term contracts to new hires.

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<sup>12</sup>Analogue to the household price index in Kaplan and Menzio (forthcoming), our firm-specific price is a measure of the “expensiveness” of the firm relative to other firms.

<sup>13</sup>In the previous subsection we define QLP (QHP) by deflating revenue by the firm’s price index, analogously to national income accounting. Here we define it on the basis of average market prices for the firm’s products. Therefore, the growth of the latter measure between any two years also reflects a change in average market prices. The next version of this paper contains a decomposition of a firm’s RLP growth (for the *homogeneous sample* into three components: (i) quantity labor productivity growth at fixed prices, (ii) average price growth, (iii) firm-specific price growth.

<sup>14</sup>Table 10 and Table 11 in Appendix A.3 reveal that some of these results are reversed without controls for industry, year or region (weighted or unweighted by firm size).

Table 2: Summary statistics for price, productivity and wage dispersion

	<i>RLP</i>	<i>QLP</i>	<i>RHP</i>	<i>QHP</i>	$\tilde{P}$	<i>W/H</i>	<i>W/E</i>
<i>RLP</i>	1.000						
<i>QLP</i>	0.633	1.000					
<i>RHP</i>	0.793	0.491	1.000				
<i>QHP</i>	0.491	0.877	0.623	1.000			
$\tilde{P}$	0.087	-0.715	0.083	-0.686	1.000		
<i>W/H</i>	0.044	0.009	0.330	0.205	0.027	1.000	
<i>W/E</i>	0.268	0.180	0.066	0.039	0.009	0.441	1.000
Std. dev.	0.134	0.183	0.115	0.166	0.139	0.063	0.089

Notes: Each variable is expressed in logs. (*QHP*) *QLP* and (*RHP*) *RLP* denote (hourly) quantity labor productivity and (hourly) revenue labor productivity respectively.  $\tilde{P}$ , *W/H*, and *W/E* denote firm-specific price, wage per hour and wage per employee, respectively. All statistics control for year, industry and region.

There is a representative household which owns all the firms. The household comprises a fixed stock of workers and it sends buyers to purchase goods. A worker can be either employed at a firm or unemployed; likewise, a buyer can be either attached to a firm or unattached. We can also interpret a “buyer” as a unit of shopping time that can be used to either buy a good from a previously known seller or to search for a purchase elsewhere. Search in both markets is competitive: Firms post employment contracts to attract job seekers, and they offer discounts on their products to attract new customers. In both markets, firms trade off higher matching rates against more profitable offers. Firms face idiosyncratic demand and productivity shocks. In the event of adverse shocks, a firm may find it optimal to layoff workers; it may also decide not to serve some of its previous customers.

We describe a stationary equilibrium in which search values of buyers and workers are constant over time, while individual firms’ employment and sales grow and shrink, depending on their idiosyncratic productivity and demand shocks. We then establish a welfare theorem which permits a tractable equilibrium characterization by a social planning problem.

### 3.1 The Environment

**Goods and preferences.** The representative household consumes the goods produced by firms, as well as a separate numeraire good. Utility of the household is  $\sum_t \beta^t [e_t + u(C_t)]$ , where

$e_t$  is consumption of the numeraire good,  $u$  is a concave utility function and  $\beta$  is the discount factor.  $C_t = \int y_t(f)c_t(f)d\mu_t(f)$  is a consumption aggregator which integrates over the measure  $\mu_t$  of active firms  $f$  in period  $t$  at which the household buys  $c_t(f)$  units of output.  $y_t(f)$  is an idiosyncratic (firm-specific) taste parameter which reflects, for instance, local preferences or quality differences between firms and over time. This specification captures the idea that goods within an industry are close substitutes, whereas it abstracts from imperfect substitutability across industries or from industry-specific taste shocks. In a stationary equilibrium,  $C_t = C$  is a constant, as is the marginal rate of substitution between the consumption aggregator and the numeraire,  $u'(C)$ . All prices and costs defined below are expressed in units of the numeraire good.

**Workers and customers.** There is a constant stock  $\bar{L}$  of workers who are members of the household. A worker can be either employed at a firm or unemployed. An unemployed worker produces  $b$  units of the numeraire good. The household also has a large number of potential buyers (or, units of shopping time) that are either attached to the customer base of a firm or that search for purchases elsewhere in the goods market. Any active buyer (shopping or searching) imposes a cost  $c$  on the household; once matched to a firm, the buyer can buy up to one unit of the good produced by the firm per period. Attached customers or employed workers do not search.

**Technology.** A firm with  $L$  workers produces  $xF(L)$  units of its output good.  $F$  is increasing and concave and it satisfies the Inada condition  $F'(L) \rightarrow 0$  for  $L \rightarrow \infty$ .  $x$  is the firm's idiosyncratic productivity. Given this reduced-form modeling of a firm's production technology, changes of  $x$  could reflect any type of supply-side shocks, such as technology changes or price changes of intermediate goods.

**Demand.** A firm with  $B$  customers can sell up to  $B$  units of output. Every customer of the firm wishes to buy one unit of the firm's good, as long as the unit price is smaller than the marginal rate of substitution between the firm's good and the numeraire good, which is  $u'(C)y$  where  $y$  is the firm-specific demand state. Because the good is non-storable, the firm is naturally constrained by  $B \leq xF(L)$  in any period. If that inequality is strict, the firm wastes some of its output.

**Shocks.** Both  $x$  and  $y$  follow a joint Markov process on a finite state space. We write  $z = (x, y) \in Z$  with finite set  $Z$  and denote  $\pi(z_+|z)$  the transition probability from  $z$  to  $z_+$ . For a firm of age  $a$ , we write  $z^a = (z_0, \dots, z_a)$  for the shock history from the entry period (firm age zero) up to the current period (firm age  $a$ ), where  $z_k = (x_k, y_k)$ ,  $k = 0, \dots, a$ .  $\pi^a(z^a)$  denotes the unconditional probability of that history event.

**Recruitment and sales activities.** For recruitment and sales, the firm spends  $r(R, L)$  and  $s(S, L)$  respectively, where  $R$  and  $S$  measure recruitment and sales effort and  $L$  is the size of the firm's workforce before it matches with new workers and customers. One can also think of recruitment and sales costs to capture the working time of managers and staff assigned to those activities. Both  $r$  and  $s$  are increasing and convex in their respective first arguments.

They are non-increasing in the size of the workforce  $L$  to capture scale effects.

**Labor market search.** Search in the labor market is directed. Recruiting firms offer long-term contracts to new hires. They are matched with unemployed workers in submarkets that differ by the offered contract values. In a given submarket, a firm hires  $m(\lambda) \leq \lambda$  workers per unit of recruitment effort, where  $\lambda$  measures unemployed workers per unit of recruitment effort in the submarket, and  $m$  is a strictly increasing and concave function. Hence,  $m(\lambda)/\lambda$  is the probability that an unemployed worker finds a job in this submarket, which decreases in  $\lambda$ . An employment contract specifies wage payments and separation probabilities contingent on realizations of firm-specific shocks. We write  $\mathcal{C}^a = (w^a(z^k), \delta_w^a(z^{k+1}))_{k \geq a}$  for the employment contract of a worker who is hired by a firm of age  $a$ .  $w^a(z^k)$  is the wage that the worker earns when the firm has age  $k \geq a$ , conditional on the shock history  $z^k$  and conditional on staying employed at this firm.  $\delta_w^a(z^{k+1})$  is the probability to separate from the firm in event history  $z^{k+1}$  with  $k+1 > a$ .

**Product market search.** Search in the product market is also directed, but firms cannot commit to long-term contracts.<sup>15</sup> Instead, firms that aim to expand the customer base offer discount prices  $p$  to new customers. In all subsequent periods, an attached customer can still purchase at this firm, but the customer anticipates that the firm charges the reservation price  $p^R$  which makes the buyer exactly indifferent between buying at this firm or remaining inactive. Unattached buyers and selling firms are matched in submarkets that differ by the buyers' match values. For each unit of sales effort, the firm attracts  $q(\varphi) \leq \varphi$  new customers, where  $\varphi$  is the measure of unattached buyers per unit of sales effort in the submarket, and  $q$  is an increasing and concave function. An unattached buyer searching for purchases is successful with probability  $q(\varphi)/\varphi$ , which is a decreasing function of  $\varphi$ .

**Entry, separations and exit.** New firms can enter the economy at cost  $K > 0$  with zero workforce and zero customer base. They draw an initial productivity and demand state  $(x, y)$  from probability distribution  $\pi^0$ . Any existing firm, depending on its supply and demand shocks, separates from workers according to the contractual commitments. Separated workers can search for jobs in the same period. The firm may also decide not to serve some of its attached customers who would then leave the firm's customer base. Workers quit the job into unemployment with exogenous probability  $\bar{\delta}_w$ , and buyers leave the customer pool of a firm with exogenous probability  $\bar{\delta}_b$ . This implies that the actual customer churn rate is bounded below by  $\delta_b \geq \bar{\delta}_b$ . Likewise, the contractual state-contingent worker separation rates are bounded

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<sup>15</sup>The assumption that firms offer long-term contracts to workers though not to customers is intended to reflect realistic features of worker-firm and customer-firm relationships. Although long-term contracts with customers are common in some industries, they tend to be rather short. For German manufacturing firms, Stahl (2010) finds that although 50% of sales are undertaken in written contracts, the average contract duration is just 9 months. With an annual calibration, the absence of price commitment seems a reasonable abstraction. Nonetheless, in subsection 3.4 we also consider the case where firms offer long-term contracts and do not price discriminate among customers.

below by  $\delta_w \geq \bar{\delta}_w$ . At the end of the period, after firms produce, sell and pay workers, any firm exits with probability  $\delta$  in which case all its workers enter the unemployment pool and all its customers become unattached.

## 3.2 Stationary Competitive Search Equilibrium

We now describe a stationary equilibrium in which search values of workers and customers, as well as the distributions of workers and customers across firm types, are constant over time. Any firm's policy then only depends on the idiosyncratic shock history  $z^a$  where  $a$  is the firm's age. Hence we identify the different firm types with  $z^a$ .

### 3.2.1 Workers

Let  $U$  denote the value of an unemployed worker and let  $W(\mathcal{C}^a, z^k)$  denote the value of an employed worker in contract  $\mathcal{C}^a$  at firm  $z^k$  with  $k \geq a$ . Those values represent the marginal contribution of the worker to the representative household's utility. Unemployed workers observe the offered contracts  $\mathcal{C}^a$  at firm types  $z^a$  and the corresponding market tightness  $\lambda$  in the submarkets in which value-equivalent contracts are traded. The unemployed worker's Bellman equation is

$$U = \max_{(W(\mathcal{C}^a, z^a), \lambda)} \frac{m(\lambda)}{\lambda} W(\mathcal{C}^a, z^a) + \left(1 - \frac{m(\lambda)}{\lambda}\right) [b + \beta U], \quad (1)$$

where maximization is over all active submarkets  $(W(\mathcal{C}^a, z^a), \lambda)$ . With probability  $m(\lambda)/\lambda$ , the worker finds employment in which case the continuation value is  $W(\mathcal{C}^a, z^a)$ . With the counter-probability, the worker earns unemployment income  $b$  and stays unemployed to the next period.

The employment value  $W(\mathcal{C}^a, z^k)$  satisfies the Bellman equation

$$W(\mathcal{C}^a, z^k) = w^a(z^k) + \beta(1 - \delta) \mathbb{E}_{z^k} W'(\mathcal{C}^a, z^{k+1}) + \beta \delta U. \quad (2)$$

This worker earns  $w^a(z^k)$  in the current period. At the end of the period, the firm exits with probability  $\delta$  in which case the worker becomes unemployed. With the counter-probability, the worker stays employed to the next period which yields continuation value  $W'(\mathcal{C}^a, z^{k+1})$  where the prime indicates the employment value *before* the firm separates from workers after demand and productivity shocks are realized:

$$W'(\mathcal{C}^a, z^k) = [1 - \delta_w^a(z^k)] W(\mathcal{C}^a, z^k) + \delta_w^a(z^k) U. \quad (3)$$

With contractual separation probability  $\delta_w^a(z^k)$ , the worker leaves the firm and can search for employment in the same period (continuation utility  $U$ ). Otherwise the worker stays employed with continuation utility  $W(\mathcal{C}^a, z^k)$ .

It is convenient to define the option value of search in submarket  $(W, \lambda)$  by

$$\rho(W, \lambda) \equiv \frac{m(\lambda)}{\lambda} (W - b - \beta U) .$$

Then, the flow utility value of unemployment satisfies

$$(1 - \beta)U = b + \rho^* , \quad (4)$$

where  $\rho^* \equiv \max \rho(W(\mathcal{C}^a, z^a), \lambda)$  is the maximal search value. It follows that any contract that attracts unemployed workers (i.e.,  $\lambda > 0$ ) yields the same search value  $\rho^*$ .

