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Abstract

We study the Cauchy problem of the Ball model for an extensible beam:

$$\rho\partial_t^2 u + \delta\partial_t u + \kappa\partial_x^4 u + \eta\partial_t\partial_x^4 u = \left(\alpha + \beta\int_{\mathbb{R}} |\partial_x u|^2 dx + \gamma\eta\int_{\mathbb{R}} \partial_t\partial_x u\partial_x u dx\right)\partial_x^2 u.$$

The aim of this paper is to investigate singular limits as $\rho \to 0$ for this problem. In the authors' previous paper [8] decay estimates of solutions u_ρ to the equation in the case $\rho > 0$ were shown. With the help of the decay estimates we describe the singular limit in the sense of the following uniform (in time) estimate:

$$\|u_\rho - u_0\|_{L^\infty([0,\infty);H^2(\mathbb{R}))} \leq C\rho.$$

Keywords: decay estimate, extensible beam, Cauchy problem, singular limit, Ball's model, Kelvin-Voigt damping

1 Introduction

We consider the initial value problem for the model of an extensible beam proposed by Ball [1] as a modified model of the Woinovsky-Krieger model [11] (see also [5]), where he assumes that the beam has linear structural (Kelvin-Voigt) and external (frictional) damping, that is, we consider the following problem:

$$\begin{cases} \rho\partial_t^2 u + \kappa\partial_x^4 u + \delta\partial_t u + \eta\partial_t\partial_x^4 u \\ \quad = \left(\alpha + \beta\int_{\mathbb{R}} |\partial_x u|^2 dx + \gamma\eta\int_{\mathbb{R}} \partial_x u \partial_t \partial_x u dx\right)\partial_x^2 u, \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}, \\ u(0,\cdot) = f, \quad \partial_t u(0,\cdot) = g, \qquad x\in\mathbb{R}, \end{cases} \tag{1.1}$$

where ρ, κ, δ, α, β and γ are positive and η is a non-negative constant. For the physical background of this model we refer to [1]. For the initial value problem the authors proved in [8] global existence and decay estimates for the solution as follows:

Theorem 1.1 ([8]). *Let $k \geq 2$ be an integer and set*

$$\theta_\ell := \min\left\{\frac{\ell}{2},\ 2\right\}, \qquad \widetilde{\theta}_\ell := \begin{cases} \frac{\ell}{2}, & \ell = 0,1,2,3,4 \\ \max_{m=3,4,\dots,\ell} \min\left\{\widetilde{\theta}_{\ell+2-m} + 1,\ \frac{m}{2}\right\} & \ell \geq 5. \end{cases} \tag{1.2}$$

1. *Let $\eta = 0$. For any $(f,g)\in H^k\times H^{k-2}$, there exists unique global solution u to* (1.1) *satisfying $u \in C([0,\infty),H^k)$ and $\partial_t u \in C([0,\infty),H^{k-2})$. Moreover, the solution satisfies*

$$\|\partial_x^\ell u(t)\|_{L^2} \leq C_k(t+1)^{-\theta_\ell} \qquad (0\leq \ell\leq k),$$
$$\|\partial_t\partial_x^m u(t)\|_{L^2} \leq C_k(t+1)^{-\theta_{m+2}} \qquad (0\leq m\leq k-2),$$

where $C_k = C(\|f\|_{H^k}, \|g\|_{H^{k-2}})$.

2. *Let $\eta > 0$. For any $(f,g)\in H^k\times H^k$ there exists unique global solution u to* (1.1) *satisfying $u\in C^1([0,\infty),H^k)$. Moreover, the solution satisfies*

$$\|\partial_x^\ell u(t)\|_{L^2} \leq \widetilde{C}_k(t+1)^{-\widetilde{\theta}_\ell}, \qquad \|\partial_t\partial_x^\ell u(t)\|_{L^2} \leq \widetilde{C}_k(t+1)^{-\widetilde{\theta}_{\ell+2}} \qquad (0\leq\ell\leq k),$$

where $\widetilde{C}_k = C(\|f\|_{H^k}, \|g\|_{H^k})$.

In this article we observe how the solutions behave as the effect of the inertial term decreases as $\rho\to 0$. We remark that our results also can be carried over to the bounded domain case with appropriate boundary conditions such as hinged boundary condition, because the decay in that setting is better than in the unbounded domain case.

Before stating our main results, we explain several related results. Singular limit problems are one of the main topics in partial differential equations. For the bounded domain case ($x\in[0,l]$), Cwiszewski and Rybakowski [4] investigated the singular limit as $\epsilon\to 0$ for the equation

$$\epsilon^2\partial_t^2 u + \kappa\partial_x^4 u + \epsilon\delta\partial_t u + \eta\partial_t\partial_x^4 u = g\left(\int_0^l |\partial_x u|^2 dx\right) + \epsilon\gamma\eta\int_0^l \partial_x u\partial_t\partial_x u dx\partial_x^2 u,$$

using topological ingredients such as the Conley index. The quasilinear equation (1.1) with $\kappa = \eta = 0$ is called Kirchhoff equation and has also been extensively studied by many authors. The problem in which $\rho \to 0$ in the Kirchhoff equation or in the wave equation is called a hyperbolic-parabolic singular perturbation problem, and it is extensively studied by many authors (see e.g. [3], [6] and [7]). Hashimoto and Yamazaki [7] gave a singular limit result for (1.1) with $\kappa = \alpha = \eta = 0$ (and $\rho \to 0$). To get an order of convergence in ρ they had to impose the assumption that ρ is small because the Kirchhoff equation is a quasilinear problem. Schöwe [9] studied the singular limit problem for the hyperbolic Navier-Stokes system from another point of view. He gave singular limit result locally in time under compatibility conditions for the data by a different method. We apply his method to our problem, but our singular limit result is global in time and in the coefficient ρ, i.e., for the order of convergence, no smallness assumption on ρ is needed.

The following result on the singular limit is the main result of this paper. We compare the solution to (1.1) with the one for the problem when $\rho = 0$:

$$\begin{cases} \delta\partial_t v + \eta\partial_t\partial_x^4 v + \kappa\partial_x^4 v \\ \qquad = \left(\alpha + \beta \int_{\mathbb{R}} |\partial_x v|^2 dx + \gamma\eta \int_{\mathbb{R}} \partial_x v \partial_t \partial_x v dx\right) \partial_x^2 v, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ v(0,\cdot) = f, & x \in \mathbb{R}, \end{cases} \tag{1.3}$$

Theorem 1.2 (Singular limit)**.** *Assume that $f \in H^2$ for $\eta > 0$ and $f \in H^6$ for $\eta = 0$. Let v be a solution for* (1.3)*, and define $g = \lim_{t\to+0} \partial_t v(t,\cdot)$. Then we have the following global in time singular limit estimate,*

$$\|u - v\|_{L^\infty H^2} \le C\rho,$$

where u is a solution to (1.1) *with data (f, g) and* $C = \begin{cases} C(\|f\|_{H^6}), & \eta = 0, \\ C(\|f\|_{H^2}), & \eta > 0. \end{cases}$

This paper is organized as follows. In Section 2 we introduce several lemmas needed later on to show the main results. In Section 3 we show the decay estimate for the nonlinear problem (1.3) with $\rho = 0$. Section 4 is devoted to the proof of the singular limit theorem which shall be done separately for $\eta = 0$ and for $\eta > 0$, respectively.

We finish the introduction by giving some notation used in this paper. We use the notation $\partial_t := \frac{\partial}{\partial t}$ and $\partial_x := \frac{\partial}{\partial x}$. We denote several positive constants by C and C_i ($i = 1, 2, 3, \ldots$); the constant may change from line to line. Important dependencies of constants are denoted by $C = C(\ldots)$. L^p and H^s are the standard Lebesgue and Sobolev spaces, respectively. We also use the following notation for space-time norms (distinguishing $1 \le p < \infty$ and $p = \infty$),

$$\|u\|_{L^pL^q} := \begin{cases} \left(\int_0^\infty \|u(t,\cdot)\|_{L^q}^p dt\right)^{\frac{1}{p}}, \\ \sup_{t\in[0,\infty)} \|u(t,\cdot)\|_{L^q}, \end{cases} \qquad \|u\|_{L_t^pL^q} := \begin{cases} \left(\int_0^t \|u(s,\cdot)\|_{L^q}^p ds\right)^{\frac{1}{p}}, \\ \sup_{s\in[0,t]} \|u(s,\cdot)\|_{L^q}. \end{cases}$$

We denote the Fourier and the Fourier inverse transforms by $\mathcal{F}$ and $\mathcal{F}^{-1}$ and the Fourier transform of a function f by $\widehat{f}$.

2 Preliminaries

In this section we prepare some definitions and introduce some useful lemmas and estimates for the linearized case.

For short, throughout this paper we often denote

$$I(u) := \int_{\mathbb{R}} |\partial_x u|^2 dx, \qquad \widetilde{I}(u) := \int_{\mathbb{R}} \partial_x u \partial_t \partial_x u dx. \tag{2.1}$$

We define the mild solution to (1.1) in the case $\rho > 0$ by the solution of the following integral equation in the L^2-sense

$$u(t) = \mathcal{K}_0(t)f + \mathcal{K}_1(t)g + \frac{1}{\rho}\int_0^t \mathcal{K}_1(t-s)\{\beta I(u) + \gamma\eta\widetilde{I}(u)\}\partial_x^2 u(s)ds, \tag{2.2}$$

where $\mathcal{K}_0(t)f := \mathcal{F}^{-1}\left[K_0(t,\xi)\widehat{f}\right]$ and $\mathcal{K}_1(t)f := \mathcal{F}^{-1}\left[K_1(t,\xi)\widehat{f}\right]$ with

$$\begin{aligned} K_0(t,\xi) := {} & e^{-\frac{1}{2}a(\xi)t}\frac{e^{\frac{\sqrt{a(\xi)^2-4b(\xi)}}{2}t} + e^{-\frac{\sqrt{a(\xi)^2-4b(\xi)}}{2}t}}{2} \\ & + \frac{a(\xi)e^{-\frac{1}{2}a(\xi)t}}{2\sqrt{a(\xi)^2-4b(\xi)}}\left(e^{\frac{\sqrt{a(\xi)^2-4b(\xi)}}{2}t} - e^{-\frac{\sqrt{a(\xi)^2-4b(\xi)}}{2}t}\right), \end{aligned} \tag{2.3}$$

$$K_1(t,\xi) := \frac{e^{-\frac{1}{2}a(\xi)t}}{\sqrt{a(\xi)^2-4b(\xi)}}\left(e^{\frac{\sqrt{a(\xi)^2-4b(\xi)}}{2}t} - e^{-\frac{\sqrt{a(\xi)^2-4b(\xi)}}{2}t}\right), \tag{2.4}$$

$$a(\xi) := \frac{\delta}{\rho} + \frac{\eta}{\rho}\xi^4, \qquad b(\xi) := \frac{\kappa}{\rho}\xi^4 + \frac{\alpha}{\rho}\xi^2. \tag{2.5}$$

Similarly, for the case $\rho = 0$ we define the mild solution to (1.3) by the solution of the following integral equation in the L^2-sense

$$v(t) = \mathcal{K}(t)f + \int_0^t (\eta\partial_x^4 + \delta)^{-1}\mathcal{K}(t-s)\{\beta I(v) + \gamma\eta\widetilde{I}(v)\}\partial_x^2 v(s)ds, \tag{2.6}$$

where $\mathcal{K}(t)f := \mathcal{F}^{-1}\left[K(t)\widehat{f}\right]$ and $K(t,\xi) = e^{-\frac{\kappa\xi^4+\alpha\xi^2}{\eta\xi^4+\delta}t}$.

