MODELS OF $Th(\mathbb{N})$ ARE IPs OF NICE $RCFs$

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Abstract. Exploring further the connection between exponentiation on real closed fields and the existence of an integer part modelling strong fragments of arithmetic, we demonstrate that each model of true arithmetic is an integer part of an exponential real closed field that is elementary equivalent to the reals with exponentiation and that each model of Peano arithmetic is an integer part of a real closed fields that admits an isomorphism between its additive and its multiplicative group of positive elements.

1. Introduction

This work originates in [3], where it was shown that a countable real closed field K has an integer part modelling PA iff it is recursively saturated. Marker (see [5]) gave a counterexample in the uncountable case by lifting exponentiation from the model to every real closed field of which it is an integer part. This was refined in [2] to the theorem that real closed fields with IPs modelling $I\Delta_0 + EXP$ always allow left exponentiation. It is natural to ask what influence a model of arithmetic has on the spectrum of real closed fields of which it is an IP. We show that models of true arithmetic are always IPs of real closed fields that are very similar to the reals with exponentiation in a model-theoretic sense: Namely, let us say that an RCF K is exponential iff there is $f : K \mapsto K$ such that $f : (K, +, 0, <) \simeq (K^{>0}, \cdot, 1, <)$ and that K is left-exponential iff there is an isomorphism from an additive group complement of the valuation ring to a multiplicative group complement of the positive group of units. Furthermore, let us say that an RCF K is a $\mathbb{R}_{exp}$-RCF iff there is a function $f : K \mapsto K$ such that $(K, +, \cdot, f, <)$ is elementary equivalent to $(\mathbb{R}, +, \cdot, exp, <)$, where exp denotes the usual exponentiation on the reals. Then each model of PA is an integer part of an exponential RCF and $Th(\mathbb{N})$ is an integer part of a $\mathbb{R}_{exp}$-RCF. We also show that this fails if one replaces true arithmetic with bounded arithmetic ($I\Delta_0$). We conjecture that Peano arithmetic is actually enough to achieve our results. We don't know where the exact benchmark is.

Our result can be seen as a further variation of one direction of a well-known theorem of Shepherdson (see [8]), according to which each model of open induction - Peano arithmetic with induction restricted to open (i.e. quantifier-free) formulas - is an integer part of an RCF. The result of [2] mentioned above implies that each model of $I\Delta_0 + EXP$ is an exponential integer part of a left exponential RCF. Together with the theorems proved in this paper, we hence get the following picture:

- $M \models IOpen \to M$ is an integer part of an RCF
- $M \models I\Delta_0 + EXP \to M$ is an (exponential) integer part of a left-exponential RCF
- $M \models PA \to M$ is integer part of an exponential RCF
- $M \models Th(\mathbb{N}) \to M$ is integer part of an $\mathbb{R}_{exp}$-RCF

Notation: If $\vec{v} = (v_1, ..., v_n)$, $\vec{v} \in M$ means that $\vec{v}$ is a sequence of elements of M.

2. The M-definable reals

The idea behind the following construction is to define M-reals as equivalence classes of convergent sequences of elements of the fraction field ff(M) of $-M \cup M$.

Definition 1. Let $M \models Th(\mathbb{N})$. A pre-real over M is a function $f : M \to (M \cup -M) \times M - \{0\}$ with definable graph, i.e. such that there is an $\mathfrak{L}_{PA}$-formula $\psi(x, y, z, \vec{p})$ and a finite sequence $\vec{v} \subseteq M$ such that $M \models \psi(x, y, z, \vec{v})$ iff $f(x) = (y, z)$.

Remark: Strictly speaking, this would not allow the first element of an element of the image to be in $-M$. This can be solved by a convention stating e.g. that $2n + 1$ denotes $-n$ while $2n$ denotes n. Since this does not cause any principal difficulties, we will, by slight abuse of notation, ignore this subtlety.
Also, when M is clear from the context, we will drop "over M". The mentioning of the parameter sequence $\vec{v}$ will usually also be dropped.

Definition 2. A pre-real over M given by some formula $\psi(x, y, z)$ is zero iff $M \models \forall m \exists n \forall k > n(\psi(k, a, b) \implies ma < b)$. It is convergent iff

$$M \models \forall m \exists n \forall k_1, k_2 > n(\psi(k_1, a_1, b_1) \wedge \psi(k_2, a_2, b_2) \implies m(a_1 b_2 - a_2 b_1) < a_2 b_2).$$

A convergent pre-real over M is an M-real. Two M-reals x_1, x_2 given by ψ_1 and ψ_2 are equivalent, written $x_1 \sim x_2$ iff $M \models \forall m \exists n \forall k > n(\psi_1(k, a, b) \wedge \psi_2(k, c, d) \implies m(ad - bc) < bd)$. Let $[x]_\sim$ denote the $\sim$-equivalence class of x when x is an M-real. Finally, we set $K_M := \{[x]_\sim | x \text{ is an } M\text{-real}\}$. If $n \in M$, then n_K denotes the equivalence class of the constant function on M which takes the value n everywhere; the subscript K is dropped wherever possible.

Definition 3. Let x and y be M-reals. Then we write $x < y$ iff there exist $m, k \in M$ such that, for all $l > k \in M$, $mx_l + 1 < my_l$.