### 3.2.2 Customers

The household can send arbitrarily many buyers to the goods market at shopping cost  $c$  per buyer. Hence the marginal contribution of any buyer to the household's utility must be zero. Searching buyers observe unit discount prices  $p$  offered by firms of different types. Buyers and firms are matched in submarkets that yield identical match values  $u'(C)y_a - p$  to the buyer. Since the firm charges the reservation price for all attached customers in subsequent periods, the continuation value beyond the matching period is zero. Let  $\varphi$  denote buyer-to-sales-effort ratio in such a submarket with matching probability  $q(\varphi)/\varphi$ . The expected gain from searching must equal the search cost:

$$c = \max_{(p, y_a, \varphi)} \frac{q(\varphi)}{\varphi} [u'(C)y_a - p] , \quad (5)$$

where maximization is over all active submarkets  $(p, y_a, \varphi)$ . Any discount price that attracts new customers (i.e.,  $\varphi > 0$ ) yields the same search value  $c$ . It follows that discount prices are linked to market tightness  $\varphi$  via

$$p = u'(C)y_a - \frac{c\varphi}{q(\varphi)} .$$

Reservation prices  $p^R$  charged on existing customers make the buyer indifferent between buying at this price after incurring the shopping cost  $c$ , or remaining inactive. Hence,

$$p^R = u'(C)y_a - c .$$

### 3.2.3 Firms

A firm of type  $z^a$  takes as given the workers hired in earlier periods,  $L^\tau$ ,  $\tau = 0, \dots, a - 1$ , together with their respective contracts  $\mathcal{C}^\tau$ .<sup>16</sup> It also takes as given the existing stock of the

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<sup>16</sup>Without loss of generality, all workers hired by a firm of a given type are hired in the same contract which is an optimal policy of the firm (see Kaas and Kircher (2015) for a formal argument).

customer base  $B_-$ . Hence the firm's state vector is  $\sigma = [(L^\tau, \mathcal{C}^\tau)_{\tau=0}^{a-1}, B_-, z^a]$ . Let  $J_a(\sigma)$  denote the value of the firm at the beginning of the period; it solves the problem

$$J_a(\sigma) = \max_{(\lambda, R, \mathcal{C}^a), (\delta_b, \varphi, S, p, p^R)} \left\{ p^R B_- (1 - \delta_b) + pq(\varphi)S - W - r(R, L_0) - s(S, L_0) + \beta(1 - \delta) \mathbb{E} J_{a+1}(\sigma_+) \right\} \quad (6)$$

subject to

$$\sigma_+ = [(L^{\tau+}, \mathcal{C}^\tau)_{\tau=0}^a, B, z^{a+1}], \quad \mathcal{C}^a = (w^a(z^k), \delta_w^a(z^{k+1}))_{k \geq a}, \quad \delta_w^a(\cdot) \geq \bar{\delta}_w, \quad (7)$$

$$L^{\tau+} = (1 - \delta_w^\tau(z^a))L^\tau, \quad \tau = 0, \dots, a-1, \quad L^{a+} = m(\lambda)R, \quad L_0 = \sum_{\tau=0}^{a-1} L^{\tau+}, \quad (8)$$

$$W = \sum_{\tau=0}^a w^\tau(z^a) L^{\tau+}, \quad (9)$$

$$B = B_- (1 - \delta_b) + q(\varphi)S, \quad \delta_b \geq \bar{\delta}_b, \quad (10)$$

$$B \leq xF(L), \quad L = \sum_{\tau=0}^a L^{\tau+}, \quad (11)$$

$$\rho^* = \rho(W(\mathcal{C}^a, z^a), \lambda) \quad \text{if } \lambda > 0, \quad (12)$$

$$p = u'(C)y_a - \frac{c}{q(\varphi)} \quad \text{if } \varphi > 0, \quad p^R = u'(C)y_a - c. \quad (13)$$

In (6), the firm's problem is to maximize revenue from sales to existing and new customers minus expenditures for wages, sales and recruitment plus the expected continuation profit. The firm is committed to separation rates  $\delta_w^\tau(z^a)$ ,  $\tau < a$ , for workers hired in the past. For workers hired in this period, it is free to commit to any future separation rates,  $\delta_w^a(z^k) \geq \bar{\delta}_w$ . Together with wages  $w^a(\cdot)$ , they define the contract  $\mathcal{C}^a$  offered to new hires. Equations (8) say how employment in different worker cohorts evolves over time.  $L_0$  is the firm's workforce *before hiring* which affects recruitment and sales costs. Equation (9) states the total wage bill of the firm. (10) says how the firm's customer stock evolves. Because the firm is not committed in the product market, it decides customer separations  $\delta_b \geq \bar{\delta}_b$  (if required) freely. Condition (11) says that the firm cannot sell more than what it produces. Regarding wage contract offers to new hires  $\mathcal{C}^a$ , as well as discount price offers  $p$  to new customers, the firm respects the search incentives of workers and customers, as expressed by constraints (12) and (13). That is, to attract more workers per recruitment effort (higher  $\lambda$ ), the firm needs to offer a more attractive employment contract. Likewise, to attract more customers per sales effort (higher  $\varphi$ ), the firm needs to offer a lower discount price. The last equation in (13) says that the firm optimally charges the reservation price  $p^R$  on existing customers.

### 3.2.4 Equilibrium

We can express all firm policy functions defined above to depend only on the firm's history  $z^a$ , ignoring the dependence on pre-committed contracts and worker cohorts. This is feasible because those firm state variables evolve endogenously as functions of the firm's past shocks and policies. Hence, all firm policies (in stationary equilibrium) are functions of the idiosyncratic state history. For a firm of type  $z^a$ , we write  $\lambda(z^a)$  and  $R(z^a)$  for the recruitment policy,  $\varphi(z^a)$  and  $S(z^a)$  for the sales policy, and so on.<sup>17</sup> We also define

$$L(z^a) = \sum_{\tau=0}^a L^\tau(z^a) , \quad (14)$$

$$B(z^a) = B(z^{a-1})[1 - \delta_b(z^a)] + q(\varphi(z^a))S(z^a) , \quad (15)$$

for the stocks of workers and customers in firm history  $z^a$ , where  $L^\tau(z^a) = L^\tau(z^{a-1})[1 - \delta_w^\tau(z^a)]$  if  $a > \tau$ ,  $L^a(z^a) = m(\lambda(z^a))R(z^a)$ , and  $B(z^{-1}) = 0$ . Further, there are

$$N(z^a) = N_0(1 - \delta)^a \pi^a(z^a) \quad (16)$$

firms of type  $z^a$  when  $N_0$  is the mass of entrant firms in any period. We are now ready to define the stationary equilibrium.

**Definition:** *A stationary competitive search equilibrium is a list of value functions  $U$ ,  $W$ ,  $W'$ ,  $J_a$ , firm policies  $\lambda$ ,  $R$ ,  $\varphi$ ,  $S$ ,  $\delta_b$ ,  $C^a = (w^a(\cdot), \delta_w^a(\cdot))$ ,  $(L^\tau)_{\tau=0}^a$ ,  $L$ ,  $B$ ,  $p$ ,  $p^R$  which are all functions of the firm type  $z^a$ , entrant firms  $N_0$ , aggregate consumption  $C$  and a search value  $\rho^*$  such that*

(a) *Workers' value functions  $U$ ,  $W$ ,  $W'$  and the search value  $\rho^*$  describe optimal search behavior, equations (1)–(4).*

(b) *Buyers search optimally, equation (5), with aggregate consumption*

$$C = \sum_{z^a} N(z^a)B(z^a) . \quad (17)$$

(c) *Firms' value functions  $J_a$  and policy functions solve problem (6)–(13), and  $L(\cdot)$ ,  $B(\cdot)$  and  $N(\cdot)$  evolve according to (14), (15) and (16).*

(d) *Firm entry is optimal. That is,  $N_0 > 0$  and*

$$K = \sum_{z^0} \pi^0(z^0)J_0(0, z^0) . \quad (18)$$

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<sup>17</sup>With abuse of notation, we do not index those functions by the firm's age.

(e) *Aggregate resource feasibility:*

$$\bar{L} = \sum_{z^a} N(z^a) \left\{ L(z^a) + [\lambda(z^a) - m(\lambda(z^a))]R(z^a) \right\}. \quad (19)$$

Aggregate resource feasibility (e) requires that any either belongs to the workforce  $L(z^a)$  at one of  $N(z^a)$  firms of type  $z^a$  or that the worker is searching for a job in the same submarket as this firm and does not find a job: Precisely,  $\lambda(z^a)R(z^a)$  workers are searching for employment per firm of type  $z^a$ , and share  $1 - m(\lambda(z^a))/\lambda(z^a)$  of those workers are not successful and hence remain unemployed.

We can also verify that the aggregate resource constraint for the numeraire good is satisfied in a stationary equilibrium. The budget constraint of the representative household in any given period<sup>18</sup> is

$$\begin{aligned} & \sum_{z^a} N(z^a) \left[ p^R(z^a)B(z^{a-1})[1 - \delta_b(z^a)] + p(z^a)q(\varphi(z^a))S(z^a) \right] + e \\ &= \sum_{z^a} N(z^a) \left[ \pi(z^a) + \sum_{\tau \leq a} L^\tau(z^a)w^\tau(z^a) \right] + b \left[ \bar{L} - \sum_{z^a} N(z^a)L(z^a) \right] \\ & \quad - KN_0 - c \sum_{z^a} N(z^a) \left\{ B(z^a) + [\varphi(z^a) - q(\varphi(z^a))]S(z^a) \right\}. \end{aligned}$$

The left-hand side expresses the household's consumption expenditures for the differentiated goods and for the numeraire  $e$ . The right-hand side gives the household's income which includes wage and profit income at all firm types  $z^a$  plus income from home production net of expenditures for the creation of new firms and for shopping. Shopping costs are paid both for searching and for attached buyers. Profit income of firm  $z^a$  is

$$\begin{aligned} \pi(z^a) &= p^R(z^a)B(z^{a-1})[1 - \delta_b(z^a)] + p(z^a)q(\varphi(z^a))S(z^a) - \sum_{\tau \leq a} L^\tau(z^a)w^\tau(z^a) \\ & \quad - r(R(z^a), L_0(z^a)) - s(S(z^a), L_0(z^a)). \end{aligned}$$

Rearranging shows that the household's consumption of the numeraire good<sup>19</sup> is identical to the home production of the numeraire good net of the costs for recruitment, sales, firm entry, and shopping which are all paid in the numeraire good:

$$\begin{aligned} e &= b \left[ \bar{L} - \sum_{z^a} N(z^a)L(z^a) \right] - \sum_{z^a} N(z^a) \left[ r(R(z^a), L_0(z^a)) + s(S(z^a), L_0(z^a)) \right] \\ & \quad - KN_0 - c \sum_{z^a} N(z^a) \left\{ B(z^a) + [\varphi(z^a) - q(\varphi(z^a))]S(z^a)B^0(z^a)(1 - \delta_b(z^a)) + \varphi(z^a)S(z^a) \right\}. \end{aligned}$$

<sup>18</sup>We assume that the interest rate equals the rate of time preference so that the household does not want to save or borrow.

<sup>19</sup>If  $e < 0$ , the household produces  $-e$  units of the numeraire good which, together with unemployment income and net of shopping costs, is sold to the firms who spend resources on entry, recruitment or sales.

### 3.3 Social Optimum and Firm Policies

A stationary competitive search equilibrium is identical to the solution of a social planning problem which maximizes the utility of the representative household, starting from the same initial distribution of customers and workers across firms. The social planner is subject to search frictions in product and labor markets and decides firms' recruitment and sales efforts and assigns workers and customers into submarkets which differ by the characteristics of the searching firms. In the Appendix we formally define the planner's problem and show that it permits a rather simple recursive formulation at the level of individual firms. Write  $\rho$  for the multiplier on the aggregate resource constraint (19) for workers. For a given firm in a given period, the planner takes as given the firm's stocks of workers and customers,  $L_-$  and  $B_-$ , as well as the current productivity and demand state  $z = (x, y)$ . Write  $G(L_-, B_-, z)$  for the social value of a firm, that is, the contribution of the firm to the representative household's utility. It satisfies the recursive equation

$$G(L_-, B_-, z) = \max_{(\lambda, R, \delta_w), (\varphi, S, \delta_b)} \left\{ u'(C)yB - bL - r(R, L_-(1 - \delta_w)) - s(S, L_-(1 - \delta_w)) - \rho[L + (\lambda - m(\lambda))R] - c[B + (\varphi - q(\varphi))S] + \beta(1 - \delta)\mathbb{E}_z G(L, B, z_+) \right\}, \quad (20)$$

subject to

$$\begin{aligned} L &= L_-(1 - \delta_w) + m(\lambda)R, \quad B = B_-(1 - \delta_b) + q(\varphi)S, \\ B &\leq xF(L), \quad \delta_w \geq \bar{\delta}_w, \quad \delta_b \geq \bar{\delta}_b. \end{aligned}$$

The flow surplus that the firm generates includes the marginal utility value of sales,  $u'(C)yB$ , net of the opportunity cost of employment,  $bL$ , net of recruitment and sales costs,  $r(\cdot)$  and  $s(\cdot)$ , net of shopping costs and net of the social costs for the workers who are linked to this firm in the given period. Regarding the latter, there are  $L$  workers employed at the firm, and  $(\lambda - m(\lambda))R$  unemployed workers who search for employment at this firm and do not find a job. Any of those workers can neither work nor search for jobs elsewhere in the economy and hence impose a social cost equal to the multiplier  $\rho$ . Shopping costs are incurred by the  $B$  customers buying at this firm, but also by  $(\varphi - q(\varphi))S$  unsuccessful customers who search for purchases in the same submarket as this firm. The planner neither needs to commit to separation rates, nor is there a need to discriminate between workers hired at different points in time (see the Appendix for details).