Next, we give the (linear) decay estimates for the limiting equation (1.3) in $\rho = 0$. Although these may be known results at least in the case $\eta = 0$, we give the proof here for self-containedness.

Proposition 2.1. *Let k be an any nonnegative integer.*

1. *If $\eta = 0$, then it holds that for $0 \le \ell, m \le k$ and $0 \le n \le k+4$*

$$\left\|\partial_x^k \mathcal{K}(t)f\right\|_{L^2} \le \frac{C}{(t+1)^{\frac{\ell}{2}}}\|\partial_x^{k-\ell} f\|_{L^2} + \frac{C}{t^{\frac{m}{4}}e^{Ct}}\|\partial_x^{k-m} f\|_{L^2}, \tag{2.7}$$

$$\left\|\partial_x^k \partial_t \mathcal{K}(t)f\right\|_{L^2} \le \frac{C}{(t+1)^{\frac{\ell}{2}+1}}\|\partial_x^{k-\ell} f\|_{L^2} + \frac{C}{t^{\frac{n}{4}}e^{Ct}}\|\partial_x^{k+4-n} f\|_{L^2}, \tag{2.8}$$

2. *If $\eta > 0$, then it holds that for $0 \le \ell, m \le k$ and $0 \le n \le \min\{k, 4\}$*

$$\left\|\partial_x^k \mathcal{K}(t) f\right\|_{L^2} \le \frac{C}{(t+1)^{\frac{\ell}{2}}} \|\partial_x^{k-\ell} f\|_{L^2} + Ce^{-Ct} \|\partial_x^k f\|_{L^2}, \tag{2.9}$$

$$\left\|\partial_x^k \partial_t \mathcal{K}(t) f\right\|_{L^2} \le \frac{C}{(t+1)^{\frac{\ell}{2}+1}} \|\partial_x^{k-\ell} f\|_{L^2} + Ce^{-Ct} \|\partial_x^k f\|_{L^2}, \tag{2.10}$$

Proof. Let us consider the case $\eta = 0$. Observe that $|\xi|^k e^{-C|\xi|^\alpha t} \le C/t^{k/a}$. For $|\xi| \le 1$ we have

$$\left||\xi|^k K(t,\xi)\right| \le \left||\xi|^k e^{-\frac{\alpha}{\delta} t\xi^2}\right| \le e^{\frac{\alpha}{\delta}} |\xi|^{k-\ell} \left||\xi|^\ell e^{-\frac{\alpha}{\delta}(t+1)\xi^2}\right| \le \frac{C}{(t+1)^{\frac{\ell}{2}}} |\xi|^{k-\ell}.$$

It holds for $|\xi| \ge 1$

$$\left||\xi|^k K(t,\xi)\right| \le |\xi|^{k-m} e^{-\frac{\alpha}{\delta} t} |\xi|^m e^{-\frac{\kappa}{\delta}\xi^4 t} \le \frac{C}{t^{\frac{m}{4}} e^{Ct}} |\xi|^{k-m}.$$

From the Plancherel theorem we obtain

$$\begin{aligned}
\|\partial_x^k \mathcal{K}(t) f\|_{L^2} &\le \left\||\xi|^k K(t,\xi)\widehat{f}\right\|_{L^2(|\xi|\le 1)} + \left\||\xi|^k K(t,\xi)\widehat{f}\right\|_{L^2(|\xi|\ge 1)} \\
&\le \frac{C}{(t+1)^{\frac{\ell}{2}}} \|\partial_x^{k-\ell} f\|_{L^2} + \frac{C}{t^{\frac{m}{4}} e^{Ct}} \|\partial_x^{k-m} f\|_{L^2},
\end{aligned}$$

which completes the proof of (2.7). Similarly, it holds for $|\xi| \le 1$

$$\left||\xi|^k \partial_t K(t,\xi)\right| \le \left||\xi|^k \frac{\kappa\xi^4 + \alpha\xi^2}{\delta} e^{-\frac{\alpha}{\delta} t\xi^2}\right| \le C|\xi|^{k-\ell} \left||\xi|^{\ell+2} e^{-\frac{\alpha}{\delta}(t+1)\xi^2}\right| \le \frac{C}{(t+1)^{\frac{\ell}{2}+1}} |\xi|^{k-\ell}.$$

and for $|\xi| \ge 1$

$$\left||\xi|^k \partial_t K(t,\xi)\right| \le C\left||\xi|^k \frac{\kappa\xi^4 + \alpha\xi^2}{\delta} e^{-\frac{\kappa\xi^4+\alpha\xi^2}{\delta} t}\right| \le C|\xi|^{k+4-n} e^{-\frac{\alpha}{\delta} t} |\xi|^n e^{-\frac{\kappa}{\delta}\xi^4 t} \le \frac{C}{t^{\frac{n}{4}} e^{Ct}} |\xi|^{k+4-n},$$

which implies (2.8).

Next we consider the case $\eta > 0$. We easily check that for $|\xi| \le 1$

$$\frac{\alpha\xi^2}{\eta + \delta} \le \frac{\kappa\xi^4 + \alpha\xi^2}{\eta\xi^4 + \delta} \le \frac{\kappa + \alpha}{\delta},$$

and that for $|\xi| \ge 1$

$$\frac{\kappa}{\eta + \delta} \le \frac{\kappa\xi^4 + \alpha\xi^2}{\eta\xi^4 + \delta} \le \frac{\kappa + \alpha}{\eta}$$

holds. Then we have for $|\xi| \le 1$

$$\left||\xi|^k \partial_t K(t,\xi)\right| = e^{\frac{\kappa\xi^4+\alpha\xi^2}{\eta\xi^4+\delta}} \left||\xi|^k e^{-\frac{\kappa\xi^4+\alpha\xi^2}{\eta\xi^4+\delta}(t+1)}\right| \le e^{\frac{\kappa+\alpha}{\delta}} |\xi|^{k-\ell} \left||\xi|^\ell e^{-\frac{\alpha}{\eta+\delta}(t+1)\xi^2}\right| \le \frac{C}{(t+1)^{\frac{\ell}{2}}} |\xi|^{k-\ell},$$

and for $|\xi| \ge 1$

$$\left||\xi|^k K(t,\xi)\right| \le |\xi|^k e^{-\frac{\kappa}{\eta+\delta} t},$$

which yields (2.9). Lastly the estimate (2.10) follows from

$$\left||\xi|^k\partial_t K(t,\xi)\right| \le e^{\frac{\kappa+\alpha}{\delta}}\left||\xi|^{k+2}\frac{\kappa+\alpha}{\delta}e^{-\frac{\alpha}{\eta+\delta}(t+1)\xi^2}\right| \le \frac{C}{(t+1)^{\frac{\ell}{2}+1}}|\xi|^{k-\ell} \qquad (|\xi|\le 1).$$

and

$$\left||\xi|^k\partial_t K(t,\xi)\right| \le |\xi|^k\frac{\kappa+\alpha}{\eta}e^{-\frac{\kappa}{\eta+\delta}t} \le C|\xi|^k e^{-\frac{\kappa}{\eta+\delta}t} \qquad (|\xi|\ge 1).$$

This completes the proof. □

By the standard contraction mapping argument the unique local existence is immediately established.

Proposition 2.2 (Local existence and uniqueness)**.** *Let $k \ge 2$ be an integer.*

1. *Let $\eta = 0$. For any $f \in H^k$, there is $T = T(\|f\|_{H^k})$ such that there exists a unique mild solution v to* (1.3) *satisfying*

$$v \in C([0,T], H^k).$$

2. *Let $\eta > 0$. For any $f \in H^k$, there is $T = T(\|f\|_{H^k})$ such that there exists a unique mild solution v to* (1.3) *satisfying*

$$v \in C^1([0,T], H^k).$$

Outline of the proof. To establish the local existence result, we define a nonlinear mapping by:

$$\Phi[u](t) := \mathcal{K}(t)f + \int_0^t (\delta+\eta\partial_x^4)^{-1}\mathcal{K}(t-s)\{\beta I(v(s)) + \gamma\eta\widetilde{I}(v(s))\}\partial_x^2 v(s)ds$$

and the ball $X_T := \{u \mid \|u\|_X \le M\}$, where

$$\|u\|_X := \begin{cases} \|u\|_{L^\infty_T H^k}, & \eta = 0, \\ \|u\|_{L^\infty_T H^k} + \|\partial_t u\|_{L^\infty_T H^k}, & \eta > 0. \end{cases}$$

We can easily show that the map Φ is a contraction mapping on X_T, with the help of Lemma 2.1. In the case $\eta > 0$, additionally one should observe the fact that

$$\begin{aligned}\partial_t\Phi[u](t) :=& \partial_t\mathcal{K}(t)f + \{\beta I(v(t)) + \gamma\eta\widetilde{I}(v(t))\}(\delta+\eta\partial_x^4)^{-1}\partial_x^2 v(t) \\ &+ \int_0^t (\delta+\eta\partial_x^4)^{-1}\partial_t\mathcal{K}(t-s)\{\beta I(v(s)) + \gamma\eta\widetilde{I}(v(s))\}\partial_x^2 v(s)ds,\end{aligned}$$

because $\mathcal{K}(0)$ is the identity operator. □

From the property of mild solution with the help of the estimates (2.8), we also easily deduce the following result (see also [2, Section 4]).

Proposition 2.3 (Regularity)**.** *Let k be an any non-negative integer. If $\eta = 0$, then the mild solution obtained in Proposition 2.2 with $f \in H^{k+4}$ satisfies $\partial_t v \in C([0,T]; H^k)$.*

The following inequalities are well-known and will use for the estimate of the nonlinear terms in next section.

