SUMS OF SQUARES IN ALGEBRAIC FUNCTION FIELDS OVER A COMPLETE DISCRETE VALUED FIELD

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Abstract. A recently found local-global principle for quadratic forms over function fields of curves over a complete discrete valued field is applied to the study of quadratic forms, sums of squares, and related field invariants.

Keywords: isotropy, local-global principle, real field, sums of squares, u-invariant, pythagoras number, valuation, algebraic function fields

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1. Introduction

Let K be a field of characteristic different from 2 and F/K an algebraic function field (i.e. a finitely generated extension of transcendence degree one). The study of quadratic forms over F is generally difficult, even in such cases where the quadratic form theory over all finite extensions of K is well understood. It can be considered complete in the cases where K is algebraically closed, real closed, or finite, but it is wide open for example when K is a number field.

A breakthrough was obtained recently in the situation where the base field K is a nondyadic local field. Parimala and Suresh [15] proved that in this case any quadratic form of dimension greater than eight over F is isotropic. Harbater, Hartmann, and Krashen [8] obtained the same result as a consequence of a new local–global principle for isotropy of quadratic forms over F. The local conditions are in geometric terms, relative to an arithmetic model for F. A less geometric version of the local–global principle, in terms of the discrete rank one valuations of F, was obtained by Colliot-Thélène, Parimala, and Suresh [4]; see (6.1) below. Both versions of the local–global principle hold more generally when K is complete with respect to a non-dyadic discrete valuation.

In this article we use the local–global principle to study sums of squares in F and to obtain further results on quadratic forms over F. This is of particular interest in the case where K is the field of Laurent series $k(\!(t)\!)$ over a (formally) real field k. In (6.2) we show that the upper bound on the dimension of anisotropic torsion forms over algebraic function fields over K is the double of the corresponding upper bound for algebraic function fields over k. In the case where k is real closed we show in (6.10) that any sum of squares in F can be expressed as a sum of three squares and further prove the finiteness of $\sum F^2/D_F(2)$, the quotient of the group of nonzero sums of squares modulo the subgroup of sums of two squares in F. We further study two conjectures, (4.9) and (4.10), on the behavior of the pythagoras number of a rational function field under extension of the field of constants, and we show in (6.9) that both conjectures are equivalent.

Our methods involve valuation theory, quadratic form theory, and some algebraic geometry. As standard references we refer to [7] for valuation theory and to [10] for quadratic form theory. Results needed from algebraic geometry are cited from [11].

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2. Valuations

For a ring R we denote by $R^\times$ its group of invertible elements.

Let K be a field. Given a valuation on K, we denote by $\mathcal{O}_v$ the valuation ring of v, by $\mathfrak{m}_v$ its maximal ideal, and by κ_v the residue field, and we call v *dyadic* if κ_v has characteristic 2, *nondyadic* otherwise. Given a local ring R contained in K, we say that a valuation v of K *dominates* R if $\mathfrak{m}_v \cap R$ is the maximal ideal of R. Given a field extension L/K, we say that a valuation v of L is *unramified over* K if $v(L^\times) = v(K^\times)$.

A valuation with value group $\mathbb{Z}$ is called a *$\mathbb{Z}$-valuation*. Any discrete valuation of rank one can be identified (via a unique isomorphism of the value groups) with a $\mathbb{Z}$-valuation. A commutative ring is the valuation ring of a $\mathbb{Z}$-valuation if and only if it is a regular local ring of dimension one (cf. [12, (11.2)]); such rings are called *discrete valuation rings*.

Lemma 2.1. *Let w_1 and w_2 be two valuations on K such that $\mathfrak{m}_{w_1} \subseteq \mathcal{O}_{w_2}$. Then $\mathcal{O}_{w_1} \subseteq \mathcal{O}_{w_2}$ or $\mathcal{O}_{w_2} \subseteq \mathcal{O}_{w_1}$.*

Proof. If $\mathfrak{m}_{w_1} \subseteq \mathfrak{m}_{w_2}$, then $\mathcal{O}_{w_1} \supseteq \mathcal{O}_{w_2}$, otherwise for any choice of $t \in \mathfrak{m}_{w_1} \setminus \mathfrak{m}_{w_2}$ we have $t^{-1} \in \mathcal{O}_{w_2}$ and $\mathcal{O}_{w_1} = t^{-1}(t\mathcal{O}_{w_1}) \subseteq t^{-1}\mathfrak{m}_{w_1} \subseteq \mathcal{O}_{w_2}$. □

The property for a valuation to be *henselian* is characterized by a list of equivalent conditions, including the statement of Hensel's Lemma, hence satisfied in particular by complete valuations; see [7, Sect. 4.1].

Proposition 2.2. *Let v be a henselian $\mathbb{Z}$-valuation on K. Then v is the unique $\mathbb{Z}$-valuation on K.*

Proof. By [7, (2.3.2)] for distinct $\mathbb{Z}$-valuations w_1 and w_2 on K one has $\mathcal{O}_{w_1} \not\subseteq \mathcal{O}_{w_2}$ and $\mathcal{O}_{w_2} \not\subseteq \mathcal{O}_{w_1}$. Consider now a $\mathbb{Z}$-valuation w on K. Since v is henselian we have $1 + \mathfrak{m}_v \subseteq K^{\times n}$ for all $n \in \mathbb{N}$ prime to the characteristic of κ_v. As $w(K^\times) = \mathbb{Z}$, this implies that $1 + \mathfrak{m}_v \subseteq \mathcal{O}_w^\times$ and thus $\mathfrak{m}_v \subseteq \mathcal{O}_w$. Now (2.1) yields that $\mathcal{O}_w = \mathcal{O}_v$. □

Let X always denote a variable over a given ring or field.

Proposition 2.3. *Let R be a local domain with maximal ideal $\mathfrak{m}$ and residue field k. Let $p \in R[X]$ be monic and such that $\overline{p} \in k[X]$, the reduction of p modulo $\mathfrak{m}$, is irreducible. Then $R[X]/(p)$ is a local domain with maximal ideal $(\mathfrak{m}[X] + (p))/(p)$ and residue field $k[X]/(\overline{p})$. The ring $R[X]/(p)$ has the same dimension as R. Moreover, if R is regular, then $R[X]/(p)$ is regular.*

Proof. Note that $\mathfrak{m}[X] + (p)$ is a maximal ideal of $R[X]$. Consider a maximal ideal M of $R[X]$ containing p and set $\mathfrak{p} = M \cap R$. Since $R[X]/(p)$ is an integral extension of R, it follows using [12, (9.3) and (9.4)] that both rings have the same dimension. Moreover, the field $R[X]/M$ is an integral extension of $R/\mathfrak{p}$, whereby $R/\mathfrak{p}$ is a field. It follows that $\mathfrak{p} = \mathfrak{m}$ and thus $M = \mathfrak{m}[X] + (p)$. This shows that $\mathfrak{m}[X] + (p)$ is the unique maximal ideal of $R[X]$ containing p. Hence, $R[X]/(p)$ is a local domain with maximal ideal $(\mathfrak{m}[X] + (p))/(p)$ and residue field $k[X]/(\overline{p})$. Any set of generators of $\mathfrak{m}$ in R yields a set of generators of $(\mathfrak{m}[X] + (p))/(p)$ in $R[X]/(p)$. In particular, if R is regular, then so is $R[X]/(p)$. □

