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Semidefinite Representation for Convex Hulls of Real Algebraic Curves*

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Abstract. We show that the closed convex hull of any one-dimensional semialgebraic subset of $\mathbb{R}^n$ is a spectrahedral shadow, meaning that it can be written as a linear image of the solution set of some linear matrix inequality. This is proved by an application of the moment relaxation method. Given a nonsingular affine real algebraic curve C and a compact semialgebraic subset K of its $\mathbb{R}$-points, the preordering $\mathcal{P}(K)$ of all regular functions on C that are nonnegative on K is known to be finitely generated. Our main result, from which all others are derived, says that $\mathcal{P}(K)$ is stable, meaning that uniform degree bounds exist for weighted sum of squares representations of elements of $\mathcal{P}(K)$. We also extend this last result to the case where K is only virtually compact. The main technical tool for the proof of stability is the archimedean local-global principle. As a consequence of our results we show that every convex semialgebraic subset of $\mathbb{R}^2$ is a spectrahedral shadow.

Key words. spectrahedral shadows, convex algebraic geometry, real algebraic curves, convex hull, linear matrix inequalities, moment relaxation, semidefinite programming, Helton–Nie conjecture

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Introduction. Let $K \subseteq \mathbb{R}^n$ be a real algebraic set, or more generally a semialgebraic set. The question of how to represent the convex hull $\text{conv}(K)$ of K has attracted growing attention in recent years. A good part of this interest originates from optimization theory, namely from the problem of optimizing a linear functional over K. One of the most promising approaches that has been discussed is to express $\text{conv}(K)$ (at least up to taking closures) as a linear image of a spectrahedron, that is, of the solution set of a linear matrix inequality. In other words, one would like to find symmetric real matrices M_i, N_j of some size (for $0 \le i \le n$, $1 \le j \le k$, and some k) such that, writing

$$M(x,y) \;=\; M_0 + \sum_{i=1}^{n} x_i M_i + \sum_{j=1}^{k} y_j N_j, \tag{1}$$

the closure of $\text{conv}(K)$ coincides with the closure of the set

$$S \;=\; \{x \in \mathbb{R}^n \colon\; \exists\, y \in \mathbb{R}^k \;\, M(x,y) \succeq 0\}. \tag{2}$$

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Here $M \succeq 0$ means that the symmetric matrix M is positive semidefinite. In view of the very efficient methods available in semidefinite programming, such a representation is perfectly well suited for optimizing linear functionals over K.

Another approach tries to understand the set $\mathrm{conv}(K)$ via the dual algebraic variety of the Zariski closure of its boundary; see [18], [19], [30] for more details.

A subset $S \subseteq \mathbb{R}^n$ is called a *spectrahedral shadow* if it can be written as in (2) with suitable symmetric matrices M_i, N_j. Any representation (2) is called a *semidefinite representation* of S. The question of characterizing spectrahedral shadows was raised by Nemirovski in his plenary address at the ICM in Madrid [10]. Spectrahedral shadows are clearly convex semialgebraic sets, and for many years no other restriction was known. In 2009, Helton and Nie [5] conjectured that conversely every convex semialgebraic set is a spectrahedral shadow. This conjecture was recently disproved by the author [28]. In the present paper, however, we prove the existence of a semidefinite representation for the closed convex hull of any one-dimensional semialgebraic set in $\mathbb{R}^n$. Using this result, we show that every convex semialgebraic subset of the plane is a spectrahedral shadow; i.e., we show that the Helton–Nie conjecture does hold in dimension two.

Our result does not extend to convex hulls of sets of dimension greater than one. Indeed, for *every* semialgebraic set $K \subseteq \mathbb{R}^n$ of dimension at least two, there exists a polynomial map $\varphi\colon \mathbb{R}^n \to \mathbb{R}^N$ (for some $N \geq 1$) such that the closed convex hull of $\varphi(K)$ in $\mathbb{R}^N$ has no semidefinite representation. This is proved in [28].

For the construction of semidefinite representations we use the moment relaxation method, introduced by Lasserre and Parrilo ([7], [13], [14]; see also [2], [8]). Computing the convex hull of a set $K \subseteq \mathbb{R}^n$ (that we assume to be basic closed semialgebraic) means to determine the linear moments of all probability measures on K for which these moments exist. By considering finite-dimensional relaxations of the K-moment problem, one obtains a nested hierarchy $K(1) \supseteq K(2) \supseteq \cdots$ of sets with explicit semidefinite representations that all contain K. Their closures $\overline{K(d)} = TH(d)$ have also been studied under the name *theta bodies* of K (see [4] and [2, Chapter 7]). When K is a compact semialgebraic set, the sets $K(d)$ approximate $\mathrm{conv}(K)$ arbitrarily closely. Moreover, the approximation becomes exact, that is, $K(d) = \mathrm{conv}(K)$ holds for some $d \geq 1$, if and only if every linear polynomial that is nonnegative on K has a weighted sum of squares representation with uniform degree bounds on the summands. See Theorem 2.4 below for a rigorous formulation.