Lemma 2.4 (see e.g. [10, Lemma 2.4])**.** *1. Let $a > 0$ and $b > 0$ with $\min\{a,b\} > 1$. It holds*

$$\int_0^t (t-s+1)^{-a}(s+1)^{-b}ds \le C(t+1)^{-\min\{a,b\}}.$$

2. Let $1 > a \ge 0$, $b > 0$ and $c > 0$. It holds

$$\int_0^t e^{-c(t-s)}(t-s)^{-a}(s+1)^{-b}ds \le C(t+1)^{-b}.$$

3 Decay estimates for the limit problem $\rho = 0$

In this part we shall show several decay estimates for the first order problem (1.3) in the case $\rho = 0$. Combining a standard energy method with Lemma 2.1, the following a priori estimates to the nonlinear problem (4.2) are derived. The decay of the energy $E_\rho(t)$, defined by

$$E_0(t) := \frac{\kappa}{2}\|\partial_x^2 v\|_{L^2}^2 + \frac{\alpha}{2}\|\partial_x v\|_{L^2}^2 + \frac{\beta}{4}\|\partial_x v\|_{L^2}^4, \tag{3.1}$$

can be proved for $\eta > 0$ and for $\eta = 0$ simultaneously.

Lemma 3.1. *Let $\eta \ge 0$. For any $f \in H^2$, the solution for (1.3) constructed in Lemma 2.2 satisfies, for $t \ge 0$,*

$$E_0(t) \le \frac{C}{t+1}. \tag{3.2}$$

Proof. Multiplying (1.3) by v yields

$$\partial_t\left(\frac{\delta}{2}\|v\|_{L^2}^2 + \frac{\eta}{2}\|\partial_x^2 v\|_{L^2}^2 + \frac{\gamma\eta}{4}\|\partial_x v\|_{L^2}^4\right) + \kappa\|\partial_x^2 v\|_{L^2}^2 + \alpha\|\partial_x v\|_{L^2}^2 + \beta\|\partial_x v\|_{L^2}^4 = 0, \tag{3.3}$$

and hence we have for any $t \ge 0$

$$\frac{\delta}{2}\|v(t)\|_{L^2}^2 + \frac{\eta}{2}\|\partial_x^2 v(t)\|_{L^2}^2 + \frac{\gamma\eta}{4}\|\partial_x v(t)\|_{L^2}^4 \le \frac{\delta}{2}\|f\|_{L^2}^2 + \frac{\eta}{2}\|\partial_x^2 f\|_{L^2}^2 + \frac{\gamma\eta}{4}\|\partial_x f\|_{L^2}^4. \tag{3.4}$$

Next, multiplying (1.3) by $\partial_t v$ we have

$$\partial_t E_0(t) + A(t) = 0, \tag{3.5}$$

where $A(t)$ is defined by

$$A(t) := \delta\|\partial_t v(t)\|_{L^2}^2 + \eta\|\partial_t\partial_x^2 v(t)\|_{L^2}^2 + \gamma\eta\left(\int_{\mathbb{R}} \partial_x v(t)\partial_t\partial_x v(t)dx\right)^2. \tag{3.6}$$

It follows from (3.3), (3.4) and (3.5) that

$$
\begin{aligned}
(2E_0(t))^2 &\le (\kappa\|\partial_x^2 v\|_{L^2}^2 + \alpha\|\partial_x v\|_{L^2}^2 + \beta\|\partial_x v\|_{L^2}^4)^2 \\
&= \Bigg(\delta\int_{\mathbb{R}} v(t)\partial_t v(t)dx + \eta\int_{\mathbb{R}} \partial_x^2 v(t)\partial_t\partial_x^2 v(t)dx \\
&\qquad + \gamma\eta\int_{\mathbb{R}} \partial_x v(t)\partial_t\partial_x v(t)dx\|\partial_x v(t)\|_{L^2}^2\Bigg)^2 \qquad (3.7)\\
&\le C\|\partial_t v(t)\|_{L^2}^2 + C\|\partial_t\partial_x^2 v(t)\|_{L^2}^2 + C\left(\int_{\mathbb{R}} \partial_x v(t)\partial_t\partial_x v(t)dx\right)^2 \\
&\le CA(t) = -C\partial_t E_0(t)
\end{aligned}
$$

Then we have

$$\partial_t E_0(t) + kE_0(s)^2 \le 0, \qquad (3.8)$$

where $k := 4/C$. By a nonlinear version of the Gronwall lemma (see e.g. Schöwe [9]), we see that

$$E_0(t) \le h(t),$$

where $h(t)$ is a solution to

$$\partial_t h(t) + kh^2(t) = 0. \qquad (3.9)$$

Since the solution for (3.9) is given by $h(t) = k/(t + 1/E_0(0))$, we conclude that

$$E_0(t) \le \frac{k}{t + 1/E_0(0)} \le \frac{C}{t+1}.$$

□

From here we split our argument to the cases $\eta = 0$ and $\eta > 0$.

Lemma 3.2. *Assume that $\eta = 0$ and let $k \ge 2$. For any data $f \in H^k$, there exists a unique global mild solution $v \in C([0,\infty); H^k)$, and the solution decays, for $t \ge 0$,*

$$\|\partial_x^p v(t)\|_{L^2} \le C_k(t+1)^{-\widetilde{\theta}_p} \qquad (0 \le p \le k) \qquad (3.10)$$

where $C_k = C(\|f\|_{H^k})$.

Proof. From Lemma 3.1, the solution decays like

$$\|\partial_x^2 v(t)\|_{L^2} \le \frac{C}{(t+1)^{\frac{1}{2}}}, \qquad \|\partial_x v(t)\|_{L^2} \le \frac{C}{(t+1)^{\frac{1}{2}}}, \qquad \|v(t)\|_{L^2} \le C. \qquad (3.11)$$

The decay of $\|\partial_x^2 v(t)\|_{L^2}$ can be shown to be faster. Since $|I(t)| \le C/(t+1)$, applying (2.7) to the Duhamel formula (2.6) yields

$$
\begin{aligned}
\|\partial_x^2 v\|_{L^2} &\le \|\partial_x^2 \mathcal{K}(t)f\|_{L^2} + \int_0^t \frac{C}{s+1}\left\|\partial_x^2\mathcal{K}(t-s)\partial_x^2 v(s)\right\|_{L^2} ds \\
&\le \frac{C}{t+1}\|f\|_{H^2} + \int_0^t \frac{C}{s+1}\left\{\frac{\|\partial_x^{4-\ell}v(s)\|_{L^2}}{(t-s+1)^{\frac{\ell}{2}}} + \frac{\|\partial_x^{4-m}v(s)\|_{L^2}}{(t-s)^{m/4}e^{C(t-s)}}\right\} ds.
\end{aligned}
$$

Taking $\ell = m = 3$, we obtain

$$
\begin{aligned}
\|\partial_x^2 v\|_{L^2} &\le \frac{C}{t+1} + \int_0^t \frac{C\|\partial_x v(s)\|_{L^2}}{(s+1)(t-s+1)^{\frac{3}{2}}} ds + \int_0^t \frac{C\|\partial_x v(s)\|_{L^2}}{(s+1)(t-s)^{\frac{3}{4}} e^{C(t-s)}} ds \\
&\le \frac{C}{t+1} + \int_0^t \frac{C}{(s+1)^{\frac{3}{2}}(t-s+1)^{\frac{3}{2}}} ds + \int_0^t \frac{C}{(s+1)^{\frac{3}{2}}(t-s)^{\frac{3}{4}} e^{C(t-s)}} ds \\
&\le \frac{C}{t+1} + \frac{C}{(t+1)^{\frac{3}{2}}} \le \frac{C}{t+1},
\end{aligned}
$$

by virtue of Lemma 2.4. Similarly, in the case $p = 3$, by choosing $\ell = m = 3$, we have

$$
\|\partial_x^3 v\|_{L^2} \le \frac{C}{(t+1)^{\frac{3}{2}}} + \int_0^t \frac{C\|\partial_x^2 v(s)\|_{L^2}}{(s+1)(t-s+1)^{\frac{3}{2}}} ds + \int_0^t \frac{C\|\partial_x^2 v(s)\|_{L^2}}{(s+1)(t-s)^{\frac{3}{4}} e^{C(t-s)}} ds \le \frac{C}{(t+1)^{\frac{3}{2}}}.
$$

In the case $p \ge 4$, we use the induction argument. We assume that $\|\partial_x^q v\|_{L^2} \le C(t+1)^{-\widetilde{\theta}_q}$ for every $q \le p-1$. By taking $m = 3$, we obtain for $\ell = 3, 4, \dots, p+2$

$$
\begin{aligned}
\|\partial_x^p v\|_{L^2} &\le \frac{C}{(t+1)^{\frac{p}{2}}} + \int_0^t \frac{C\|\partial_x^{p+2-\ell} v(s)\|_{L^2}}{(s+1)(t-s+1)^{\frac{\ell}{2}}} ds + \int_0^t \frac{C\|\partial_x^{p-1} v(s)\|_{L^2}}{(s+1)(t-s)^{\frac{3}{4}} e^{C(t-s)}} ds \\
&\le \frac{C}{(t+1)^{\frac{p}{2}}} + \frac{C}{(t+1)^{\min\{\widetilde{\theta}_{p+2-\ell}+1,\ \ell/2\}}} + \frac{C}{(t+1)^{\widetilde{\theta}_{p-1}+1}}
\end{aligned}
$$

As we have already mentioned in [8], we see that

$$
\max_{\ell=3,4,\dots,p+2} \min\left\{\widetilde{\theta}_{p+2-\ell}+1,\ \frac{\ell}{2}\right\} = \widetilde{\theta}_p, \qquad \widetilde{\theta}_p \le \min\left\{\frac{p}{2},\ \widetilde{\theta}_{p-1}+1\right\}.
$$

Then we obtain

$$
\|\partial_x^p v\|_{L^2} \le \frac{C}{(t+1)^{\widetilde{\theta}_p}},
$$

which completes the proof. □

From Lemma 3.2 and Proposition 2.2 we can construct a unique global mild solution $v \in C([0,\infty), H^6)$ for (4.2).

Next we shall give the corresponding estimate for $\eta > 0$. Similarly as above, we extend the solution globally in time by the following a priori estimates. The key of proof is to obtain the decay estimate for $\|\partial_t v(t)\|_{L^2}$ because the nonlinear term in the case $\eta > 0$ includes a derivative of v with respect to the time variable.

Lemma 3.3. *Assume that $\eta > 0$ and let $k \ge 2$. For any data $f \in H^k$, there exists a unique global mild solution $v \in C^1([0,\infty); H^k)$, and the solution decays, for $t \ge 0$,*

$$
\|\partial_x^\ell v(t)\|_{L^2} \le C_k(t+1)^{-\widetilde{\theta}_\ell}, \quad \|\partial_x^\ell \partial_t v(t)\|_{L^2} \le C_k(t+1)^{-\widetilde{\theta}_{\ell+2}} \quad (0 \le \ell \le k), \tag{3.12}
$$

where $C_k = C(\|f\|_{H^k})$.

Proof. From the energy decay estimate (Lemma 2.2) the solution has the decay

$$\|\partial_x^2 v(t)\|_{L^2} \le \frac{C}{(t+1)^{\frac{1}{2}}}, \qquad \|\partial_x v(t)\|_{L^2} \le \frac{C}{(t+1)^{\frac{1}{2}}}, \qquad \|v(t)\|_{L^2} \le C, \tag{3.13}$$

which implies the cases $\ell = 0$ and $\ell = 1$ of (3.12).