Corollary 2.4. *Let T be a discrete valuation ring of K with residue field k. Let $p \in T[X]$ be monic and such that $\overline{p} \in k[X]$ is irreducible. Then $T[X]/(p)$ is a discrete valuation ring with field of fractions $K[X]/(p)$ and residue field k-isomorphic to $k[X]/(\overline{p})$.*

Proof. Since a discrete valuation ring is the same as a regular local ring of dimension one, the statement follows from (2.3). □

We want to mention the following partial generalization of (2.4).

Proposition 2.5. *Let T be a valuation ring of K with residue field k and let ℓ/k be a finite field extension. There exists a finite field extension L/K with $[L:K]=[\ell:k]$ and a valuation v on L dominating T and unramified over K whose residue field is k-isomorphic to ℓ.*

Proof. It suffices to consider the case where $\ell = k[x]$ for some $x\in\ell$. Let $\mathfrak{m}$ denote the maximal ideal of T. Let $p\in T[X]$ be a monic polynomial whose residue $\overline{p}$ in $k[X]$ is the minimal polynomial of x over k. Then p is irreducible in $K[X]$, so $L=K[X]/(p)$ is a field. We obtain from (2.3) that $R=T[X]/(p)$ is a local domain with maximal ideal $M=(\mathfrak{m}[X]+(p))/(p)$ and residue field $k[X]/(\overline{p})$. Let v be a valuation on L dominating T. Then $T\subseteq R\subseteq \mathcal{O}_v$, and as M is generated by $\mathfrak{m}$, it follows that v dominates R. Hence, $k[X]/(\overline{p})$ embeds naturally into κ_v. In particular $[\kappa_v:k]\geq \deg(\overline{p})=\deg(p)=[L:K]$. Using the Fundamental Inequality [7, (3.3.4)] we conclude that v is unramified over K and $[\kappa_v:k]=\deg(\overline{p})=[L:K]$, whereby κ_v is k-isomorphic to $k[X]/(\overline{p})$ and therefore to ℓ. □

3. Valuations on algebraic function fields

In this section we want to relate algebraic function fields over a valued field to algebraic function fields over the corresponding residue field. In particular we show in (3.4) that an algebraic function field over the residue field of a valuation on K can be realized as the residue field of an unramified extension to some algebraic function field over K, and we refine this statement in (3.5) for rational function fields.

In the sequel let T denote a valuation ring, K its field of fractions, and k the residue field of T. (That is, we have $T=\mathcal{O}_v$ for a valuation v on K and $k=\kappa_v$.) We consider the residue fields of valuations dominating T. (The reader may observe that we avoid to speak of extensions of valuations, as this can lead to confusion about the corresponding value groups.) For a field extension F/K and a valuation v on F dominating T, the field k is naturally embedded in the residue field κ_v. We often identify residue fields of valuations dominating T up to k-isomorphism, in order to simplify the language.

A finitely generated field extension F/K of transcendence degree one is called an *algebraic function field*. We say that F/K is *algebro-rational* if $F=L(x)$ for a finite extension L/K and some element $x\in F$ that is transcendental over K.

Proposition 3.1. *Let F/K be an algebraic function field and v a valuation on F dominating T. The extension κ_v/k is either algebraic or an algebraic function field.*

Proof. This is a special case of the Dimension Inequality [7, (3.4.3))]. □

In the sequel x denotes a transcendental element over K. The following gives an improvement of (3.1) for $F=K(x)$.

Theorem 3.2 (Ohm-Nagata)**.** *Let v be a valuation on $K(x)$ dominating T. Then κ_v/k is either an algebraic or algebro-rational.*

Proof. This generalization of [13, Theorem 1] is shown in [14, Theorem]. □

We recall a construction to extend a valuation to a rational function field; in [7, Sect. 2.2] this is called the 'Gauss extension'.

Proposition 3.3. *Let T' be the localization of $T[x]$ with respect to the prime ideal $\mathfrak{m}[x]$ where $\mathfrak{m}$ is the maximal ideal of T. Then T' is a valuation ring with field of fractions $K(x)$. The residue $\overline{x}$ of x modulo $\mathfrak{m}[x]$ is transcendental over k, and the residue field of T' is $k(\overline{x})$. The corresponding valuation v on $K(x)$ with $\mathcal{O}_v=T'$, uniquely determined up to equivalence, is unramified over K.*

Proof. This follows from [7, (2.2.2)]. □

Proposition 3.4. *Let E/k be an algebraic function field. There exists an algebraic function field F/K and a valuation v on F dominating T and unramified over K whose residue field is E.*

Proof. We consider the valuation ring T' given in (3.3) and identify $\overline{x}$ with some element of E transcendental over k. Then $E/k(\overline{x})$ is a finite extension. By (2.5) there exists a finite field extension $F/K(x)$ with $[F : K(x)] = [E : k(\overline{x})]$ and a valuation v on F dominating T' and unramified over $K(x)$ with residue field E. Using (3.3) it follows that v is also unramified over K. □

Theorem 3.5. *Assume that $T \neq K$. Let ℓ/k be a finite separable field extension. There exists a valuation v on $K(x)$ dominating T and unramified over K for which κ_v/k is an algebro-rational function field with field of constants ℓ.*

Proof. Let $\alpha \in \ell$ be such that $\ell = k(\alpha)$. Let $q \in T[Y]$ be monic and such that the residue $\overline{q}$ in $k[Y]$ is the minimal polynomial of α. Let $\mathfrak{m}$ be the maximal ideal of T. We choose $m \in \mathfrak{m} \setminus \{0\}$ and set $z = m^{-1}q(x) \in K(x)$. Note that z is transcendental over K. Let T' be the localization of $T[z]$ with respect to $\mathfrak{m}[z]$. Let $\mathfrak{m}'$ be the maximal ideal of T'. By (3.3) T' is a valuation ring with field of fractions $K(z)$ and residue field $k(\overline{z})$, and $\overline{z}$ is transcendental over k. Note that $\overline{q}$ remains irreducible in $k(\overline{z})[Y]$.