The Inada condition  $F'(\infty) = 0$  and standard dynamic programming techniques imply that problem (20) has a solution  $G : [0, L^{\max}] \times [0, B^{\max}] \times Z \rightarrow \mathbb{R}$  which is continuous in  $(L_-, B_-) \in [0, L^{\max}] \times [0, B^{\max}]$  for some appropriately specified upper bounds  $L^{\max}$  and  $B^{\max}$ .

Because  $G(0, 0, z)$  denotes the social firm value upon entry, socially optimal entry requires that

$$K = \sum_z \pi^0(z)G(0, 0, z). \quad (21)$$

We show that a joint solution of (20) and (21) together with the resource constraint (19) indeed gives rise to a stationary planning solution. Moreover we prove a welfare theorem: the stationary planning solution corresponds to a stationary competitive search equilibrium with identical firm policies and where the social multipliers on the resource constraint coincides with the equilibrium search value of workers:  $\rho = \rho^*$ .<sup>20</sup>

**Proposition 1** *Suppose that  $(\rho, G, N_0, C)$  solves the recursive social planning problem (20) together with (21), aggregate consumption (17) and the aggregate resource constraint (19), where  $N(z^a)$  is defined by (16), and  $L(z^a)$  and  $B(z^a)$  are recursively defined by iterating over the policy functions of problem (20). Then:*

- (a) *The firm policies solve the sequential social planning problem which maximizes the discounted household's utility, starting from the initial distribution  $(N(z^a), L(z^a), B(z^a))_{z^a}$ .*
- (b)  *$(\rho, G, N_0, C)$  corresponds to a stationary competitive search equilibrium with identical firm policies and search value  $\rho^* = \rho$ .*

The welfare theorem (b) extends well-known efficiency results for competitive search economies (cf. Moen (1997)) to a setting with two-sided market frictions and firms with multiple workers and multiple buyers. Kaas and Kircher (2015) prove a similar result for multi-worker firms in an environment without product market frictions (and without demand shocks). The main intuition for efficiency is that the private search values of workers and customers in the competitive search equilibrium reflect their social values; by either posting long-term contracts (to workers) or discounts (to customers), firms fully internalize all congestion externalities of search. They also internalize the trade-off between costly search effort and higher matching rates. We elaborate on this trade-off in the next paragraph. Long-term contingent contracts further implement the socially efficient worker separation rates.

The socially optimal recruitment and sales policies permit a straightforward characterization which link the effort of search (recruitment and sales expenditures) to the matching rates, resembling earlier findings of Kaas and Kircher (2015) and Gourio and Rudanko (2014). Write  $\gamma$  for the multiplier on the constraint  $B \leq xF(L)$ . Then, the first-order conditions for  $R$ ,  $\lambda$ ,

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<sup>20</sup>Assuming alternatively that the household has a fixed stock of buyers (or shopping time)  $\bar{B}$ , rather than a large potential stock of buyers, does not change this characterization of a stationary equilibrium by a planning solution. The only difference is that  $c$  stands for the multiplier on a resource constraint for buyers, defined similarly as (19), and in a stationary equilibrium the buyers' search value  $c^*$  coincides with the social value  $c$ . This alternative formulation makes a difference, however, for comparative statics or for the analysis of aggregate shocks.

$S$ , and  $\varphi$ , for positive recruitment and sales activities, are:

$$-r'_1 - \rho\lambda + [\beta(1 - \delta)\mathbb{E}G'_1(+)] + \gamma xF' - b]m = 0 , \quad (22)$$

$$-\rho R + [\beta(1 - \delta)\mathbb{E}G'_1(+)] + \gamma xF' - b]m'R = 0 , \quad (23)$$

$$-s'_1 - c\varphi + [\beta(1 - \delta)\mathbb{E}G'_2(+)] + (yu'(C) - \gamma)]q = 0 , \quad (24)$$

$$-cS + [\beta(1 - \delta)\mathbb{E}G'_2(+)] + (yu'(C) - \gamma)]q'S = 0 . \quad (25)$$

Here  $\mathbb{E}G'_i(+)$ ,  $i = 1, 2$ , denotes the derivative of  $\mathbb{E}G(L, B, z_+)$  with respect to the first/second argument. Equations (22) and (23) can be combined to obtain

$$r'_1(R, L_-(1 - \delta_w)) = \rho \left[ \frac{m(\lambda)}{m'(\lambda)} - \lambda \right] . \quad (26)$$

Similarly, (24) and (25) yield a relation between  $\varphi$  and  $S$ :

$$s'_1(S, L_-(1 - \delta_w)) = c \left[ \frac{q(\varphi)}{q'(\varphi)} - \varphi \right] . \quad (27)$$

Condition (26) says that across firms (of a given size) recruitment effort and matching rates are positively related: if the planner wishes that a firm grows faster, this is achieved by spending more on recruitment (higher  $R$ ) but also by assigning more workers to find employment at this type of firm. In the decentralization with competitive search, faster growing firms spend more on recruitment and they offer higher salaries, attracting more workers (cf. Kaas and Kircher (2015)). Condition (27) expresses a similar relation in the product market: firms that spend more on sales also have lower discount prices (cf. Gourio and Rudanko (2014)).

Another straightforward insight of the first-order conditions of problem (20) is that the firm does not recruit workers and fire workers at the same time, i.e.  $R > 0$  and  $\delta_w > \bar{\delta}_w$  are mutually exclusive. To see this formally, it follows from (22) that  $R > 0$  requires that

$$\rho < \frac{m(\lambda)}{\lambda} [\beta(1 - \delta)\mathbb{E}G'_1(+)] + \gamma xF' - b] .$$

For  $\delta_w > \bar{\delta}_w$ , the first-order condition is

$$\rho = \beta(1 - \delta)\mathbb{E}G'_1(+)] + \gamma xF' - b - (r'_2 + s'_2) \geq \beta(1 - \delta)\mathbb{E}G'_1(+)] + \gamma xF' - b ,$$

where the inequality follows since  $r$  and  $s$  are non-increasing in employment. Because of  $m(\lambda)/\lambda \leq 1$ , the two conditions are mutually exclusive. By a similar argument, the firm does not reject existing customers and attract new buyers simultaneously. It may however be possible that the firm hires new workers and still rejects customers. Conversely, it is conceivable that the firm fires workers and attracts new customers at the same time.

### 3.4 Revenue, Prices and Wages

Productivity and demand shocks impact the joint dynamics of the firms' revenue (total sales value) and employment in distinct ways. While the firms' employment dynamics  $L(z^a)$  follows directly from the solution of the social planning problem, the sales dynamics requires the calculation of equilibrium prices in the decentralized competitive search equilibrium. Using (13), the revenue of firm  $z^a$  is

$$\begin{aligned} Re(z^a) &\equiv p^R(z^a)B(z^{a-1})(1 - \delta_b(z^a)) + p(z^a)q(\varphi(z^a))S(z^a) \\ &= u'(C)y_a B(z^a) - c \left[ B(z^a) + [\varphi(z^a) - q(\varphi(z^a))]S(z^a) \right] . \end{aligned}$$

Note that both prices  $p$  and  $p^R$  are increasing in firm-specific demand  $y$ . Via (27), there is however a countervailing effect of  $y$  on the discount price  $p$ : if a firm experiences a positive demand shock, it increases sales effort  $S$  and it wants to attract more buyers per unit of effort which is achieved by a lower discount price ( $q(\varphi)/\varphi$  falls). The firm's (average) price level is  $P(z^a) \equiv Re(z^a)/B(z^a)$ , because  $B(z^a)$  is the quantity of output units sold.

Other decentralizations without price discrimination (albeit with commitment) are also possible. Suppose for example that each firm charges the same price  $p(z^a)$  for all its customers who also know that they are separated from firms with identical probability  $\delta_b(z^a)$ . Then optimal search requires that

$$c = \frac{q(\varphi(z^a))}{\varphi(z^a)} Q(z^a) ,$$

where the value of a customer  $Q(z^a)$  buying from firm  $z^a$  satisfies the Bellman equation.

$$Q(z^a) = u'(C)y_a - p(z^a) + \beta(1 - \delta) \mathbb{E}_{z^a} \left( [1 - \delta_b(z^{a+1})][Q(z^{a+1}) - c] \right) .$$

Given firm policies  $\varphi(z^a)$  and  $\delta_b(z^a)$ , these two equations can be directly solved for non-discriminatory prices  $p(z^a)$  and for the firms' revenue  $Re(z^a) = p(z^a)B(z^a)$ .

We can also solve for wages in the competitive search equilibrium. As in the social planning problem specified in the previous subsection, separation rates for all workers in a firm are assumed identical:<sup>21</sup>  $\delta_w^\tau(z^a) = \delta_w(z^a)$ . Furthermore, we consider a particular decentralization in which each firm pays the same wage to all its workers, i.e.  $w^\tau(z^a) = w(z^a)$  for all  $\tau \leq a$ . In this case, worker values  $W$  and  $W'$  do not depend on the particular contract  $\mathcal{C}^a$  and can therefore be written  $W(z^a)$  and  $W'(z^a)$ , so that (1)–(4) become

$$\begin{aligned} U &= \frac{m(\lambda(z^a))}{\lambda(z^a)} W(z^a) + \left( 1 - \frac{m(\lambda(z^a))}{\lambda(z^a)} \right) [b + \beta U] , \\ W(z^a) &= w(z^a) + \beta(1 - \delta) \mathbb{E}_{z^a} W'(z^{a+1}) + \beta \delta U , \\ W'(z^a) &= [1 - \delta_w(z^a)] W(z^a) + \delta_w(z^a) U , \\ U &= b + \rho + \beta U . \end{aligned}$$

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<sup>21</sup>Other implementations are feasible as long as total separations at the firm are unchanged.

These equations can be solved for the worker surplus

$$W(z^a) - U = \rho \left[ \frac{\lambda(z^a) - m(\lambda(z^a))}{m(\lambda(z^a))} \right] \equiv S^w(z^a) ,$$

and for wages:

$$w(z^a) = b + \rho + S^w(z^a) - \beta(1 - \delta) \mathbb{E}_{z^a} \left( [1 - \delta_w(z^{a+1})] S^w(z^{a+1}) \right) . \quad (28)$$

Alternative wage schedules are feasible as well since the model does not pin down individual wage profiles. For example, it is conceivable that firms offer flat wage contracts to all workers, thus paying different wages to different cohorts; see Appendix C in Kaas and Kircher (2015) for the procedure how to calculate wages in that case. Another (less plausible) alternative is the absence of commitment: here the firm pays a hiring bonus upfront and (identical) reservation wages afterwards to all workers.

## 4 Quantitative Evaluation

In this section we examine the role of supply and demand for the dynamics of firms and for the dispersion of prices and wages. To this end we calibrate the model to match certain features of dynamics of German firms that we outline in Section 2. We then conduct a number of counterfactual experiments: (1) How do product market reforms affect the labor market? (2) How do aggregate demand and supply shocks impact the cross-sectional properties of wages, prices, job creation and destruction?

### 4.1 Parameterization

We calibrate the model at annual frequency and hence set the discount factor to  $\beta = 0.96$ . The production function is Cobb-Douglas  $F(L) = L^\alpha$  where  $\alpha = 0.7$  gives rise to a labor income share of about 70 percent. As Felbermayr et al. (2014), we set the firm exit rate to  $\delta = 0.05$ , and we choose the exogenous worker separation rate to  $\bar{\delta}_w = 0.02$  so that the total separation rate in equilibrium is around 7 percent.<sup>22</sup> The customer separation rate is set to  $\bar{\delta}_b = 0.43$ ; this corresponds to the finding of Stahl (2010) that regular customers account for 57% of the annual sales in German manufacturing firms. This choice implies rather high customer turnover so that no firm in equilibrium voluntarily decides to separate from any customers.