Next, we show the decay of $A(t)$. Multiplying (1.3) by $\partial_t^2 v$ yields

$$\begin{aligned} &\frac{1}{2}\partial_t A(t) + \rho\|\partial_t^2 v\|_{L^2}^2 + \kappa\partial_t \int_{\mathbb{R}} \partial_t\partial_x^2 v \partial_x^2 v dx + \alpha\partial_t \int_{\mathbb{R}} \partial_t\partial_x v \partial_x v dx + \beta\partial_t(I(v)\widetilde{I}(v)) \\ &\quad = \kappa\|\partial_t\partial_x^2 v\|_{L^2}^2 + \alpha\|\partial_t\partial_x v\|_{L^2}^2 + 2\beta\widetilde{I}(v)^2 + \beta I(v)\|\partial_t\partial_x v\|_{L^2}^2 + \gamma\eta\widetilde{I}(v)\|\partial_t\partial_x v\|_{L^2}^2. \end{aligned} \tag{3.14}$$

Integrating the resulting equation over $[0,t]$ yields

$$\begin{aligned} \frac{1}{2}A(t) \le \frac{1}{2}&A(0) + \kappa\|\partial_t\partial_x^2 v(t)\|_{L^2}\|\partial_x^2 v(t)\|_{L^2} + \kappa\|\partial_x^2 g\|_{L^2}\|\partial_x^2 f\|_{L^2} \\ &+ \alpha\|\partial_t\partial_x^2 v(t)\|_{L^2}\|v(t)\|_{L^2} + \alpha\|\partial_x g\|_{L^2}\|\partial_x f\|_{L^2} + \beta|I(v)\widetilde{I}(v)|(t) \\ &+ \beta|I(v)\widetilde{I}(v)|(0) + \kappa\int_0^t \|\partial_t\partial_x^2 v(s)\|_{L^2}^2 ds + \alpha\int_0^t \|\partial_t\partial_x^2 v(s)\|_{L^2}\|\partial_t v(s)\|_{L^2} ds \\ &+ 2\beta\int_0^t \widetilde{I}(v(s))^2 ds + \beta\int_0^t I(v(s))\|\partial_t\partial_x^2 v(s)\|_{L^2}\|\partial_t v(s)\|_{L^2} ds \\ &+ \gamma\eta\int_0^t \widetilde{I}(v(s))\|\partial_t\partial_x^2 v(s)\|_{L^2}\|\partial_t v(s)\|_{L^2} ds. \end{aligned}$$

It follows from (3.3) that

$$\int_0^t A(s)ds \le C \qquad (C \text{ being independent of } t).$$

Then, from the estimate (3.13), we have

$$A(t) \le C + C\|\partial_t\partial_x^2 v(t)\|_{L^2},$$

which implies

$$A(t) \le C \qquad (t \ge 0), \tag{3.15}$$

where the positive constant C is independent of t. We shall show that for any $t \ge 1$

$$A(t) \le \frac{C}{t+1}. \tag{3.16}$$

By the mean value theorem, there exists $\tau_3 \in [t, t+1/2]$ satisfying

$$A(\tau_3) = 2\int_t^{t+1/2} A(s)ds = 2\{E_0(t) - E_0(t+1/2)\} \le 2E_0(t) \le \frac{C}{t+1} \tag{3.17}$$

due to (3.5) and (3.2). Using (3.14) again, we have for any $\tau \in [\tau_3, t+1]$

$$
\begin{aligned}
\frac{1}{2}A(\tau) \leq \frac{1}{2}A(\tau_3) &+ \kappa \int_{\mathbb{R}} \partial_t \partial_x^2(\tau) v \partial_x^2 v(\tau) dx - \kappa \int_{\mathbb{R}} \partial_t \partial_x^2(\tau_3) v \partial_x^2 v(\tau_3) dx \\
&+ \alpha \int_{\mathbb{R}} \partial_t \partial_x v(\tau) \partial_x v(\tau) dx - \alpha \int_{\mathbb{R}} \partial_t \partial_x v(\tau_3) \partial_x v(\tau_3) dx \\
&+ \beta(I(v)\widetilde{I}(v))(\tau) - \beta(I(v)\widetilde{I}(v))(\tau_3) \\
&+ \kappa \int_{\tau_3}^{\tau} \|\partial_t \partial_x^2 v\|_{L^2}^2(s) ds + \alpha \int_{\tau_3}^{\tau} \|\partial_t \partial_x v\|_{L^2}^2(s) ds + 2\beta \int_{\tau_3}^{\tau} \widetilde{I}(v(s))^2 ds \\
&+ \beta \int_{\tau_3}^{\tau} I(v(s)) \|\partial_t \partial_x v\|_{L^2}^2(s) ds + \gamma\eta \int_{\tau_3}^{\tau} \widetilde{I}(v(s)) \|\partial_t \partial_x v\|_{L^2}^2(s) ds.
\end{aligned}
$$

Here from (3.13) we see that for any $\tau \in [t, t+1]$

$$
\begin{aligned}
\kappa \int_{\mathbb{R}} \partial_t \partial_x^2(\tau) v \partial_x^2 v(\tau) dx &\leq \epsilon \|\partial_t \partial_x^2 v(\tau)\|_{L^2}^2 + C_\epsilon \|\partial_x^2 v(\tau)\|_{L^2}^2 \leq \epsilon A(\tau) + \frac{C_\epsilon}{t+1}, \\
\kappa \int_{\mathbb{R}} \partial_t \partial_x^2(\tau_3) v \partial_x^2 v(\tau_3) dx &\leq CA(\tau_3) + \frac{C}{t+1}, \\
\alpha \int_{\mathbb{R}} \partial_t \partial_x v(\tau) \partial_x v(\tau) dx &\leq \epsilon \|\partial_t v(\tau)\|_{L^2}^2 + C_\epsilon \|\partial_x^2 v(\tau)\|_{L^2}^2 \leq \epsilon A(\tau) + \frac{C_\epsilon}{t+1}, \\
\alpha \int_{\mathbb{R}} \partial_t \partial_x v(\tau_3) \partial_x v(\tau_3) dx &\leq CA(\tau_3) + \frac{C}{t+1}, \\
|I(v)\widetilde{I}(v)|(\tau) &\leq C\|\partial_x v\|_{L^2}^2 \|\partial_t v\|_{L^2} \|\partial_x^2 v\|_{L^2}(\tau) \leq \epsilon A(\tau) + \frac{C_\epsilon}{(t+1)^3}, \\
|I(v)\widetilde{I}(v)|(\tau_3) &\leq C\|\partial_t v(\tau_3)\|_{L^2}^2 + C\|\partial_x v(\tau_3)\|_{L^2}^4 \|\partial_x^2 v(\tau_3)\|_{L^2}^2 \leq CA(\tau_3) + \frac{C}{(t+1)^3}, \\
\kappa \int_{\tau_3}^{\tau} \|\partial_t \partial_x^2 v\|_{L^2}^2(s) ds &\leq \int_t^{\tau} A(s) ds \leq C(E_0(t) - E_0(\tau)) \leq CE_0(t) \leq \frac{C}{t+1}, \\
\alpha \int_{\tau_3}^{\tau} \|\partial_t \partial_x v\|_{L^2}^2(s) ds &\leq \alpha \int_{\tau_3}^{\tau} \|\partial_t \partial_x^2 v\|_{L^2}(s) \|\partial_t v\|_{L^2}(s) ds \leq C \int_t^{\tau} A(s) ds \leq \frac{C}{t+1}, \\
2\beta \int_{\tau_3}^{\tau} \widetilde{I}(v(s))^2 ds &\leq C \int_{\tau_3}^{\tau} \|\partial_x^2 v(s)\|_{L^2}^2 \|\partial_t v(x)\|_{L^2}^2 ds \leq \frac{C}{t+1} \int_t^{\tau} A(s) ds \leq \frac{C}{(t+1)^2},
\end{aligned}
$$

and from $I(v(t)) \leq C/(t+1)$ and $|\widetilde{I}(v(t))| \leq C$ (see (3.15)) that

$$
\begin{aligned}
\beta \int_{\tau_3}^{\tau} I(v(s)) \|\partial_t \partial_x v\|_{L^2}^2(s) ds &\leq \frac{C}{t+1} \int_t^{\tau} A(s) ds \leq \frac{C}{(t+1)^2}, \\
\gamma\eta \int_{\tau_3}^{\tau} |\widetilde{I}(v(s))| \|\partial_t \partial_x v\|_{L^2}^2(s) ds &\leq C \int_t^{\tau} \|\partial_t \partial_x v\|_{L^2}^2(s) ds \leq \frac{C}{t+1}.
\end{aligned}
$$

Consequently, from (3.17) we obtain for any $\tau \in [\tau_3, t+1]$

$$
A(\tau) \leq CA(\tau_3) + \frac{C}{t+1} \leq \frac{C}{t+1},
$$

which implies

$$
A(\tau) \leq \frac{C}{t+1} \qquad (\tau \in [t+1/2, t+1]).
$$

By replacing t by $t-\frac{1}{2}$, we also obtain

$$A(\tau) \le \frac{C}{t+1/2} \le \frac{C}{t+1} \qquad (\tau \in [t, t+1/2]),$$

with the help of $1/(t+1/2) \le 2/(t+1)$. Thus we conclude that

$$\sup_{s\in[t,t+1]} A(s) \le \frac{C}{t+1},$$

which implies (3.16). Then the decay rate for $\widetilde{I}$ can be improved to

$$|\widetilde{I}(t)| \le \|\partial_t u(t)\|_{L^2} \|\partial_x^2 u(t)\|_{L^2} \le \frac{C}{t+1}, \tag{3.18}$$

and hence $|I(t)| + |\widetilde{I}(t)| \le C(t+1)^{-1}$.

