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MAXIMAL L_p-REGULARITY OF NON-LOCAL BOUNDARY VALUE PROBLEMS

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Abstract. We investigate the $\mathscr{R}$-boundedness of operator families belonging to the Boutet de Monvel calculus. In particular, we show that weakly and strongly parameter-dependent Green operators of nonpositive order are $\mathscr{R}$-bounded. Such operators appear as resolvents of non-local (pseudodifferential) boundary value problems. As a consequence, we obtain maximal L_p-regularity for such boundary value problems. An example is given by the reduced Stokes equation in waveguides.

1. Introduction

During the last decade, the theory of maximal L_p-regularity turned out to be an important tool in the theory of nonlinear partial differential equations and boundary value problems. Roughly speaking, maximal regularity in the sense of well-posedness of the linearized problem is the basis for a fixed-point approach to show (local in time) unique solvability for the nonlinear problem. Here, the setting of L_p-Sobolev spaces with $p \neq 2$ is helpful in treating the nonlinear terms, due to better Sobolev embedding results. Meanwhile, a large number of equations from mathematical physics has been successfully treated by this method, in particular in fluid dynamics and for free boundary problems. Exemparily, we mention [Ama05] for the general concept of maximal regularity and [EPS03] for one of the first applications in fluid mechanics.

A densely defined closed operator $A : \mathscr{D}(A) \subset X \to X$ in a Banach space X is said to have maximal L_p-regularity, $1 < p < \infty$, in the interval $I = (0, T)$ with $0 < T \leq \infty$ if the Cauchy problem

$$u'(t) - Au(t) = f(t) \quad (t \in I), \qquad u(0) = 0,$$

has, for any right-hand side $f \in L_p(I, X)$, a unique solution u satisfying

$$\|u'\|_{L_p(I,X)} + \|Au\|_{L_p(I,X)} \leq C\|f\|_{L_p(I,X)}$$

with a constant C independent of f. Here, $W_p^1(I, X)$ refers to the standard X-valued first-order Sobolev space. If I is finite or A is invertible an equivalent formulation is that the map

$$\frac{d}{dt} - A : {}_0W_p^1(I, X) \cap L_p(I, \mathscr{D}(A)) \longrightarrow L_p(I, X)$$

is an isomorphism, where ${}_0W_p^1(I, X)$ denotes the space of all elements in $W_p^1(I, X)$ with vanishing time trace at $t = 0$. Note that non-zero initial values can be treated by an application of related trace theorems. A standard approach to prove maximal regularity is based on operator-valued Mikhlin type results due to Weis [Wei01] and the concept of $\mathscr{R}$-boundedness (see [DHP03], [KW04]). For a short introduction to $\mathscr{R}$-boundedness, see Section 2 of this paper.

In many applications, the operator A is given as the L_p-realization of a differential boundary value problem. Under appropriate ellipticity and smoothness assumptions, maximal regularity is known to hold in this case (see, e.g., [DHP03]). However, several applications demand for generalizations to non-local (pseudodifferential) operators and boundary value problems. For instance, the Dirichlet-to-Neumann map in a bounded domain leads to a pseudodifferential operator on the boundary, i.e. on a closed manifold. An example for a non-local boundary value problem is obtained by the pseudodifferential approach to the Stokes equation as developed by Grubb and Solonnikov [GS91] (see also [Gru95] and [GK93]), which was also one of our motivations.

In the present short note we analyze the $\mathscr{R}$-boundedness of operator families belonging to the so-called Boutet de Monvel calculus with parameter. This is a pseudodifferential calculus containing, in particular, the resolvents to a vast class of non-local boundary value problems which allows to describe in great detail the micro-local fine structure of such resolvents. An exemplary application of the calculus is the following theorem (which, in fact, is a simplified version of Theorem 3.2.7 of Grubb [Gru86]):

Theorem 1.1. *Let $A(\mu)$, $\mu \in \Sigma$ (an angular subsector of the complex plane), be a parameter-dependent second order differential operator on a compact manifold M with smooth boundary and $G(\mu)$ be a weakly parameter-dependent Green operator of order and type less than or equal to 2 and regularity at least $1/2$. Let γ_0 and γ_1 denote Dirichlet and Neumann boundary conditions, respectively. If the parameter-dependent boundary value problem*

$$\begin{pmatrix} A(\mu)+G(\mu) \\ \gamma_j \end{pmatrix} :\ H_p^s(M) \longrightarrow \begin{matrix} H_p^{s-2}(M) \\ \oplus \\ B_{pp}^{s-j-1/p}(\partial M) \end{matrix}, \qquad s > 1+1/p,$$

with $p \in (1,\infty)$ is parameter-elliptic then it is an isomorphism for $|\mu|$ sufficiently large, and

$$\begin{pmatrix} A(\mu)+G(\mu) \\ \gamma_j \end{pmatrix}^{-1} = \begin{pmatrix} P(\mu) & K(\mu) \end{pmatrix}, \tag{1.1}$$

with $P(\mu) \in B^{-2,0,\nu}(M;\Sigma)$ and a parameter-dependent Poisson operator $K(\mu)$ of order $-j$.

The involved operator classes as well as the meaning of parameter-ellipticity will be explained in the sequel; the mentioned Green operators are certain non-local operators that are smoothing in the interior of M, but on the whole manifold with boundary have a finite order. As a consequence of (1.1),

$$A(\mu)+G(\mu) : \{u \in H_p^2(M) \mid \gamma_j u = 0\} \subset L_p(M) \longrightarrow L_p(M)$$

is invertible for large μ with inverse $P(\mu) \in B^{-2,0,\nu}(M;\Sigma)$. Making use of this specific pseudodifferential structure we shall derive, in particular, that $\{(1+|\mu|)^2 P(\mu) \mid \mu \in \Sigma\} \subset \mathscr{L}(L_p(M))$ is $\mathscr{R}$-bounded, cf. Theorem 4.4. For the proof we also adopt a tensor-product argument first used in [DK07] in the analysis of the $\mathscr{R}$-boundedness of parameter-dependent families of "scattering" or "SG-psudodifferential" operators (which, roughly speaking, allows to reduce considerations to constant coefficient operators) and use general results of Kalton, Kunstmann and Weis [KKW06] on the behaviour of $\mathscr{R}$-boundedness under interpolation and duality.

There are different versions of Boutet de Monvel's calculus, one with a *strong*, the other with a *weak* parameter-dependence. The first calculus is essentially designed to handle fully differential problems, and is described, for example, in Schrohe, Schulze [SS94]. The second is a broader calculus developed by Grubb allowing the investigation of certain non-local problems, see Grubb [Gru86] and Grubb, Kokholm [GK93] for instance. Actually, we shall blend these two versions and consider operator families depending on two parameters, where one enters in the strong way and the other only weakly; for details see Section 3. Though this combination cannot be found explicitly in the literature, we shall use it freely and avoid giving any proofs, since these are quite standard (though laborious if done with all necessary details). Our main result is Theorem 4.4 stating that such operator families are $\mathscr{R}$-bounded as operator families in the L_p-space of the bounded manifold. An application is provided in Section 5 where we consider a resolvent problem for the Stokes operator in a wave guide (i.e. cylindrical domain) with compact, smoothly bounded cross-section.

Boutet de Monvel's calculus can also be exploited to demonstrate existence of bounded imaginary powers and even of a bounded H_∞-calculus, cf. Duong [Duo90], Abels [Abe05] and Coriasco, Schrohe, Seiler [CSS07] for example; as it turns out, the strategy of proof we use in the present work is closely related to that of [Abe05]. On the one hand bounded H_∞-calculus is stronger than $\mathscr{R}$-boundedness, on the other hand the concept of $\mathscr{R}$-boundedness applies to operator-families more general than the resolvent of a fixed operator.

2. A short rewiew of $\mathscr{R}$-boundedness

We will briefly recall the definition of $\mathscr{R}$-boundedness and some results that will be important for our purpose. For more detailed expositions we refer the reader to [DHP03] and [KW04]. Throughout this section we let X, Y, Z denote Banach spaces.

A set $T \subset \mathscr{L}(X,Y)$ is called $\mathscr{R}$-bounded if there exists a $q \in [1,\infty)$ such that

$$\mathscr{R}_q(T) := \sup\Big\{\Big(\sum_{z_1,\ldots,z_N=\pm1}\Big\|\sum_{j=1}^N z_j A_j x_j\Big\|^q\Big)^{1/q}\Big(\sum_{z_1,\ldots,z_N=\pm1}\Big\|\sum_{j=1}^N z_j x_j\Big\|^q\Big)^{-1/q}\Big\}$$

is finite, where the supremum is taken over all $N \in \mathbb{N}$, $A_j \in T$ and $x_j \in X$ (for which the denominator is different from zero, of course). The number $\mathscr{R}_q(T)$ is called the $\mathscr{R}$-bound of T. It is a consequence of Kahane's inequality that finiteness of $\mathscr{R}_q(T)$ for a particular q implies finiteness for any other choice of $q \geq 1$. Therefore q is often suppressed from the notation. Clearly an $\mathscr{R}$-bounded set is norm bounded and its norm-bound is majorized by its $\mathscr{R}$-bound. In case both X and Y are *Hilbert spaces* $\mathscr{R}$-boundedness is equivalent to norm-boundedness.

If $S, T \subset \mathscr{L}(X, Y)$ and $R \subset \mathscr{L}(Y, Z)$ are $\mathscr{R}$-bounded then $S + T$ and RS are $\mathscr{R}$-bounded, too, with

$$\mathscr{R}(S+T) \leq \mathscr{R}(S) + \mathscr{R}(T), \qquad \mathscr{R}(RS) \leq \mathscr{R}(R)\mathscr{R}(S).$$

Under mild assumptions on the involved Banach spaces $\mathscr{R}$-boundedness is well behaved under duality and interpolation:

Theorem 2.1 (Proposition 3.5 in [KKW06])**.** *Let T be an $\mathscr{R}$-bounded subset of $\mathscr{L}(X, Y)$ and assume that X is B-convex*[1]*. Then*

$$T' := \big\{A' \mid A \in T\big\} \qquad (\textit{set of dual operators})$$

is an $\mathscr{R}$-bounded subset of $\mathscr{L}(Y', X')$ with $\mathscr{R}(T') \leq C\mathscr{R}(T')$ with a constant $C \geq 0$ not depending on T.