For recruitment and sales costs we adopt the constant-returns specifications  $r(R, L) = \frac{r_0}{3} \left(\frac{R}{L}\right)^2 R$  and  $s(S, L) = \frac{s_0}{3} \left(\frac{S}{L}\right)^2 S$  which are cubic functions of recruitment and sales effort; division by employment size makes sure that larger firms with proportionally higher recruitment and sales

<sup>22</sup>These targets are based on Fuchs and Weyh (2010) who measure plant-level job creation and destruction rates from the IAB Establishment History Panel for the period 2000–2006.

effort incur the same unit costs (cf. Merz and Yashiv (2007)). As in Kaas and Kircher (2015), convex recruitment costs give rise to sluggish employment adjustment with some variation in job-filling rates and wages across firms. Similarly in the product market, convex sales costs allow for variation in discount prices across firms. The scale parameters  $r_0$  and  $s_0$  are set to match plausible shares of spending on recruitment and advertising; specifically we target recruitment (advertising) expenditures to be 1 (2.5) percent of GDP as in Christiano et al. (2013) (Arseneau and Chugh (2007)). For the labor market matching function, we choose the Cobb-Douglas form  $m(\lambda) = m_0\lambda^{0.5}$ . The elasticity of 0.5 is a standard value (e.g. Petrongolo and Pissarides (2001)) and the scale parameter  $m_0$  is set to match a stationary unemployment rate of 7 percent. In the absence of suitable empirical studies, we follow Arseneau and Chugh (2007) and Mathä and Pierrard (2011) and adopt the same Cobb-Douglas functional form  $q(\varphi) = q_0\varphi^{0.5}$  in the product market, and we set parameter  $q_0$  such that the average matching probability of a shopper is 50 percent. This reflects that consumers visit on average 1.1–3 stores for a given purchase (cf. Lehmann and Van der Linden (2010)), so that roughly half of all visits do not result in a match.<sup>23</sup>

The unemployment income parameter is set to 60 percent of the average wage to capture the unemployment replacement rate in Germany (see Krebs and Scheffel (2013)). According to the time use survey of the German Statistical Office, the average person in Germany spends 46 minutes shopping per day (including transit time) and it spends five hours on market and non-market work (excluding shopping). This leads us to set  $c$  to equal 15 percent of the average wage.

The utility function has constant elasticity,  $u(C) = \frac{u_0}{1-\sigma}C^{1-\sigma}$  with  $\sigma \geq 0$ . Lacking comparable estimates for Germany, we set  $\sigma = 2/3$  so that the elasticity of industry demand corresponds to the mean estimate for U.S. manufacturing industries of Chang et al. (2009). This parameter does not affect any steady state outcomes of our model, it only matters for the analysis of aggregate changes. The marginal valuation of a good in the model equals  $yu'(C)$  in units of the numeraire good. As the unit of measurement is arbitrary, we normalize the average value of  $yu'(C)$  to unity by setting the mean value of the demand shock to  $y = \bar{y} = 1$  and adjusting  $u_0$  accordingly.

Firm productivity is  $x = \bar{x}x'$  where  $\bar{x} \in \{\bar{x}^s, \bar{x}^l\}$  is a permanent component which differs between two firm size classes: small firms with less than 50 employees and larger firms.<sup>24</sup>  $\bar{x} = \bar{x}^i$  is drawn upon entry with probability  $\pi^i$  and constant over time. Demand shocks  $y$  and transitory productivity shocks  $x'$  follow AR(1) processes  $\ln(y_{t+1}) = \rho^y \ln(y_t) + \sigma^y \varepsilon_{t+1}^y$  and  $\ln(x'_{t+1}) = \rho^x \ln(x'_t) + \sigma^x \varepsilon_{t+1}^x$ , with standard normally distributed  $\varepsilon^x$  and  $\varepsilon^y$ . The autocorrelations  $\rho^y$  and  $\rho^x$  and standard deviations  $\sigma^y$  and  $\sigma^x$  are decisive for the dynamics of firm-level

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<sup>23</sup>For both matching functions, we make sure that the matching rates of workers and shoppers ( $m(\lambda)/\lambda$  and  $q(\varphi)/\varphi$  resp.) do not exceed one; that is we set  $m(\lambda) = \min(\lambda, m_0\lambda^{0.5})$  and  $q(\varphi) = \min(\varphi, q_0\varphi^{0.5})$ .

<sup>24</sup>As the focus of this paper is about firm dynamics, this parsimonious assumption is not meant to capture a realistic firm-size distribution but merely to study the implications of firm size for some model outcomes.

prices and output. We choose the following calibration targets to pin down these four parameters:<sup>25</sup> (i) the standard deviation of log price growth of 0.099; (ii) the standard deviation of log output growth of 0.16; (iii) the correlation of log price growth and log output growth of -0.364; (iv) the fraction of firms where price growth is within the interval  $[-0.02, +0.02]$ . The entry cost parameter  $K$  (and thereby the endogenous variable  $\rho$ ) is set so that the average firm employs 50 workers which is the average firm size in our data.<sup>26</sup>

Table 3: Parameter choices

Parameter	Value	Explanation/Target
$\beta$	0.96	Annual interest rate 4%
$\alpha$	0.7	Labor income share
$\delta$	0.05	Firm exit rate (Fuchs and Weyh (2010))
$\bar{\delta}_w$	0.02	Worker separation rate 7%
$\bar{\delta}_b$	0.43	Customer separation rate 57%
$r_0$	0.334	Recruitment costs 1% of output
$s_0$	293.6	Sales costs 2.5% of output
$m_0$	0.593	Unemployment rate 7%
$q_0$	1.423	Customer matching rate 50%
$b$	0.113	Unemployment income 60% of mean wage
$c$	0.070	Shopping costs 15% of mean wage
$K$	35.89	Entry cost (average firm size = 50)
$(\bar{x}^s, \pi^s)$	(0.8, 0.84)	Firm and employment shares for firms with $< 50$ workers
$(\bar{x}^l, \pi^l)$	(1.5, 0.16)	Firm and employment shares for firms with $\geq 50$ workers
$\sigma^x$	0.109	Standard deviation of quantity growth
$\sigma^y$	0.045	Standard deviation of price growth
$\rho^x$	-0.238	Correlation price and quantity growth
$\rho^y$	0.070	price changes within $[-0.02, 0.02]$

Table 3 summarizes all parameter choices. Table 4 shows how the model fits the cross-sectional features of firm dynamics. Although the calibration targets (first four rows) are not hit exactly, the model presents a reasonably close approximation to the data. The other rows of the table

<sup>25</sup>All these data moments are residuals after controlling for year, industry and region (cf. Table 4).

<sup>26</sup>Parameter  $K$  cannot be identified independently of the average values of productivity  $\bar{x}^s$  and  $\bar{x}^l$  because firm-level value functions are linearly homogeneous in the vector  $(x, b, r_0, s_0^{-2}, \rho, K)$  (see problem (20), together with the assumed functional forms), so that all firm-level policies are independent of scaling transformations.

show that the model accounts rather well for the volatility and cross-correlations of employment growth.

## 4.2 Cross-sectional features

Table 4: Firm dynamics: Data and model

	Data	Model
$\sigma(\hat{P})$	0.099	0.081
$\sigma(\hat{Q})$	0.16	0.207
$\rho(\hat{P}, \hat{Q})$	-0.374	-0.215
Share $\hat{P} \in [-0.02, 0.02]$	0.51	0.53
$\sigma(\hat{E})$	0.062	0.080
$\rho(\hat{P}, \hat{E})$	0.021	0.274
$\rho(\hat{Q}, \hat{E})$	0.227	0.408

Note: Standard deviations ( $\sigma$ ) and correlations ( $\rho$ ) after controlling for year, industry and region.

Table 5 shows how well the model matches the dispersion of productivity, prices and wages. While the dispersions of revenue and quantity labor productivity are of similar magnitudes as in the data, the gap between the two productivity measures is too small in the model. As in the data, firm-level prices correlate weakly positively with revenue productivity and negatively with quantity productivity. Nonetheless, price dispersion in the model is less than half of what it is in the data, which explains why quantity productivity is not more dispersed.

Table 5: Dispersion: Data and model

	Data	Model
$\sigma(RLP)$	0.115	0.128
$\sigma(QLP)$	0.166	0.135
$\sigma(\tilde{P})$	0.139	0.057
$\sigma(w/h)$	0.063	0.037
$\rho(RLP, \tilde{P})$	0.083	0.106
$\rho(QLP, \tilde{P})$	-0.686	-0.325

Note: Standard deviations ( $\sigma$ ) and correlations ( $\rho$ ) after controlling for year, industry and region.

Figure 2 compares quantity and revenue productivity as well as prices between small firms (up to 50 employees) and larger firms (more than 50 employees). In both productivity measures, larger firms are more productive, but the gap is bigger in terms of quantity productivity. The difference is accounted for by firm-level prices which are lower at larger firms. The right graph in the figure shows that the model accounts for these differences rather well although the gaps in productivity and prices are considerably larger in the model.

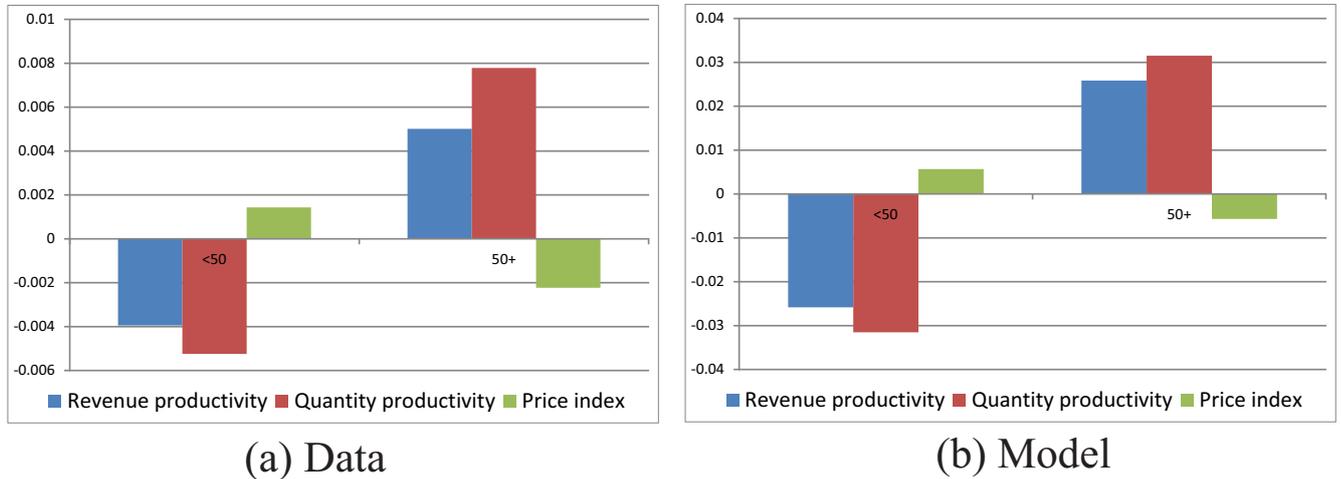


Figure 2: Productivity and prices in small (< 50 workers) and large firms.

Note: Data statistics are average residuals after controlling for year, industry and region. Model statistics are deviations from the respective mean values.

Figure 3 illustrates the growth of firms in the absence of shocks. It takes about 15–20 years for an entrant firm to reach its long-run optimal size. Young firms charge high discounts, but prices converge much faster than output or employment. Firms which are more productive (higher  $x$ ) or firms with higher demand (higher  $y$ ) have higher employment and revenue. Higher productivity is not reflected in the firm’s price but only in its output level, whereas higher demand induces the firm to raise both output and price.

## 5 Conclusions

To be written.