Next, we show higher-order estimates. With the same arguments as in the proof of Lemma 2.1, we easily show the following (linear) estimates: for any integer $n \in [0, 4]$,

$$\left\|\partial_x^k(\delta+\eta\partial_x^4)^{-1}\mathcal{K}(t)f\right\|_{L^2} \le \frac{C}{(t+1)^{\frac{\ell}{2}}}\|\partial_x^{k-\ell} f\|_{L^2} + Ce^{-Ct}\|\partial_x^{k-n} f\|_{L^2}, \tag{3.19}$$

$$\left\|\partial_x^k(\delta+\eta\partial_x^4)^{-1}\partial_t\mathcal{K}(t)f\right\|_{L^2} \le \frac{C}{(t+1)^{\frac{\ell}{2}+1}}\|\partial_x^{k-\ell} f\|_{L^2} + Ce^{-Ct}\|\partial_x^{k-n} f\|_{L^2}. \tag{3.20}$$

Then from Lemma 2.1, (3.18) and these estimates we have

$$\begin{aligned}
\|\partial_x^2 v\|_{L^2} &\le \|\partial_x^2\mathcal{K}(t)f\|_{L^2} + \int_0^t \frac{C}{s+1}\left\|\partial_x^2(\delta+\eta\partial_x^4)^{-1}\mathcal{K}(t-s)\partial_x^2 v(s)\right\|_{L^2} ds \\
&\le \frac{C}{t+1}\|f\|_{H^2} + \int_0^t \frac{C}{s+1}\left\{\frac{\|\partial_x^{4-\ell} v(s)\|_{L^2}}{(t-s+1)^{\frac{\ell}{2}}} + \frac{\|\partial_x^{4-n} v(s)\|_{L^2}}{e^{C(t-s)}}\right\} ds.
\end{aligned}$$

Taking $\ell = n = 3$, we obtain

$$\begin{aligned}
\|\partial_x^2 v\|_{L^2} &\le \frac{C}{t+1} + \int_0^t \frac{C\|\partial_x v(s)\|_{L^2}}{(s+1)(t-s+1)^{\frac{3}{2}}} ds \\
&\le \frac{C}{t+1} + \int_0^t \frac{C}{(s+1)^{\frac{3}{2}}(t-s+1)^{\frac{3}{2}}} ds \\
&\le \frac{C}{t+1} + \frac{C}{(t+1)^{\frac{3}{2}}} \le \frac{C}{t+1},
\end{aligned}$$

by virtue of Lemma 2.4. In a similar manner, by taking $\ell = n = 1$, we have

$$\begin{aligned}
\|\partial_t v(t)\|_{L^2} &\le \|\partial_t\mathcal{K}(t)f\|_{L^2} + \frac{C}{t+1}\left\|(\delta+\eta\partial_x^4)^{-1}\partial_x^2 v(t)\right\|_{L^2} \\
&\quad + \int_0^t \frac{C}{s+1}\left\|\partial_t(\delta+\eta\partial_x^4)^{-1}\mathcal{K}(t-s)\partial_x^2 v(s)\right\|_{L^2} ds \\
&\le \frac{C}{t+1}\|f\|_{L^2} + \frac{C}{t+1}\|\partial_x^2 v(t)\|_{L^2} + \int_0^t \frac{C}{s+1}\left\{\frac{\|\partial_x v(s)\|_{L^2}}{(t-s+1)^{\frac{1}{2}+1}} + \frac{\|\partial_x v(s)\|_{L^2}}{e^{C(t-s)}}\right\} ds \\
&\le \frac{C}{t+1} + \frac{C}{(t+1)^2} + \frac{C}{(t+1)^{\frac{3}{2}}} \le \frac{C}{t+1}.
\end{aligned}$$

In the case $k \geq 3$, by the same argument as Lemma 3.2, we see that $\|\partial_x^k v(t)\|_{L^2} \leq C(t+1)^{-\widetilde{\theta}_k}$. Lastly we show the second part of (3.12). It follows that

$$\frac{C}{t+1}\left\|(\delta+\eta\partial_x^4)^{-1}\partial_x^{k+2}v(t)\right\|_{L^2} \leq \frac{C}{t+1}\left\|\partial_x^k v(t)\right\|_{L^2} \leq \frac{C}{(t+1)^{\widetilde{\theta}_k+1}},$$

and from (3.19) that

$$\begin{aligned}
&\int_0^t \frac{C}{s+1}\left\|\partial_x^k\partial_t(\delta+\eta\partial_x^4)^{-1}\mathcal{K}(t-s)\partial_x^2 v(s)\right\|_{L^2} ds \\
&\leq \int_0^t \frac{C}{s+1}\left(\frac{\|\partial_x^{k+2-\ell}u(s)\|_{L^2}}{(t-s+1)^{\frac{\ell}{2}+1}} + \frac{\|\partial_x^k u(s)\|_{L^2}}{e^{C(t-s)}}\right) ds \\
&\leq \int_0^t \frac{C}{(s+1)^{\widetilde{\theta}_{k+2-\ell}+1}(t-s+1)^{\frac{\ell}{2}+1}} ds + \int_0^t \frac{C}{(s+1)^{\widetilde{\theta}_k+1}e^{C(t-s)}} ds
\end{aligned}$$

for $\ell = 3, 4, \ldots, k+2$. Then we obtain

$$\begin{aligned}
\|\partial_x^k\partial_t v(t)\|_{L^2} &\leq \frac{C}{(t+1)^{\frac{k}{2}+1}} + \frac{C}{(t+1)^{\widetilde{\theta}_k+1}} + \frac{C}{(t+1)^{\widetilde{\theta}_{k+2}}} + \frac{C}{(t+1)^{\widetilde{\theta}_k+1}} \\
&\leq \frac{C}{(t+1)^{\widetilde{\theta}_{k+2}}},
\end{aligned}$$

where we have used the facts that $\max_{\ell=3,4,\ldots,k+2}\min\left\{\widetilde{\theta}_{k+2-\ell}+1,\ \frac{\ell}{2}+1\right\}$ and that $\widetilde{\theta}_{k+2} \leq \widetilde{\theta}_k + 1$. This completes the proof. □

Now, similarly to the previous subsection, the local mild solution constructed in Proposition 2.2 can be extended to a global one.

4 Singular limit problem

In this section we prove Theorem 1.2. We present a convergence estimate which is global in time and uniform in ρ by a simple proof assuming the compatibility conditions for the data. We discuss the cases $\eta = 0$ and $\eta > 0$ separately.

4.1 The equation with frictional damping and $\eta = 0$

In this subsection we compare the solution to (1.1) with $\eta = 0$,

$$\begin{cases} \rho\partial_t^2 u + \delta\partial_t u + \kappa\partial_x^4 u = \left(\alpha + \beta\int_{\mathbb{R}}|\partial_x u|^2 dx\right)\partial_x^2 u, \\ u(0,x) = f(x), \qquad \partial_t u(0,x) = g(x), \end{cases} \tag{4.1}$$

with the solution to (1.3) with $\eta = 0$:

$$\begin{cases} \delta\partial_t v + \kappa\partial_x^4 v = \left(\alpha + \beta\int_{\mathbb{R}}|\partial_x v|^2 dx\right)\partial_x^2 v, \\ v(0,x) = f(x), \end{cases} \tag{4.2}$$

under the compatibility condition $g = \lim_{t\to+0}\partial_t v(t,\cdot)$. For the reader's convenience we restate Theorem 1.2 in the case $\eta = 0$.

Theorem 4.1 (Singular limit)**.** *Consider the case $\eta = 0$. Let v be a solution to (4.2) for $f \in H^6$ and define $g := \lim_{t\to+0} \partial_t v(t,\cdot)$ then we have the following global in time singular limit estimate:*

$$\|u - v\|_{L^\infty H^2} \le C\rho,$$

where u is a solution to (4.1) for the data (f, g) and $C = C(\|f\|_{H^6})$.

We remark that the fact $g \in H^2$ is assured by Proposition 2.3 thanks to the assumption $f \in H^6$. We use the following space-time a priori estimates in our proof of the singular limit result.

Lemma 4.2 (Higher-order estimates)**.** *The solution v to (4.2) with $f \in H^6$ satisfies for some $C = C(\|g\|_{L^2})$ that*

$$\|\partial_t v\|^2_{L^\infty L^2} + \|\partial_t\partial_x^2 v\|^2_{L^2L^2} + \|\partial_t\partial_x v\|^2_{L^2L^2} + \int_0^\infty \left(\int \partial_t\partial_x v\partial_x v dx\right)^2 dt \le C,$$

and for some $C = C(\|f\|_{H^2}, \|g\|_{H^2})$ that

$$\|\partial_t\partial_x^2 v\|^2_{L^\infty L^2} + \|\partial_t\partial_x v\|^2_{L^\infty L^2} + \|\partial_t^2 v\|^2_{L^2L^2} + \|\partial_t\partial_x^4 v\|^2_{L^2L^2} + \|\partial_t\partial_x^3 v\|^2_{L^2L^2}. \le C.$$

Proof. Differentiating the equation (4.2) with respect to the time variable yields

$$\delta\partial_t^2 v + \kappa\partial_t\partial_x^4 v - \alpha\partial_t\partial_x^2 v - \beta\partial_t(I(v)\partial_x^2 v) = 0. \tag{4.3}$$

Multiplying $\partial_t v$ to the resulting equation, we have

$$\begin{aligned}\frac{\delta}{2}\|\partial_t v\|^2_{L^\infty L^2} + \kappa\|\partial_t\partial_x^2 v\|^2_{L^2L^2} + \alpha\|\partial_t\partial_x v\|^2_{L^2L^2} + \beta\int_0^\infty \|\partial_x v\|^2_{L^2}\|\partial_t\partial_x v\|^2_{L^2} dt\\ + 2\beta\int_0^\infty \left(\int \partial_t\partial_x v\partial_x v dx\right)^2 dt = \frac{\delta}{2}\|g\|^2_{L^2},\end{aligned} \tag{4.4}$$

which is the first assertion.

Next, multiplying (4.3) by $\partial_t^2 v + \partial_t\partial_x^4 v$ yields

$$\begin{aligned}&\partial_t\left(\left(\frac{\kappa}{2} + \frac{\delta}{2}\right)\|\partial_t\partial_x^2 v\|^2_{L^2} + \frac{\alpha}{2}\|\partial_t\partial_x v\|^2_{L^2}\right) + \delta\|\partial_t^2 v\|^2_{L^2} + \kappa\|\partial_t\partial_x^4 v\|^2_{L^2} + \alpha\|\partial_t\partial_x^3 v\|^2_{L^2}\\ &= \beta\int \partial_t\left(\|\partial_x v\|^2_{L^2}\partial_x^2 v\right)\cdot(\partial_t^2 v + \partial_t\partial_x^4 v)dx\\ &\le \frac{\delta}{2}\|\partial_t^2 v\|^2_{L^2} + \frac{\kappa}{2}\|\partial_t\partial_x^4 v\|^2_{L^2} + \left(\frac{1}{2\delta} + \frac{1}{2\kappa}\right)\beta^2\|\partial_t\left(\|\partial_x v\|^2_{L^2}\partial_x^2 v\right)\|^2_{L^2}.\end{aligned} \tag{4.5}$$