Theorem 2.2 (Proposition 3.7 in [KKW06])**.** *Let (X_0, X_1) and (Y_0, Y_1) be two interpolation couples with both X_0 and X_1 being B-convex. Let $T \subset \mathscr{L}(X_0{+}X_1, Y_0{+}Y_1)$ such that $T \subset \mathscr{L}(X_j, Y_j)$ is $\mathscr{R}$-bounded with $\mathscr{R}$-bound κ_j for $j = 0, 1$. Then*

$$T \subset \mathscr{L}\big((X_0, X_1)_{\theta,p}, (Y_0, Y_1)_{\theta,p}\big), \qquad 0 < \theta < 1, \quad 1 < p < \infty,$$

is $\mathscr{R}$-bounded with $\mathscr{R}$-bound $\kappa \leq \kappa_0^{1-\theta}\kappa_1^{\theta}$, where $(\cdot,\cdot)_{\theta,p}$ refers to the real interpolation method.

The following result (Proposition 3.3 in [DHP03]) is very useful in analyzing the $\mathscr{R}$-boundedness of families of integral operators.

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^n$ be open, $1 < p < \infty$, and assume that*

$$(K_0 f)(\omega) = \int_\Omega k_0(\omega, \omega') f(\omega)\, d\omega$$

defines an integral operator $K_0 \in \mathscr{L}(L_p(\Omega))$. Let $\big\{k_\lambda : \Omega \times \Omega \to \mathscr{L}(X, Y) \mid \lambda \in \Lambda\big\}$ be a family of measurable integral kernels and $T = \big\{K_\lambda \mid \lambda \in \Lambda\big\}$ be the set of associated integral operators. If

$$\mathscr{R}_p\Big(\big\{k_\lambda(\omega, \omega') \mid \lambda \in \Lambda\big\}\Big) \leq k_0(\omega, \omega') \qquad \textit{for all } \omega, \omega' \in \Omega$$

then $T \subset \mathscr{L}\big(L_p(\Omega, X), L_p(\Omega, Y)\big)$ is $\mathscr{R}$-bounded with

$$\mathscr{R}_p\Big(\big\{K_\lambda \mid \lambda \in \Lambda\big\}\Big) \leq \|K_0\|_{\mathscr{L}(L_p(\Omega))}.$$

[1] For a definition of B-convexity we refer the reader to [KKW06]. For us it will be sufficient to know that L_p-spaces with $1 < p < \infty$ are B-convex.

Now consider $a : \mathbb{R}^\ell \times \mathbb{R}^\ell \to \mathscr{L}(X,Y)$ and let

$$[\mathrm{op}(a)u](y) = \int e^{iy\eta} a(y,\eta)\widehat{u}(\eta)\, đ\eta, \qquad u \in \mathscr{S}(\mathbb{R}^\ell, X),$$

denote the associated pseudodifferential operator, where $đ\eta = (2\pi)^{-\ell/2}d\eta$. Under suitable assumptions on a we have $\mathrm{op}(a) : \mathscr{S}(\mathbb{R}^\ell, X) \to L_p(\mathbb{R}^\ell, Y)$, say. One may ask when this operator induces a continuous map $L_p(\mathbb{R}^\ell, X) \to L_p(\mathbb{R}^\ell, Y)$. In answering this question the concept of $\mathscr{R}$-boundedness plays a decessive role. For example, Girardi and Weis [GW03] have shown the following:

Theorem 2.4. *Let both X and Y have properties $(\mathcal{HT})$ and (α).*[2] *Let $T \subset \mathscr{L}(X,Y)$ be $\mathscr{R}$-bounded. Then*

$$\Big\{op(a) \mid a \in \mathscr{C}^\ell(\mathbb{R}^\ell_\eta \setminus \{0\}, \mathscr{L}(X,Y)) \text{ with } \eta^\alpha D^\alpha_\eta a(\eta) \in T \text{ for all } \eta \neq 0 \text{ and } \alpha \in \{0,1\}^\ell\Big\}$$

is an $\mathscr{R}$-bounded subset of $\mathscr{L}\big(L_p(\mathbb{R}^\ell, X), L_p(\mathbb{R}^\ell, Y)\big)$ with $\mathscr{R}$-bound less than or equal to $C\mathscr{R}(T)$ for some constant C not depending on T.

In other words, this Theorem of Girardi and Weis is the operator-valued generalization of the classical theorem of Lizorkin on the continuity of Fourier multipliers in L_p-spaces. As an immediate consequence one obtains:

Corollary 2.5. *Denote by $S^d_{\mathscr{R}}(\mathbb{R}^\ell; X, Y)$, $d \in \mathbb{R}$, the space of all smooth functions $a : \mathbb{R}^\ell_\eta \to \mathscr{L}(X,Y)$ such that $T_\alpha(a) := \big\{\langle\eta\rangle^{-d+|\alpha|} D^\alpha_\eta a(\eta) \mid \eta \in \mathbb{R}^\ell\big\}$ is an $\mathscr{R}$-bounded subset of $\mathscr{L}(X,Y)$ for any choice of the multi-index α. As shown in* [DK07], *this is a Fréchet space, by taking as semi-norms the $\mathscr{R}$-bounds of $T_\alpha(a)$. If both X and Y have properties $(\mathcal{HT})$ and (α) then* op *induces a continuous mapping*

$$S^0_{\mathscr{R}}(\mathbb{R}^\ell; X, Y) \longrightarrow \mathscr{L}\big(L_p(\mathbb{R}^\ell, X), L_p(\mathbb{R}^\ell, Y)\big).$$

For the interested reader, we refer to Portal-Strkalj [PŠ06] for a more general result on the L_p-continuity of pseudodifferential operators with symbols in operator valued $S^0_{\varrho,\delta}$-classes of Hörmander type.

3. Boutet de Monvel's calculus with parameters

In this section, we will present some elements of a parameter-dependent version of Boutet de Monvel's calculus [BdM71]which we use to describe solution operators of parameter-elliptic boundary value problems subject to homogeneous boundary conditions. The elements of this calculus are operators of the form

$$P(\tau,\mu) = A_+(\tau,\mu) + G(\tau,\mu) :\ \mathscr{S}(\mathbb{R}^n_+) \longrightarrow \mathscr{S}(\mathbb{R}^n_+) \tag{3.1}$$

(extending by continuity to Sobolev spaces), where $\mathbb{R}^n_+$ denotes the half-space

$$\mathbb{R}^n_+ = \Big\{x = (x', x_n) \in \mathbb{R}^n \mid x_n > 0\Big\}$$

[2]For the definition of these properties we refer the reader to [KW04] or [DHP03]. For us it is sufficient to know that scalar-valued L_p-spaces, $1 < p < \infty$, have these properties.

and $\mathscr{S}(\mathbb{R}^n_+)$ consists of all functions obtained by restricting rapidly decreasing functions from $\mathbb{R}^n$ to the half-space $\mathbb{R}^n_+$ (this space is a Fréchet space by identification with the quotient space $\mathscr{S}(\mathbb{R}^n)/N$, where $N := \{u \in \mathscr{S}(\mathbb{R}^n_+) \mid u = 0 \text{ on } \mathbb{R}^n_+\}$ is a closed subspace of $\mathscr{S}(\mathbb{R}^n)$).

In (3.1) $A_+(\tau,\mu)$ is a parameter-dependent pseudodifferential operator and $G(\tau,\mu)$ a so-called parameter-dependent Green operator (one also speaks of *singular* Green operators; however for convenience we omit the term 'singular'). We shall consider two classes of Green operators which are *weakly* and *strongly* parameter-dependent, respectively.

In the following, we let Σ denote a closed sector in the two-dimensional plane with vertex at the origin. We call a function smooth on Σ provided all partial derivatives exist in the interior and extend continuously to Σ.

We shall frequently make use of pseudodifferential symbols taking values in Fréchet spaces. To this end, let us give the following definition:

Definition 3.1. *Let E be a Fréchet space with a system $\{p_j \mid j \in \mathbb{N}\}$ of semi-norms determining its topology. We let $S^d(\mathbb{R}^m;E)$, $d \in \mathbb{R}$, denote the space of all smooth functions $a : \mathbb{R}^m \to E$ satisfying uniform estimates*

$$q_{j,\alpha}(a) := \sup_{y\in\mathbb{R}^m} p_j\big(\langle y\rangle^{|\alpha|-d} D^\alpha_y a(y)\big) < \infty \tag{3.2}$$

for every j and every multi-index α. These semi-norms make $S^d(\mathbb{R}^m;E)$ a Fréchet space. In case $E = \mathbb{C}$ we suppress E from the notation.

The subspace $S^d_{\mathrm{cl}}(\mathbb{R}^m;E)$ consists, by definition, of those symbols that have an expansion into homogeneous components: there exist $a_{(d-\ell)} \in \mathscr{C}^\infty(\mathbb{R}^m \setminus \{0\}, E)$ satisfying

$$a_{(d-\ell)}(ty) = t^{d-\ell} a_{(d-\ell)}(y), \qquad t > 0, \quad y \neq 0,$$

such that

$$R_N(a)(y) := a(y) - \sum_{\ell=0}^{N-1} \chi(y) a_{(d-\ell)}(y) \;\in\; S^{d-N}(\mathbb{R}^m;E)$$

for any $N \in \mathbb{N}$, where χ denotes an arbitrary zero-excision function.