Consider $p = q - q(x) \in T'[Y]$. As $q(x) = mz$, taking residues modulo $\mathfrak{m}'[Y]$ we have $\overline{p} = \overline{q}$ in $k(\overline{z})[Y]$. It follows by (2.3) that $R = T'[Y]/(p)$ is a local ring with maximal ideal lying over $\mathfrak{m}'$, with field of fractions $K(z)[Y]/(p)$, and residue field $k(\overline{z})[Y]/(\overline{p})$. Note that $K(z)[Y]/(p)$ is $K(z)$-isomorphic to $K(x)$. Using Chevalley's Theorem [7, (3.1.1)], we obtain a valuation v on $K(x)$ that dominates T'. Then v also dominates T. As $p(x) = 0$, we have that x is integral over T', whereby $v(X) = 0$. We obtain that $\overline{q}(\overline{x}) = \overline{p}(\overline{x}) = 0$. Hence, $\overline{z}, \overline{x} \in \kappa_v$ and $\overline{x}$ is algebraic over k. As $\overline{q}$ is irreducible in $k(\overline{z})[Y]$ we obtain that

$$[\kappa_v : k(\overline{z})] \geq [k(\overline{z})[\overline{x}] : k(\overline{z})] = \deg(\overline{p}) = \deg(p) = [K(x) : K(z)]\,.$$

By the Fundamental Inequality [7, (3.3.4)], it follows that v is unramified over $K(z)$ and $\kappa_v = k(\overline{z})[\overline{x}] = k[\overline{x}](\overline{z})$. Using (3.3) we obtain that v is unramified over K. Since $q(\overline{x}) = 0 = \overline{q}(\alpha)$ and since we consider residue fields up to k-isomorphism, we can identify $\ell = k[\alpha]$ with $k[\overline{x}]$. □

Together (3.2) and (3.5) give a full description of the non-algebraic extensions of k that occur as residue fields of valuations on $K(x)$ dominating T.

Assume that the valuation ring T is discrete and consider an algebraic function field F/K. By a *regular model for F/T* we mean a 2-dimensional integral regular projective flat T-scheme $\mathcal{X}$ whose function field is K-isomorphic to F. Given a regular model $\mathcal{X}$ for F/K we denote by $\mathcal{X}_k$ its special fiber; by [11, (8.3.3)] $\mathcal{X}_k$ is a curve.

Given an integral scheme $\mathcal{X}$, a point $P \in \mathcal{X}$, and a valuation v on the function field of $\mathcal{X}$, we say that v is *centered at P* if v dominates $\mathcal{O}_{\mathcal{X},P}$, the local ring at P.

Proposition 3.6. *Assume that T is a discrete valuation ring. Let F/K be an algebraic function field. Let $\mathcal{X}$ be a regular model for F/T. Let v be a $\mathbb{Z}$-valuation on F dominating T. Then v is centered at a point P of $\mathcal{X}$ lying in $\mathcal{X}_k$. Moreover, if the extension κ_v/k is neither algebraic nor algebro-rational, then $\mathcal{O}_v = \mathcal{O}_{\mathcal{X},P}$ where P is the generic point of an irreducible component of $\mathcal{X}_k$.*

Proof. By [11, (8.3.17)] v is centered at a point P of the special fiber $\mathcal{X}_k$. Since $\mathcal{X}_k$ is a curve, P is either a closed point or the generic point of an irreducible component $\mathcal{X}_k$. In either case $\mathcal{O}_{\mathcal{X},P}$ is a regular local ring.

If P is a closed point of $\mathcal{X}_k$, then by [1, Proposition 3] the extension κ_v/k is either algebraic or algebro-rational. Assume now that P is a generic point of $\mathcal{X}_k$.

Then P has codimension one in $\mathcal{X}$, so $\mathcal{O}_{\mathcal{X},P}$ is a regular local ring of dimension one and thus a discrete valuation ring. As $\mathcal{O}_{\mathcal{X},P}$ is dominated by $\mathcal{O}_v$ and both are discrete valuation rings with the same field of fractions, it follows by [7, (2.3.2)] that $\mathcal{O}_v = \mathcal{O}_{\mathcal{X},P}$. □

Proposition 3.7. *Assume that T is a complete discrete valuation ring. Let F/K be an algebraic function field. Then there exists a regular model for F/T.*

Proof. There exists a regular projective curve over K whose function field is K-isomorphic to F. From this we obtain (e.g. following the first steps in [11, (10.1.8)]) a 2-dimensional projective T-scheme $\mathcal{X}$ with function field F. Since the structure morphism $\mathcal{X} \longrightarrow \mathrm{Spec}(T)$ is surjective, by [11, (8.3.1)] it is flat. By [11, (8.2.40)] T is an excellent ring. Since $\mathcal{X}$ is locally of finite type over T, it follows by [11, (8.2.39)] that $\mathcal{X}$ is excellent.

Let $\mathcal{X}' \longrightarrow \mathcal{X}$ be the normalization of $\mathcal{X}$. Since $\mathcal{X}$ is excellent and projective over T, the normalization $\mathcal{X}' \longrightarrow \mathcal{X}$ is a finite projective birational morphism, by [11, (8.2.39) and (8.3.47)]. The singular locus of $\mathcal{X}'$ is closed in $\mathcal{X}'$, by [11, (8.2.38)]. We consider the blowing-up $\mathcal{X}'' \longrightarrow \mathcal{X}'$ along the singular locus of $\mathcal{X}'$. By [11, (8.1.12) and (8.1.22)] the blowing-up is a birational projective morphism.

We may alternate normalizalization and blowing-up until we reach a scheme that is regular. At each step we obtain a flat projective 2-dimensional T-scheme whose function field is F. By Lipman's Desingularization Theorem [11, (8.3.44)], after finitely many steps we come to a situation where the T-scheme is regular. □

Corollary 3.8. *Assume that T is a complete discrete valuation ring. Let F/K be an algebraic function field. Then there exist only finitely many $\mathbb{Z}$-valuations v on F dominating T for which the extension κ_v/k is neither algebraic nor algebro-rational.*

Proof. By (3.7) there exists a regular model for F/T. The statement follows by applying (3.6) to any such model. □

The result (3.8) can be extended to the situation where T is an arbitrary discrete valuation ring. Moreover, one may ask to characterize the $\mathbb{Z}$-valuations on an algebraic function field that dominate a given discrete valuation ring of the base field and for which the residue field extension is neither algebraic nor algebro-rational. We intend to develop these topics in a forthcoming article.

4. Sums of squares and valuations

From now on let K be a field of characteristic different from 2. We denote by $\sum K^2$ the subgroup of nonzero sums of squares in K and, for $n \in \mathbb{N}$, by $D_K(n)$ the set of nonzero elements that can be written as sums of n squares in K. One calls

$$s(K) = \inf\{n \in \mathbb{N} \mid -1 \in D_K(n)\} \in \mathbb{N} \cup \{\infty\}$$

the *level of* K. Recall that K is *real* if $s(K) = \infty$ and *nonreal* otherwise, and in the latter case $s(K)$ is a power of two (cf. [10, Chap. XI, Sect. 2]).