We already know that $0 \le \|\partial_x v\|^2_{L^2} \le C_2$, for some $C_2 = C_2(\|f\|_{H^2})$ being independent of t (see (3.2)), implying

$$\int_0^\infty (\partial_t\|\partial_x v\|^2_{L^2})^2 dt \le 4\int_0^\infty \|\partial_t\partial_x v\|^2_{L^2}\|\partial_x v\|^2_{L^2} dt \le \frac{2\delta}{\alpha}C_2\|g\|^2_{L^2}, \tag{4.6}$$

due to (4.4). Then we have

$$\begin{aligned}\int_0^\infty \|\partial_t \left(\|\partial_x v\|_{L^2}^2 \partial_x^2 v\right)\|_{L^2}^2 dt &\le \int_0^\infty \left\|\partial_t \|\partial_x v\|_{L^2}^2 \cdot \partial_x^2 v + \|\partial_x v\|_{L^2}^2 \partial_t \partial_x^2 v\right\|_{L^2}^2 dt \\ &\le 2\int_0^\infty \left(\partial_t \|\partial_x v\|_{L^2}^2\right)^2 \|\partial_x^2 v\|_{L^2}^2 dt + 2\int_0^\infty \|\partial_x v\|_{L^2}^4 \|\partial_t \partial_x^2 v\|_{L^2}^2 dt \\ &\le 2\|\partial_x^2 v\|_{L^\infty L^2}^2 \int_0^\infty (\partial_t \|\partial_x v\|_{L^2}^2)^2 dt + 2\|\partial_x v\|_{L^\infty L^2}^4 \|\partial_t \partial_x^2 v\|_{L^2 L^2}^2 \\ &\le \left(\frac{4}{\alpha} + \frac{1}{\kappa}\right) \delta C_2^2 \|g\|_{L^2}^2.\end{aligned}$$

By integrating (4.5) over t, we have

$$\begin{aligned}&\left(\frac{\kappa}{2} + \frac{\delta}{2}\right) \|\partial_t \partial_x^2 v\|_{L^\infty L^2}^2 + \frac{\alpha}{2}\|\partial_t \partial_x v\|_{L^\infty L^2}^2 + \frac{\delta}{2}\|\partial_t^2 v\|_{L^2 L^2}^2 \\ &+ \frac{\kappa}{2}\|\partial_t \partial_x^4 v\|_{L^2 L^2}^2 + \alpha \|\partial_t \partial_x^3 v\|_{L^2 L^2}^2 \\ &\le \left(\frac{\kappa}{2} + \frac{\delta}{2}\right) \|\partial_x^2 g\|_{L^2}^2 + \frac{\alpha}{2}\|\partial_x g\|_{L^2}^2 + \left(\frac{1}{2\delta} + \frac{1}{2\kappa}\right)\left(\frac{4}{\alpha} + \frac{1}{\kappa}\right) \beta^2 \delta C_2^2 \|g\|_{L^2}^2.\end{aligned} \tag{4.7}$$

□

In the procedure of the proof of Proposition 2.3, we easily see that $\|g\|_{H^k} \le C(\|f\|_{H^{k+4}})$. Then the constant of the second assertion in the above lemma is rewritten as $C = C(\|f\|_{H^6})$.

Proof of Theorem 4.1. Let us denote by u the solution to

$$\rho \partial_t^2 u + \delta \partial_t u + \kappa \partial_x^4 u - (\alpha + \beta I(u))\, \partial_x^2 u = 0, \tag{4.8}$$

and by v the solution to

$$\delta \partial_t v + \kappa \partial_x^4 v - (\alpha + \beta I(v))\, \partial_x^2 v = 0. \tag{4.9}$$

Calculating $((4.9) + \frac{\rho}{\delta}\partial_t(4.9))$ yields

$$\rho \partial_t^2 v + \delta \partial_t v + \kappa \partial_x^4 v - \alpha \partial_x^2 v - \beta I(v) \partial_x^2 v = -\frac{\rho\kappa}{\delta}\partial_t \partial_x^4 v + \frac{\rho\alpha}{\delta}\partial_t \partial_x^2 v + \frac{\rho\beta}{\delta}\partial_t (I(v)\partial_x^2 v). \tag{4.10}$$

We set $w = u - v$. Subtracting (4.8) from (4.10), we see

$$\begin{aligned}\rho \partial_t^2 w + \delta \partial_t w + \kappa \partial_x^4 w - \alpha \partial_x^2 w &= \frac{\rho\kappa}{\delta}\partial_t \partial_x^4 v - \frac{\rho\alpha}{\delta}\partial_t \partial_x^2 v \\ &\quad + \beta (I(u)\partial_x^2 u - I(v)\partial_x^2 v) - \frac{\rho\beta}{\delta}\partial_t (I(v)\partial_x^2 v).\end{aligned} \tag{4.11}$$

Multiplying (4.11) by $\partial_t w$ yields

$$\begin{aligned}&\partial_t \left(\frac{\rho}{2}\|\partial_t w\|_{L^2}^2 + \frac{\kappa}{2}\|\partial_x^2 w\|_{L^2}^2 + \frac{\alpha}{2}\|\partial_x w\|_{L^2}^2\right) + \delta \|\partial_t w\|_{L^2}^2 = \frac{\rho\kappa}{\delta}\int_{\mathbb{R}} \partial_t \partial_x^4 v \partial_t w dx \\ &- \frac{\rho\alpha}{\delta}\int_{\mathbb{R}} \partial_t \partial_x^2 v \partial_t w dx + \beta \int_{\mathbb{R}} \left(I(u)\partial_x^2 u - I(v)\partial_x^2 v\right) \partial_t w dx - \frac{\rho\beta}{\delta}\int_{\mathbb{R}} \partial_t (I(v)\partial_x^2 v)\partial_t w dx.\end{aligned}$$

Since

$$\|\partial_x u\|_{L^2}^2 - \|\partial_x v\|_{L^2}^2 \le \|\partial_x w\|_{L^2} \left(\|\partial_x u\|_{L^2} + \|\partial_x v\|_{L^2}\right),$$

we have

$$\begin{aligned}
&\int_{\mathbb{R}} \left(I(u)\partial_x^2 u - I(v)\partial_x^2 v\right) \partial_t w dx \\
&= \int_{\mathbb{R}} \|\partial_x u\|_{L^2}^2 \partial_x^2 w \partial_t w dx + \int_{\mathbb{R}} \left(\|\partial_x u\|_{L^2}^2 - \|\partial_x v\|_{L^2}^2\right) \partial_x^2 v \partial_t w dx \\
&\le \epsilon \|\partial_t w\|_{L^2}^2 + C_\epsilon \|\partial_x u\|_{L^2}^4 \|\partial_x^2 w\|_{L^2}^2 \\
&\qquad + C_\epsilon \left(\|\partial_x u\|_{L^2}^2 + \|\partial_x v\|_{L^2}^2\right) \|\partial_x^2 v\|_{L^2}^2 \|\partial_x w\|_{L^2}^2 \\
&\le \epsilon \|\partial_t w\|_{L^2}^2 + \frac{C_\epsilon}{(t+1)^2} \left(\|\partial_x^2 w\|_{L^2}^2 + \|\partial_x w\|_{L^2}^2\right),
\end{aligned}$$

due to decay estimates given in Theorem 1.1 and Lemma 3.2. On the other hand, from the higher-order regularity estimate (Lemma 4.2) for the solution v of (4.9), we have

$$\rho \int_0^t \int_{\mathbb{R}} \partial_t w \partial_t \partial_x^4 v dx ds \le \epsilon \|\partial_t w\|_{L_t^2 L^2}^2 + \frac{1}{4\epsilon} \rho^2 \|\partial_t \partial_x^4 v\|_{L^2 L^2}^2 \le \epsilon \|\partial_t w\|_{L_t^2 L^2}^2 + C_\epsilon \rho^2,$$

$$\rho \int_0^t \int_{\mathbb{R}} \partial_t (I(v)\partial_x^2 v) \partial_t w dx ds \le \epsilon \|\partial_t w\|_{L_t^2 L^2}^2 + \frac{1}{4\epsilon} \rho^2 \|\partial_t (I(v)\partial_x^2 v)\|_{L^2 L^2}^2 \le \epsilon \|\partial_t w\|_{L^2 L^2}^2 + C_\epsilon \rho^2,$$

where in the second line we have used

$$\begin{aligned}
\|\partial_t (I(v)\partial_x^2 v)\|_{L^2 L^2}^2 &\le 2\|\partial_t I(v) \partial_x^2 v\|_{L^2 L^2}^2 + 2\|I(v)\partial_t \partial_x^2 v\|_{L^2 L^2}^2 \\
&\le 8 \int_0^\infty \left(\int \partial_t \partial_x v \partial_x v dx\right)^2 dt \|\partial_x^2 v\|_{L^\infty L^2}^2 + 2\|\partial_x v\|_{L^\infty L^2}^4 \|\partial_t \partial_x^2 v\|_{L^2 L^2}^2 \\
&\le 8 \|\partial_t \partial_x^2 v\|_{L^2 L^2}^2 \|v\|_{L^\infty L^2}^2 \|\partial_x^2 v\|_{L^\infty L^2}^2 + 2\|\partial_x v\|_{L^\infty L^2}^4 \|\partial_t \partial_x^2 v\|_{L^2 L^2}^2 \\
&\le C(\|f\|_{H^2}, \|g\|_{L^2}),
\end{aligned}$$

with the help of (3.4). Consequently, setting

$$D_\rho(t) := \frac{\rho}{2} \|\partial_t w\|_{L^2}^2 + \frac{\kappa}{2} \|\partial_x^2 w\|_{L^2}^2 + \frac{\alpha}{2} \|\partial_x w\|_{L^2}^2 \tag{4.12}$$

and choosing ϵ small and absorbing $3\epsilon \|\partial_t w\|_{L_t^2 L^2}^2$ to the left-hand side, we have

$$D_\rho(t) \le \int_0^t \frac{C}{(s+1)^2} D_\rho(s) ds + C\rho^2.$$

By the Gronwall lemma (see e.g. [2]) we obtain

$$\begin{aligned}
D_\rho(t) &\le C\rho^2 \exp\left(C \int_0^t (s+1)^{-2} ds\right) \\
&= C\rho^2 \exp(C - C(t+1)^{-1}) \\
&\le C\rho^2.
\end{aligned}$$

Therefore we conclude

$$\sup_t D_\rho(t) \le C\rho^2.$$

□

4.2 The equation with Kelvin-Voigt damping $\eta > 0$

In this subsection we consider the problem with $\eta > 0$, that is, the comparison between the solution to (1.1) and the one to (1.3) under the compatibility condition $g = \lim_{t\to+0} \partial_t v(t)$. In this subsection we shall show the following theorem.

Theorem 4.3 (Singular limit, $\eta > 0$)**.** *Let v be a solution for* (1.3) *with $f \in H^2$. We define $g = \lim_{t\to+0} \partial_t v(t,\cdot)$. Let u be a solution for* (1.1) *with (f,g). Then we have the following global in time singular limit estimate:*

$$\|u - v\|_{L^\infty H^2} \le C\rho.$$

In the case $\eta > 0$, from the assumption $f \in H^2$ we have shown the existence of a local solution for (1.3) satisfying $\partial_t v \in C([0,T], H^2)$. Then $g \in H^2$ is automatically satisfied. Compared to Theorem 4.1, we do not need any regularity assumption for g in the Kelvin-Voigt problem.

We also use the following space-time estimates in our proof of the singular limit result.