Note that the space of smooth positively homogeneous functions $\mathbb{R}^m \setminus \{0\} \to E$ of a fixed degree is canonically isomorphic to $\mathscr{C}^\infty(\mathbb{S}^{m-1},E)$, the smooth E-valued functions on the unit-sphere in $\mathbb{R}^m$. We then equip $S^d_{\mathrm{cl}}(\mathbb{R}^m;E)$ with the projective topology with respect to the maps

$$\begin{aligned} a \mapsto a_{(d-\ell)} &: \; S^d_{\mathrm{cl}}(\mathbb{R}^m;E) \longrightarrow \mathscr{C}^\infty(\mathbb{S}^{m-1},E), \\ a \mapsto R_N(a) &: \; S^d_{\mathrm{cl}}(\mathbb{R}^m;E) \longrightarrow S^{d-N}(\mathbb{R}^m;E), \end{aligned}$$

where N and ℓ run through the non-negative integers. It will be of some importance for us that

$$S^d_{\mathrm{cl}}(\mathbb{R}^m;E) = S^d_{\mathrm{cl}}(\mathbb{R}^m)\,\widehat{\otimes}_\pi\, E, \tag{3.3}$$

where $F\widehat{\otimes}_\pi E$ denotes the completed projective tensor-product of the two Fréchet spaces E and F, see for example [Trè67]. In other words, $S^d_{\rm cl}(\mathbb{R}^m;E)$ can be identified with the closure of the algebraic tensor product

$$S^d_{\rm cl}(\mathbb{R}^m)\otimes E=\Big\{\sum_{i=1}^N a_ie_i\;\Big|\;N\in\mathbb{N},\ a_i\in S^d_{\rm cl}(\mathbb{R}^m),\ e_i\in E\Big\}$$

with respect to the system of semi-norms

$$\widehat{q}_{j,\alpha}(a)=\inf\Big\{\sum_{i=1}^N q_\alpha(a_i)p_j(e_i)\;\Big|\;a=\sum_{i=1}^N a_ie_i\Big\},$$

where q_α is as in (3.2) with $E=\mathbb{C}$.

3.1. **Parameter-dependent pseudodifferential operators.** Let us denote by

$$S^d_{\rm const}(\mathbb{R}^n\times\mathbb{R}\times\Sigma),\qquad d\in\mathbb{R}, \tag{3.4}$$

the space of all smooth functions $a:\mathbb{R}^n_\xi\times\mathbb{R}_\tau\times\Sigma_\mu\longrightarrow\mathbb{C}$ satisfying

$$\sup_{(\xi,\tau,\mu)\in\mathbb{R}^n\times\mathbb{R}\times\Sigma}\big|D^\alpha_\xi D^k_\tau D^\gamma_\mu a(\xi,\tau,\mu)\big|\langle\xi,\tau,\mu\rangle^{|\alpha|+|\gamma|+k-d}<\infty$$

for every order of derivatives. This is a Fréchet space and we can define

$$S^d(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}\times\Sigma):=S^0_{\rm cl}(\mathbb{R}^n_x;S^d_{\rm const}(\mathbb{R}^n\times\mathbb{R}\times\Sigma)). \tag{3.5}$$

With a symbol a from (3.5) we associate a family of pseudodifferential operators

$$A(\mu,\tau)=\mathrm{op}(a)(\mu,\tau):\mathscr{S}(\mathbb{R}^n)\to\mathscr{S}(\mathbb{R}^n)$$

in the standard way, i.e.,

$$[A(\mu,\tau)u](x)=\int e^{ix\xi}a(x,\xi,\tau,\mu)\widehat{u}(\xi)\,đ\xi.$$

This map can be extended to a map $\mathscr{S}'(\mathbb{R}^n)\to\mathscr{S}'(\mathbb{R}^n)$ in the space of tempered distributions. Now let

$$e_+:\mathscr{S}(\mathbb{R}^n_+)\longrightarrow\mathscr{S}'(\mathbb{R}^n),\qquad r_+:\mathscr{S}'(\mathbb{R}^n)\longrightarrow\mathscr{D}'(\mathbb{R}^n_+),$$

be the operators of extension by 0 and restriction to the half-space, respectively. For $A(\mu,\tau)$ as above we set

$$A_+(\mu,\tau)=\mathrm{op}(a)_+(\tau,\mu):=r_+\circ A(\mu,\tau)\circ e_+.$$

This gives rise to a map $\mathscr{S}(\mathbb{R}^n_+)\to\mathscr{C}^\infty(\mathbb{R}^n_+)$, for example. If $d=0$ it induces maps

$$A_+(\mu,\tau):L_p(\mathbb{R}^n_+)\longrightarrow L_p(\mathbb{R}^n_+),\qquad 1<p<\infty. \tag{3.6}$$

It is this mapping (3.6) we will be most interested in, and we shall analyze it below for the symbol class we have just introduced. However, for motivations of the calculus (for example, to ensure that $A_+(\tau,\mu)$ preserves the space $\mathscr{S}(\mathbb{R}^n_+)$ and that the operators behave nicely under standard operations like composition) one actually needs to require an additional property of the symbols: the so-called two-sided *transmission condition* with respect to the boundary of $\mathbb{R}^n_+$. For a symbol a of order d as above the condition requires that, for any choice of $k\in\mathbb{N}_0$,

$$\mathscr{F}^{-1}_{\xi_n\to z}D^k_{x_n}p(x',0,\xi',\langle\xi',\tau,\mu\rangle\xi_n,\tau,\mu)\Big|_{\pm z>0}\in S^0\Big(\mathbb{R}^{n-1}_{x'};S^d\big(\mathbb{R}^{n-1}_{\xi'}\times\mathbb{R}\times\Sigma;\mathscr{S}(\mathbb{R}_\pm)\big)\Big),$$

i.e., the restriction of the distribution $\mathscr{F}^{-1}_{\xi_n\to z} D^k_{x_n} p(x',0,\xi',\langle\xi',\tau,\mu\rangle\xi_n,\tau,\mu)\in\mathscr{S}'(\mathbb{R}_z)$ to the half-space $\mathbb{R}_+$ or $\mathbb{R}_-$ can be extended to a rapidly decreasing function on $\mathbb{R}$, and the other variables enter as parameters in the indicated specific way. Here, $\langle\xi',\tau,\mu\rangle := (1+|\xi'|^2+\tau^2+|\mu|^2)^{1/2}$. Symbols with the transmisssion condition form a closed subspace of $S^d(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}\times\Sigma)$ that we shall denote by

$$S^d_{\mathrm{tr}}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}\times\Sigma). \tag{3.7}$$

Remark 3.2. *The operator $A_+(\mu,\tau)=\mathrm{op}(a)_+(\tau,\mu)$ does not depend on the values of the symbol a for $x_n<0$. Hence, if we define $S^d_-(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}\times\Sigma)$ as the closed subspace of symbols from $S^d_{\mathrm{tr}}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}\times\Sigma)$ whose x-support is contained in half-space $\{x\in\mathbb{R}^n\mid x_n\le 0\}$, then the class of operators is isomorphic to the quotient*

$$S^d_{\mathrm{tr}}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}\times\Sigma)\big/S^d_-(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}\times\Sigma),$$

yielding a natural Fréchet topology.

3.2. **Parameter-dependent Green operators.** We shall use the splitting $\mathbb{R}^n_+=\mathbb{R}^{n-1}\times\mathbb{R}_+$ and write $x=(x',x_n)$ and, correspondingly, $\xi=(\xi',\xi_n)$ for the co-variable ξ to x. Roughly speaking, Green operators in tangential direction (i.e., on $\mathbb{R}^{n-1}$) act like pseudodifferential operators while in normal direction (i.e., on $\mathbb{R}_+$) they act like integral operators with smooth kernel. However, there is a certain twisting between the two directions which reflects in a specific structure of the operators. To describe this structure we shall need the function ϱ defined by

$$\varrho(\xi',\tau,\mu):=\langle\mu\rangle\langle\xi',\tau,\mu\rangle^{-1}. \tag{3.8}$$

Note that $0<\varrho\le 1$. Now let

$$R^{d,\nu}_{\mathrm{const}}(\mathbb{R}^{n-1}\times\mathbb{R}\times\Sigma),\qquad d,\nu\in\mathbb{R}, \tag{3.9}$$

denote the space of all smooth scalar-valued functions $k(\xi',\tau,\mu;x_n,y_n)$ satisfying uniform estimates

$$\begin{aligned}\big\|x_n^\ell D^{\ell'}_{x_n} y_n^m D^{m'}_{y_n} D^{\alpha'}_{\xi'} D^k_\tau D^\gamma_\mu k(\xi',\tau,\mu;x_n,y_n)\big\|_{L^2(\mathbb{R}_{+,x_n}\times\mathbb{R}_{+,y_n})}\\ \le C_{\alpha',\ell,\ell',m,m'}(k)\Big(\varrho(\xi',\tau,\mu)^{\nu-[\ell-\ell']_+-[m-m']_+-|\alpha'|-k}+1\Big)\times\\ \times\langle\xi',\tau,\mu\rangle^{d-\ell+\ell'-m+m'-|\alpha'|-k-|\gamma|},\end{aligned} \tag{3.10}$$

for every order of derivatives and any $\ell,m\in\mathbb{N}_0$; here $[s]_+=\max(s,0)$ for any real number s. We call such a k a *weakly parameter-dependent* symbol kernel of order d and regularity ν (with constant coefficients), see also [Gru86]. The best constants define a system of semi-norms, yielding a Fréchet topology. We set

$$R^{d,\nu}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}\times\mathbb{R}\times\Sigma)=S^0_{\mathrm{cl}}(\mathbb{R}^{n-1}_{x'};R^{d,\nu}_{\mathrm{const}}(\mathbb{R}^{n-1}\times\mathbb{R}\times\Sigma)) \tag{3.11}$$

The class of *strongly parameter-dependent* symbol kernels

$$R^d_{\mathrm{const}}(\mathbb{R}^{n-1}\times\mathbb{R}\times\Sigma):=\bigcap_{\nu\in\mathbb{R}}R^{d,\nu}_{\mathrm{const}}(\mathbb{R}^{n-1}\times\mathbb{R}\times\Sigma) \tag{3.12}$$

consists of those symbol kernels satisfying the uniform estimates

$$\begin{aligned}(3.13)\qquad &\left\|x_n^\ell D_{x_n}^{\ell'} y_n^m D_{y_n}^{m'} D_{\xi'}^\alpha D_\tau^k D_\mu^\gamma k(\xi',\tau,\mu;x_n,y_n)\right\|_{L_2(\mathbb{R}_{+,x_n}\times\mathbb{R}_{+,y_n})}\\ &\qquad\le C_{\alpha',\ell,\ell',m,m'}(k)\langle\xi',\tau,\mu\rangle^{d-\ell+\ell'-m+m'-|\alpha|-|\gamma|-k}\end{aligned}$$

for any order of derivatives and any $\ell, m\in\mathbb{N}_0$. Again this is a Fréchet space, and we have

$$\begin{aligned}(3.14)\qquad R^d(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}\times\mathbb{R}\times\Sigma) &:= \bigcap_{\nu\in\mathbb{R}} R^{d,\nu}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}\times\mathbb{R}\times\Sigma)\\ &= S^0_{\mathrm{cl}}(\mathbb{R}^{n-1}_{x'}; R^d_{\mathrm{const}}(\mathbb{R}^{n-1}\times\mathbb{R}\times\Sigma)).\end{aligned}$$