Lemma 4.1. *Let v be a valuation on K and $n \in \mathbb{N}$. Then $s(\kappa_v) \geq n$ if and only if $v(a_1^2 + \cdots + a_n^2) = 2\min\{v(a_1), \ldots, v(a_n)\}$ holds for all $a_1, \ldots, a_n \in K$.*

Proof. Both conditions are easily seen to be equivalent to having that any sum of n squares of elements in $\mathcal{O}_v^\times$ lies in $\mathcal{O}_v^\times$. □

Let $\Omega(K)$ denote the set of nondyadic $\mathbb{Z}$-valuations on K. For $v \in \Omega(K)$, let K^v denote the corresponding completion of K. For $S \subseteq \Omega(K)$ we define a homomorphism

$$\Phi_S : K^\times \longrightarrow \mathbb{Z}^S, x \longmapsto (v(x))_{v\in S}\,.$$

If $S \subseteq \Omega(K)$ is a finite subset, then it follows from the Approximation Theorem (cf. [7, (2.4.1)] or [11, (9.1.9)]) that Φ_S is surjective.

Proposition 4.2. *Let S be a finite subset of $\Omega(K)$ and $n \in \mathbb{N}$. Then*
$$\Phi_S(D_K(n)) = \{(e_v)_{v\in S} \in \mathbb{Z}^S \mid e_v \in 2\mathbb{Z} \text{ for } v \in S \text{ with } s(\kappa_v) \geq n\}\,.$$

Proof. For $v \in \Omega(K)$ with $s(\kappa_v) \geq n$ we have $v(D_K(n)) \subseteq 2\mathbb{Z}$ by (4.1). This shows that
$$\Phi_S(D_K(n)) \subseteq \{(e_v)_{v\in S} \in \mathbb{Z}^S \mid e_v \in 2\mathbb{Z} \text{ for } v \in S \text{ with } s(\kappa_v) \geq n\}\,.$$
It remains to show the other inclusion. Consider a tuple $(e_v)_{v\in S} \in \mathbb{Z}^S$ such that $e_v \in 2\mathbb{Z}$ for all $v \in S$ with $s(\kappa_v) \geq n$. The aim is to find an element $x \in D_K(n)$ with $\Phi_S(x) = (e_v)_{v\in S}$. We explain how to obtain such an element, using the Approximation Theorem (cf. [7, (2.4.1)] or [11, (9.1.9)]) several times.

For $v \in S$ with $e_v \notin 2\mathbb{Z}$, as $s(\kappa_v) < n$ we may choose $x_{v,2}, \dots, x_{v,n} \in \mathcal{O}_v$ such that $v(1 + x_{v,2}^2 + \cdots + x_{v,n}^2) > 0$. For $v \in S$ with $e_v \in 2\mathbb{Z}$ we set $x_{v,2} = \cdots = x_{v,n} = 0$. For $i = 2, \dots, n$ we choose $x_i \in K^\times$ such that $v(x_i - x_{v,i}) > 0$ for all $v \in S$. We set $y = x_2^2 + \cdots + x_n^2$. For $v \in S$ we have $v(1 + y) = 0$ if $e_v \in 2\mathbb{Z}$ and $v(1+y) > 0$ otherwise. We choose $t \in K^\times$ such that, for all $v \in S$, we have $v(t) = 1$ if $v(1+y) > 1$, and $v(t) > 1$ otherwise. Note that $(1+t)^2+y \in D_K(n)$. For any $v \in S$ the value $v((1 + t)^2 + y)$ is either 0 or 1 and such that $v((1 + t)^2 + y) \equiv e_v \bmod 2\mathbb{Z}$. Choose now $z \in K^\times$ such that $2v(z) = e_v - v((1 + t)^2 + y)$ for all $v \in S$ and set $x = z^2((1 + t)^2 + y)$. Then $x \in D_K(n)$ and $\Phi_S(x) = (v(x))_{v\in S} = (e_v)_{v\in S}$. □

We say that a valuation v on K is *real* or *nonreal*, respectively, if the residue field κ_v has the corresponding property.

Corollary 4.3. *Let $v \in \Omega(K)$. If v is real, then $v(\sum K^2) = 2\mathbb{Z}$, otherwise $v(\sum K^2) = \mathbb{Z}$.*

Proof. This follows from (4.2) applied to $S = \{v\}$ and all $n \in \mathbb{N}$. □

Corollary 4.4. *Let n be a positive integer and S a finite subset of $\Omega(K)$ such that $s(\kappa_v) = 2^n$ for all $v \in S$. Then Φ_S induces a surjective homomorphism $D_K(2^{n+1})/D_K(2^n) \longrightarrow (\mathbb{Z}/2\mathbb{Z})^S$. In particular, $|D_K(2^{n+1})/D_K(2^n)| \geq 2^{|S|}$.*

Proof. By the hypotheses on S and by (4.2), we have $\Phi_S(D_K(2^{n+1})) = \mathbb{Z}^S$ and $\Phi_S(D_K(2^n)) = (2\mathbb{Z})^S$. From this the statement follows. □

The *pythagoras number of K* is defined as
$$p(K) \;=\; \inf\{n \in \mathbb{N} \mid D_K(n) = \textstyle\sum K^2\} \;\in\; \mathbb{N} \cup \{\infty\}\,.$$
Case distinctions in statements involving valuations and pythagoras numbers can often be avoided when $p(K)$ is replaced by $s(K) + 1$ in case K is nonreal. We therefore set
$$p'(K) = \begin{cases} p(K) & \text{if } K \text{ is real,} \\ s(K) + 1 & \text{if } K \text{ is nonreal.} \end{cases}$$
Note that for nonreal fields K, we always have $s(K) \leq p(K) \leq s(K) + 1 = p'(K)$.

Proposition 4.5. *Let $v \in \Omega(K)$. Then $p'(K) \geq p(K) \geq p'(\kappa_v)$. Moreover, if v is henselian, then $p'(K) = p(K) = p'(\kappa_v)$.*

Proof. Note that $p(K) \geq p(\kappa_v)$. If v is real, then κ_v and K are real, and we obtain that $p'(K) = p(K) \geq p(\kappa_v) = p'(\kappa_v)$. Assume that v is nonreal. Applying (4.2) with $S = \{v\}$ we obtain $x_0, \dots, x_s \in \mathcal{O}_v^\times$ with $s = s(\kappa_v)$ such that $v(x_0^2 + \cdots + x_s^2) = 1$, and then (4.1) shows that $x_0^2 + \cdots + x_s^2 \notin D_K(s)$. Hence $p(K) \geq s + 1 = p'(\kappa_v)$.

Assume finally that v is henselian. Then $s(K) = s(\kappa_v)$, and further $p(K) = p(\kappa_v)$ in case v is real. This yields that $p'(K) = p'(k)$. □

For example, by (4.5) we have that $p'(K(\!(t)\!)) = p(K(\!(t)\!)) = p'(K)$.