From now on, we are almost exclusively interested in arithmetic formulas $\phi(\vec{v}, x, y)$ that define an M-real for every $\vec{v}$, i.e. such that

$$Th(\mathbb{N}) \models \forall \vec{v} \text{`} \phi(\vec{v}, x, y)$$

defines a total function from naturals to pairs of naturals with second element $\neq 0$ and this function gives rise to a convergent sequence'. Let us call such a formula ϕ a 'safe' formula.
What we would like to do is assume that all occuring formulas, unless stated otherwise, are of this kind. However, we have to ensure that by this restriction, we do not loose any M-reals.

Lemma 4. For any $\mathfrak{L}_{PA}$-formula $\phi(\vec{v}, x, y)$, there exists an $\mathfrak{L}_{PA}$-formula $\phi'(\vec{v}, x, y)$ such that it is a theorem of $Th(\mathbb{N})$ that, for every parameter $\vec{v}$, ϕ and ϕ' define the same function if ϕ defines a convergent total function and otherwise ϕ' defines the constant 0 function.

Proof. We abbreviate by $tot(\phi, \vec{v})$ the $\mathfrak{L}_{PA}$-formula expressing that $\phi(\vec{v}, x, y)$ defines a total function such that $\phi(\vec{v}, a, b) \wedge \pi_2(b) = c$ implies that $b \neq 0$. (Here π_2 is the function for obtaining the second element of a coded pair.) Then, $conv(\phi, \vec{v})$ expresses that $tot(\phi, \vec{v})$ and that $((a, b)|x \in M \wedge \phi(\vec{v}, a, b))$ defines a convergent sequence. Now let $\phi'(\vec{v}, x, y)$ be

$$(conv(\phi, \vec{v}) \wedge \phi(\vec{v}, x, y)) \vee (\neg conv(\phi, \vec{v}) \wedge \pi_1(y) = 0 \wedge \pi_2(y) = 1)$$

. This is obviously as desired. □

Corollary 5. If x is an M-real, then there exist a safe formula ϕ and a finite sequence $\vec{v} \subseteq M$ such that $\phi(\vec{v}, i, j)$ defines x.

Proof. Immediate from the last lemma. (Take the corresponding safe formula.) □

Proposition 6. There is an $\mathfrak{L}_{PA}[X, Y]$-formula $\phi_<(X, Y)$ such that, for all $x, y \in K_M$, $\phi(X \mapsto x, Y \mapsto y)$ holds iff $x < y$.

Proof. Immediate from the definition. □

Definition 7. Let $x = (x_i)_{i\in M}$ and $y = (y_i)_{i\in M}$ be M-reals with $x_i = \frac{a_i}{b_i}$ and $y_i = \frac{c_i}{d_i}$. We define $x +_M y$ by $(x_i + y_i)_{i\in M}$, where $x_i + y_i = \frac{a_i d_i + b_i c_i}{b_i d_i}$. Furthermore, we define $x \cdot_M y$ by $(x_i y_i)_{i\in M}$, where $x_i \cdot y_i = \frac{a_i c_i}{b_i d_i}$. The subscript M is dropped whenever there is no danger of confusion.

Proposition 8. K_M is closed under $+$ and $\cdot$.

Proof. Trivial. □

Lemma 9. $(K_M, +, \cdot, <)$ is an ordered field.

Proof. It is clear from the definition that $(K_M, +, <)$ and $(K^M - [0]_\sim, \cdot, <)$ are ordered abelian groups. The distributivity of $\cdot$ over $+$ is also immediate.
We proceed by showing that, for all $x \in K_M$, we have $x > 0$ iff there exists y such that $x = y^2$.
To see this, let $x \in K_M^{>0}$ be arbitrary, say $x = (\frac{p_i}{q_i})_{i\in M}$. As $x > 0$ and x is convergent, there must exist some $m \in M$ such that $p_i > 0$

for $i > m$. As $M \models Th(\mathbb{N})$, it holds in M that, for every $k \in M^{>0}$, there exists k' such that $k'^2 \leq k < (k'+1)^2$. Let $x' := (\frac{p'_{i+m}}{q'_{i+m}})_{i\in M}$. Then $M \models |\frac{p_{i+m}}{q_{i+m}} - (\frac{p'_i}{q'_i})^2| < \frac{3}{q_i}$. Since x is convergent, $(\frac{3}{q_i})_{i\in M}$ is also convergent, hence $x'^2 \sim x$. So x' is as desired.
In order to see that K_M is an ordered field, we finally show that -1 is not a sum of squares. Otherwise, let $-1 = x_1^2 + ... + x_n^2$ with $x_1, ..., x_n \in K_M$ By definition of K_M, there are formulas $\phi_1, ..., \phi_n$ and parameters $\vec{v}_1, ..., \vec{v}_n$ such that $\phi_i(\vec{v}_i, x)$ codes the M-real x_i. Hence $M \models \exists \vec{v}_1, ..., \vec{v}_n(x_1^2 + ... + x_n^2 = -1)$ (the term in the brackets appropriately expressed). By elementary equivalence, $\mathbb{N}$ is a model of the same statement. Hence -1 is a sum of squares in the reals, a contradiction. □

Theorem 10. Let $X_1, ..., X_n \in K_{\mathbb{N}}$, and let $Y : \mathbb{N} \to \mathbb{Z} \times \mathbb{N}^{>0}$.
(1) If Y is recursive in $X_1, ..., X_n$ and convergent, then $Y \in K_{\mathbb{N}}$.
(2) $K_{\mathbb{N}}$ is closed under the Turing-Jump, i.e. for $Y \in K_{\mathbb{N}}$, $n \in \mathbb{N}$, we have $Y^{(n)} \in K_{\mathbb{N}}$.