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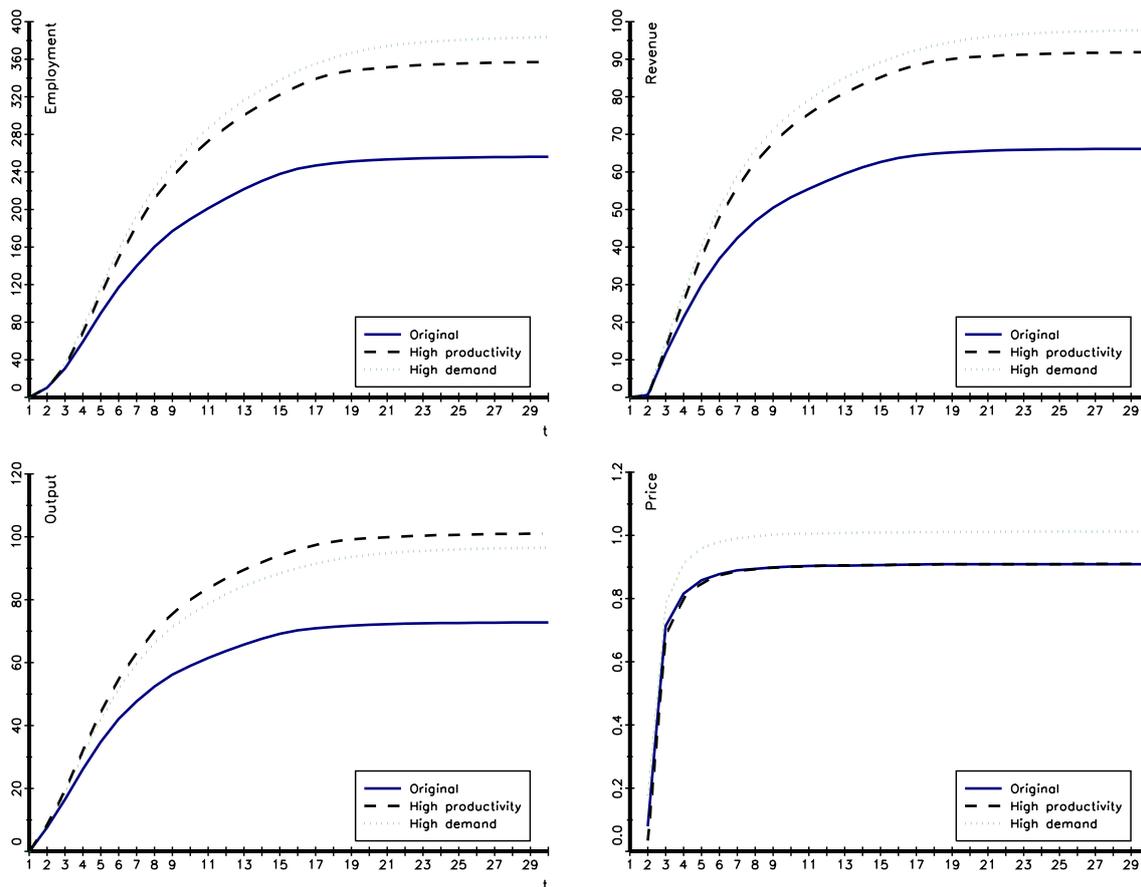


Figure 3: Firm growth.

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# Appendices

## A Data

### A.1 Further details

Here we describe further details about the construction of the *full* and *homogeneous samples*. Reportage of any statistics is meaningful provided that the products that remain in either sample contribute significantly to the total sales value of the firm, as in the original uncleaned sample. This is why we keep both in the *full sample* and in the *homogeneous sample* only those firm-year observations for which the sample sales value is at least 50 percent of the total sales value of the firms, as in Foster et al. (2008). We further follow their procedure and adjust proportionately the sales values (and quantities) of the firm’s products so that the sales

value in the cleaned sample equals the total sales value before removing any goods from the analysis.<sup>27</sup>

*Full sample.* The full sample is restricted to firms whose products with valid quantity information make at least 50% of its total sales values. Across firms, the sales share of valid products is 98.9% at the 25th percentile and 100% at the 50th percentile, which reflects that many of the firms produce only one good. We drop as outliers price, quantity, employment, and wage growth rates (as defined below) that are beyond the 2nd and 98th percentiles. After these adjustments, the full sample has 149,228 product-years and 60,732 firm-years. On average in the full sample, firms produce about 2.44 products and more than 70% of the firms produce one or two products. On the other hand, a given nine-digit product is on average produced by 11.32 firms whereas the median product is produced by four firms.

*Homogeneous sample.* To construct the homogeneous sample, we start at the cleaned sample *before* truncating it to obtain the full sample. We then drop all products which are not measured in physical units (length, area, volume or weight) and those which are produced by up to five firms, in order to be able to compute a meaningful average price for each product. Across firms producing at least one of those products, the valid products contribute 54.8% of the total sales value at the 25th percentile and 100% at the 50th percentile. Again we restrict the sample to firms whose valid products make at least 50% of the total sales value. We drop as outliers all log firm-specific prices and log quantity labor productivity (as defined below) beyond the 2nd and 98th percentiles to arrive at 87,724 product-years and 35,457 firm-years. Averaged over the years 2005–2007, firms produce on average 2.47 products and 72.3% of the firms produce one or two products. A given product is on average produced by 23.52 firms, whereas the median product is produced by ten firms.

## A.2 Empirical Methodology

Here we describe the empirical methodology how we construct our growth and dispersion measures.

Let  $I$  denote the set of firms in the data panel,  $J$  the set of products, and  $T$  the set of years covered by the panel. Let  $J_0 \subset J$  be the set of products in either the *full sample* or the *homogeneous sample* described above. For each firm-product-year pair  $(i, j, t) \in I \times J_0 \times T$ , calculate unit price,  $P_{jit}$ , as sales,  $S_{jit}$ , divided by quantity,  $Q_{jit}$ . Define the quantity-weighted average price of good  $j$  at time  $t$  as

$$\bar{P}_{jt} = \frac{\sum_i P_{jit} Q_{jit}}{\sum_i Q_{jit}}.$$

Denote by  $S_{it} = \sum_{j \in J} S_{jit}$  the total sales value and define adjusted quantities

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<sup>27</sup>The underlying assumption is that the firm's production technology is as if it produces the sample products in the same proportion as any other products with its labor input.

$\tilde{Q}_{jit} = Q_{jit} \frac{S_{it}}{\sum_{j \in J_0} P_{jit} Q_{jit}}$  so that  $S_{it} = \sum_{j \in J_0} P_{jit} \tilde{Q}_{jit}$ . This adjustment is valid modification of the data if the goods in sample  $J_0$  are sufficiently representative for the set of goods  $J$  so that a firm's price and output policies are well proxied by goods  $j \in J_0$ . In order to simplify notation, we denote with  $Q_{jit}$  these adjusted quantities in the following.

### A.2.1 Measuring Price and Output Growth

We begin by splitting firm's sales into a price and a quantity component. We replace a firm's product price with the average market price of that good whenever the firm's product portfolio changes between two periods (so that a firm's price for that product is not observed in that year). That is, for any  $(i, t, t+1)$  and  $j \in J_0$  define

$$\hat{P}_{ji\tau} = \begin{cases} P_{ji\tau} & \text{if } Q_{ji\tau} > 0, \\ \bar{P}_{j\tau} & \text{else.} \end{cases}, \quad \tau = t, t+1.$$

Define the firm's Paasche price index by  $P_{i,t,t} = 1$  and

$$P_{i,t+1,t} = \frac{\sum_j \hat{P}_{j,i,t+1} Q_{j,i,t+1}}{\sum_j \hat{P}_{j,i,t} Q_{j,i,t}}.$$

Define the firm's quantities (real output) by

$$Q_{i,t,t} = \frac{S_{it}}{P_{i,t,t}} = \sum_j \hat{P}_{j,i,t} Q_{j,i,t}, \quad Q_{i,t+1,t} = \frac{S_{i,t+1}}{P_{i,t+1,t}} = \sum_j \hat{P}_{j,i,t+1} Q_{j,i,t+1}.$$

Therefore sales growth is split into a price and quantity component:

$$\frac{S_{i,t+1}}{S_{i,t}} = \frac{P_{i,t+1,t}}{P_{i,t,t}} \frac{Q_{i,t+1,t}}{Q_{i,t,t}}. \quad (29)$$

Write  $\hat{X}_t = \ln(X_t/X_{t-1})$  for the log growth rate of variable  $X$ . Therefore

$$\hat{S}_{i,t} = \hat{P}_{i,t} + \hat{Q}_{i,t}.$$

Define the wage (per employee)  $w_{i,t} = W_{i,t}/E_{i,t}$ , the hourly wage  $wh_{i,t} = W_{i,t}/H_{i,t}$ , quantity labor productivity  $QLP_{it} = Q_{it}/E_{it}$ , hourly quantity labor productivity  $QHP_{it} = Q_{it}/H_{it}$ , revenue labor productivity  $RLP_{it} = S_{it}/E_{it}$ , and hourly revenue labor productivity  $RHP_{it} = S_{it}/H_{it}$ , where  $E_{it}$  and  $H_{it}$  are employment and working hours.

There is another decomposition that is applicable to the homogeneous sample and which allows decomposing sales growth into a component reflecting growth in firm-specific prices, average prices, and output growth. Start with the set of firms  $i \in I$  producing goods  $j \in J_1$  where  $J_1 \subset J$  is the set of products in the homogeneous sample and define a firm-specific price level:

$$\tilde{P}_{it} = \frac{\sum_{j \in J_1} P_{jit} Q_{jit}}{\sum_{j \in J_1} \bar{P}_{jt} Q_{jit}}, \quad (30)$$

where as before  $\bar{P}_{jt}$  is the quantity-weighted average price defined over goods  $j \in J_1$ . The denominator expresses the firm's hypothetical revenue if the firm would charge the average price for all its products.  $\tilde{P}_{it} - 1$  measures the “expensiveness” of firm  $i$  relative to other firms. Define

$$\bar{Q}_{it} = \sum_{j \in J_1} \bar{P}_{jt} Q_{jit}$$

for firm  $i$ 's output measured in average date  $t$  prices. Total sales of firm  $i$  are therefore

$$S_{it} = \tilde{P}_{it} \bar{Q}_{it} ,$$

the product between firm  $i$ 's relative price and its output.

We can decompose sales growth as

$$\hat{S}_{i,t+1} = \frac{S_{i,t+1}}{S_{it}} = \underbrace{\frac{\tilde{P}_{i,t+1}}{\tilde{P}_{i,t}}}_{=\hat{P}_{i,t+1}} \cdot \frac{\bar{Q}_{i,t+1}}{\bar{Q}_{it}} .$$

The latter part can be split into

$$\frac{\bar{Q}_{i,t+1}}{\bar{Q}_{it}} = \underbrace{\frac{\sum_{j \in J_1} \bar{P}_{jt} Q_{ji,t+1}}{\sum_{j \in J_1} \bar{P}_{jt} Q_{jit}}}_{=\hat{Q}_{i,t+1}} \cdot \underbrace{\frac{\sum_{j \in J_1} \bar{P}_{j,t+1} Q_{ji,t+1}}{\sum_{j \in J_1} \bar{P}_{jt} Q_{ji,t+1}}}_{=\hat{P}_{i,t+1}} .$$

Hence, sales growth is

$$\hat{S}_{i,t+1} = \hat{P}_{i,t+1} \cdot \hat{P}_{i,t+1} \cdot \hat{Q}_{i,t+1} .$$

The first two components are price growth: the first is the change of firm  $i$ 's relative price, the second is the general growth of prices that firm  $i$  produces. The last component is firm  $i$ 's quantity growth.

Note the difference to the decomposition (29) that we undertake for the full sample. There we measure firm  $i$ 's quantity growth evaluated at firm  $i$ 's prices in period  $t$ . Here  $\hat{Q}_{i,t+1}$  is measured with average goods prices in period  $t$ . For firms producing one good, both are clearly identical.

### A.2.2 Measuring Productivity and Price Dispersion

Since this analysis is done for a given year, the time index is dropped from all variables. Again write  $J_1$  for the products in the homogeneous sample. Define revenue and quantity productivity:

$$RLP_i = \frac{\sum_{j \in J_1} Q_{ji} P_{ji}}{E_i} , \quad RHP_i = \frac{\sum_{j \in J_1} Q_{ji} P_{ji}}{H_i} ,$$

$$QLP_i = \frac{\sum_{j \in J_1} Q_{ji} \bar{P}_j}{E_i}, \quad QHP_i = \frac{\sum_{j \in J_1} Q_{ji} \bar{P}_j}{H_i}.$$

Revenue labor productivity can be split into quantity productivity and the firm's price level, as defined in (30):

$$RLP_i = QLP_i \cdot \tilde{P}_i. \quad (31)$$

### A.2.3 Measuring Wage Dispersion

In the module *Verdienste* we have data on workers for establishments  $i \in I^V$  which includes the service sector and hence is distinct from set  $I$ . When we relate wage dispersion with price, quantity and productivity, we look at the intersection of firms contained in  $I_1$  and  $I^V$ , labeled  $I_\cap = I_1 \cap I^V$ .

For all  $i \in I^V$  and a (sub)sample of workers  $k \in K(i)$ , we have information on hourly wages  $w_k$  and working hours  $h_k$ , together with other worker and firm characteristics. We run four wage regressions (based on workers  $k \in K(i)$  in all establishments  $i \in I^V$ ), separately for males and females located in East and West German states:

$$\ln(w_k) = \beta_0 + \beta_1 X_k + \varepsilon_k,$$

where vector  $X_k$  contains age, age<sup>2</sup>, tenure, tenure<sup>2</sup>, education dummies, sector union dummy, firm-union dummy, apprentice, temporary contract, old-age part-time, minijob. We drop all workers for which education, age or tenure is missing. Then we remove outliers where the standardized residuals have absolute value greater than five, and we run the regressions again with the remaining observations.