Theorem 4.6. *Let K be a real field. For $n \in \mathbb{N}$ the following are equivalent:*

(*i*) $p(K(X)) \leq 2^n$.
(*ii*) $p(L) < 2^n$ *for all finite real extensions* L/K.
(*iii*) $s(L) \leq 2^{n-1}$ *for all finite nonreal extensions* L/K.
(*iv*) $p'(L) < 2^n$ *for all finite extensions* L/K *with* $-1 \notin L^{\times 2}$.

Proof. See [2, Theorem 3.3] for the equivalence of (*i*)–(*iii*); the equivalence of these conditions with (*iv*) is obvious. □

Corollary 4.7. *Let* $n \in \mathbb{N}$ *be such that* $p(K(X)) \leq 2^n$. *Then* $p(L(X)) \leq 2^n$ *for any finite field extension* L/K.

Proof. If K is nonreal, then $p(L(X)) = s(L) + 1 \leq s(K) + 1 = p(K(X)) \leq 2^n$. If K is real and L is nonreal, then $s(L) \leq 2^{n-1}$ by (4.6) and thus $p(L(X)) \leq 2^n$. If L is real, then since any finite real extension of L is a finite real extension of K, the equivalence of (*i*) and (*ii*) in (4.6) allows us to conclude that $p(L(X)) \leq 2^n$. □

Theorem 4.8. *Let* K *be henselian with respect to a* $\mathbb{Z}$*-valuation with residue field* k. *If* $n \in \mathbb{N}$ *is such that* $p(k(X)) \leq 2^n$, *then* $p(k(X)) \leq p(K(X)) \leq 2^n$.

Proof. If K is nonreal, then $p(K(X)) = s(K) + 1 = s(k) + 1 = p(k(X))$, and there remains nothing to show. Assume now that K is real. Then k and $k(X)$ are real. Let v' denote the $\mathbb{Z}$-valuation on $K(X)$ whose valuation ring $\mathcal{O}_{v'}$ is the localization of $\mathcal{O}_v[X]$ with respect to the prime ideal $\mathfrak{m}_v[X]$, as described in (3.3). As $\kappa_{v'} = k(X)$, we obtain by (4.5) that $p(K(X)) \geq p'(k(X)) \geq p(k(X))$. This shows the first inequality.

Let $n \in \mathbb{N}$ be such that $p(k(X)) \leq 2^n$. By (4.6), to prove that $p(K(X)) \leq 2^n$ it suffices to show that $p'(L) < 2^n$ for all finite extensions L/K with $-1 \notin L^{\times 2}$. Consider such an extension L/K. Since v is henselian, it extends uniquely to a valuation w on L. This extension is henselian and equivalent to a $\mathbb{Z}$-valuation, and its residue field κ_w is a finite extension of k. Since w is henselian, we have that $p'(L) = p'(\kappa_w)$ by (4.5) and $-1 \notin \kappa_w^{\times 2}$. Hence, $p'(L) = p'(\kappa_w) < 2^n$, by (4.6). □

The last two statements motivate us to formulate the following two conjectures.

Conjecture 4.9. *For any finite field extension* L/K, *one has* $p(L(X)) \leq p(K(X))$.

Conjecture 4.10. *If* K *is complete with respect to a nondyadic* $\mathbb{Z}$*-valuation with residue field* k, *then* $p(K(X)) = p(k(X))$.

We shall see in (6.9) that these two conjectures are equivalent.

5. The u-invariant for algebraic function fields

We refer to [10] for basic facts and terminology from the theory of quadratic forms over fields of characteristic different from two. The u-invariant of K was defined by Elman and Lam [5] as

$$u(K) = \sup\{\dim(\varphi) \mid \varphi \text{ anisotropic torsion form over } K\} \in \mathbb{N} \cup \{\infty\},$$

where a *torsion form* is a regular quadratic form that corresponds to a torsion element in the Witt ring.

Proposition 5.1. *Let* $v \in \Omega(K)$. *Let* ψ *be a torsion form over* κ_v. *There exist* $n \in \mathbb{N}$, $a_1, \dots, a_n \in \mathcal{O}_v^\times$, *and* $t \in K^\times$ *with* $v(t) = 1$ *such that* $\langle 1, -t\rangle \otimes \langle a_1, \dots, a_n\rangle$ *is a torsion form over* K *and such that* ψ *is Witt equivalent to* $\langle \overline{a}_1, \dots, \overline{a}_n\rangle$.

Proof. Assume first that v is nonreal. Then by (4.3) there exists $t \in \sum K^2$ with $v(t) = 1$. For $n = \dim(\psi)$ and $a_1, \dots, a_n \in \mathcal{O}_v^\times$ such that ψ is isometric to $\langle \overline{a}_1, \dots, \overline{a}_n\rangle$, we obtain that $\langle 1, -t\rangle \otimes \langle a_1, \dots, a_n\rangle$ is a torsion form over K.

Assume now that v is real. Then ψ is Witt equivalent to a sum of binary torsion forms over κ_v (cf. [16, Satz 22]). Every binary torsion form over κ_v is of the shape $\langle \overline{a}_1, \overline{a}_2 \rangle$ with $a_1, a_2 \in \mathcal{O}_v^\times$ such that $-a_1a_2 \in \sum K^2$. Hence, there exist $r \in \mathbb{N}$ and $a_1, \dots, a_{2r} \in \mathcal{O}_v^\times$ such that ψ is Witt equivalent to $\langle \overline{a}_1, \dots, \overline{a}_{2r} \rangle$ and $-a_{2i-1}a_{2i} \in \sum K^2$ for $i = 1, \dots, r$. Then $\langle a_1, \dots, a_{2r} \rangle$ is torsion form over K. We choose any $t \in K^\times$ with $v(t) = 1$. Then also $\langle 1, -t \rangle \otimes \langle a_1, \dots, a_{2r} \rangle$ is a torsion form over K. □

The following statement was independently obtained in [19, Proposition 5] using different arguments, based on the theory of spaces of orderings.

Proposition 5.2. *For $v \in \Omega(K)$ we have $u(K) \geq 2u(\kappa_v)$.*

Proof. Let $v \in \Omega(K)$. To prove the statement it suffices to show that to any anisotropic torsion form ψ over κ_v there exists an anisotropic torsion form φ over K with $\dim(\varphi) \geq 2\dim(\psi)$. Let ψ be an anisotropic torsion form over κ_v. We choose $n \in \mathbb{N}$, $a_1, \dots, a_n \in \mathcal{O}_v^\times$, and $t \in K^\times$ with $v(t) = 1$ as in (5.1). Then $\langle 1, -t \rangle \otimes \langle a_1, \dots, a_n \rangle$ is a torsion form over K. Let φ denote its anisotropic part. Then φ is a torsion form and isometric to $\langle b_1, \dots, b_s \rangle \perp -t\langle c_1, \dots, c_r \rangle$ for certain $r, s \in \mathbb{N}$ and $c_1, \dots, c_r, b_1, \dots, b_s \in \mathcal{O}_v^\times$. Applying residue homomorphisms (cf. [18, Chap. 6, §2]), it follows that the forms $\langle \overline{b}_1, \dots, \overline{b}_s \rangle$ and $\langle \overline{c}_1, \dots, \overline{c}_r \rangle$ over κ_v are Witt equivalent to ψ. As ψ is anisotropic we conclude that $\dim(\varphi) \geq r+s \geq 2\dim(\psi)$. □

A generalization of (5.2) for arbitrary nondyadic valuations is given in [3, (5.2)].