Definition 3.3. *A weakly parameter-dependent Green operator $G(\tau,\mu)=\mathrm{op}(k)(\tau,\mu)$ of order $d\in\mathbb{R}$,* type $r=0$, *and regularity ν is of the form*

$$(3.15)\qquad [G(\tau,\mu)u](x)=\int e^{ix'\xi'}\int_0^\infty k(x',\xi',\tau,\mu;x_n,y_n)\mathcal{F}_{y'\to\xi'}u(\xi',y_n)\,dy_n đ\xi'$$

where $k\in R^{d,\nu}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}\times\mathbb{R}\times\Sigma)$ is a weakly parameter-dependent symbol kernel of order d and regularity ν as introduced above. Parameter-dependent Green operators of order $d\in\mathbb{R}$, type $r\in\mathbb{N}$, and regularity ν have the form

$$(3.16)\qquad G(\tau,\mu)=G_0(\tau,\mu)+\sum_{j=1}^r G_j(\tau,\mu)D^j_{x_n}$$

where each G_j has order $d-j$, type 0, and regularity ν. We shall denote this class of operators by $G^{d,r,\nu}(\mathbb{R}^n_+;\mathbb{R}\times\Sigma)$. Analogously, we obtain the classes $G^{d,r}(\mathbb{R}^n_+;\mathbb{R}\times\Sigma)$ of strongly parameter-dependent Green operators, using strongly parameter-dependent symbols kernel. The subclasses $G^{d,r,\nu}_{\mathrm{const}}$ and $G^{d,r}_{\mathrm{const}}$ refer to symbol kernels that do not depend on the x'-variable.

All the previously introduced spaces of Green operators inherit a Fréchet topology from the underlying spaces of symbol kernels (factoring out the ambiguity of representing Green operators as different linear combinations).

Below we shall make use of an alternative characterisation of strongly parameter-dependent Green operators (see Theorem 3.7 in [Sch01], for example):

Proposition 3.4. *Any strongly parameter-dependent Green operator of order d and type 0 has a symbol kernel of the form*

$$k(x',\xi',\tau,\mu;x_n,y_n)=\widetilde{k}(x',\xi',\tau,\mu;\langle\xi',\tau,\mu\rangle x_n,\langle\xi',\tau,\mu\rangle y_n).$$

Here,

$$\widetilde{k}(x',\xi',\tau,\mu;s_n,t_n)\in S^0_{\mathrm{cl}}\Big(\mathbb{R}^{n-1}_{x'};S^{d+1}\big(\mathbb{R}^{n-1}\times\mathbb{R}\times\Sigma;\mathscr{S}(\mathbb{R}_{+,s_n}\times\mathbb{R}_{+,t_n})\big)\Big),$$

where $\mathscr{S}(\mathbb{R}_+\times\mathbb{R}_+)=\mathscr{S}(\mathbb{R}^2)\big|_{\mathbb{R}_+\times\mathbb{R}_+}$ and $S^d(\mathbb{R}^{n-1}\times\mathbb{R}\times\Sigma;E)$ for a Fréchet space E is defined as in Definition 3.1, replacing $\mathbb{R}^m$ by $\mathbb{R}^{n-1}\times\mathbb{R}\times\Sigma$.

3.3. **Some elements of the calculus.** Having described parameter-dependent pseudodifferential and Green operators let us introduce the spaces

$$B^{d,r}_{(\mathrm{const})}(\mathbb{R}^n_+;\mathbb{R}\times\Sigma),\qquad B^{d,r,\nu}_{(\mathrm{const})}(\mathbb{R}^n_+;\mathbb{R}\times\Sigma)$$

consisting of operators $A_+(\tau,\mu)+G(\tau,\mu)$ with a parameter-dependent pseudodifferential operator of order $d\in\mathbb{Z}$ as in Section 3.1 and a – strongly or weakly – parameter-dependent Green operator of order d, type $r\in\mathbb{N}_0$, and regularity $\nu\ge 0$ as described in Section 3.2. Using the topologies of both pseudodifferential operators and Green operators introduced above we obtain natural topologies as non-direct sums of Fréchet spaces. Considering the parameter-dependent operators as families of operators $\mathscr{S}(\mathbb{R}^n_+)\to\mathscr{S}(\mathbb{R}^n_+)$, the following results are true:

Theorem 3.5 (Theorem 5.1 in [GK93])**.** *The pointwise composition of operators induces continuous mappings*

$$B^{d_0,r_0,\nu_0}(\mathbb{R}^n_+;\mathbb{R}\times\Sigma)\times B^{d_1,r_1,\nu_1}(\mathbb{R}^n_+;\mathbb{R}\times\Sigma)\longrightarrow B^{d,r,\nu}(\mathbb{R}^n_+;\mathbb{R}\times\Sigma)$$

where

$$d=d_0+d_1,\quad r=\max\{r_1,r_0+d_1\},\quad \nu=\min\{\nu_0,\nu_1\}.$$

Moreover, the subclass of Green operators forms an ideal, i.e., is preserved under composition from the left or the right by operators of the full class. Similar statements hold for the classes of strongly parameter-dependent operators.

Theorem 3.6 (Theorem 5.1 in [GK93])**.** *If $d\le 0$, taking the (formal) adjoint with respect to the $L_2(\mathbb{R}^n_+)$-inner products induces continuous mappings*

$$B^{d,0;\nu}(\mathbb{R}^n_+;\mathbb{R}\times\Sigma)\longrightarrow B^{d,0;\nu}(\mathbb{R}^n_+;\mathbb{R}\times\Sigma).$$

The subclasses of Green operators are preserved under taking adjoints.

It has been shown in [GK93] that the operators extend by continuity from the spaces of Schwarz functions to L_p-Sobolev spaces. In fact, if we set, with $s\in\mathbb{R}$ and $1<p<\infty$,

$$H^s_p(\mathbb{R}^n_+)=\Big\{u|_{\mathbb{R}^n_+}\mid u\in H^s_p(\mathbb{R}^n)\Big\}\cong H^s_p(\mathbb{R}^n)/N^s_p,$$

where

$$N^s_p:=\Big\{u\in H^s_p(\mathbb{R}^n)\mid \operatorname{supp}u\subset\mathbb{R}^{n-1}\times(-\infty,0]\Big\},$$

then any element of $B^{d,r,\nu}(\mathbb{R}^n_+;\mathbb{R}\times\Sigma)$, $\nu\ge 1/2$, induces pointwise (i.e., for any value of (τ,μ)) mappings

$$H^s_p(\mathbb{R}^n_+)\longrightarrow H^{s-d}_p(\mathbb{R}^n_+),\qquad s>r-1+\frac{1}{p}. \tag{3.17}$$

4. $\mathscr{R}$-boundedness of families from Boutet de Monvel's calculus

Due to (3.17), operators of non-positive order and type zero induce families of continuous operators in L_p-spaces. We are now going to analyze the $\mathscr{R}$-boundedness of these families. First we consider strongly parameter-dependent Green operators. They can be treated using their particular symbol kernel structure exhibited in Proposition 3.4.

Theorem 4.1. *Let $d \le 0$. Then*

$$G^{d,0}(\mathbb{R}^n_+;\mathbb{R}\times\Sigma) \hookrightarrow S^d_{\mathscr{R}}(\mathbb{R}\times\Sigma;L_p(\mathbb{R}^n_+),L_p(\mathbb{R}^n_+))$$

(*where the latter space is defined as in Corollary 2.5, replacing $\mathbb{R}^\ell$ by $\mathbb{R}\times\Sigma$*).

Proof. For convenience we shall use the short-hand notations $G^{d,0}$, $G^{d,0}_{\rm const}$, and $S^d_{\mathscr{R}}$.

Step 1. We first consider operators with symbol kernel independent of x'. Let $G \in G^{d,0}_{\rm const}$ have symbol kernel k. Define $\mathfrak{g}(\xi',\tau,\mu): L_p(\mathbb{R}_+) \to L_p(\mathbb{R}_+)$ by

$$[\mathfrak{g}(\xi',\tau,\mu)u](x_n) = \int_0^\infty k(\xi',\tau,\mu;x_n,y_n)u(y_n)\,dy_n. \tag{4.1}$$

Then $G(\tau,\mu)$ can be understood as the Fourier multiplier with symbol $\mathfrak{g}(\cdot,\tau,\mu)$. In view of Theorem 2.4 it suffices to show that

$$\Big\{\langle\tau,\mu\rangle^{-d+k+|\gamma|}\langle\xi'\rangle^{|\alpha|}D^\alpha_{\xi'}D^k_\tau D^\gamma_\mu\mathfrak{g}(\xi',\tau,\mu) \mid (\xi',\tau,\mu)\in\mathbb{R}^{n-1}\times\mathbb{R}\times\Sigma\Big\} \tag{4.2}$$

is an $\mathscr{R}$-bounded subset of $\mathscr{L}(L_p(\mathbb{R}_+))$. Since

$$\langle\tau,\mu\rangle^{-d+k+|\gamma|}\langle\xi'\rangle^{|\alpha|} \le \langle\xi',\tau,\mu\rangle^{-d+|\alpha|+k+|\gamma|},$$

this follows with Kahane's contraction principle[3] if we show the $\mathscr{R}$-boundedness of

$$\Big\{\langle\xi',\tau,\mu\rangle^{-d+|\alpha|+k+|\gamma|}D^\alpha_{\xi'}D^k_\tau D^\gamma_\mu\mathfrak{g}(\xi',\tau,\mu) \mid (\xi',\tau,\mu)\in\mathbb{R}^{n-1}\times\mathbb{R}\times\Sigma\Big\}. \tag{4.3}$$

Since $\langle\xi',\tau,\mu\rangle^{-d+|\alpha|+k+|\gamma|}D^\alpha_\xi D^k_\tau D^\gamma_\mu\mathfrak{g}$ is a finite linear combination of symbols like $\mathfrak{g}$ we may assume without loss of generality that $|\alpha| = k = |\gamma| = 0$. Then we can estimate

$$\begin{aligned}\big|k(\xi',\tau,\mu;x_n,y_n)\big| &= \big|\widetilde{k}(\xi',\tau,\mu;\langle\xi',\tau,\mu\rangle x_n,\langle\xi',\tau,\mu\rangle y_n)\big|\\ &\le C\,\langle\xi',\tau,\mu\rangle^{d+1}\big(\langle\xi',\tau,\mu\rangle(x_n+y_n)\big)^{-1}\\ &\le C\,\langle\tau,\mu\rangle^d\frac{1}{x_n+y_n},\end{aligned}$$

since $\widetilde{k}$ behaves like a symbol of order $d+1$ in (ξ',τ,μ) and is rapidly decreasing in (s_n,t_n). Now the $\mathscr{R}$-boundedness of (4.3) follows from Theorem 2.3.