Write  $\bar{w}_k = e^{\beta_0 + \beta_1 X_k}$  for the predicted hourly wage of worker  $k$  (i.e. the part of the wage explained by worker observables). Define for each firm  $i \in I^V$  the firm-specific wage level which expresses how much more (or less) firm  $i$  pays to its workers relative to what observationally equivalent workers would earn in other firms:

$$\tilde{W}_i = \frac{\sum_{k \in K(i)} w_k h_k}{\sum_{k \in K(i)} \bar{w}_k h_k}.$$

This index is the equivalent to the firm price level  $\tilde{P}_i$  defined before. Note that

$$\tilde{W}_i - 1 = \frac{\sum_{k \in K(i)} (e^{\varepsilon_k} - 1) \bar{w}_k h_k}{\sum_{k \in K(i)} \bar{w}_k h_k},$$

which shows that the firm's wage level (minus one) is an earnings-weighted average of the wage residuals ( $e^{\varepsilon_k} - 1 \approx \varepsilon_k$ ). It should be stressed that this firm-specific wage captures both a direct firm effect on wages but also any unobserved worker characteristics that correlate between different workers in the same firm. If those unobserved characteristics were uncorrelated across

workers in the same firm, they should not matter much (and they should be negligible if the firm is sufficiently large). But if those characteristics correlate (positive sorting *between workers in the same firm*), then they matter for the overall index. This issue cannot be addressed with our data which have no panel dimension on the worker side.

Define the firm's average wage (per hour)

$$WH_i = \frac{\sum_{k \in K(i)} w_k h_k}{\sum_{k \in K(i)} h_k} .$$

The firm's wage can be decomposed into the wage predicted by worker observables  $\overline{WH}_i$  and into the firm-specific wage level:

$$WH_i = \overline{WH}_i \cdot \widetilde{W}_i , \text{ with } \overline{WH}_i \equiv \frac{\sum_{k \in K(i)} \bar{w}_k h_k}{\sum_{k \in K(i)} h_k} . \quad (32)$$

This decomposition says that a firm pays higher wages either because it employs more valuable workers (based on worker observables) or because its wage is higher than in other firms (which may be due to the firm's wage policy but also due to correlated unobservable worker characteristics). Note that this decomposition is similar to the decomposition (31) of revenue labor productivity into quantity labor productivity and the firm's price level. See Figure 6 for kernel density plots (unweighted and weighted by firm size) for these three measures.

### A.3 Additional Tables

### A.4 Kernel Density Plots

Table 6: Variance decomposition

	$Var(\widehat{P})$	$Var(\widehat{Q})$	$2Cov(\widehat{P}, \widehat{Q})$
$Var(\widehat{S}) = 0.0253$	0.0109	0.0257	-0.0114
	$Var(\widehat{Q})$	$Var(\widehat{E})$	$2Cov(\widehat{Q}, \widehat{E})$
$Var(\widehat{QLP}) = 0.0244$	0.0257	0.0058	-0.0071
	$Var(\widehat{Q})$	$Var(\widehat{H})$	$2Cov(\widehat{Q}, \widehat{H})$
$Var(\widehat{QHP}) = 0.0259$	0.0250	0.0103	-0.0094

	$Var(\widehat{P})$	$Var(\widehat{Q})$	$Var(\widehat{E})$	$2Cov(\widehat{P}, \widehat{Q})$	$2Cov(\widehat{P}, \widehat{E})$	$2Cov(\widehat{Q}, \widehat{E})$
$Var(\widehat{RLP}) = 0.0233$	0.0109	0.0257	0.0058	-0.0114	-0.0007	-0.0071
	$Var(\widehat{P})$	$Var(\widehat{Q})$	$Var(\widehat{H})$	$2Cov(\widehat{P}, \widehat{Q})$	$2Cov(\widehat{P}, \widehat{H})$	$2Cov(\widehat{Q}, \widehat{H})$
$Var(\widehat{RHP}) = 0.0246$	0.0108	0.0250	0.0103	-0.0111	-0.0010	-0.0094

Notes: Each variable is expressed in logs. See Table 1 for the definition of the variables.

Estimates are weighted by average employment of the firm over all years of observation.

Table 7: Summary statistics for growth dynamics (unweighted)

	$\hat{S}$	$\hat{P}$	$\hat{Q}$	$\hat{E}$	$\hat{W}$	$\hat{H}$	$\widehat{W/E}$	$\widehat{W/H}$	$\widehat{QLP}$	$\widehat{QHP}$	$\widehat{RLP}$	$\widehat{RHP}$
$\hat{S}$	1.000											
$\hat{P}$	0.284	1.000										
$\hat{Q}$	0.795	-0.356	1.000									
$\hat{E}$	0.307	0.035	0.276	1.000								
$\hat{W}$	0.342	0.050	0.302	0.686	1.000							
$\hat{H}$	0.299	0.035	0.269	0.595	0.551	1.000						
$\widehat{W/E}$	0.114	0.027	0.094	-0.207	0.570	0.069	1.000					
$\widehat{W/H}$	-0.009	0.009	-0.014	-0.013	0.338	-0.599	0.469	1.000				
$\widehat{QLP}$	0.666	-0.381	0.890	-0.193	-0.018	-0.008	0.194	-0.009	1.000			
$\widehat{QHP}$	0.579	-0.366	0.796	-0.107	-0.055	-0.369	0.047	0.363	0.864	1.000		
$\widehat{RLP}$	0.883	0.276	0.685	-0.177	0.015	0.015	0.220	-0.003	0.784	0.652	1.000	
$\widehat{RHP}$	0.782	0.255	0.601	-0.089	-0.026	-0.361	0.066	0.383	0.656	0.807	0.853	1.000
Std. dev.	0.172	0.109	0.176	0.087	0.100	0.112	0.080	0.099	0.176	0.178	0.170	0.172
Mean	0.071	0.017	0.054	0.026	0.037	0.029	0.011	0.007	0.028	0.030	0.045	0.047
Median	0.070	0.002	0.050	0.019	0.038	0.021	0.017	0.014	0.030	0.032	0.047	0.049

Notes:

Table 8: Summary statistics for growth dynamics (weighted)

	$\hat{S}$	$\hat{P}$	$\hat{Q}$	$\hat{E}$	$\hat{W}$	$\hat{H}$	$\widehat{W/E}$	$\widehat{W/H}$	$\widehat{QLP}$	$\widehat{QHP}$	$\widehat{RLP}$	$\widehat{RHP}$
$\hat{S}$	1.000											
$\hat{P}$	0.321	1.000										
$\hat{Q}$	0.782	-0.339	1.000									
$\hat{E}$	0.339	0.047	0.306	1.000								
$\hat{W}$	0.360	0.066	0.315	0.715	1.000							
$\hat{H}$	0.325	0.046	0.293	0.632	0.558	1.000						
$\widehat{W/E}$	0.108	0.037	0.083	-0.172	0.566	0.041	1.000					
$\widehat{W/H}$	-0.016	0.012	-0.024	-0.019	0.343	-0.589	0.505	1.000				
$\widehat{QLP}$	0.648	-0.374	0.890	-0.163	-0.016	0.000	0.169	-0.016	1.000			
$\widehat{QHP}$	0.564	-0.362	0.798	-0.098	-0.042	-0.343	0.056	0.347	0.874	1.000		
$\widehat{RLP}$	0.887	0.315	0.674	-0.133	0.029	0.032	0.198	-0.008	0.762	0.642	1.000	
$\widehat{RHP}$	0.791	0.292	0.594	-0.069	0.000	-0.322	0.082	0.364	0.648	0.786	0.867	1.000
Std. dev.	0.159	0.105	0.160	0.076	0.090	0.101	0.066	0.090	0.156	0.161	0.153	0.157
Mean	0.075	0.018	0.056	0.021	0.040	0.024	0.019	0.016	0.036	0.034	0.054	0.052
Median	0.077	0.004	0.052	0.015	0.041	0.019	0.023	0.022	0.036	0.036	0.056	0.054

Notes:

Table 9: Variance decomposition (unweighted)

	$Var(\widehat{P})$	$Var(\widehat{Q})$	$2Cov(\widehat{P}, \widehat{Q})$
$Var(\widehat{S}) = 0.0295$	0.0120	0.0310	-0.0135
	$Var(\widehat{Q})$	$Var(\widehat{E})$	$2Cov(\widehat{Q}, \widehat{E})$
$Var(\widehat{QLP}) = 0.0311$	0.0310	0.0076	-0.0075
	$Var(\widehat{Q})$	$Var(\widehat{H})$	$2Cov(\widehat{Q}, \widehat{H})$
$Var(\widehat{QHP}) = 0.0318$	0.0296	0.0126	-0.0104

	$Var(\widehat{P})$	$Var(\widehat{Q})$	$Var(\widehat{E})$	$2Cov(\widehat{P}, \widehat{Q})$	$2Cov(\widehat{P}, \widehat{E})$	$2Cov(\widehat{Q}, \widehat{E})$
$Var(\widehat{RLP}) = 0.0289$	0.0120	0.0310	0.0076	-0.0135	-0.0006	-0.0075
	$Var(\widehat{P})$	$Var(\widehat{Q})$	$Var(\widehat{H})$	$2Cov(\widehat{P}, \widehat{Q})$	$2Cov(\widehat{P}, \widehat{H})$	$2Cov(\widehat{Q}, \widehat{H})$
$Var(\widehat{RHP}) = 0.0295$	0.0119	0.0296	0.0126	-0.0134	-0.0008	-0.0104

Notes:

Table 10: Summary statistics for firm dispersion (unweighted)

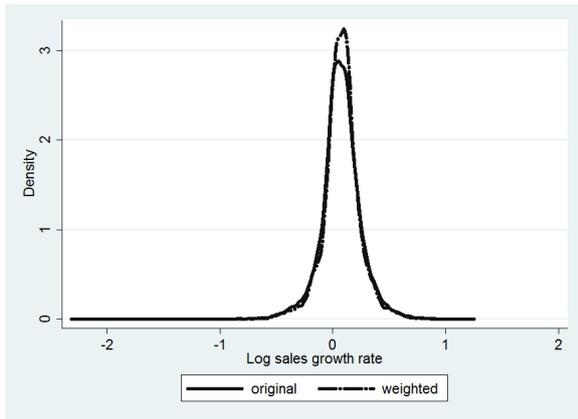
	<i>RLP</i>	<i>QLP</i>	<i>RHP</i>	<i>QHP</i>	$\widetilde{P}$	<i>E</i>	<i>W/H</i>	<i>W/E</i>
<i>RLP</i>	1.000							
<i>QLP</i>	0.792	1.000						
<i>RHP</i>	0.962	0.753	1.000					
<i>QHP</i>	0.761	0.981	0.775	1.000				
$\widetilde{P}$	-0.127	-0.706	-0.108	-0.712	1.000			
<i>E</i>	0.280	0.222	0.293	0.229	-0.035	1.000		
<i>W/H</i>	0.476	0.329	0.558	0.383	0.017	0.308	1.000	
<i>W/E</i>	0.611	0.453	0.549	0.406	-0.028	0.300	0.863	1.000
Std. dev.	0.732	1.023	0.697	0.986	0.629	0.871	0.374	0.395
Mean	11.608	11.316	4.278	3.995	0.292	4.037	2.829	10.169
Median	11.641	11.395	4.298	4.065	0.202	3.892	2.867	10.235

Notes:

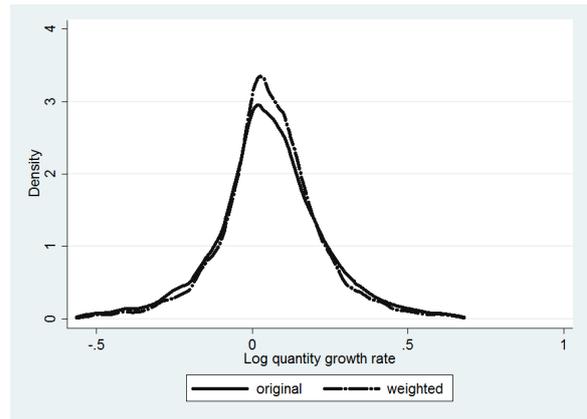
Table 11: Summary statistics for firm dispersion (weighted)

	<i>RLP</i>	<i>QLP</i>	<i>RHP</i>	<i>QHP</i>	$\tilde{P}$	<i>E</i>	<i>W/H</i>	<i>W/E</i>
<i>RLP</i>	1.000							
<i>QLP</i>	0.801	1.000						
<i>RHP</i>	0.966	0.769	1.000					
<i>QHP</i>	0.775	0.983	0.790	1.000				
$\tilde{P}$	-0.156	-0.716	-0.142	-0.719	1.000			
<i>E</i>	0.355	0.293	0.387	0.315	-0.070	1.000		
<i>W/H</i>	0.447	0.316	0.543	0.381	-0.000	0.439	1.000	
<i>W/E</i>	0.579	0.430	0.548	0.406	-0.034	0.412	0.877	1.000
Std. dev.	0.698	0.985	0.670	0.954	0.594	1.132	0.362	0.360
Mean	11.850	11.594	4.518	4.269	0.255	5.051	2.968	10.309
Median	11.868	11.699	4.521	4.368	0.147	5.009	3.015	10.379