Corollary 5.3. *Let k be the residue field of a non-dyadic $\mathbb{Z}$-valuation on K. For every algebraic function field F/K there exists an algebraic function field E/k such that $u(F) \geq 2u(E)$.*

Proof. Let T denote the discrete valuation ring with field of fractions K and residue field k. Let F/K be an algebraic function field. Choose $x \in F$ transcendental over K. Consider the valuation ring T' in $K(x)$ described in (3.3). Note that T' is a discrete valuation ring. Since $F/K(x)$ is a finite extension, there exists a $\mathbb{Z}$-valuation v on F dominating T'. The residue field E of v is a finite extension of $k(\overline{x})$, hence an algebraic function field over k. By (5.2) we obtain that $u(F) \geq 2u(E)$. □

We define

$$\hat{u}(K) = \tfrac{1}{2}\sup\{u(F) \mid F/K \text{ algebraic function field }\}.$$

For nonreal fields $\hat{u}$ coincides with the *strong u-invariant* defined in [8, Definition 1.2], by the following result.

Corollary 5.4. *For any algebraic extension L/K we have*

$$u(L) \leq \tfrac{1}{2}u(K(X)) \leq \hat{u}(K).$$

Proof. If L is a field of odd characteristic p, then the Frobenius homomorphism given by $x \longmapsto x^p$ shows that any quadratic form over L is obtained by scalar extension from a quadratic form defined over L^p. Therefore every torsion form defined over an algebraic extension of K comes from a torsion form defined over a finite separable extension of K. Since any finite separable extension of K is the residue field of a $\mathbb{Z}$-valuation v on $K(X)$, the first inequality now follows from (5.2). The second inequality is obvious. □

6. Function fields over complete discrete valued fields

In this section we assume that K is the field of fractions of a complete discrete valuation ring T with residue field k of characteristic different from 2. We want to apply the following reformulation of the local-global principle in [4, (3.1)] to the study of the u-invariant and the pythagoras number of algebraic function fields over K.

Theorem 6.1 (Colliot-Thélène, Parimala, Suresh)**.** *Let F be an algebraic function field over K. A regular quadratic form over F of dimension at least 3 is isotropic if and only if it is isotropic over F^v for every $v \in \Omega(F)$.*

Proof. This slightly more general version of [4, (3.1)] follows from [9, (9.10)]. □

We now can extend the result [8, Theorem 4.10] to the current setting, thus covering real function fields. C. Scheiderer independently gave a more geometric proof in [19, Theorem 3].

Theorem 6.2. *We have $\hat{u}(K) = 2\hat{u}(k)$.*

Proof. For any algebraic function field E/k, by (3.4) there exists an algebraic function field F/K and a $\mathbb{Z}$-valuation on F with residue field E, and using (5.2) we obtain that $u(E) \leq \frac{1}{2}u(F) \leq \hat{u}(K)$. This yields that $2\hat{u}(k) \leq \hat{u}(K)$.

To prove the converse inequality, we need to show for an arbitrary algebraic function field F/K that $u(F) \leq 4\hat{u}(k)$ holds. Fix F/K. By (6.1), any anisotropic form over F remains anisotropic over F^w for some $w \in \Omega(F)$. It thus suffices to show that $u(F^w) \leq 4\hat{u}(k)$ for every $w \in \Omega(F)$. Fix $w \in \Omega(F)$. As $u(F^w) = 2u(\kappa_w)$, it suffices to show that $u(\kappa_w) \leq 2\hat{u}(k)$. This is clear by the definition of $\hat{u}$ in case κ_w/k is an algebraic function field. If the extension κ_w/k is algebraic, it follows from (5.4) that $u(\kappa_w) \leq \hat{u}(k)$. In the remaining case, κ_w is a finite extension of K. Then κ_w is complete with respect to a nondyadic $\mathbb{Z}$-valuation, and the corresponding residue field ℓ is a finite extension of k. Then $u(\kappa_w) = 2u(\ell) \leq 2\hat{u}(k)$, by (5.4). □

Corollary 6.3. *Let $m \in \mathbb{N}$. If $u(E) = m$ for every algebraic function field E/k, then $u(F) = 2m$ for every algebraic function field F/K.*

Proof. Let F/K be an algebraic function field over K. Using (6.2) we obtain that $u(F) \leq 2\hat{u}(K) = 4\hat{u}(k)$. By (5.3) there exists an algebraic function field E/k with $u(F) \geq 2u(E)$. If we assume that $u(E) = m$ holds for every algebraic function field E/k, we obtain that $2\hat{u}(k) = m$ and conclude that $u(F) = 2m$. □

Theorem 6.4. *We have $u(K(X)) = 2 \cdot \sup\{u(\ell(X)) \mid \ell/k \text{ finite field extension}\}$.*

Proof. Let $F = K(X)$. As $u(F) \geq 2$, it follows from (6.1) that

$$u(F) \leq \sup\{u(F^v) \mid v \in \Omega(F)\}.$$

Consider $v \in \Omega(F)$. We have $u(F^v) = 2u(\kappa_v)$. If v is trivial on K, then κ_v is a finite extension of K, hence complete with respect to a $\mathbb{Z}$-valuation w with $\mathcal{O}_w \cap K = T$ and whose residue field κ_w is a finite extension of k, so that $u(\kappa_v) = 2u(\kappa_w) \leq u(k(X))$ by (5.4) and thus $u(F^v) \leq 2u(k(X))$. If v is nontrivial on K, then by (2.2) and (3.2) κ_v/k is either an algebraic extension or algebro-rational. In any case we obtain that $u(\kappa_v) \leq u(\ell(X))$ and thus $u(F^v) \leq 2u(\ell(X))$ for a finite extension ℓ/k. This shows that

$$u(F) \leq 2 \cdot \sup\{u(\ell(X)) \mid \ell/k \text{ finite field extension}\}.$$

Given a finite field extension ℓ/k, it follows from (3.4) that there exists a $\mathbb{Z}$-valuation on $K(X)$ with residue field $\ell(X)$, which by (5.2) implies that $u(K(X)) \geq 2u(\ell(X))$. This shows the claimed equality. □

We turn to the study of sums of squares and the pythagoras number.