Since $G^{d,0}_{\rm const}$ is continuously embedded in $\mathscr{C}(\mathbb{R}\times\Sigma;\mathscr{L}(L_p(\mathbb{R}^n_+))$, the Closed Graph Theorem implies the continuity of the embedding into $S^d_{\mathscr{R}}$.

[3]This principle states that the inequality

$$\sum_{z_1,\ldots,z_N=\pm1}\Big\|\sum_{j=1}^N z_j\alpha_jx_j\Big\|^q \le 2^q\sum_{z_1,\ldots,z_N=\pm1}\Big\|\sum_{j=1}^N z_j\beta_jx_j\Big\|^q$$

holds true whenever $\alpha_j,\beta_j\in\mathbb{C}$ with $|\alpha_j|\le|\beta_j|$ and $x_1,\ldots,x_N\in X$ with arbitrary N.

Step 2. Due to Step 1, $G^{d,0}_{\text{const}} \hookrightarrow S^d_{\mathscr{R}}$. In other words, for any semi-norm $p(\cdot)$ of $S^d_{\mathscr{R}}$ there exists a semi-norm $q(\cdot)$ of $G^{d,0}_{\text{const}}$ such that $p(G) \le q(G)$ for any $G \in G^{d,0}_{\text{const}}$. For a function $f \in S^0_{\text{cl}} := S^0_{\text{cl}}(\mathbb{R}^{n-1}_{x'})$ let M_f denote the operator of multiplication, $M_f \in \mathscr{L}(L_p(\mathbb{R}^n_+))$. By (3.14) and (3.3) we have the identification $G^{d,0} = S^0_{\text{cl}} \widehat{\otimes}_\pi G^{d,0}_{\text{const}}$. Now let

$$G = \sum_{j=1}^N M_{f_j} G_j, \qquad f_j \in S^0_{\text{cl}}, \quad G_j \in G^{d,0}_{\text{const}}.$$

Then G belongs to $S^d_{\mathscr{R}}$ and, with $p(\cdot)$ and $q(\cdot)$ as above,

$$p(G) \le \sum_{j=1}^N p\big(M_{f_j} G_j\big) = \sum_{j=1}^N \|f_j\|_\infty p(G_j) \le \sum_{j=1}^N \|f_j\|_\infty q(G_j),$$

where $\|\cdot\|_\infty$ is the supremum-norm. By passing to the infimum over all possibilities to represent G as such a linear combination we get

$$p(G) \le \inf\Big\{ \sum_{j=1}^N \|f_j\|_\infty q(G_j)\big) \;\Big|\; G = \sum_{j=1}^N M_{f_j} G_j \Big\} =: \widehat{q}(G).$$

However, $\widehat{q}(\cdot)$ induces a continuous semi-norm on the projective tensor product $S^0_{\text{cl}} \widehat{\otimes}_\pi G^{d,0}_{\text{const}}$, cf. the discussion after (3.3). Since $p(\cdot)$ was arbitrary, we conclude that $S^0_{\text{cl}} \widehat{\otimes}_\pi G^{d,0}_{\text{const}} \hookrightarrow S^d_{\mathscr{R}}$. ◻

It seems that the direct proof of Theorem 4.1 does not generalize to the case of weakly parameter-dependent Green operators. Thus we proceed differently, combining results of Grubb-Kokholm [GK93] on mapping properties of Green operators in weighted L_2-spaces and the stability of $\mathscr{R}$-boundedness under interpolation. To this end we shall make use of the spaces

$$L^\delta_2(\mathbb{R}_+) = L^2(\mathbb{R}_+, t^{2\delta} dt), \qquad \delta \in \mathbb{R}. \tag{4.4}$$

Theorem 4.2 (Theorem 1.9 of [GK93])**.** *Let $p \ge 2$ be given. Then, for any choice of $0 < \delta' < \frac12 - \frac1p < \delta < 1$,*

$$\big(H^{\delta'}_2(\mathbb{R}_+), H^\delta_2(\mathbb{R}_+)\big)_{\theta,p} \hookrightarrow L_p(\mathbb{R}_+) \hookrightarrow \big(L^{-\delta'}_2(\mathbb{R}_+), L^{-\delta}_2(\mathbb{R}_+)\big)_{\theta,p}$$

where θ is chosen such that $\theta\delta + (1-\theta)\delta' = \frac12 - \frac1p$ and $(\cdot,\cdot)_{\theta,p}$ refers to real interpolation.

Moreover, let us introduce the Fréchet space $S^d_{\mathscr{R},w}(\mathbb{R}\times\Sigma; X, Y)$ of smooth functions $a : \mathbb{R}\times\Sigma \to \mathscr{L}(X,Y)$ for which the sets

$$T_{k,\gamma}(a) := \big\{ \langle\tau,\mu\rangle^{-d} \langle\tau\rangle^k \langle\mu\rangle^{|\gamma|} D^k_\tau D^\gamma_\mu a(\tau,\mu) \mid (\tau,\mu) \in \mathbb{R}\times\Sigma \big\}$$

are $\mathscr{R}$-bounded for any choice of k and γ. The semi-norms are defined as the $\mathscr{R}$-bounds of the sets $T_{k,\gamma}$.

Theorem 4.3. *If $d \le 0$ and $\nu \ge 1/2$ then*

$$G^{d,0;\nu}(\mathbb{R}^n_+; \mathbb{R}\times\Sigma) \hookrightarrow S^d_{\mathscr{R},w}(\mathbb{R}\times\Sigma; L_p(\mathbb{R}^n_+), L_p(\mathbb{R}^n_+)).$$

Proof. By Theorem 2.1 and Theorem 3.6 we may assume that $p \geq 2$. Using a tensor product argument as in the second step of the proof of Theorem 4.1 reduces the proof to showing that $G^{d,0;\nu}_{\text{const}} \hookrightarrow S^d_{\mathscr{R},w}$.

Thus let $G \in G^{d,0;\nu}_{\text{const}}$. We have to show that $\{\langle\tau,\mu\rangle^{-d}\langle\tau\rangle^k\langle\mu\rangle^{|\gamma|}D^k_\tau D^\gamma_\mu G(\tau,\mu) \mid (\tau,\mu) \in \mathbb{R}\times\Sigma\}$ is an $\mathscr{R}$-bounded subset of $\mathscr{L}(L_p(\mathbb{R}^n_+))$. To this end represent G as a Fourier multiplier with symbol $\mathfrak{g}(\xi',\tau,\mu)$ as done in the proof of Theorem 4.1. Due to Theorem 2.4 it suffices to show that

$$\Big\{\langle\tau,\mu\rangle^{-d}\langle\tau\rangle^k\langle\mu\rangle^{|\gamma|}\langle\xi'\rangle^{|\alpha|}D^\alpha_{\xi'}D^k_\tau D^\gamma_\mu \mathfrak{g}(\xi',\tau,\mu) \mid (\xi',\tau,\mu) \in \mathbb{R}^{n-1}\times\mathbb{R}\times\Sigma\Big\}.$$

is an $\mathscr{R}$-bounded subset of $\mathscr{L}(L_p(\mathbb{R}_+))$. Observing that

$$\langle\tau,\mu\rangle^{-d}\langle\tau\rangle^k\langle\mu\rangle^{|\gamma|}\langle\xi'\rangle^{|\alpha|} \leq \langle\xi',\tau,\mu\rangle^{-d+|\gamma|}\langle\xi',\tau\rangle^{k+|\alpha|},$$

that $\langle\xi',\tau,\mu\rangle^{|\gamma|}D^\gamma_\mu\mathfrak{g}$ has the same structure as $\mathfrak{g}$, and using Kahane's contraction principle, we may assume $\gamma = 0$ and show that

$$M_{\alpha,k} := \Big\{\langle\xi',\tau,\mu\rangle^{-d}\langle\xi',\tau\rangle^{k+|\alpha|}D^\alpha_{\xi'}D^k_\tau\mathfrak{g}(\xi',\tau,\mu) \mid (\xi',\tau,\mu) \in \mathbb{R}^{n-1}\times\mathbb{R}\times\Sigma\Big\} \tag{4.5}$$

is an $\mathscr{R}$-bounded subset of $\mathscr{L}(L_p(\mathbb{R}_+))$. We know from Theorem 4.1.(5) of [GK93] (see actually (4.15) in its proof) that for any $0 < \varepsilon < \frac{1}{2}$

$$M_{\alpha,k} \subset \mathscr{L}(L_2^{-\varepsilon}(\mathbb{R}_+), H_2^\varepsilon(\mathbb{R}_+)) \tag{4.6}$$

is a bounded set. Since the involved spaces are Hilbert spaces, boundedness coincides with $\mathscr{R}$-boundedness. Then using Theorem 4.2 (with $\varepsilon = \delta$ and $\varepsilon = \delta'$ where $0 < \delta' < \frac{1}{2} - \frac{1}{p} < \delta < \frac{1}{2}$, respectively) and Theorem 2.2 we obtain the $\mathscr{R}$-boundedness of $M_{\alpha,k}$ in $\mathscr{L}(L_p(\mathbb{R}_+))$.