Proof. (1) Let P be a Turing programm such that $P^{\oplus_{i=1}^n X_i}(k) = y_k$ (the k-th bit of y) for all $k \in \mathbb{N}$. Let $\phi_P(v_1, v_2, X_1, ..., X_n)$ be a formula of $\mathfrak{L}_{PA}$ amended with n extra predicates such that, for all $i, j \in \mathbb{N}$, $Z_1, ..., Z_n \in \mathbb{R}$, $\phi_P(i, j, Z_1, ..., Z_n)$ holds in $\mathbb{N}$ iff $P^{\oplus_{i=1}^n Z_i}(i) \downarrow j$. Now consider $\tilde{\phi}_P(x, y)$ obtained by eliminating the X_i using their definition in K_M. (I.e. $X_1(t)$ would be replaced by $\exists \tilde{t}\phi_1(\tilde{t})$, where ϕ_1 defines X_1.) Then $\tilde{\phi}_P$ is an $\mathfrak{L}_{PA}$-formula defining Y. Hence $Y \in K_{\mathbb{N}}$.
(2) By arithmetical definability of the Turing jump. □

Proposition 11. $M_K := \{n_K | n \in M\} \subseteq K_M$.

Proof. Immediate, as constant functions are obviously definable over M. □

Proposition 12. $(M_K, 0_K, 1_K, +_K, \cdot_K, <_K) \equiv_{el} (M, 0, 1, +, \cdot, <)$.

Proof. : Obvious. □

Lemma 13. M_K is an integer part of K_M.

Proof. : For $(a, b) \in M \times M - \{0\}$, define $\lfloor \frac{a}{b} \rfloor$ to be the unique $k \in M$ such that $kb \leq a < (k+1)b$. If ψ defines a real r over M, then $\phi(x) \equiv \forall n \exists k > n \exists a, b(\psi(k, a, b) \wedge x = \lfloor \frac{a}{b} \rfloor)$ defines a subset S of M (which is clearly non-empty, as $\lfloor \frac{a}{b} \rfloor$ exists for all $a, b \in M$ since $M \models Th(\mathbb{N})$). As M is a model of true arithmetic and hence of full induction, S must have a least element s. By definition, there must be $k' \in M$ such that from k' on, the floor functions of the elements of r never drop below s. Also, there is some k'' such that, from k'' on, the elements of r are at most $\frac{1}{2}$ apart. If $k > max\{k', k''\}$, it follows that from k on, the only possible values of the floor function are s and $s+1$. We now distinguish

the following cases:
(1) From some point on, the floor function becomes constantly s. Then all elements or r eventually lie between s and $s+1$, hence $s_K \leq r < s_K +_K 1_K$.
(2) The floor function alternates cofinally many times between s and $s+1$. As r converges, this implies that the elements of r get arbitrarily close to $s+1$, so that $r \sim (s+1)_K$.
In both cases, r can be rounded down to an element of M_K. □

Proposition 14. : Let K be a real closed field, let Q be a dense subset of K, ε a positive element of K and let p be a polynomial such that, for all $q \in Q$, we have $p(q) \geq \varepsilon$. Then p has no zero in K.

Proof. : As K is an RCF, it inherits from $\mathbb{R}$ the property that polynomials are continuous. Hence, when we get arbitrarily close to a zero, the image has to become arbitrarily small, yet, by assumption, it remains above $\varepsilon > 0$, a contradiction. □

Convention: If $\phi(x, y, z, \vec{p})$ is an $\mathfrak{L}_{PA}$-formula and $\vec{v} \subseteq M$ is such that $\phi(x, y, z, \vec{v})$ defines an M-real, then this M-real is denoted by $x_\phi^{\vec{v}}$.

Lemma 15. : K_M is closed under square roots for positive elements, i.e. if $0 < c \in K_M$, then there exists $d \in K_M$ such that $c = d^2$.

Proof. : For every ϕ, there exists ψ such that $\mathbb{N} \models \forall \vec{v} \exists \vec{p} (x_\psi^{\vec{p}})^2 = x_\phi^{\vec{v}}$ by Theorem 10 since the square root of any $x \in \mathbb{R}$ is recursive in x. Hence M is a model of the same statement. Now, every $x \in K_M$ is defined by some ϕ and some parameters from M, it follows that K_M is closed under square roots of positive elements. □

Lemma 16. K_M is real closed.

Proof. : It suffices to show that K_M is formally real, closed under square roots for positive elements and that, for every $n \in \mathbb{N}$ and $c_0, ..., c_{2n+1} \in K_M$ with $c_{2n+1} \neq 0$, the polynomial $p(x) = \Sigma_{i=0}^{2n+1} c_i x^i$ has a root in K_M. We have already shown that K_M is closed under square roots for positive elements and formally real.
The proof that polynomials of odd degree have roots is similar to the proof of root-closure for positive elements: Such a root is (over $\mathbb{R}$) recursive in the coefficients of the polynomial. Hence, for every $n \in \mathbb{N}$ and every sequence $(\phi_0, ..., \phi_{2n+1})$ of formulas, there exists a formula ψ such that we have

$$\mathbb{N} \models \forall \vec{v}_0, ..., \vec{v}_{2n+1}((x_{\phi_{2n+1}}^{\vec{v}_{2n+1}} \neq 0) \implies (\exists \vec{v}(\Sigma_{i=1}^{2n+1} x_{\phi_i}^{\vec{v}_i} (x_\phi^{\vec{v}})^i = 0))).$$

So M is a model of the same statement. Thus every polyomial of odd degree over K_M has a root in K_M. □

3. Functions on K_M

In this section, we start considering analysis on K_M. To this purpose, we need to define functions on K_M. If properties of these functions are to be preserved between different K_Ms, these will have to be sufficiently explicitely definable in M. This is made precise by the following definition.