Theorem 6.5. *Let F/K be an algebraic function field. For any $m \geq 2$ we have that $D_F(m) = F^\times \cap (\bigcap_{v \in \Omega(K)} D_{F^v}(m))$. Moreover, $p(F) = \sup\{p'(\kappa_v) \mid v \in \Omega(F)\}$.*

Proof. Applying (6.1) to the quadratic forms $m \times \langle 1\rangle \perp \langle -a\rangle$ for $a \in F^\times$ shows for any $m \geq 2$ the claimed equality of sets. Note that $\Omega(F)$ contains a nonreal valuation v, and we have that $p(F^v) = s(\kappa_v) + 1 \geq 2$. As $p(F) \geq 2$, we obtain that

$$\begin{aligned} p(F) &= \inf\{m \geq 2 \mid D_F(m) = D_F(m+1)\} \\ &= \inf\{m \geq 2 \mid D_{F^v}(m) = D_{F^v}(m+1) \text{ for all } v \in \Omega(F)\} \\ &= \sup\{p(F^v) \mid v \in \Omega(F)\}\,. \end{aligned}$$

Moreover, by (4.5) we have $p(F^v) = p'(\kappa_v)$ for every $v \in \Omega(F)$. □

Theorem 6.6. *Let $n \in \mathbb{N}$ and assume that $p(k(X)) \leq 2^n$ and that $\sum E^2/D_E(2^n)$ is finite for every algebraic function field E/k. Then $p(K(X)) \leq 2^n$ and $\sum F^2/D_F(2^n)$ is finite for every algebraic function field F/K.*

Proof. By (4.8) we have $p(K(X)) \leq 2^n$. Consider an algebraic function field F/K. By (6.5) the natural homomorphism

$$\textstyle\sum F^2/D_F(2^n) \longrightarrow \prod\limits_{v\in\Omega(F)} \sum (F^v)^2/D_{F^v}(2^n)$$

is injective. To prove that $\sum F^2/D_F(2^n)$ is finite, it thus suffices to show that the set

$$S = \{v \in \Omega(F) \mid p(F^v) > 2^n\}$$

is finite and that $\sum F^{v\,2}/D_{F^v}(2^n)$ is finite for each $v \in S$. Let

$$\Omega_T(F) \;=\; \{v \in \Omega(F) \mid \mathcal{O}_v \cap K = T \text{ and } \kappa_v/k \text{ is transcendental}\}.$$

Consider $v \in \Omega(F) \setminus \Omega_T(F)$. Then v is trivial on K, so κ_v is a finite extension of K. Hence κ_v is complete with respect to a $\mathbb{Z}$-valuation whose residue field ℓ is a finite extension of k. We conclude that $p(F^v) = p'(\kappa_v) = p'(\ell) \leq 2^n$. This shows that $S \subseteq \Omega_T(F)$.

Consider now $v \in \Omega_T(F)$. Then κ_v/k is an algebraic function field, in particular $|\sum (F^v)^2/D_F(2^n)| \leq 2 \cdot |\sum (\kappa_v)^2/D_{\kappa_v}(2^n)|$, which is finite by the hypothesis. If κ_v/k is algebro-rational, then $p'(\kappa_v) \leq p(K(X)) \leq 2^n$, thus $p(F^v) = p'(\kappa_v) \leq 2^n$. The finiteness of S thus follows from (3.8). □

Theorem 6.7. *Assume that $n \in \mathbb{N}$ is such that $p(E) \leq 2^n$ for any algebraic function field E/k. Let F/K be an algebraic function field. Then $p(F) \leq 2^n + 1$ and the set $S = \{v \in \Omega(F) \mid s(\kappa_v) = 2^n\}$ is finite with $|\sum F^2/D_F(2^n)| = 2^{|S|}$. Moreover, $\Phi_S : F^\times \longrightarrow \mathbb{Z}^S$ induces an isomorphism $\sum F^2/D_F(2^n) \longrightarrow (\mathbb{Z}/2\mathbb{Z})^S$.*

Proof. For any algebraic extension ℓ/k we have that $p'(\ell) \leq p(\ell(X)) \leq 2^n$. In particular, we have $p'(E) \leq 2^n$ for any algebro-rational function field E/k. Note further that $p'(\kappa_v) = 2^n + 1$ for any $v \in S$.

Consider $v \in \Omega(F)$. If $v|_K$ is trivial, then κ_v is a finite extension of K and therefore complete with respect to a $\mathbb{Z}$-valuation w whose residue field κ_w is a finite extension of k, whence $p'(\kappa_v) = p'(\kappa_w) \leq 2^n$ by (4.6) and in particular $v \notin S$. Suppose that $v|_K$ is nontrivial. By (2.2), $\mathcal{O}_v \cap K$ is thus the complete discrete valuation ring on K, and the extension κ_v/k is either algebraic or an algebraic function field. If κ_v/k is algebraic then $p'(\kappa_v) \leq 2^n$ and in particular $v \notin S$. If κ_v/k is an algebraic function field, then $p(\kappa_v) \leq 2^n$ and thus $p'(\kappa_v) \leq 2^n + 1$, with equality holding if and only if $v \in S$, and in this case κ_v/k is not algebro-rational. Hence, for $v \in \Omega(F)$ we have $p'(\kappa_v) = 2^n + 1$ if $v \in S$ and $p'(\kappa_v) \leq 2^n$ otherwise.

By (6.5) we conclude that $p(F) \leq p'(F) \leq 2^n + 1$ and furthermore

$$\textstyle\sum F^2 = \Big(\bigcap\limits_{v\in S} D_{F^v}(2^n{+}1)\Big) \cap \Big(\bigcap\limits_{v\in S^c} D_{F^v}(2^n)\Big),$$

where $S^c = \Omega(F)\setminus S$. Moreover, using (3.8) we obtain that S is finite. By (4.4) then $\Phi_S : F^\times \longrightarrow \mathbb{Z}^S$ induces a surjective homomorphism $\sum F^2/D_F(2^n) \longrightarrow (\mathbb{Z}/2\mathbb{Z})^S$.