Lemma 4.4 (Higher-order estimates)**.** *Let $f \in H^2$ and $g \in H^2$. We assume $g = \lim_{t\to+0} \partial_t v(t,\cdot)$. Then there exists a constant $C = C(\|f\|_{H^2})$ such that*

$$\|\partial_t v\|^2_{L^2L^2} + \|\partial_t\partial_x^2 v\|^2_{L^2L^2} + \int_0^\infty \left(\int \partial_t\partial_x v \partial_x v dx\right)^2 dt \le C,$$

and a constant $C = C(\|f\|_{H^2}, \|g\|_{H^2})$ such that

$$\|\partial_t v\|^2_{L^\infty L^2} + \|\partial_t\partial_x^2 v\|^2_{L^\infty L^2} + \|\partial_t\partial_x v\|^2_{L^\infty L^2} + \|\partial_t^2 v\|^2_{L^2L^2} + \|\partial_t^2\partial_x^2 v\|^2_{L^2L^2} + \sup_t |\widetilde{I}(v)|^2 \le C.$$

Proof. From (3.5) we conclude

$$\delta\|\partial_t v\|^2_{L^2L^2} + \eta\|\partial_t\partial_x^2 v\|^2_{L^2L^2} + \gamma\eta\int_0^\infty |\widetilde{I}(v)(t)|^2 dt \le \widetilde{k},$$

where $\widetilde{k} := \frac{\kappa}{2}\|\partial_x^2 f\|^2_{L^2} + \frac{\alpha}{2}\|\partial_x f\|^2_{L^2} + \frac{\beta}{4}\|\partial_x f\|^4_{L^2}$. This implies the first assertion. Differentiating the equation with respect to the time variable yields

$$\delta\partial_t^2 v + \eta\partial_t^2\partial_x^4 v + \kappa\partial_t\partial_x^4 v - \alpha\partial_t\partial_x^2 v - \partial_t(\beta I(v)\partial_x^2 v + \gamma\eta\widetilde{I}(v)\partial_x^2 v) = 0. \tag{4.13}$$

Multiplying $\partial_t v$ to the resulting equation, we have

$$\begin{aligned}
&\partial_t\left(\frac{\delta}{2}\|\partial_t v\|^2_{L^2} + \frac{\eta}{2}\|\partial_t\partial_x^2 v\|^2_{L^2} + \frac{\gamma\eta}{2}(\widetilde{I}(v))^2\right)\\
&\quad + \kappa\|\partial_t\partial_x^2 v\|^2_{L^2} + \alpha\|\partial_t\partial_x v\|^2_{L^2} + \beta I(v)\|\partial_t\partial_x v\|^2_{L^2} + 2\beta(\widetilde{I}(v))^2 = -\gamma\eta\widetilde{I}(v)\|\partial_t\partial_x v\|^2_{L^2}\\
&\quad \le \frac{\eta\gamma^2}{2\epsilon}(\widetilde{I}(v))^2 + \frac{\eta}{2}\epsilon\|\partial_t v\|^2_{L^2}\|\partial_t\partial_x^2 v\|^2_{L^2},
\end{aligned}$$

since

$$\|\partial_t\partial_x v\|^2_{L^2} \le \|\partial_t v\|_{L^2}\|\partial_t\partial_x^2 v\|_{L^2}. \tag{4.14}$$

By integrating over t, we obtain

$$
\begin{aligned}
&\frac{\delta}{2}\|\partial_t v\|_{L^\infty L^2}^2 + \frac{\eta}{2}\|\partial_t\partial_x^2 v\|_{L^\infty L^2}^2 + \sup_t \frac{\gamma\eta}{2}(\widetilde{I}(v))^2 \\
&\quad\le \frac{\delta}{2}\|g\|_{L^2}^2 + \frac{\eta}{2}\|\partial_x^2 g\|_{L^2}^2 + \frac{\gamma\eta}{2}\|\partial_x^2 g\|_{L^2}^2\|f\|_{L^2}^2 \\
&\qquad + \frac{\eta\gamma^2}{2\epsilon}\int_0^\infty (\widetilde{I}(v))^2 dt + \frac{\eta}{2}\epsilon\|\partial_t v\|_{L^2L^2}^2\|\partial_t\partial_x^2 v\|_{L^\infty L^2}^2 \\
&\quad\le \frac{\delta}{2}\|g\|_{L^2}^2 + \frac{\eta}{2}\|\partial_x^2 g\|_{L^2}^2 + \frac{\gamma\eta}{2}\|\partial_x^2 g\|_{L^2}^2\|f\|_{L^2}^2 + \frac{\gamma}{2\epsilon}\widetilde{k} + \frac{\eta}{2\delta}\widetilde{k}\epsilon\|\partial_t\partial_x^2 v\|_{L^\infty L^2}^2.
\end{aligned}
$$

Choosing ϵ satisfying $\widetilde{k}\epsilon/\delta = 1/2$, we have

$$
\frac{\delta}{2}\|\partial_t v\|_{L^\infty L^2}^2 + \frac{\eta}{4}\|\partial_t\partial_x^2 v\|_{L^\infty L^2}^2 + \sup_t \frac{\gamma\eta}{2}(\widetilde{I}(v))^2 \le C(\|f\|_{H^2}, \|g\|_{H^2}). \tag{4.15}
$$

Next, multiplying $\partial_t^2 v$ to (4.13) yields

$$
\begin{aligned}
&\partial_t\left(\frac{\kappa}{2}\|\partial_t\partial_x^2 v\|_{L^2}^2 + \frac{\alpha}{2}\|\partial_t\partial_x v\|_{L^2}^2\right) + \delta\|\partial_t^2 v\|_{L^2}^2 + \eta\|\partial_t^2\partial_x^2 v\|_{L^2}^2 \\
&\quad = \beta\int \partial_t\left(I(v)\partial_x^2 v\right)\cdot\partial_t^2 v dx + \gamma\eta\int \partial_t\left(\widetilde{I}(v)\partial_x^2 v\right)\cdot\partial_t^2 v dx.
\end{aligned} \tag{4.16}
$$

Here, since $\partial_t\widetilde{I}(v) = -\int \partial_x^2 v\partial_t^2 v dx + \|\partial_t\partial_x v\|_{L^2}^2$, we have

$$
\begin{aligned}
&\gamma\eta\int \partial_t\left(\widetilde{I}(v)\partial_x^2 v\right)\cdot\partial_t^2 v dx \\
&\quad = \gamma\eta\partial_t\widetilde{I}(v)\int \partial_x^2 v\partial_t^2 v dx + \gamma\eta\widetilde{I}(v)\int \partial_t\partial_x^2 v\partial_t^2 v dx \\
&\quad \le \gamma\eta\|\partial_t\partial_x v\|_{L^2}^2\|\partial_t^2 v\|_{L^2}\|\partial_x^2 v\|_{L^2} + \gamma\eta|\widetilde{I}(v)|\|\partial_t\partial_x^2 v\|_{L^2}\|\partial_t^2 v\|_{L^2} \\
&\quad \le \epsilon\|\partial_t^2 v\|_{L^2}^2 + \frac{1}{2\epsilon}(\gamma\eta)^2\|\partial_x^2 v\|_{L^2}^2\|\partial_t v\|_{L^2}^2\|\partial_t\partial_x^2 v\|_{L^2}^2 + \frac{1}{2\epsilon}(\gamma\eta)^2|\widetilde{I}(v)|^2\|\partial_t\partial_x^2 v\|_{L^2}^2,
\end{aligned}
$$

where we have used (4.14) again. Then it follows from (4.15) that

$$
\begin{aligned}
\gamma\eta\int_0^\infty\int \partial_t\left(\widetilde{I}(v)\partial_x^2 v\right)\cdot\partial_t^2 v dx dt &\le \epsilon\|\partial_t^2 v\|_{L^2L^2}^2 + C_\epsilon\|\partial_x^2 v\|_{L^\infty L^2}^2\|\partial_t v\|_{L^\infty L^2}^2\|\partial_t\partial_x^2 v\|_{L^2L^2} \\
&\qquad + C_\epsilon\sup_t|\widetilde{I}(v)|^2\|\partial_t\partial_x^2 v\|_{L^2L^2} \\
&\le \epsilon\|\partial_t^2 v\|_{L^2L^2}^2 + C_\epsilon(\|f\|_{H^2}, \|g\|_{H^2}).
\end{aligned}
$$

As the same as the proof of Lemma 4.2, we see that

$$
\begin{aligned}
\int_0^\infty \|\partial_t\left(\|\partial_x v\|_{L^2}^2\partial_x^2 v\right)\|_{L^2}^2 dt &\le C\|\partial_t\partial_x^2 v\|_{L^2L^2}^2\|v\|_{L^\infty L^2}^2\|\partial_x^2 v\|_{L^\infty L^2}^2 + C\|\partial_x v\|_{L^\infty L^2}^4\|\partial_t\partial_x^2 v\|_{L^2L^2} \\
&\le C \qquad (\text{ independent of } t).
\end{aligned}
$$

By integrating (4.16) over t, we have

$$
\|\partial_t\partial_x v\|_{L^\infty L^2} + \|\partial_t^2 v\|_{L^2L^2} + \|\partial_t^2\partial_x^2 v\|_{L^2L^2} \le C(\|f\|_{H^2}, \|g\|_{H^2}). \tag{4.17}
$$

From (4.15) and (4.17) the second assertion is proved. □

Proof of Theorem 4.3. Let us denote by u the solution to

$$\rho\partial_t^2 u + \delta\partial_t u + \eta\partial_t\partial_x^4 u + \kappa\partial_x^4 u - \left(\alpha + \beta I(u) + \gamma\eta\widetilde{I}(u)\right)\partial_x^2 u = 0, \tag{4.18}$$

and by v the solution of

$$\delta\partial_t v + \eta\partial_t\partial_x^4 v + \kappa\partial_x^4 v - \left(\alpha + \beta I(v) + \gamma\eta\widetilde{I}(v)\right)\partial_x^2 v = 0. \tag{4.19}$$

Calculating $((4.19) + \rho\partial_t(4.19))$ yields

$$\begin{aligned} \rho\partial_t^2 v &+ \delta\partial_t v + \eta\partial_t\partial_x^4 v + \kappa\partial_x^4 v - \left(\alpha + \beta I(v) + \gamma\eta\widetilde{I}(v)\right)\partial_x^2 v \\ &= -\frac{\rho\kappa}{\delta}\partial_t\partial_x^4 v - \frac{\rho\eta}{\delta}\partial_t^2\partial_x^4 v + \frac{\rho\alpha}{\delta}\partial_t\partial_x^2 v + \frac{\rho}{\delta}\partial_t\left\{(\beta I(v) + \gamma\eta\widetilde{I}(v))\partial_x^2 v\right\}. \end{aligned} \tag{4.20}$$