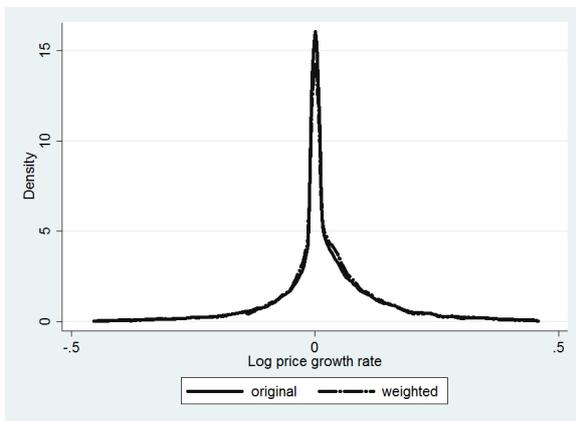
Notes:



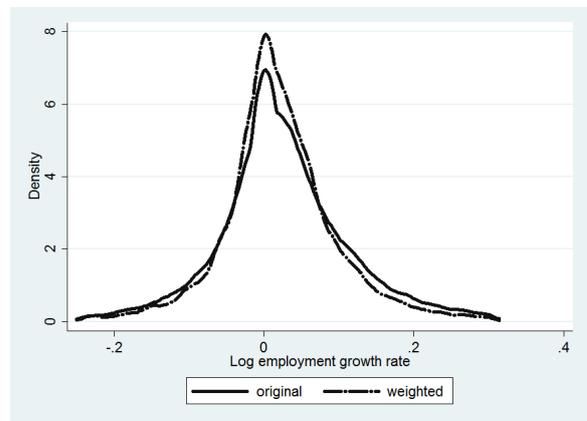
(a) Sales growth



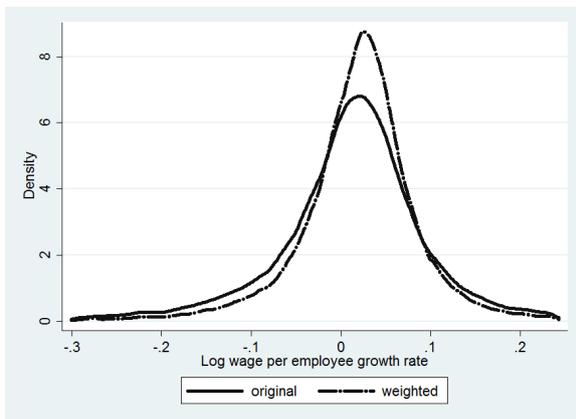
(b) Quantity growth



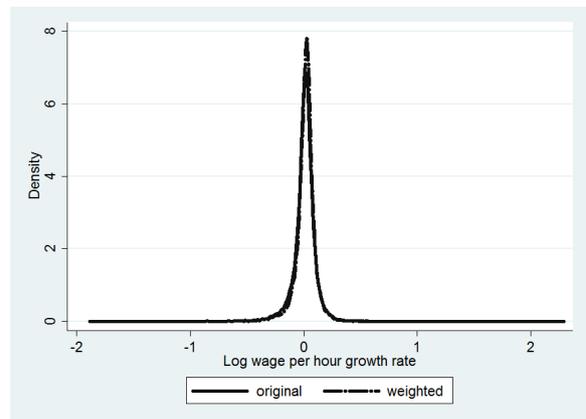
(c) Price growth



(d) Employment growth

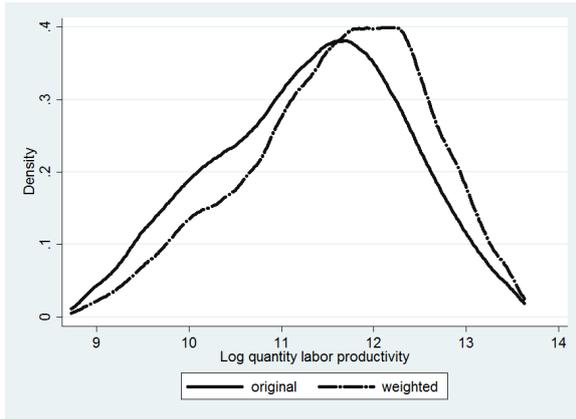


(e) Wage per employee growth

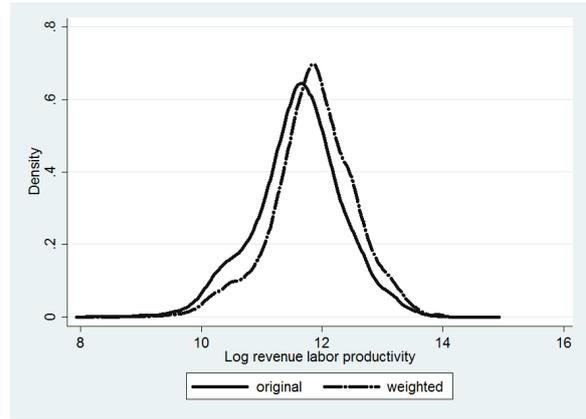


(f) Wage per hour growth

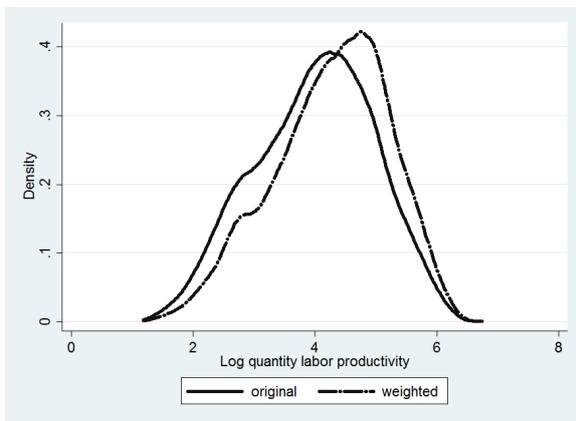
Figure 4: Distribution of growth rates in the *full sample*



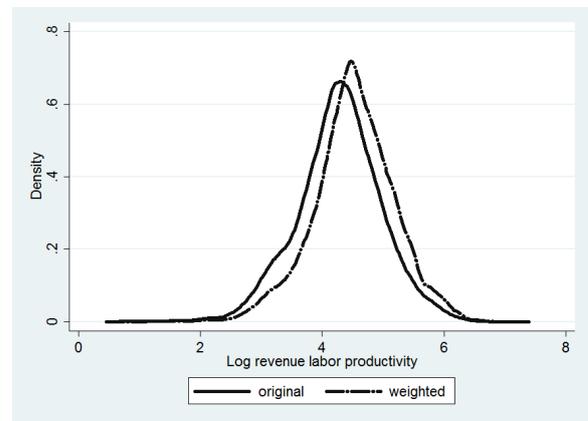
(a) QLP



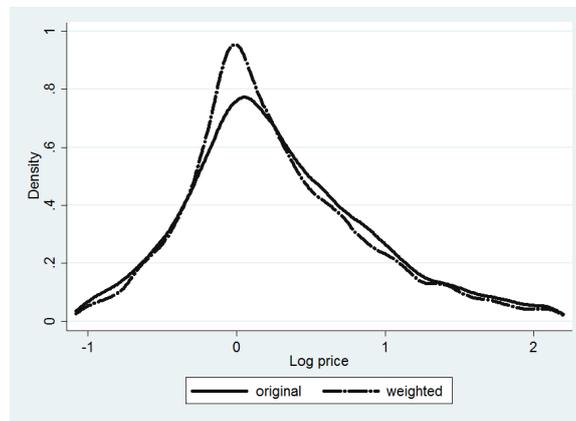
(b) RLP



(c) QHP

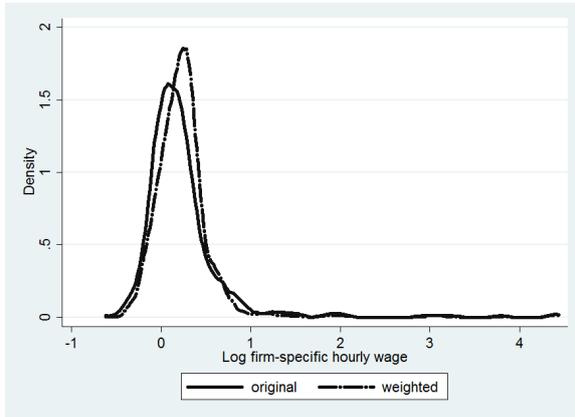


(d) RHP

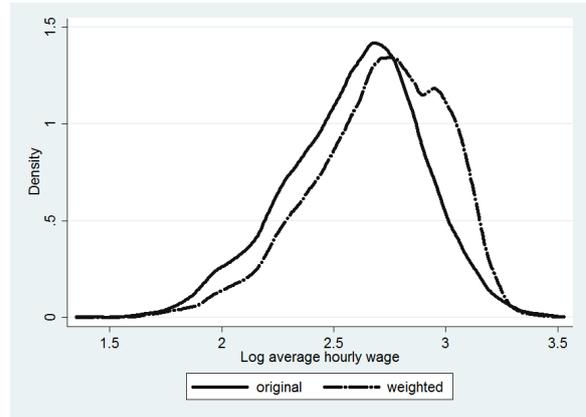


(e) Firm-specific price

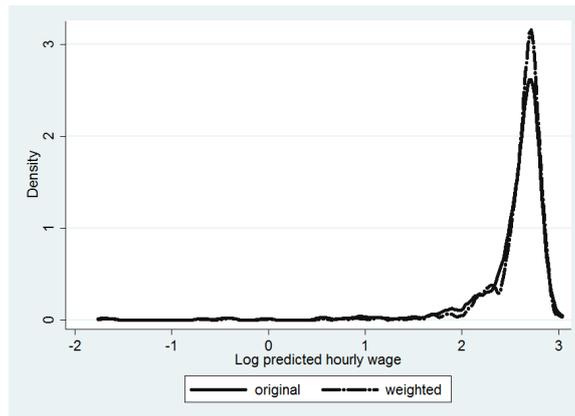
Figure 5: Quantity and revenue (hourly) labor productivity and firm-specific price in the *homogeneous sample*



(a) Firm-specific wage  $\tilde{W}$



(b) Average hourly wage  $WH$



(c) Predicted hourly wage  $\overline{WH}$

Figure 6: Wage distributions including module *Verdienste* in the *homogeneous sample*

## B Proofs

### Proof of Proposition 1:

Part (a).

Consider first the sequential planning problem to maximize the discounted household utility for a given initial distribution of workers and customers among heterogeneous firms. For any time  $t$  and any firm's age  $a$ , write  $z_{a,t} = (x_{a,t}, y_{a,t})$  for the firm's productivity and demand state, and write  $z^{a,t} = (z_{0,t-a}, z_{1,t-a+1}, \dots, z_{a,t})$  for the idiosyncratic state history. At time  $t = 0$ , the planner takes as given the initial firm distribution  $(N(z^{a-1,-1}), L(z^{a-1,-1}), B(z^{a-1,-1}))_{a \geq 1, z^{a-1,-1}}$ . The planner decides for all periods  $t \geq 0$  and state-contingent firm histories  $z^{a,t}$  the firm policies  $\lambda(z^{a,t})$ ,  $R(z^{a,t})$ ,  $\varphi(z^{a,t})$ ,  $S(z^{a,t})$ ,  $\delta_w(z^{a,t})$ ,  $\delta_b(z^{a,t})$ , as well as entrant firms  $N_t$  so as to maximize discounted household utility

$$\sum_{t \geq 0} \beta^t \left\{ u \left( \sum_{z^{a,t}} N(z^{a,t}) y_{a,t} B(z^{a,t}) \right) + b\bar{L} - KN_t \right. \\ \left. - \sum_{z^{a,t}} N(z^{a,t}) \left[ bL(z^{a,t}) + r(R(z^{a,t}), L_0(z^{a,t})) + s(S(z^{a,t}), L_0(z^{a,t})) \right. \right. \\ \left. \left. + c \left( B(z^{a,t}) + [\varphi(z^{a,t}) - q(\varphi(z^{a,t}))] \right) \right] \right\}.$$

subject to

$$\begin{aligned} L(z^{a,t}) &= L(z^{a-1,t-1})[1 - \delta_w(z^{a,t})] + m(\lambda(z^{a,t}))R(z^{a,t}), \\ B(z^{a,t}) &= B(z^{a-1,t-1})[1 - \delta_b(z^{a,t})] + q(\varphi(z^{a,t}))S(z^{a,t}), \\ L_0(z^{a,t}) &= L(z^{a-1,t-1})[1 - \delta_w(z^{a,t})], \\ N(z^{a,t}) &= (1 - \delta)\pi(z_{a,t}|z_{a-1,t-1})N(z^{a-1,t-1}), \end{aligned}$$

for  $t \geq 0$  and  $a \geq 1$ ,

$$\begin{aligned} L(z^{0,t}) &= m(\lambda(z^{0,t}))R(z^{0,t}), \quad B(z^{0,t}) = q(\varphi(z^{0,t}))S(z^{0,t}), \\ N(z^{0,t}) &= \pi^0(z^{0,t})N_t, \end{aligned}$$

for  $t \geq 0$ ,

$$B(z^{a,t}) \leq x_{a,t}F(L(z^{a,t})), \quad \delta_w(z^{a,t}) \geq \bar{\delta}_w, \quad \delta_b(z^{a,t}) \geq \bar{\delta}_b$$

for  $t \geq 0$  and  $a \geq 0$ , and subject to the resource constraint for all  $t \geq 0$ ,

$$\bar{L} \geq \sum_{z^{a,t}} N(z^{a,t}) \left[ L(z^{a,t}) + \left( \lambda(z^{a,t}) - m(\lambda(z^{a,t})) \right) \right], \quad (33)$$