We consider a nonsingular affine algebraic curve C over $\mathbb{R}$ and a compact semialgebraic subset K of $C(\mathbb{R})$, the set of real points on C. We work in $\mathbb{R}[C]$, the affine coordinate ring of C. Let $\mathcal{P}(K)$ be the saturated preordering of K, i.e., the set of all elements of $\mathbb{R}[C]$ that are nonnegative on K. It is known [22] that $\mathcal{P}(K)$ is finitely generated as a preordering. This means that there exist finitely many elements $1 = h_0, h_1, \dots, h_r \in \mathcal{P}(K)$ such that every $f \in \mathcal{P}(K)$ has a representation

$$f = \sum_{i=0}^{r} \sum_j p_{ij}^2 h_i \tag{3}$$

with $p_{ij} \in \mathbb{R}[C]$. Fixing C, K, and the h_i, the main result of this paper (Corollary 4.4) says that there exist uniform degree bounds for such representations. That is, every $f \in \mathcal{P}(K)$

has some representation (3) in which the degrees of the summands $p_{ij}^2 h_i$ are bounded above by some number that only depends on $\deg(f)$. (We are using degrees here to simplify the exposition, and so we tacitly assume that C is given with a fixed embedding in some affine space.) Technically, this result is expressed by saying that the preordering $\mathcal{P}(K)$ is stable. From this it follows that, for any morphism $\varphi\colon C \to \mathbb{A}^n$ into affine space of any dimension, the relaxation process for the convex hull of $\varphi(K)$ in $\mathbb{R}^n$ becomes exact. In fact, this latter property is equivalent to stability of $\mathcal{P}(K)$.

Our method for proving stability of $\mathcal{P}(K)$ may be of interest in that we do not show the existence of degree bounds directly. Rather we establish the following equivalent fact: For any real closed field R containing $\mathbb{R}$, the preordering generated by the h_i in $R[C] = \mathbb{R}[C] \otimes R$ is again saturated (Theorem 4.3). This fact, in turn, is proved by an application of the archimedean local-global principle [23], which allows us to reduce the problem to local rings. At first sight this may seem impossible since the field R is non-archimedean. We get around this problem by working in the ring $B[C] = \mathbb{R}[C] \otimes B$, rather than in $R[C]$, where B is the smallest convex subring of R that contains $\mathbb{R}$ (so B is a non-noetherian valuation ring). We believe that this way of applying the local-global principle is novel and perhaps somewhat unexpected.

In the case where C has genus one and $K = C(\mathbb{R})$ is the full real curve (assumed to be compact), our main result was already known by the author [27]. In that paper, using geometric arguments of Riemann–Roch type, we had given degree bounds of quite explicit nature, resulting in bounds for the sizes of the derived exact semidefinite representations. For all curves of higher genus, as well as for genus one and $K \neq C(\mathbb{R})$, our results are new. In contrast to the method of [27], the techniques used in the present paper unfortunately do not seem to give any explicit degree bounds.

From Corollary 4.4 we deduce the existence of a semidefinite representation for the convex hull of any compact semialgebraic set $S \subseteq \mathbb{R}^n$ with $\dim(S) \leq 1$ (Theorem 5.1). For this, one first desingularizes via normalization and then uses the moment relaxation process. This case in turn implies the existence of such a representation for the closed convex hull of any semialgebraic set S with $\dim(S) \leq 1$, not necessarily compact (Theorem 6.1). From this we establish the Helton–Nie conjecture in dimension two (Theorem 6.8).

On the other hand, we extend the stability result to certain noncompact cases. Namely, when C is a nonsingular affine curve and $K \subseteq C(\mathbb{R})$ is a closed semialgebraic set that is merely virtually compact (meaning that there exists $f \in \mathbb{R}[C]$ that is nonconstant and bounded on K), the saturated preordering $\mathcal{P}(K)$ is still finitely generated and stable (Theorem 7.3). Again, this is proved by a reduction to the compact case.

We would like to point out that our main theorem on degree bounds for (weighted) sum of squares representations does not extend to dimensions bigger than one, according to the results of [24]. For example, it was shown there that, for any nonsingular affine $\mathbb{R}$-variety V with $V(\mathbb{R}) \neq \varnothing$ compact and $\dim(V) \geq 2$, degree bounds for sums of squares in $\mathbb{R}[V]$ cannot exist.