We set $w = u - v$. Subtracting (4.18) to (4.20), we see

$$\begin{aligned} &\rho\partial_t^2 w + \delta\partial_t w + \eta\partial_t\partial_x^4 w + \kappa\partial_x^4 w - \alpha\partial_x^2 w \\ &\quad = \beta(I(u)\partial_x^2 u - I(v)\partial_x^2 v) + \gamma\eta(\widetilde{I}(u)\partial_x^2 u - \widetilde{I}(v)\partial_x^2 v) \\ &\qquad + \frac{\rho\kappa}{\delta}\partial_t\partial_x^4 v + \frac{\rho\eta}{\delta}\partial_t^2\partial_x^4 v - \frac{\rho\alpha}{\delta}\partial_t\partial_x^2 v - \frac{\rho}{\delta}\partial_t\left\{(\beta I(v) + \gamma\eta\widetilde{I}(v))\partial_x^2 v\right\}. \end{aligned} \tag{4.21}$$

Multiplying $\partial_t w$ to (4.21) yields

$$\begin{aligned} &\partial_t\left(\frac{\rho}{2}\|\partial_t w\|_{L^2}^2 + \frac{\kappa}{2}\|\partial_x^2 w\|_{L^2}^2 + \frac{\alpha}{2}\|\partial_x w\|_{L^2}^2\right) + \delta\|\partial_t w\|_{L^2}^2 + \eta\|\partial_t\partial_x^2 w\|_{L^2}^2 \\ &\quad = \int_{\mathbb{R}} (\text{right-hand side of (4.21)}) \cdot \partial_t w dx. \end{aligned}$$

Observe that

$$\int (\widetilde{I}(u)\partial_x^2 u - \widetilde{I}(v)\partial_x^2 v)\partial_t w dx = \int \widetilde{I}(u)\partial_x^2 w\partial_t w dx + \int (\widetilde{I}(u) - \widetilde{I}(v))\partial_x^2 v\partial_t w dx.$$

We have

$$\gamma\eta\int \widetilde{I}(u)\partial_x^2 w\partial_t w dx \le \frac{\epsilon}{2}\|\partial_t w\|_{L^2}^2 + \frac{(\gamma\eta)^2}{2\epsilon}(\widetilde{I}(u))^2\|\partial_x^2 w\|_{L^2}^2,$$

and

$$\begin{aligned} \gamma\eta\int (\widetilde{I}(u) - \widetilde{I}(v))\partial_x^2 v\partial_t w dx &= \gamma\eta\int (\partial_t\partial_x u\partial_x u - \partial_t\partial_x v\partial_x v)dx \cdot \int \partial_x^2 v\partial_t w dx \\ &= \gamma\eta\left\{-\int \partial_t u\partial_x^2 w dx - \int \partial_t w\partial_x^2 v dx\right\} \cdot \int \partial_x^2 v\partial_t w dx \\ &\le \gamma\eta\|\partial_t u\|_{L^2}\|\partial_x^2 w\|_{L^2}\|\partial_x^2 v\|_{L^2}\|\partial_t w\|_{L^2} - \gamma\eta\left(\int \partial_x^2 v\partial_t w dx\right)^2 \\ &\le \frac{(\gamma\eta)^2}{2\epsilon}\|\partial_t u\|_{L^2}^2\|\partial_x^2 w\|_{L^2}^2\|\partial_x^2 v\|_{L^2}^2 + \frac{\epsilon}{2}\|\partial_t w\|_{L^2}^2. \end{aligned}$$

Then it follows from Theorem 1.1 and Lemma 3.3 that

$$\begin{aligned} \gamma\eta\int (\widetilde{I}(u)\partial_x^2 u - \widetilde{I}(v)\partial_x^2 v)\partial_t w dx &\le \epsilon\|\partial_t w\|_{L^2}^2 + \frac{(\gamma\eta)^2}{2\epsilon}(\widetilde{I}(u))^2\|\partial_x^2 w\|_{L^2}^2 \\ &\quad + \frac{(\gamma\eta)^2}{2\epsilon}\|\partial_t u\|_{L^2}^2\|\partial_x^2 w\|_{L^2}^2\|\partial_x^2 v\|_{L^2}^2 \\ &\le \epsilon\|\partial_t w\|_{L^2}^2 + \frac{C_\epsilon}{(t+1)^4}\|\partial_x^2 w\|_{L^2}^2. \end{aligned}$$

In the same manner as in the proof of Theorem 4.1, we have

$$\int_{\mathbb{R}} \left(I(u)\partial_x^2 u - I(v)\partial_x^2 v\right) \partial_t w dx \le \epsilon \|\partial_t w\|_{L^2}^2 + \frac{C_\epsilon}{(t+1)^2} \left(\|\partial_x^2 w\|_{L^2}^2 + \|\partial_x w\|_{L^2}^2\right).$$

From the higher-order regularity estimate for the solution v of (4.19), we have

$$\begin{aligned}
\frac{\rho\alpha}{\delta} \int_0^\infty \int_{\mathbb{R}} \partial_t w \partial_t \partial_x^2 v dx ds &\le \epsilon \|\partial_t w\|_{L^2L^2}^2 + C_\epsilon \rho^2 \|\partial_t \partial_x^2 v\|_{L^2L^2}^2, \\
\frac{\rho\kappa}{\delta} \int_0^\infty \int_{\mathbb{R}} \partial_t w \partial_t \partial_x^4 v dx ds &\le \tilde{\epsilon} \|\partial_t \partial_x^2 w\|_{L^2L^2}^2 + C_{\tilde{\epsilon}} \rho^2 \|\partial_t \partial_x^2 v\|_{L^2L^2}^2, \\
\frac{\rho\beta}{\delta} \int_0^\infty \int_{\mathbb{R}} \partial_t (I(v)\partial_x^2 v) \partial_t w dx ds &\le \epsilon \|\partial_t w\|_{L^2L^2}^2 + C_\epsilon \rho^2 \|\partial_t (I(v)\partial_x^2 v)\|_{L^2L^2}^2, \\
\frac{\rho\eta}{\delta} \int_0^\infty \int_{\mathbb{R}} \partial_t^2 \partial_x^4 v \partial_t w dx &\le \tilde{\epsilon} \|\partial_t \partial_x^2 w\|_{L^2L^2}^2 + C_{\tilde{\epsilon}} \rho^2 \|\partial_t^2 \partial_x^2 v\|_{L^2L^2}^2, \\
\frac{\rho\gamma\eta}{\delta} \eta \int_0^\infty \int_{\mathbb{R}} \partial_t (\widetilde{I}(v)\partial_x^2 v) \partial_t w dx ds &\le \tilde{\epsilon} \|\partial_t \partial_x^2 w\|_{L^2L^2}^2 + C_{\tilde{\epsilon}} \rho^2 \|\partial_t (\widetilde{I}(v) v)\|_{L^2L^2}^2.
\end{aligned}$$

Observe that

$$\begin{aligned}
\|\partial_t (\widetilde{I}(v)v)\|_{L^2L^2} &\le \left\| (\|\partial_t^2 v\|_{L^2} \|\partial_x^2 v\|_{L^2} + \|\partial_t \partial_x v\|_{L^2}^2) v \right\|_{L^2L^2} + \left\| \|\partial_t v\|_{L^2} \|\partial_x^2 v\|_{L^2} \partial_t v \right\|_{L^2L^2} \\
&\le \|\partial_t^2 v\|_{L^2L^2} \|\partial_x^2 v\|_{L^\infty L^2} \|v\|_{L^\infty L^2} + \|\partial_t v\|_{L^\infty L^2} \|\partial_t \partial_x^2 v\|_{L^2L^2} \|v\|_{L^\infty L^2} \\
&\qquad + \|\partial_t v\|_{L^\infty L^2} \|\partial_x^2 v\|_{L^\infty L^2} \|\partial_t v\|_{L^2L^2} \\
&\le C,
\end{aligned}$$

and from the same calculation as Theorem 4.1 that

$$\|\partial_t (I(v)\partial_x^2 v)\|_{L^2L^2} \le C \|\partial_t \partial_x^2 v\|_{L^2L^2} \|v\|_{L^\infty L^2} \|\partial_x^2 v\|_{L^\infty L^2} + C \|\partial_x v\|_{L^\infty L^2}^2 \|\partial_t \partial_x^2 v\|_{L^2L^2} \le C.$$

Choosing $\epsilon = \delta/4$ and $\tilde{\epsilon} = \eta/3$, we have

$$\begin{aligned}
&\frac{\rho}{2} \|\partial_t w(t)\|_{L^2}^2 + \frac{\kappa}{2} \|\partial_x^2 w(t)\|_{L^2}^2 + \frac{\alpha}{2} \|\partial_x w(t)\|_{L^2}^2 \\
&\le C \int_0^t \frac{1}{(s+1)^2} \left(\|\partial_x^2 w(s)\|_{L^2}^2 + \|\partial_x w(s)\|_{L^2}^2\right) ds + C\rho^2.
\end{aligned}$$

Using the definition (4.12) again, we have

$$D_\rho(t) \le C \int_0^t \frac{1}{(s+1)^2} D_\rho(s) ds + C\rho^2.$$

In the same fashion as Theorem 4.1 we conclude that

$$\sup_t D_\rho(t) \le C\rho^2.$$

This completes the proof. □

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References

[1] J. M. Ball, Initial-boundary value problems for an extensible beam, J. Math. Anal. Appl., **42** (1973), 61-90.

[2] T. Cazenave and A. Haraux, *An introduction to semilinear evolution equations,* Oxford Lecture Series in Mathematics and its Applications, 13. The Clarendon Press, Oxford University Press, New York, 1998.

[3] R. Chill and A. Haraux, An optimal estimate for the time singular limit of an abstract wave equation, Funkcial. Ekvac. **47** (2004), 277-290.

[4] A. Cwiszewski and K.P. Rybakowski, Singular dynamics of strongly damped beam equation, J. Differential Equations, **247** (2009), 3202–3233.

[5] J. G. Eisley, Nonlinear vibration of beams and rectangular plates, ZAMP, **15** (1964), 167-175.

[6] M. Gobbino, Singular perturbation hyperbolic-parabolic for degenerate nonlinear equations of Kirchhoff type, Nonlinear Anal., **44** (2001), 361-374. ,

[7] H. Hashimoto and T. Yamazaki, Hyperbolic-parabolic singular perturbation for quasilinear equations of Kirchhoff type, J. differential Equations, **237** (2007), 491-525.

[8] R. Racke and S. Yoshikawa, Decay estimates for global solutions of the Cauchy problem for the damped extensible beam equation, submitted.

[9] A. Schöwe, A quasilinear delayed hyperbolic Navier-Stokes system: global solution, asymptotics and relaxation limit, Methods Appl. Anal., **19** (2012), 99-118.

[10] H. Takeda and S. Yoshikawa, On the initial value problem of the semilinear beam equation with weak damping I: Smoothing effect, J. Math. Anal. Appl., **401** (2013) 244–258.

[11] S. Woinovsky-Krieger, The effect of axial force on the vibration of hinged bars, J. Appl. Mech. **17** (1950), 35-36.