Write  $\beta^t \rho_t$  for the multiplier on constraint (33). The Lagrange function of the planning problem is

$$\begin{aligned} \mathcal{L} = & \sum_{t \geq 0} \beta^t \left\{ u \left( \sum_{z^{a,t}} N(z^{a,t}) y_{a,t} B(z^{a,t}) \right) - K N_t \right. \\ & - \sum_{z^{a,t}} N(z^{a,t}) \left[ bL(z^{a,t}) + r(R(z^{a,t}), L_0(z^{a,t})) + s(S(z^{a,t}), L_0(z^{a,t})) \right. \\ & \left. \left. + c \left( B(z^{a,t}) + [\varphi(z^{a,t}) - q(\varphi(z^{a,t}))] \right) + \rho_t \left( L(z^{a,t}) + [\lambda(z^{a,t}) - m(\lambda(z^{a,t}))] \right) \right] \right\}. \end{aligned} \quad (34)$$

The derivative of the Lagrangian with respect to firm  $z^{a,t}$ 's output  $B(z^{a,t})$  is

$$\frac{d\mathcal{L}}{dB(z^{a,t})} = \beta^t N(z^{a,t}) u'(C_t) y_{a,t},$$

with aggregate consumption  $C_t \equiv \sum_{z^{a,t}} N(z^{a,t}) y_{a,t} B(z^{a,t})$ . Therefore, the number of firms of type  $z^{a,t}$ ,  $N(z^{a,t})$ , enters *linearly* all first-order conditions of the sequential planning problem with respect to firm-level policies, namely  $B(\cdot)$ ,  $L(\cdot)$ ,  $\lambda(\cdot)$ ,  $\varphi(\cdot)$ ,  $S(\cdot)$ ,  $R(\cdot)$ . The number of firm types is thus irrelevant for the planner's firm-level policies which solve the firm-level problem, defined recursively for a given (bounded) sequence  $(\rho_t, C_t)_{t \geq 0}$  by

$$\begin{aligned} G_t(L_-, B_-, z) = & \max_{(\lambda, R, \delta_w), (\varphi, S, \delta_b)} \left\{ u'(C_t) y B - bL - r(R, L_-(1 - \delta_w)) - s(S, L_-(1 - \delta_w)) \right. \\ & \left. - \rho_t [L + (\lambda - m(\lambda))R] - c[B + (\varphi - q(\varphi))S] + \beta(1 - \delta) \mathbb{E}_z G_{t+1}(L, B, z_+) \right\}, \end{aligned} \quad (35)$$

subject to

$$\begin{aligned} L &= L_-(1 - \delta_w) + m(\lambda)R, \quad B = B_-(1 - \delta_b) + q(\varphi)S, \\ B &\leq xF(L), \quad \delta_w \geq \bar{\delta}_w, \quad \delta_b \geq \bar{\delta}_b. \end{aligned}$$

A solution  $(G_t)_{t \geq 0}$  for this problem exists with functions  $G_t : [0, L^{\max}] \times [0, B^{\max}] \times Z \rightarrow \mathbb{R}$  for appropriately define upper bounds  $L^{\max}$  and  $B^{\max}$ . The proof of this assertion follows the same lines as in Lemma A.4, part (a), of Kaas and Kircher (2015).

To prove part (a) of Proposition 1, suppose that  $(\rho, G, N_0, C)$  solves the recursive social planning problem (20) together with (21), aggregate consumption (17) and the aggregate resource constraint (19) are satisfied. Then, for the constant sequences  $\rho_t = \rho$  and  $C_t = C$ , value functions  $G_t = G$  for all  $t \geq 0$  also solves problem (35). If  $N_t = N_0$  for all  $t$ , the resource constraint (33) is satisfied in all periods  $t$  because the distribution of firm types and the distribution of workers across firms is stationary:  $N(z^{a,t}) = N(z^a)$ ,  $L(z^{a,t}) = L(z^a)$ , provided

that  $(N(z^a), L(z^a), B(z^a))$  is the initial firm distribution. Because individual firm policies solve problem (35), they also maximize the Lagrange function (34). Condition (21) further says that  $K = \sum_z \pi^0(z) G_t(0, 0, z)$ . On the other hand, the first-order condition of (34) with respect to  $N_t$  is

$$\begin{aligned} 0 &= -\beta^t K + \sum_{a \geq 0} \beta^{t+a} (1 - \delta)^a \pi(z^a) \left\{ u'(C) y_a B(z^a) - bL(z^a) - r(R(z^a), L_0(z^a)) \right. \\ &\quad \left. - s(S(z^a), L_0(z^a)) - c \left( B(z^a) + [\varphi(z^a) - q(\varphi(z^a))] \right) - \rho \left( L(z^a) + [\lambda(z^a) - m(\lambda(z^a))] \right) \right\} \\ &= \beta^t \left[ -K + \sum_z \pi^0(z^0) G_t(0, 0, z^0) \right]. \end{aligned}$$

Condition (21) then implies that  $N_t = N_0$ ,  $t \geq 0$ , solve the Lagrange problem. Since aggregate resource feasibility is satisfied, these firm policies solve the sequential planning problem where the multiplier on (33) is  $\beta^t \rho$ . For a rigorous argument that no other feasible allocation dominates the one defined by the individual firm's problem, see the proof of part (b) of Lemma A.4 in Kaas and Kircher (2015).

Part (b).

Consider  $(\rho, G, N_0, C)$  where  $G$  solves the recursive social planning problem (20) together with (21). Further aggregate consumption is (17) and the aggregate resource constraint (19) is satisfied when  $L(z^a)$ ,  $B(z^a)$  etc. are defined by the policy functions of problem (20). Define candidate equilibrium contracts  $\mathcal{C}^{a*} = (w^{a*}(z^k), \delta_w^{a*}(z^{k+1}))_{k \geq a}$  with separation rates  $\delta_w^{a*}(z^k) \equiv \delta_w(z^k)$  from the policy functions of problem (20) (hence, separations are independent of the tenure in the firm). Equilibrium wages can be defined such that all workers within the firm earn the same:  $w^{a*}(z^k) = w(z^k)$ , where  $w(z^k)$  is defined as in (28). As in Section 3.2.3, define the generic state vector of the firm as  $\sigma = [(L^\tau, \mathcal{C}^\tau)_{\tau=0}^{a-1}, B_-, z^a]$ , and let  $G_a(\sigma)$  denote the *social* value of firm type  $z^a$ , assuming that the firm takes as given previous worker cohorts  $L^\tau$  and the precommitted separation rates as specified in contracts  $\mathcal{C}^\tau$ ,  $\tau < a$ . For the contracts  $(\mathcal{C}^{\tau*})_{\tau=0}^{a-1}$  in the candidate equilibrium (and the corresponding worker cohorts  $L^{\tau*}$ ) write  $\sigma^*$  for the firm's state vector.

The recursive problem to maximize social firm value is

$$\begin{aligned} G_a(\sigma) &= \max_{(\lambda, R, \mathcal{C}^a), (\varphi, S, \delta_b)} \left\{ u'(C) y_a B - bL - r(R, L_0) - s(S, L_0) \right. \\ &\quad \left. - \rho [L + (\lambda - m(\lambda))R] - c [B + (\varphi - q(\varphi))S] + \beta(1 - \delta) \mathbb{E} G_{a+1}(\sigma_+) \right\}, \end{aligned} \quad (36)$$

subject to (7), (8), (10) and (11). Wages in contracts  $\mathcal{C}^\tau$  are clearly irrelevant for that problem. The same policies that solve problem (20), and in particular contracts  $\mathcal{C}^{a*}$  for all  $a \geq 0$ , also solve problem (36). The only difference between those two problems is that the firm is precommitted to separation rates for existing workers in the latter but not in the former

problem. But since policies for the latter problem are time consistent, both problems have the same solutions. Hence it remains to show that those policies not only solve problem (36) but that they also maximize the *private* value of the firm, as specified in the recursive problem (6)–(13), provided that  $\rho^* = \rho$ .

Substitution of (13) shows that

$$u'(C)y_a B - c[B + (\varphi - q(\varphi))S] = p^R B_-(1 - \delta_b) + pq(\varphi)S .$$

Hence, the left-hand side of that term in problem (36) can be replaced by the right-hand side together with constraint (13). Further, we can write the social labor costs

$$bL + \rho[L + (\lambda - m(\lambda))R] = (b + \rho)L_0 + [b + \rho \frac{\lambda}{m(\lambda)}]m(\lambda)R . \quad (37)$$

Given the precommitted contracts  $\mathcal{C}^{\tau^*}$ ,  $\tau < a$ , the first term can be written

$$\begin{aligned} (b + \rho)L_0 &= \sum_{\tau=0}^{a-1} [1 - \delta_w^{\tau^*}(z^a)]L^\tau \cdot (b + \rho) \\ &= \sum_{\tau=0}^{a-1} [1 - \delta_w^{\tau^*}(z^a)]L^\tau \left[ w^{\tau^*}(z^a) - [W(\mathcal{C}^{\tau^*}, z^a) - U] + \beta(1 - \delta)\mathbb{E}[W'(\mathcal{C}^{\tau^*}, z^{a+1}) - U] \right] \\ &= - \sum_{\tau=0}^{a-1} L^\tau [W'(\mathcal{C}^{\tau^*}, z^a) - U] + \sum_{\tau=0}^{a-1} L^{\tau+} w^{\tau^*}(z^a) + \beta(1 - \delta)\mathbb{E} \sum_{\tau=0}^{a-1} L^{\tau+} [W'(\mathcal{C}^{\tau^*}, z^{a+1}) - U] . \end{aligned}$$

For any contract  $\mathcal{C}^a = (w^a(z^k), \delta_w^a(z^{k+1}))_{k \geq a}$  offered to new hires  $m(\lambda)R = L^{a+}$ , the second term in (37) can be written

$$\begin{aligned} [b + \rho \frac{\lambda}{m(\lambda)}]m(\lambda)R &= [W(\mathcal{C}^a, z^a) - \beta U]m(\lambda)R \\ &= w^a(z^a)L^{a+} + \beta(1 - \delta)\mathbb{E}[W'(\mathcal{C}^a, z^{a+1}) - U]L^{a+} . \end{aligned}$$

Substituting those expressions into (36) at  $\sigma = \sigma^*$  shows

$$\begin{aligned} G_a(\sigma^*) &= \max_{(\lambda, R, \mathcal{C}^a), (\varphi, S, p, p^R, \delta_b)} \left\{ p^R B_-(1 - \delta_b) + pq(\varphi)S - W \right. \\ &\quad + \sum_{\tau=0}^{a-1} L^\tau [W'(\mathcal{C}^{\tau^*}, z^a) - U] - r(R, L_0) - s(S, L_0) \\ &\quad \left. + \beta(1 - \delta)\mathbb{E} \left\{ G_{a+1}(\sigma_+^*) - \sum_{\tau=0}^{a-1} L^{\tau+} [W'(\mathcal{C}^{\tau^*}, z^{a+1}) - U] - L^{a+} [W'(\mathcal{C}^a, z^{a+1}) - U] \right\} \right\} , \end{aligned} \quad (38)$$

where maximization is subject to (8)–(13) with  $\sigma_+^* = [(L^\tau, \mathcal{C}^{\tau^*})_{\tau=0}^{a-1}, (L^{a+}, \mathcal{C}^a), B, z^{a+1}]$ . In this maximization problem, the term  $\sum_{\tau=0}^{a-1} L^\tau [W'(\mathcal{C}^{\tau^*}, z^a) - U]$  is predetermined and thus not

subject to the maximization. Therefore, we can define the private firm value

$$J_a(\sigma) \equiv G_a(\sigma) - \sum_{\tau=0}^{a-1} L^\tau [W'(\mathcal{C}^\tau, z^a) - U] ,$$

i.e. the difference between the social value of firm  $z^a$  and the surplus of the existing workers. Then problem (38) (at given state vector  $\sigma^*$ ) is equivalent to problem (6). In particular, the firm policies  $\lambda$ ,  $R$ ,  $\varphi$ ,  $S$ ,  $p$  and  $p^R$  and  $\mathcal{C}^{a*}$  that solve (38) also solve (6). Moreover, because of  $G(0, 0, z) = J_0(0, z)$ , socially optimal entry (21) implies the equilibrium condition (18). Since resource constraints are satisfied, the stationary planning solution gives rise to a stationary competitive search equilibrium.  $\square$