Since from [GK93] we know that the norm-bound of $M_{\alpha,k}$ can be estimated in terms of semi-norms of G, an application of the Closed Graph Theorem yields the continuity of the embedding. □

Finally, let us consider a family of pseudodifferential operators

$$A_+(\tau,\mu) = \text{op}_+(a)(\tau,\mu) : L_p(\mathbb{R}^n_+) \longrightarrow L_p(\mathbb{R}^n_+)$$

with a symbol $a \in S^d(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}\times\Sigma)$ with $d \leq 0$, cf. 3.5. Since we consider the operator between L_p-spaces only (and not between Sobolev spaces of higher regularity) it is now not necesssary to require the transmission property for a. We will show that

$$A_+ \in S^d_{\mathscr{R}}(\mathbb{R}\times\Sigma; L_p(\mathbb{R}^n_+), L_p(\mathbb{R}^n_+)).$$

Since $\text{op}_+(a) = r_+\text{op}(a)e_+$ with the continuous operators $e_+ : L_p(\mathbb{R}^n_+) \to L_p(\mathbb{R}^n)$ and $r_+ : L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n_+)$ of extension and restriction, respectively, it suffices to show that

$$\text{op}(a) \in S^d_{\mathscr{R}}(\mathbb{R}\times\Sigma; L_p(\mathbb{R}^n), L_p(\mathbb{R}^n)).$$

Again by a tensor product argument analogous to that of Step 2 in the proof of Theorem 4.1, we can assume that a has constant coefficients, i.e., $a \in S^d_{\text{const}}$. However, then the statement follows immediately from Theorem 2.4, choosing there $X = Y = \mathbb{C}$. Thus we can conclude:

Theorem 4.4. *Let $d \leq 0$, $\nu \geq 1/2$, and $1 < p < \infty$. Then*

$$B^{d,0}(\mathbb{R}^n_+;\mathbb{R}\times\Sigma) \hookrightarrow S^d_{\mathscr{R}}(\mathbb{R}\times\Sigma;L_p(\mathbb{R}^n_+),L_p(\mathbb{R}^n_+)),$$
$$B^{d,0;\nu}(\mathbb{R}^n_+;\mathbb{R}\times\Sigma) \hookrightarrow S^d_{\mathscr{R},w}(\mathbb{R}\times\Sigma;L_p(\mathbb{R}^n_+),L_p(\mathbb{R}^n_+)).$$

Recalling the definition of the spaces $S^d_{\mathscr{R}}$ and $S^d_{\mathscr{R},w}$ this means that if $P(\tau,\mu) \in B^{d,0}(\mathbb{R}^n_+;\mathbb{R}\times\Sigma)$ and $Q(\tau,\mu)\in B^{d,0;\nu}(\mathbb{R}^n_+;\mathbb{R}\times\Sigma)$ then

$$\big\{\langle\tau,\mu\rangle^{-d+k+|\gamma|}D^k_\tau D^\gamma_\mu P(\tau,\mu) \mid (\tau,\mu)\in\mathbb{R}\times\Sigma\big\},$$
$$\big\{\langle\tau,\mu\rangle^{-d}\langle\tau\rangle^k\langle\mu\rangle^{|\gamma|}D^k_\tau D^\gamma_\mu Q(\tau,\mu) \mid (\tau,\mu)\in\mathbb{R}\times\Sigma\big\}$$

are $\mathscr{R}$-bounded subsets of $\mathscr{L}(L_p(\mathbb{R}^n_+))$.

Corollary 4.5. *Let $P(\tau,\mu)\in B^{d,0}(\mathbb{R}^n_+;\mathbb{R}\times\Sigma)$ or $P(\tau,\mu)\in B^{d,0,\nu}(\mathbb{R}^n_+;\mathbb{R}\times\Sigma)$ with $d\leq 0$, $\nu\geq 1/2$, and $p\in(1,\infty)$. Define $(\mathrm{op}_\tau P)(\mu) := \mathscr{F}^{-1}_{\tau\to r}P(\tau,\mu)\mathscr{F}_{r\to\tau}$. Then we have*

$$\mathrm{op}_\tau P \in S^d_{\mathscr{R}}\big(\Sigma;L_p(\mathbb{R}^{n+1}_+,L_p(\mathbb{R}^{n+1}_+))\big).$$

Proof. Due to Theorem 2.4, we have to show that for $k=0,1$ and all $\gamma\in\mathbb{N}^n_0$, the set

$$\big\{\langle\mu\rangle^{-d+|\gamma|}D^\gamma_\mu\tau^kD^k_\tau P(\tau,\mu) \mid \tau\in\mathbb{R}\setminus\{0\},\mu\in\Sigma\big\}$$

is $\mathscr{R}$-bounded. But in both cases, this follows from Kahane's inequality and Theorem 4.4, as

$$\langle\mu\rangle^{-d+|\gamma|}\tau^k \leq \langle\tau,\mu\rangle^{-d+k+|\gamma|}, \qquad \langle\tau,\mu\rangle^{-d}\langle\tau\rangle^k\langle\mu\rangle^{|\gamma|} \leq \langle\tau,\mu\rangle^{-d+k+|\gamma|}.$$

□

In applications, the complex parameter μ is related to the spectral parameter λ appearing in the resolvent of the L_p-realization of a non-local boundary value problem. We included a second parameter $\tau\in\mathbb{R}$ in order to be able to treat additional parameters arizing from the problem itself, e.g., in the form of a covariable in the unbounded direction of a waveguide. In this case, Corollary 4.5 leads to maximal L_p-regularity by an application of the Theorem of Weis [Wei01].

5. Maximal L_p-regularity for non-local BVPs in a wave-guide

We will study non-local boundary value problems in a wave-guide, i.e., on a cylinder $\mathbb{R}\times M$ whose cross-section is a smooth compact manifold M with boundary ∂M. For this, we need to provide some material on Boutet de Monvel's calculus on manifolds and the corresponding concept of parameter-ellipticity. We follow [Gru86] and [GK93]. As an application, we study the reduced Stokes problem in a waveguide in Section 5.2.

5.1. **Manifolds with boundary and parameter-ellipticity.** In this section we indicate how the calculus can be modified to cover domains with smooth boundary and how it is used to describe solution operators of certain non-local boundary value problems. In the sequel we let M denote a smooth compact manifold with boundary. In view of the formulation of parameter-ellipticity given below, we need to descibe a refined subclass of the class of Green operators introduced in Section 3.2 as well as to introduce another type of operators, the so-called Poisson operators.

5.1.1. *Polyhomogeneous Green operators.* Let $G(\tau,\mu)$ be a weakly parameter-dependent Green operator of order d, type 0, and regularity ν as described in Definition 3.3. We call $G(\tau,\mu)$ polyhomogeneous or classical if there exists a sequence of Green operators $G_{d-j}(\tau,\mu)$, $j\in\mathbb{N}_0$, such that, for any $N\in\mathbb{N}_0$,

$$G(\tau,\mu)-\sum_{j=0}^{N-1}G_{d-j}(\tau,\mu)\;\in\;G^{d-N,0,\nu-N}(\mathbb{R}^n_+;\mathbb{R}\times\Sigma),$$

and if k_{d-j} are the symbol kernels associated with G_{d-j} as in (3.15) and (3.10) (with d replaced by $d-j$) it holds

$$k_{d-j}(x',t\xi',t\tau,t\mu;x_n/t,y_n/t)=t^{d-j}k_{d-j}(x',\xi',\tau,\mu;x_n,y_n) \tag{5.1}$$

whenever $t\ge 1$ and $|\xi'|\ge 1$. Extension by homogeneity allows us to associate with k_{d-j} a symbol kernel k^h_{d-j} defined for $\xi'\neq 0$ and satisfying (5.1) whenever $t>0$ and $\xi'\neq 0$. With this symbol kernel we associate an operator-valued function $\mathfrak{g}^h_{d-j}(x',\xi',\tau,\mu)$, $\xi'\neq 0$, as in (4.1). The component of highest degree, $\mathfrak{g}^h_d$, is called the principal boundary symbol of G.

If G is strongly parameter-dependent, the previous definitions are slightly modified, asking the equality in (5.1) to hold whenever $t\ge 1$ and $|(\xi',\tau,\mu)|\ge 1$. Then all $\mathfrak{g}^h_{d-j}(x',\xi',\tau,\mu)$ are defined for $(\xi',\tau,\mu)\neq 0$. We denote the resulting classes by $G^{d,0,\nu}_{\mathrm{cl}}(\mathbb{R}^n_+;\mathbb{R}\times\Sigma)$ and $G^{d,0}_{\mathrm{cl}}(\mathbb{R}^n_+;\mathbb{R}\times\Sigma)$, respectively.

Forming finite sums as in (3.16) yields operators of type $r\in\mathbb{N}$. In this case the principal boundary symbol is given by

$$\mathfrak{g}^h_d(x',\xi',\tau,\mu)=\mathfrak{g}^h_{0,d}(x',\xi',\tau,\mu)+\sum_{j=1}^{r}\mathfrak{g}^h_{j,d-j}(x',\xi',\tau,\mu)D^j_{x_n}. \tag{5.2}$$

Definition 5.1. *A weakly parameter-dependent negligible Green operator C of type $r=0$ and regularity $\nu'\in\mathbb{R}$ on M is an integral-operator with kernel*

$$k(\tau,\mu;x,x')\;\in\;\mathscr{C}^\infty(\mathbb{R}\times\Sigma\times M\times M)$$

(*smoothness up to the boundary*) *that satisfies estimates*

$$p\big(D^k_\tau D^\alpha_\mu k(\tau,\mu;\cdot,\cdot)\big)\le C_{p\alpha kN}\langle\mu\rangle^{\frac12-\nu'-|\alpha|}\langle\tau\rangle^{-N}$$

for any continuous semi-norm p of $\mathscr{C}^\infty(M\times M)$, all orders of derivatives and all $N\in\mathbb{N}$. In case of strong parameter-dependence we ask that k is rapidly decreasing in (τ,μ),

$$k(\tau,\mu;x,x')\;\in\;\mathscr{S}\big(\mathbb{R}\times\Sigma,\mathscr{C}^\infty(M\times M)\big).$$

Negligible operators of general type $r \in \mathbb{N}$ are of the form

$$C(\tau,\mu) = \sum_{j=0}^{r} C_j(\tau,\mu) D^j,$$

where the C_j are negligible of type 0 and regularity ν' and D denotes a first order differential operator on M which in a collar neighborhood of the boundary coincides with the derivative in normal direction.