Definition 17. For $n \in \mathbb{N}$, $f : K_M^n \to K_M$ is M-definable iff there are $\phi[X_1, ..., X_n] \in \mathfrak{L}_{PA}[X_1, ..., X_n]$ (language of arithmetic with n extra predicate symbols $X_1, ..., X_n$) and $\vec{v} \in M$ such that, for any $\vec{x} \in K_M^n$, $\phi(X_1 \mapsto x_1, X_2 \mapsto x_2, ..., X_n \mapsto x_n, \vec{v}, i, j, k)$ defines an M-real y such that $f(x) = y$. Denote by $\text{Def}^n(M)$ the set of n-ary M-definable functions and let $\text{Def}(M) := \bigcup_{i \in \mathbb{N}} \text{Def}^i(M)$.

Proposition 18. If f is M-definable, then it is, for each ψ, uniformly definable (in the parameter $\vec{v}$) for all M-reals definable by ψ, i.e. there is a formula ψ', depending on ψ but not on $\vec{v}$, such that $\psi'(\vec{v}, ...)$ defines the image of each x if x is of the form $x_\psi^{\vec{v}}$ for some $\vec{v} \in M$.

Proof. Simply plug in the definition instead of the second-order variable X from the definition above. □

Proposition 19. $\text{Def}(M)$ contains all constant functions and is closed under composition.

Proof. Trivial. □

Definition 20. The exponential function $exp_M : K_M \to K_M$ (with base 2) is defined as follows:
For elements of $M^{>0}$, exponentiation with arbitrary bases is given by the usual arithmetical definition.
Now, for $a, b, n \in M^{>0}$, we let $appr(n, a, b)$ be the largest $m \in M$ such that $m^b \leq n^b a$.
Next, for $K_M^{>0} \ni x = (\frac{a_i}{b_i})_{i \in M}$, we assume without loss of generality that a_i and b_i are positive for all $i \in M$ and set
$exp(x) := (\frac{appr(i, exp(a_i), b_i)}{i})_{i \in M}$. Finally, if $x \in K_M^{<0}$, we suppose without loss of generality that for all i, $a_i < 0$ and $b_i > 0$ and let
$exp_M(x) = (\frac{i}{appr(i, exp(a_i), b_i)})_{i \in M}$.

Convention: Whenever possible without causing confusion, we will drop the subscripts.

Lemma 21. $exp_{\mathbb{N}} = exp_2 | K_{\mathbb{N}}$, where exp_2 is the usual real exponential function with base 2.

Proof. Trivial. □

Lemma 22. For every $M \models Th(\mathbb{N})$, exp_M is continuous.

Proof. We first note that continuity holds on the quotient field of M: As $\mathbb{N} \models \forall\varepsilon > 0\exists\delta > 0\forall p, q, r, s(|\frac{p}{q} - \frac{r}{s}| < \delta \implies |exp(\frac{p}{q}) - exp(\frac{r}{s})| < \varepsilon)$, M is a model of the same statement.
Now we show that, for any $\mathfrak{L}_{PA}$-formula ϕ, every $\vec{v} \in M$ and every $q \in \text{ff}(M)$, $q < x_\phi^{\vec{v}}$ implies $exp_M(q) < exp_M(x_\phi^{\vec{v}})$ and $x_\phi^{\vec{v}} < q$ implies $exp_M(x_\phi^{\vec{v}}) < exp_M(q)$. This follows from the fact that, for all $\phi \in \mathfrak{L}_{PA}$, we have $\mathbb{N} \models \forall\vec{v}\forall p\forall q \neq 0((x_\phi^{\vec{v}} < \frac{p}{q}) \implies (exp(x_\phi^{\vec{v}_1}) < exp(\frac{p}{q})))$ so that the same statement holds in M (and similarly for the other inequality). As every $x \in K_M$ is presentable as some $x_\phi^{\vec{v}}$, it follows that for all $x \in K_M$, $q \in \text{ff}(M)$, we have that $x < q$ implies $exp_M(x) < exp_M(q)$ and that $q < x$ implies $exp_M(q) < exp_M(x)$. But now, as $\text{ff}(M)$ is dense in K_M (since M is an IP of K_M), if $x, y \in K_M$ are such that $x < y$, then there exists $q \in \text{ff}(M)$ such that $x < q < y$. It follows that $exp_M(x) < exp_M(q) < exp_M(y)$, so $exp_M(x) < exp_M(y)$. Hence exp_M is monotonic.
The proof that exp_M is continuous is now quite straightforward: Let $x \in K_M$, then $exp(q) < exp(x) < exp(p)$ for all $q, p \in \text{ff}(M)$ with $q < x < p$. Let $\varepsilon > 0$ be given. Pick $\delta > 0$ such that, for all $\frac{p}{q}$ ($p, q \in M$) with $|x - \frac{p}{q}| < \delta$, we have that $|exp(x) - exp(\frac{p}{q})| < \varepsilon$.
To see that such a δ exists, let $x = x_\phi^{\vec{v}}$, where ϕ is safe. Clearly, we have

$$\mathbb{N} \models \forall\vec{p}\forall m > 0\exists n > 0\forall p, q \neq 0(|x - \frac{p}{q}| < \frac{1}{n} \to |exp(x) - exp(\frac{p}{q})| < \frac{1}{m}).$$