It remains to show that this homomorphism is also injective. In view of (6.5) and the above equality for $\sum F^2$, it suffices to verify that $\Phi_S^{-1}(2\mathbb{Z}^S) \subseteq \bigcap_{v\in S} D_{F^v}(2^n)$. Consider $x \in F^\times$ and $v \in S$ with $v(x) \in 2\mathbb{Z}$. Then $x = t^2 y$ with $t \in F^\times$ and $y \in \mathcal{O}_v^\times \cap (\sum F^2)$, so that $y + \mathfrak{m}_v \in \sum {\kappa_v}^2$. Since F^v is complete and $p(\kappa_v) \leq 2^n$, it follows that $x = t^2 y \in D_{F^v}(2^n)$. This shows the claim. $\square$

Theorem 6.8. *Let F/K be an algebraic function field. There exists an algebraic function field E/k such that $p'(E) \geq p'(F)$. Moreover, if F/K is algebro-rational, then one may choose E/k to be algebro-rational.*

Proof. If $p'(F) \leq p'(k(X))$, we put $E = k(X)$. Now assume that $p'(F) > p'(k(X))$. Then $p(k(X)) < \infty$ and thus $p(K(X)) < \infty$ by (4.8). Since F is K-isomorphic to a finite extension of K, it follows by [17, Chap. 7, (1.13)] that $p(F) < \infty$. By (6.5) there exists $v \in \Omega(F)$ such that $p(F) = p'(\kappa_v) = p(F^v)$.

Assume first that $v|_K$ is trivial. Then κ_v is a finite extension of K and thus carries a complete $\mathbb{Z}$-valuation w whose residue field κ_w is a finite extension of k. We obtain that $p'(\kappa_v) = p'(\kappa_w)$ and thus choose $E = \kappa_w(X)$ to have an algebro-rational function field E/k with $p'(E) \geq p'(F)$.

Assume now that $v|_K$ is nontrivial. Then by (2.2) v dominates T. If κ_v/k is an algebraic function field, we may choose $E = \kappa_v$ to have that $p'(E) \geq p'(F)$. By (3.2) if F/K is algebro-rational, then so is E/k. Consider finally the case where κ_v/k is an algebraic extension. Since $p'(\kappa_v) = p'(F) < \infty$, there exists a finite extension ℓ/k contained in κ_v/k with $p'(\ell) \geq p'(\kappa_v)$, and thus we may choose $E = \ell(X)$ to have $p'(E) \geq p'(\ell) \geq p'(F)$. $\square$

Corollary 6.9. *We have $p'(K(X)) = \sup\{p'(\ell(X)) \mid \ell/k \text{ finite field extension}\}$.*

Proof. The statement is trivial if k is nonreal. Assume that k is real. Given an arbitrary finite extension ℓ/k, by (3.5) there is a $\mathbb{Z}$-valuation on $K(X)$ with residue field $\ell(X)$, whereby (4.5) yields that $p'(\ell(X)) \leq p'(K(X))$. On the other hand, by (6.8), there exists a finite extension ℓ/k with $p'(K(X)) \leq p'(\ell(X))$. $\square$

Note that (6.9) shows the equivalence of the two conjectures (4.9) and (4.10).

Recall that the field K is said to be *hereditarily quadratically closed* if $L^\times = L^{\times 2}$ for every finite field extension L/K. The following result applies in particular to the situation where R is a real closed field.

Theorem 6.10. *Let $n \in \mathbb{N}$ and $K = R(\!(t_1)\!)\dots(\!(t_n)\!)$ for a field R such that $R(\sqrt{-1})$ is hereditarily quadratically closed. Let F/K be an algebraic function field. Then $u(F) = 2^{n+1}$, $2 \leq p(F) \leq 3$, and the group $\sum F^2/D_F(2)$ is finite.*

Proof. As F is a finite extension of a rational function field, it follows by [10, Chap. VIII, (5.7)] that $p'(F) \geq p(F) \geq 2$. We prove the statement by induction on n. For $n = 0$ we obtain from [6, Theorem] that $u(F) = 2$ and conclude by [10, Chap. XI, (6.26)] that $p(F) = 2$, hence $\sum F^2 = D_F(2)$ and $p'(F) \leq 3$. Assume that $n > 0$. Applying the induction hypothesis to all algebraic function fields over $k = R(\!(t_1)\!)\dots(\!(t_{n-1})\!)$, we obtain by (6.3) that $u(F) = 2^{n+1}$, by (6.8) that $p'(F) \leq 3$, and by (6.6) that $\sum F^2/D_F(2)$ is finite. $\square$

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References

[1] S. Abhyankar. On the valuations centered in a local domain. *Am. J. Math.* **78** (1956): 321–348.

[2] K.J. Becher, J. Van Geel. Sums of squares in function fields of hyperelliptic curves. *Math. Z.* **261** (2009): 829–844.

[3] K.J. Becher, D.B. Leep. *Real fields, valuations, and quadratic forms. Preprint* (2011).

[4] J.-L. Colliot-Thélène, R. Parimala, V. Suresh. Patching and local-global principles for homogeneous spaces over function fields of p-adic curves. *Preprint 2009*, `arXiv:0812.3099v3`; *Comentarii Mathematici Helvetici*, to appear.

[5] R. Elman, T.Y. Lam. *Quadratic forms and the u-invariant I* , Math. Z. **131** (1973): 283–304.

[6] R. Elman, A. Wadsworth. Hereditarily quadratically closed fields. *J. Algebra* **111** (1987): 475–482.

[7] A.J. Engler, A. Prestel. *Valued fields.* Springer Monographs in Mathematics, Berlin, 2005.

[8] D. Harbater, J. Hartmann, D. Krashen. Applications of patching to quadratic forms and central simple algebras. *Invent. Math.* **178** (2009): 231–263.

[9] D. Harbater, J. Hartmann, D. Krashen. Local-global principles for torsors over arithmetic curves, *Preprint 2011*, `arXiv:1108.3323`.

[10] T.Y. Lam. *Introduction to Quadratic Forms over Fields.* Graduate Studies in Mathematics, Vol. 67, American Mathematical Society, Providence, RI (2005).

[11] Q. Liu. *Algebraic Geometry and Arithmetic Curves.* Oxford Graduate Texts in Mathematics, Oxford University Press, Oxford, 2002.

[12] H. Matsumura. *Commutative ring theory.* Cambridge Studies in Advanced Mathematics Vol. 8, Cambridge University Press, Cambridge, 1986.

[13] M. Nagata. A theorem on valuation rings and its applications. *Nagoya Math. J.* **29** (1967): 85–91.

[14] J. Ohm. The ruled residue theorem for simple transcendental extensions of valued fields. *Proc. Am. Math. Soc.* **89** (1983): 16–18.

[15] R. Parimala and V. Suresh. The u-invariant of the function fields of p-adic curves. *Ann. of Math.* **172** (2010): 1391–1405.

[16] A. Pfister. Quadratische Formen in beliebigen Körpern. *Invent. Math.* **1** (1966): 116–132.

[17] A. Pfister. *Quadratic Forms with Applications to Algebraic Geometry and Topology.* LMS Lecture Notes Series, **217**. Cambridge University Press. Cambridge, 1995.

[18] W. Scharlau. *Quadratic and Hermitian forms.* Grundlehren der Mathematischen Wissenschaften, **270**. Springer-Verlag, Berlin, 1985.

[19] C. Scheiderer. The u-invariant of one-dimensional function fields over real power series fields. *Arch. Math.* **93** (2009): 245–251.

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