By a partition of unity on M, we can now define the classes of (global) parameter-dependent Green operators $G^{d,r,\nu}_{\mathrm{cl}}(M;\mathbb{R}\times\Sigma)$ and $G^{d,r}_{\mathrm{cl}}(M;\mathbb{R}\times\Sigma)$, using the correponding classes on the half-space and the negligible operators of the previous definition, where in case of finite regularity ν the negligible remainders are required to have regularity $\nu' = \nu - d$. With any such operator we can associate a principal boundary symbol, using the local principal boundary symbols, which is defined on $(T^*\partial M \setminus \{0\}) \times \mathbb{R} \times \Sigma$ in case of weak parameter-dependence and on $(T^*\partial M \times \mathbb{R} \times \Sigma) \setminus \{0\}$ in case of strong parameter-dependence. Here, T^*M denotes the cotangent bundle of ∂M.

5.1.2. *Poisson Operators.* Parameter-dependent Poisson operators on the half-space are of the form

$$[K(\tau,\mu)u](x) = \int e^{ix'\xi'} k(x',\xi',\tau,\mu;x_n)\widehat{u}(\xi')\,đ\xi' \tag{5.3}$$

where $u(x')$ is defined on the boundary of $\mathbb{R}^n_+$ and the symbol kernel has a specific structure. Poisson operators have an order d and a regularity ν, but there is no type needed. The mentioned structure of a Poisson operator of order d and finite or infinite regularity ν is obtained by repeating all the constructions of Section 3.2 concerning Green operators of type $r = 0$ and regularity ν by simply eliminating the y_n-variable and replacing d by $d - 1/2$. Such a Poisson operator induces (pointwise, for each (τ,μ)) continuous maps

$$B^{s+d-1/p}_{pp}(\mathbb{R}^{n-1}) \longrightarrow H^s_p(\mathbb{R}^n_+), \qquad s \in \mathbb{R}.$$

To obtain polyhomogeneous Poisson operators one needs to repeat the construction of the previous Section 5.1.1, again cancelling the y_n-variable.

Again these constructions can be generalized to the case of a manifold M, using a partition of unity and an analogue of Definition 5.1, replacing $\mathscr{C}^\infty(M\times M)$ by $\mathscr{C}^\infty(M\times\partial M)$. The resulting classes we shall denote by $P^{d,\nu}_{\mathrm{cl}}(M;\mathbb{R}\times\Sigma)$ and $P^{d}_{\mathrm{cl}}(M;\mathbb{R}\times\Sigma)$, respectively.

5.1.3. *Parameter-elliptic boundary value problems.* Let

$$A(\tau,\mu) = \sum_{j+k+\ell+|\alpha|\le 2} a_{jk\ell\alpha}(x',x_n)\tau^k\mu^\ell D^\alpha_{x'} D^j_{x_n}, \qquad (\tau,\mu)\in\mathbb{R}\times\Sigma,$$

be a parameter-dependent differential operator on the half-space $\mathbb{R}^n_+$ with coefficients that are smooth up to the boundary. We associate with $A(\tau,\mu)$ two principal symbols, the usual homogeneous principal symbol

$$\sum_{j+k+\ell+|\alpha|=2} a_{j\alpha}(x',x_n)\tau^k\mu^\ell {\xi'}_{x'}^{\alpha}\xi_n^j, \qquad (\xi,\tau,\mu)\neq 0,$$

and the principal boundary symbol

$$\mathfrak{a}_2^h(x',\xi',\tau,\mu) = \sum_{j+k+\ell+|\alpha|=2} a_{j\alpha}(x',0)\tau^k\mu^\ell {\xi'}_{x'}^{\alpha}D_{x_n}^j, \qquad (\xi',\tau,\mu)\neq 0.$$

We call A (interior) parameter-elliptic if its usual homogeneous principal symbol is pointwise invertible. Similar constructions make sense on the manifold M, leading to principal symbols on $(T^*M\times\mathbb{R}\times\Sigma)\setminus\{0\}$ and on $(T^*\partial M\times\mathbb{R}\times\Sigma)\setminus\{0\}$, respectively.

The following theorem is a very special version of results due to Grubb. We have chosen to only state this special version, since it suffices for our application to the reduced Stokes problem in the next section and since in this way we can keep the exposition shorter. In fact, one may admit more general classes of pseudodifferential operators $A(\tau,\mu)$ of arbitrary order acting between vector bundles as well as more general boundary conditions. For details we refer to [Gru86] and [GK93].

Theorem 5.2. *Let $A(\tau,\mu)$ be a second order parameter-dependent differential operator on M, $G(\tau,\mu)\in G_{\mathrm{cl}}^{2,r,\nu}(M;\mathbb{R}\times\Sigma)$ a weakly parameter-dependent polyhomogeneous Green operator of type $r\leq 2$ and regularity $\nu\geq 1/2$. Let γ_0 and γ_1 denote Dirichlet and Neumann boundary conditions on M, respectively. The boundary value problem*

$$\begin{pmatrix} A(\tau,\mu)+G(\tau,\mu) \\ \gamma_j \end{pmatrix} :\; H_p^s(M) \longrightarrow \begin{matrix} H_p^{s-2}(M) \\ \oplus \\ B_{pp}^{s-j-1/p}(\partial M) \end{matrix}, \qquad s>1+1/p, \tag{5.4}$$

is called parameter-elliptic if $A(\tau,\mu)$ is interior parameter-elliptic and, whenever $\xi'\neq 0$, the initial value problem

$$\begin{aligned} (\mathfrak{a}_2^h(x',\xi',\tau,\mu)+\mathfrak{g}_2^h(x',\xi',\tau,\mu))u &= 0 \qquad \text{on } \mathbb{R}_+, \\ (1-j)u(0)+ju'(0) &= 0 \end{aligned} \tag{5.5}$$

has only the trivial solution $u=0$ in $\mathscr{S}(\mathbb{R}_+)$. In this case (5.4) is an isomorphism for $|(\tau,\mu)|$ sufficiently large, and

$$\begin{pmatrix} A(\tau,\mu)+G(\tau,\mu) \\ \gamma_j \end{pmatrix}^{-1} = \begin{pmatrix} P_j(\tau,\mu) & K_j(\tau,\mu) \end{pmatrix}, \tag{5.6}$$

with an operator $P_j(\tau,\mu)\in B^{-2,0,\nu}(M;\mathbb{R}\times\Sigma)$ as described in Section 3.3[4] *and a Poisson operator $K_j(\tau,\mu)\in P_{\mathrm{cl}}^{-j,\nu}(M;\mathbb{R}\times\Sigma)$.*

[4]The operator class on M instead of the half-space is again obtained by using a partition of unity and taking into account the global smoothing remainders defined in Definition 5.1.

Corollary 5.3. *In the situation of Theorem 5.2, assume that* $A(\tau,\mu)=\mu^2+\widetilde{A}(\tau)$ *and that* $G(\tau,\mu)=G(\tau)$ *is independent of* μ*. Let* $p\in(1,\infty)$ *and* $T>0$*. Define the operator* $\mathbf{A}$ *in* $L_p(Z)$ *with* $Z:=\mathbb{R}\times M$ *by*

$$\begin{aligned}\mathscr{D}(\mathbf{A})&:=\big\{u\in W_p^2(Z)\,|\,\gamma_j u=0 \text{ on } \partial Z\big\},\\ \mathbf{A}u&:=\mathrm{op}_\tau\widetilde{A}(\tau)u+\mathrm{op}_\tau G(\tau)u\quad(u\in\mathscr{D}(\mathbf{A})).\end{aligned}$$

If the boundary value problem (5.4) is parameter-elliptic in the sector $\Sigma:=\{\mu\in\mathbb{C}\setminus\{0\}:|\arg\mu|\le\frac{\pi}{4}\}\cup\{0\}$*, then* $\mathbf{A}$ *has maximal* L_q*-regularity for every* $q\in(1,\infty)$*, i.e., the mapping*

$$\partial_t+\mathbf{A}\colon W_q^1\big((0,T);L_p(Z)\big)\cap L_q\big((0,T);\mathscr{D}(\mathbf{A})\big)\to L_q\big((0,T);L_p(Z)\big)$$

is an isomorphism of Banach spaces.

Proof. Define the operator $A_M(\tau)$ for $\tau\in\mathbb{R}$ by $\mathscr{D}(A_M(\tau)):=\{v\in W_p^2(M):\gamma_j v=0\}$ and $A_M(\tau)v:=\widetilde{A}(\tau)+G(\tau)$. By Theorem 5.2, the resolvent $(\mu^2+A_M(\tau))^{-1}$ exists for sufficiently large $\mu\in\Sigma$ and is given by $P_j(\tau,\mu)\in B^{-2,0,\nu}(M;\mathbb{R}\times\Sigma)$. Choosing $\lambda_0>0$ sufficiently large, we obtain

$$\mu^2\big(\mu^2+\lambda_0+A_M(\tau)\big)^{-1}\in B^{0,0,\nu}(M;\mathbb{R}\times\Sigma).$$

Setting $\lambda=\mu^2$, Corollary 4.5 yields

$$\lambda\big(\lambda+\lambda_0+\mathbf{A}\big)^{-1}=\mathrm{op}_\tau\big[\mu^2(\mu^2+\lambda_0+A_M(\tau))^{-1}\big]\in S^0_{\mathscr{R}}(\Sigma;L_p(Z),L_p(Z)).$$

By the Theorem of Weis [Wei01], $\mathbf{A}+\lambda_0$ has maximal L_q-regularity for all $q\in(1,\infty)$. As the time interval $(0,T)$ is assumed to be finite, this gives maximal L_q-regularity for $\mathbf{A}$. □

As indicated at the end of Section 4, the analog results hold for τ-independent operators, i.e., for parameter-elliptic boundary value problems of the form

$$\begin{aligned}(\lambda+A+G)u&=f\quad\text{in } M,\\ \gamma_j u&=0\quad\text{on }\partial M,\end{aligned}$$

where A and G are (parameter-independent) pseudodifferential and Green operators, respectively.