Hence M is a model of the same statement. If we take $0 < m \in M$ large enough such that $\frac{1}{m} < \varepsilon$ - which is possible since M is an integer part of K_M - and take $\vec{p} = \vec{v}$, this guarantees the existence of some $\delta \in K_M$ as desired.
Now, by monotonicity, it holds for $y \in K_M \cap]x - \delta, x + \delta[$ that $|exp(x) - exp(y)| < |exp(x) - exp(\frac{a}{b})| < \varepsilon$, where $a, b \in M$ are such that either $x - \delta < \frac{a}{b} < y < x$ oder $x < y < \frac{a}{b} < x + \delta$. (That such a choice of $\frac{a}{b}$ is always possible is again clear as $\text{ff}(M)$ is dense in K_M.) Hence δ is such that, for all $y \in K_M$, $|x - y| < \delta$ implies $exp_M(x) - exp_M(y) < \varepsilon$. As x and ε were arbitrary, it follows that exp_M is continuous.

□

Remark: The monotonicity is crucial in this argument; it can, however, be relaxed for other functions by splitting K_M into intervalls on which they are monotonic. This is particularly useful when one wants to turn to other functions.

Lemma 23. Let $f_1, ..., f_n, g$ be M-definable continuous functions. Then $g(f_1, ..., f_n)$ is also M-definable and continuous. Consequently, every function obtained from $+, \cdot, exp$ by composition is continuous.

Proof. M-definability of $g(f_1, ..., f_n)$ is obvious by substituting formulas. Continuity is also clear, as compositions of continuous functions are continuous. □

Theorem 24. $(K_{\mathbb{N}}, +, \cdot, exp, <) \equiv_{el} (\mathbb{R}, +, \cdot, exp, <)$.

Proof. By Wilkie's theorem (see [9]), the theory T_{exp} of $\mathbb{R}_{exp}$ is model complete and hence axiomatized by its A_2 (i.e. universal existential or $\forall\exists$) -consequences by Proposition 9.3 from [7]. It hence suffices to show that every A_2-formula that holds in $(\mathbb{R}, +, \cdot, exp, <)$ also holds in $(K_{\mathbb{N}}, +, \cdot, exp, <)$.

So let ϕ be an A_2-formula in the language of exponential rings that holds in $\mathbb{R}$, say $\phi \equiv \forall x_1, ..., x_n \exists y_1, ..., y_m(\psi(x_1, ..., x_n, y_1, ..., y_m)$, where $\psi(x_1, ..., x_n, y_1, ..., y_m)$ is a Boolean combination of statements of the form $t(x_1, ..., x_n, y_1, .., y_m) = 0$ and $t(x_1, ..., x_n, y_1, ..., y_m) > 0$ with t a term in the language of exponential rings.

We write ψ in disjunctive normal form, i.e. in the form

$$\bigvee_{i=1}^{N}(\bigwedge_{j=1}^{l_i} t_{ij}(x_1, ..., x_n, y_1, ..., y_m) = 0 \wedge \\ \bigwedge_{j=1}^{k_i} t'_{ij}(x_1, ..., x_n, y_1, ..., y_m) > 0). \ (*)$$

Note that we can eliminate negation by rewriting e.g. $(\neg t = 0 \wedge \psi)$ as $(t > 0 \wedge \psi) \vee (-t > 0 \wedge \psi)$ or $(\neg t > 0 \wedge \psi)$ as $(t = 0 \wedge \psi) \vee (-t > 0 \wedge \psi)$, so we will assume without loss of generality that only positive atomic formulas occur and that ψ is already written in this form.

Now fix $x_{\phi_1}^{\vec{v}_1}, ..., x_{\phi_n}^{\vec{v}_n} \in K_{\mathbb{N}}$. Since ϕ holds in $\mathbb{R}$, there exist $r_1, ..., r_m \in \mathbb{R}$ such that $\mathbb{R} \models \psi(x_{\phi_1}^{\vec{v}_1}, ..., x_{\phi_n}^{\vec{v}_n}, r_1, ..., r_m)$. Let us assume without loss of generality that it is the first disjunct

$$\bigwedge_{j=1}^{l_1} t_{1j}(x_{\phi_1}^{\vec{v}_1}, ..., x_{\phi_n}^{\vec{v}_n}, r_1, ..., r_m) = 0 \wedge \\ \bigwedge_{j=1}^{k_1} t'_{1j}(x_{\phi_1}^{\vec{v}_1}, ..., x_{\phi_n}^{\vec{v}_n}, r_1, ..., r_m) > 0$$

that is satisfied. By Lemma 23, every term in the language of exponential rings gives rise to a continuous function on $K_{\mathbb{N}}$. Hence, the t'_{1j} are continuous. Therefore, there are rational numbers $q_1, ..., q_m, p_1, ..., p_m$ such that $q_i < r_i < p_i$ for all $1 \leq i \leq m$, $1 \leq j \leq N$ and such that $t'_{1j}(x_{\phi_1}^{\vec{v}_1}, ..., x_{\phi_n}^{\vec{v}_n}, z_1, ..., z_m) > 0$ for all $(z_1, ..., z_m) \in \times_{i=1}^{m}[p_i, q_i]$, $1 \leq j \leq k_1$.