5.2. **The reduced Stokes problem.** Let $\Sigma:=\{\mu\in\mathbb{C}\setminus\{0\}:|\arg\mu|\le\theta\}\cup\{0\}$ with $\theta\in(\frac{\pi}{4},\frac{\pi}{2})$. For $\mu\in\Sigma$, we consider in the waveguide $Z:=\mathbb{R}\times M$ the resolvent problem

$$\begin{aligned}\mu^2u-Au+\nabla p&=f\qquad\text{in } Z,\\ \operatorname{div}u&=0\qquad\text{in } Z,\\ \gamma_0u&=0\qquad\text{on }\partial Z,\end{aligned}\tag{5.7}$$

where γ_0 denotes the operator of restriction to the boundary and $A=\Delta-\nabla\mathrm{div}$. Note that due to the divergence condition obviously we can replace A by the Laplacian Δ without changing the problem. However, using A (a 'trick' going back to Grubb, Solonnikov [GS91]) turns out to be essential for the procedure below;

roughly speaking the subtraction of ∇div eliminates some second order derivatives. Observe that $\nu = (0, \nu_M)$ for the (inner) unit normal vector ν of Z and that $\Delta = \partial_r^2 + \Delta_M$ on Z where r is the variable of $\mathbb{R}$ and the subscript M indicates the corresponding objects on M. Moreover, let us write $u = (u_1, \underline{u})$ with $u_1 : Z \to \mathbb{R}$ and $\underline{u} : Z \to \mathbb{R}^n$ and analogously $f = (f_1, \underline{f})$.

Defining the boundary operators γ_ν and γ_1 on Z by $\gamma_\nu u = \nu \cdot \gamma_0 u$ and $\gamma_1 p = \gamma_\nu(\nabla p)$ respectively, (5.7) implies for f with $\operatorname{div} f = 0$

$$\begin{aligned} \Delta p &= 0 && \text{in } Z, \\ \gamma_1 p &= \gamma_\nu A u + \gamma_\nu f && \text{on } \partial Z. \end{aligned} \tag{5.8}$$

By direct calculation one sees that

$$\gamma_\nu A u = \gamma_{1,M} \partial_r u_1 + \gamma_{\nu_M} A_M \underline{u}, \qquad A_M = \Delta_M - \nabla_M \operatorname{div}_M.$$

Substituting this in (5.8) and passing to the Fourier transform with respect to the variable r (denoting the corresponding co-variable by τ) yields that

$$\begin{aligned} (\Delta_M - \tau^2) P &= 0 \qquad \text{in } M, \\ \gamma_{1,M} P &= \gamma_{\nu_M} A_M \underline{U} + i\tau \gamma_{1,M} U_1 + \gamma_{\nu_M} \underline{F}, \qquad \text{on } \partial M, \end{aligned} \tag{5.9}$$

where capital letters indicate the Fourier transform of the respective function in the first variable. We conclude with Theorem 3.3.1 of [Gru86] that

$$P = K(\tau)\big(\gamma_{\nu_M} A_M \underline{U} + i\tau\gamma_{1,M} U_1 + \gamma_{\nu_M} \underline{F}\big)$$

with a strongly parameter-dependent potential operator $K(\tau)$ of order $-3/2$ on M (i.e., $K(\tau)$ satisfies $(\Delta_M - \tau^2)K(\tau) = 0$ and $\gamma_{1,M} K(\tau) = \mathrm{id}$). Now

$$Au = \begin{pmatrix} \Delta u_1 - \partial_r(\partial_r u_1 + \operatorname{div}_M \underline{u}) \\ A_M \underline{u} - \partial_r \nabla_M u_1 + \partial_r^2 \underline{u} \end{pmatrix} = \begin{pmatrix} \Delta u_1 - \partial_r(\partial_r u_1 + \operatorname{div}_M \underline{u}) \\ \Delta_M \underline{u} + \partial_r^2 \underline{u} \end{pmatrix},$$

where we have used that $0 = \operatorname{div} u = \partial_r u_1 + \operatorname{div}_M \underline{u}$. By writing $\nabla p = (\partial_r p, \nabla_M p)$ and passing to the Fourier transform in r, we derive from (5.7) that

$$\mu^2 U_1 - \Delta_M U_1 + i\tau \operatorname{div}_M \underline{U} + i\tau P = F_1, \qquad \gamma_{0,M} U_1 = 0, \tag{5.10}$$

$$\mu^2 \underline{U} + \tau^2 \underline{U} - \Delta_M \underline{U} + \nabla_M P = \underline{F}, \qquad \gamma_{0,M} \underline{U} = 0. \tag{5.11}$$

Inserting the above representation of p, the second last equation can be rewritten as

$$C(\tau,\mu) U_1 = B(\tau)\underline{U} + F_1, \qquad \gamma_{0,M} U_1 = 0,$$

where

$$B(\tau) := i\tau\big(K(\tau)\gamma_{\nu_M} A_M + \operatorname{div}_M\big), \qquad C(\tau,\mu) := \mu^2 - \Delta_M - \tau^2 K(\tau)\gamma_{1,M}.$$

Now $B(\tau)$ is an operator of Boutet's calculus of order and type 2 with strong parameter-dependence on τ, while $C(\tau,\mu)$ has order and type 2 as well and is weakly parameter-dependent with regularity $\nu = 1/2$; for the latter see (2.3.55) in Proposition 2.3.14 of [Gru86]. By parameter-ellipticity and Theorem 5.2 we can find

$$\begin{pmatrix} C(\tau,\mu) \\ \gamma_{0,M} \end{pmatrix}^{-1} =: \begin{pmatrix} D(\tau,\mu) & \widetilde{K}(\tau,\mu) \end{pmatrix}$$

with D of order -2, $\widetilde{K}$ of order 0, and both having type 0 and regularity $\nu = 1/2$. Therefore

$$U_1 = E(\tau,\mu)\underline{U} + D(\tau,\mu)F_1, \qquad E(\tau,\mu) := D(\tau,\mu)B(\tau); \tag{5.12}$$

note that $E(\tau,\mu)$ is weakly parameter-dependent of zero order, type 2, and regularity $\nu = 1/2$. Inserting this above we find the equation

$$(\mu^2 + \tau^2 - \Delta_M)\underline{U} + G(\tau,\mu)\underline{U} = \underline{F} - D(\tau,\mu)F_1 \tag{5.13}$$

for $\underline{U}$, where

$$G(\tau,\mu) = \nabla_M K(\tau)\gamma_{\nu,M}A_M + i\tau\nabla_M K(\tau)\gamma_{1,M}E(\tau,\mu)$$

is a weakly parameter-dependent singular Green operator of order and type 2, with regularity $\nu = 1/2$. Using the parameter-ellipticity of $(\mu^2+\tau^2-\Delta_M+G(\tau,\mu),\ \gamma_{0,M})$ we can resolve (5.13) for $\underline{U}$ and substitute $\underline{U}$ in (5.12), resulting in

$$U(\tau,\cdot) = P(\tau,\mu)F(\tau,\cdot), \qquad |(\tau,\mu)| \geq R,$$

for some sufficiently large $R \geq 0$ and with $P(\tau,\mu)$ being a $(n+1)\times(n+1)$-matrix with components belonging to $B^{-2,0,1/2}(M;\mathbb{R}\times\Sigma)$. Passing to the inverse Fourier transform with respect to τ, we see that $u = R(\mu)f$ is the unique solution of (5.7) for $\mu \in \Sigma$ sufficiently large, where the solution operator $R(\mu)$ is defined by $R(\mu) := \mathscr{F}^{-1}_{\tau\to r}P(\tau,\mu)\mathscr{F}_{r\to\tau}$. Due to Theorem 4.4, the we have for sufficiently large $\mu_0 > 0$

$$\mathscr{R}\Big(\big\{\langle\mu\rangle^{2+|\alpha|}\partial_\mu^\alpha R(\mu) : \mu\in\Sigma, |\mu|\geq\mu_0\big\}\Big) < \infty. \tag{5.14}$$

Therefore, we have seen that for sufficiently large $\mu \in \Sigma$ problem (5.7) is uniquely solvable, and (5.14) gives a resolvent estimate even in the $\mathscr{R}$-bounded version. This enables us to apply Theorem 2.4 and get maximal regularity for the Stokes operator. In particular, we obtain the following result.

Theorem 5.4. *Let $M \subset \mathbb{R}^n$ be a bounded smooth domain, and $Z := \mathbb{R}\times M$. Let $p \in (1,\infty)$, $T > 0$, and let P_p be the Helmholtz projection in $L_p(Z)$ (see [Far03]). Define the Stokes operator $\mathbf{A}$ by*

$$\mathscr{D}(\mathbf{A}) := W_p^2(Z)\cap W_{p,0}^1(Z)\cap L_{p,\sigma}(Z),$$
$$\mathbf{A}u := P_p\Delta u \quad (u\in\mathscr{D}(\mathbf{A})).$$

Here, $L_{p,\sigma}(Z)$ stands for the standard space of solenoidal L_p-vector fields. Then $\mathbf{A}$ has maximal L_q-regularity for every $q \in (1,\infty)$ in the time interval $(0,T)$.

Proof. Due to the Helmholtz decomposition of $L_p(Z)$ (see [Far03]), we see that for $f \in L_{p,\sigma}(Z)$ the solvability of $(\lambda-\mathbf{A})u = f$ in $L_{p,\sigma}(Z)$ is equivalent to the solvability of (5.7) with $\mu^2 = \lambda$. By the considerations above, we know that for large $\mu \in \Sigma$, (5.7) is uniquely solvable, and the resolvent of $\mathbf{A}$ is given by $(\lambda-\mathbf{A})^{-1} = R(\mu)$ with $R(\mu)$ being defined above. Due to the $\mathscr{R}$-boundedness result in (5.14), the result on maximal regularity for finite time intervals follows from Theorem 2.4 in the same way as in the proof of Corollary 4.5. □

Let us finally remark that more general results on maximal regularity for the Stokes operator in cylindrical domains have been obtained, e.g., by Farwig and Ri [FMH08] under weaker smoothness assumptions on the domain. For the existence of a bounded H^∞-calculus (which also implies the statement of Theorem 5.4), we also refer to Abels [Abe05]. However, the intention of the present text was not to recover or even improve these results but to outline that maximal regularity is obtained without much effort by employing the $\mathscr{R}$-boundedness of operator-families belonging to Boutet de Monvel's calculus.

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