This holds in particular for all elements of $\mathbb{Q}$. Hence

$$\mathbb{N} \models \forall \vec{v}_1, ..., \vec{v}_m \forall a_1, ..., a_m \forall b_1, ..., b_m \neq 0 \exists \varepsilon > 0((\bigwedge_{\iota=1}^{m}(p_i < \frac{a_\iota}{b_\iota} < \\ q_i) \implies (\bigwedge_{j=1}^{k_1} t'_{1j}(x_{\phi_1}^{\vec{v}_1}, ..., x_{\phi_n}^{\vec{v}_n}, \frac{a_1}{b_1}, ..., \frac{a_m}{b_m}) > \varepsilon))$$

holds for all n-tuples of $\mathfrak{L}_{PA}$-formulas.

Now we define zeros for the t_{1j} in $\times_{i=1}^{m}[p_i, q_i]$, depending on $\phi_1, ..., \phi_n$, but not on the parameters $\vec{v}_1, ..., \vec{v}_n$: To do this, we define a sequence $(s_i)_{i\in\mathbb{N}}$ of m-tuples of rational intervalls as follows: $s_0 := ([p_j, q_j])_{j=1}^{m}$, and, for all $i \geq 0$, if $s_i = ([p_j^i, q_j^i]_{j=1}^{m})$, we let s_{i+1} be the first (in some

natural, e.g. lexicographic ordering) of the 2^m tuples $\{[r_1,s_1],...,[r_m,s_m]\}$ with $[r_j,s_j]\in\{[p_j^i,p_i^j+\frac{q_i^j}{2}],[p_i^j+\frac{q_i^j}{2},q_i^j]\}$ for all $1\le j\le m$ which contains, for every $\bar{n}\in\mathbb{N}$, a tuple of rationals $q_1,...,q_m$ such that $|t_{1j}(x_{\phi_1}^{\vec{v}_1},...,x_{\phi_n}^{\vec{v}_n},q_1,...,q_m)|<\frac{1}{m}$ for all $1\le j\le l_1$.
It is easy to see that this sequence of m-tuples is definable in $\mathbb{N}$ and converges to a simultanous solution to the k_1 equations in question. Hence ϕ holds in $K_{\mathbb{N}}$.
This implies that the A_2-theory of $(\mathbb{R},+,\cdot,exp,<)$ holds in $(K_{\mathbb{N}},+,\cdot,\exp,<)$. By the model completeness of the former, it follows that $\mathbb{R}$ and $K_{\mathbb{N}}$ are elementary equivalent. □

Theorem 25. For any $M\models Th(\mathbb{N})$, we have $(K_M,+,\cdot,exp,<)\equiv_{el}(K_{\mathbb{N}},+,\cdot,exp,<)$.

Proof. By Theorem 24, the theory of $(K_{\mathbb{N}},+,\cdot,exp,<)$ is just T_{exp}, the theory of real exponentiation. It hence suffices to show that all A_2-formulas that hold in $K_{\mathbb{N}}$ also hold in K_M. Hence, let ϕ be an A_2-statement as in the proof of Theorem 24 and suppose that $K_{\mathbb{N}}\models\phi$. This means that, for all $\phi_1,...,\phi_n\in\mathfrak{L}_{PA}$ and all $\vec{v}_1,...,\vec{v}_n\in K_{\mathbb{N}}$, there are $\phi_1',...,\phi_m'\in\mathfrak{L}_{PA}$ and $\vec{w}_1,...,\vec{w}_m\in K_{\mathbb{N}}$ such that $K_{\mathbb{N}}\models\psi(x_{\phi_1}^{\vec{v}_1},...,x_{\phi_n}^{\vec{v}_n},x_{\phi_1'}^{\vec{w}_1},...,x_{\phi_m'}^{\vec{w}_m})$. Note that statements of the form (*) above and hence of the form

$$\forall\vec{v}_1,...,\vec{v}_n\exists\vec{w}_1,...,\vec{w}_m\psi(x_{\phi_1}^{\vec{v}_1},...,x_{\phi_n}^{\vec{v}_n},x_{\phi_1'}^{\vec{w}_1},...,x_{\phi_m'}^{\vec{w}_m})$$

can be expressed as $\mathfrak{L}_{PA}$-formulas: Basically, the proof of Theorem 24 shows that, for every A_2-formula ϕ as above true in $K_{\mathbb{N}}$ and every n-tuple of $\mathfrak{L}_{PA}$-formulas $\phi_1,...,\phi_n$, there are $\mathfrak{L}_{PA}$-formulas $\phi_1',...,\phi_m'$ such that

$$\forall\vec{v}_1,...,\vec{v}_n\exists\vec{w}_1,...,\vec{w}_m\psi(x_{\phi_1}^{\vec{v}_1},...,x_{\phi_n}^{\vec{v}_n},x_{\phi_1'}^{\vec{w}_1},...,x_{\phi_m'}^{\vec{w}_m})$$

holds in $\mathbb{N}$. Consequently, the same holds in M. However, the formulas depend on the rational parameters, whose existence has to be carried over to M as well. We achieve this as follows: Let $\psi_1,...,\psi_n$ be $\mathfrak{L}_{PA}$-formulas. As ϕ holds in $K_{\mathbb{N}}$, we have

$$\begin{gathered}\mathbb{N}\models\forall\vec{v}_1,...\vec{v}_n\exists a_1,...,a_n,a_1',...,a_n'\exists b_1,...,b_n,b_1',...,b_n'\neq 0\exists C>0\\ \bigvee_{i=1}^{N}(((\forall c_1,...,c_n\forall d_1,...,d_n\neq 0\bigwedge_{j=1}^{n}\frac{a_j}{b_j}<\frac{c_j}{d_j}<\frac{a_j'}{b_j'}\implies\\ (\bigwedge_{j=1}^{k_i}t_{ij}'(x_{\psi_1}^{\vec{v}_1},...,x_{\psi_n}^{\vec{v}_n},\frac{c_1}{d_1},...,\frac{c_n}{d_n})>\frac{1}{C}))\wedge\\ (\forall C'>0\exists c_1,...,c_n\exists d_1,...,d_n\neq 0((\bigwedge_{j=1}^{n}\frac{a_j}{b_j}<\frac{c_j}{d_j}<\frac{a_j'}{b_j'})\wedge\\ (\bigwedge_{j=1}^{l_i}|t_{ij}(x_{\psi_1}^{\vec{v}_1},...,x_{\psi_n}^{\vec{v}_n},\frac{c_1}{d_1},...,\frac{c_n}{d_n})|<\frac{1}{C'})))))\end{gathered}$$

(i.e. for all choices of the parameters, there are a positive ε and a rational box B such that, for at least one of the disjoints in ψ (the quantifier-free part of ϕ, see the proof of Theorem 24), all t' are bigger than ε in B while the absolute values of the t at rational numbers in

B have no positive lower bound).
The same statement hence holds in M. That ϕ holds in K_M now follows from the continuity of the t and the t' in K_M.

□

We note the following useful consequence of the proof of Theorem 25:

Lemma 26. (The A_2-uniformisation lemma)
Let $K_M \models \forall x_1, x_2, ..., x_m \exists y_1, ..., y_n \psi(x_1, ..., x_m, y_1, ..., y_n, \vec{v})$, where ψ is quantifier-free and $\vec{v} \subseteq K_M$ is finite. Then, for every m-tuple $(\phi_1, ..., \phi_m)$ of formulas in the language $\mathfrak{L}_{exp}$ of ordered exponential rings, there exists an n-tuple $(\psi_1, ..., \psi_n)$ of $\mathfrak{L}_{exp}$-formulas such that $K_M \models \forall \vec{v}_1, ..., \vec{v}_m \exists \vec{w}_1, ..., \vec{w}_n \psi(x_{\phi_1}^{\vec{v}_1}, ..., x_{\phi_m}^{\vec{v}_m}, x_{\psi_1}^{\vec{w}_1}, ..., x_{\psi_n}^{\vec{w}_n}, \vec{v})$.

Proof. The proof of Theorem 25 shows how to obtain such formulas. □

Corollary 27. For every $M \models Th(\mathbb{N})$, we have

$$(K_M, +, \cdot, exp, <) \equiv_{el} (\mathbb{R}, +, \cdot, exp, <).$$

Consequently, every model of true arithmetic is an IP of a real closed exponential field modelling T_{exp}.

Proof. Immediate from Theorem 24, Theorem 25 and the fact that M is an IP of K_M that we proved above. □

3.1. **Weakening the base theory.** A considerable portion of the arguments from the preceeding section still goes through when we replace true arithmetic with PA. In this case, we get at least:

Theorem 28. Let $M \models PA$. Then there exists an RCF K such that $-M \cup M$ is an integer part of K and K carries an algebraic exponential exp, i.e. an isomorphism from its additive group to its multiplicative group of positive elements.

Proof. Construct K_M as in the last section, and define exp as there. $E(x + y) = E(x)E(y)$ is easily provable in PA for all $x, y \in K_M$, since it is provable component-wise and hence holds for the M-reals. □

A closer inspection of the argument reveals that further weakenings of the base theory are possible:

Corollary 29. Let $M \models I\Delta_0 + EXP$. Then there exists an exponential RCF K such that $-M \cup M$ is an integer part of K and K carries a full exponential exp, i.e. an isomorphism from its additive group to its multiplicative group of positive elements.

We can further strengthen this. Recall the following definition (see e.g. [6]):

Definition 30. A real closed exponential field K is a real closed field with an exponential function 2^x such that, for all $x, y \in K$, $n \in \mathbb{N}$

- $2^1 = 2$
- $2^{x+y} = 2^x 2^y$
- $x < y$ implies $2^x < 2^y$
- $x > n^2$ implies $x^n < 2^x$
- If $x > 0$, then there is $z \in K$ such that $2^z = x$

Theorem 31. Let $M \models PA$. Then there exists a real closed exponential field K with an exponential function 2^x such that $-M \cup M$ is an integer part of K and $2^x|M$ is the base 2 exponentiation of M.

Proof. This works along the lines of the proof of Theorem 28. □

4. A counterexample for bounded arithmetic

The results of the preceeding sections about true arithmetic and Peano Arithmetic stand in sharp contrast with the situation for the weaker fragment of bounded arithmetic ($I\Delta_0$). In this case, already quite weak notions of exponential may fail to occur.

Theorem 32. There is a model $M \models I\Delta_0$ such that, for no RCF K which has M as an IP, there exists $g : K \to K$ such that

$$(K, +, \cdot, g, <) \equiv_{el} (\mathbb{R}, +, \cdot, exp, <).$$

Proof. Let M be a bounded nonstandard model of $I\Delta_0$, i.e. there exists a nonstandard element $a \in M$ such that $\{a^i | i \in \omega\}$ is cofinal in M. As $(\mathbb{R}, +, \cdot, exp, <) \models \forall x > 1(exp(x) > x)$, we have that $g(a) > a$. Also, exp is monotonic and hence g is monotonic. It follows that $g(a^2) > a^i$ for all $i \in \omega$, as, in fact, we have $g(a^2) = g(aa) > g(ax)$ for all $x < a$, and since a is nonstandard, this holds in particular for all finite x. But as $\{a^i | i \in \omega\}$ is cofinal in K, such an element does not exist in K, hence exponentiation is not total in K, a contradiction. □

In fact, we can strengthen this further. The following definition comes from [4].

Definition 33. Let K be an RCF. A GA-exponential f on K is an isomorphism between $(K, +, <)$ and $(K^{>0}, \cdot, <)$ such that, for all $a \in K$ and $n \in \mathbb{N}$, we have that $a \geq n^2$ implies $f(a) > a^n$.

Theorem 34. There is a model $M \models I\Delta_0$ such that, for no RCF K which has M as an IP, there exists $g : K \to K$ such that g is a GAT-exponential on K.

Proof. Let M again be a bounded nonstandard model of $I\Delta_0$ as above and assume for a contradiction that f is a GA-exponential on M. Let $a \in M$ be nonstandard such that $\{a^i | i \in \mathbb{N}\}$ is cofinal in M and hence in K. As a is nonstandard, we have $a > n^2$ for all $n \in \mathbb{N}$. Hence, since f is a GA-exponential, we have $f(a) > a^n$ for every $n \in \mathbb{N}$. But this implies that $f(a)$ is strictly greater than every element of K, a contradiction. □

5. Real closures do not support M-compatible exponentials

In this section, we demonstrate that real closures of models of true arithmetic never support a 'real-like' exponential. This gives an upper limit for the effect that even an IP modelling true arithmetic has on an RCF. Note that this is not quite surprising, as even considering the standard model $\mathbb{N}$, we get $\mathbb{Q}^{rc}$ as the real closure of its fraction field, which does not admit an exponential.

Definition 35. Let $M \models PA$, let K be an RCF such that $-M \cup M$ is an IP of K and let $f : K \to K$. Then f is an M-compatible exponential on K iff:

(1) $(K, +, \cdot, f, <) \equiv_{el} (\mathbb{R}, +, \cdot, exp, <)$
(2) $f|M$ is the usual base-b-exponentiation on M for some $b \in M$.

To study real closures of models M of $Th(\mathbb{N})$, we consider how the real closure of ff$(-M \cup M)$ is coded in M.

Definition 36. Let $M \models Th(\mathbb{N})$, $Z = -M \cup M$ and $Q = \text{ff}(Z)$. An algebraic M-real is an M-real r such that, for some $p \in Q[X]$, $M \models p(r) = 0$.

Let us now fix an effective coding of polynomials over the fraction field, i.e. some function f with $dom(f) = M$ such that all finite sequences of elements of Q are in the range of f.

Lemma 37. There exists an $\mathfrak{L}_{PA}$-formula $\phi_{root}(c, i, j, q)$ such that, for every $M \models Th(\mathbb{N})$, any polynomial code c in M (which does not code the zero polynomial), every positive natural number i and every $j \in M$ there exists exactly one $q = q(j) \in Q$ such that $M \models \phi_{root}(c, i, j, q)$ and such that $r := (q(j)|j \in M)$ is an M-real such that $p(r) = 0$ and there are exactly $i - 1$ M-reals below r with this property, where p denotes the polynomial coded by c.

Proof. The formula ϕ_{root} is obtained by first finding L and U in M such that the polynomial coded by c is positive and increasing or negative and decreasing beyond U and negative and increasing or positive and decreasing before L. Then the possible zeros must lie in the interval $[L, U]$. This can be searched effectively by an appropriate Turing machine, whose description can then be coded into ϕ_{root}. □

Let us recall the Gelfond-Schneider theorem:

Theorem 38. Let α and β be algebraic and assume that $\beta \notin \mathbb{Q}$. Then α^{β} is transcendental.

Proof. See e.g. [1]. □

Theorem 39. Let $M \models Th(\mathbb{N})$, and let $R := [\text{ff}(-M \cup M)]^{rc}$. Then there is no M-compatible exponential f on R.

Proof. It is easy to see that M codes R: Simply take codes for polynomials over $\text{ff}(-M \cup M)$, then define sequences defining their roots. As $\mathbb{N}$ knows that for all $b \in \mathbb{N}$ and for the positive root r of $x^2 = 2$, b^r is transcendental (by the Gelfond-Schneider theorem), the same will hold in M. But $[\text{ff}(-M \cup M)]^{rc}$ only contains elements that are algebraic over $\text{ff}(-M \cup M)$, so $f(r)$ is undefined; since we clearly have $r \in R$ this contradicts the totality of f. □

Remark: As Gelfond-Schneider is likely to be provable in PA, and the rest of this section should go through as well, we can accordingly weaken the assumption of this theorem.

Question: In the situation of the last theorem, suppose $(R, +, \cdot, f, <) \equiv_{el} (\mathbb{R}, +, \cdot, exp, <)$ and $f[M] \subseteq M$. Does it follow that $f|M$ is an exponential in the sense of M?

6. Further work

The arguments from section 3 can be extended to any functions that are definable over M and preserve the model-completeness. In particular, it is tempting to apply Gabrielov's result on the model completeness of the reals with restricted analytic functions (see [QUELLE]) and Wilkies extensions thereof (see [9]).
Furthermore, the general picture is still quite rough; it would be nice to refine it by considering further fragments of arithmetic and the impact they have on possible exponentials of real closed fields containing models thereof as integer parts.

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