For practical matters our results imply the following. Suppose we are given a compact semialgebraic set $K \subseteq \mathbb{R}^n$, $\dim(K) = 1$, and a polynomial $f \in \mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n]$ and want to find $f_* := \min f(K)$. For simplicity assume that $K = C(\mathbb{R})$ is a real algebraic curve without singularities (the more general case can be reduced to this one). For every degree d consider

$$c_d := \max\{c \in \mathbb{R}\colon f - c \text{ is modulo } I_C \text{ a sum of squares of polynomials of } \deg \le d\}$$

($I_C :=$ ideal of C in $\mathbb{R}[x]$). Then c_d is the optimum of an explicit semidefinite program, and $c_d \uparrow f_*$ by the general results of [7]. Our results imply that we in fact have finite convergence, i.e., $f_* = c_d$ for some $d \in \mathbb{N}$ which *depends only on* C *and* $\deg(f)$, but not on f. If C has genus $g \le 1$, upper bounds for d are known explicitly ([14], [6] for $g = 0$ and [27] for $g = 1$), but unfortunately not otherwise.

For both theoretical and practical reasons it would be highly desirable to have a more constructive approach to the results of this paper. In particular, one would like to have some information on the nature of the degree bounds whose existence is proved here.

The paper is organized as follows. In section 2 we give a brief account of the relaxation method for constructing semidefinite representations of convex hulls, in the generality that is needed here. Section 3 contains auxiliary results for working in the ring $\mathbb{R}[C] \otimes B$. This ring plays a key role in the proof of stability of $\mathcal{P}(K)$ in the compact case (section 4). The existence of semidefinite representations for compact convex hulls is deduced in section 5, and the extension to closed convex hulls of arbitrary one-dimensional sets is discussed in section 6. Finally, section 7 contains the proof of stability in the virtually compact case.

This paper was originally written in 2012. At that time the Helton–Nie conjecture was still open, and the results of this paper were considered as additional support for this conjecture. The present form is a slightly revised and updated version.

1. Notation and preliminaries.

1.1. Let k be a field. By an algebraic k-variety (or simply k-variety) we mean a reduced and separated k-scheme of finite type. Most algebraic varieties and schemes in this paper will be affine. An affine k-variety is therefore the Zariski spectrum $V = \mathrm{Spec}(A)$ of a k-algebra A which is finitely generated and reduced (no nonzero nilpotent elements). Following common practice, we also write $A = k[V]$ and call this ring the affine coordinate ring of V. If E is any k-algebra, then $V(E) = \mathrm{Hom}_k(A, E)$ denotes the set of E-valued points of V. Given $\xi \in V(E)$ and $f \in A$, we usually write $f(\xi)$ (rather than $\xi(f)$) for the result of evaluating the homomorphism ξ on f.

A curve over k is a k-variety all of whose irreducible components have dimension one. An affine curve C over k is irreducible (resp., irreducible and nonsingular) if and only if the ring $k[C]$ is an integral domain (resp., a Dedekind domain).

1.2. We need to employ the real spectrum, and we briefly recall the basic notions. See [3], [9], [16], or [25] for full details and background. All rings are assumed to be commutative and to have a unit. The real spectrum of the ring A, denoted by $\mathrm{Sper}(A)$, is the set consisting of all pairs $\alpha = (\mathfrak{p}, \omega)$, where $\mathfrak{p} \in \mathrm{Spec}(A)$ and ω is an ordering of the residue field of $\mathfrak{p}$. The prime ideal $\mathfrak{p}$ is called the support of α, written as $\mathfrak{p} = \mathrm{supp}(\alpha)$.

For $f \in A$ and $\alpha = (\mathfrak{p}, \omega) \in \mathrm{Sper}(A)$, the notation "$f(\alpha) \ge 0$" (resp., "$f(\alpha) > 0$") indicates that the residue class f mod $\mathfrak{p}$ is nonnegative (resp., positive) with respect to ω. The *(Harrison) topology* on $\mathrm{Sper}(A)$ is defined to have the collection of sets $U(f) = \{\alpha \in \mathrm{Sper}(A)\colon f(\alpha) > 0\}$, $f \in A$, as a subbasis of open sets. The support map $\mathrm{supp}\colon \mathrm{Sper}(A) \to$

Spec(A) is continuous. A subset of Sper(A) is called *constructible* if it is a finite boolean combination of sets $U(f)$, $f \in A$, that is, if it can be described by imposing sign conditions on finitely many elements of A. Given $\alpha, \beta \in \mathrm{Sper}(A)$, one says that α specializes to β (or that β is a specialization of α) if β lies in $\overline{\{\alpha\}}$, the closure of the set $\{\alpha\}$. Any ring homomorphism $\varphi\colon A \to B$ induces a continuous map $\varphi^*\colon \mathrm{Sper}(B) \to \mathrm{Sper}(A)$ in a natural and functorial way.

A convenient alternate way to think of the real spectrum is to observe that every point of Sper(A) is represented by a ring homomorphism $A \to R$ into some real closed field R. Two homomorphisms $A \to R_i$ $(i = 1, 2)$ represent the same point of Sper(A) if and only if there exists a third homomorphism $A \to R$ into a real closed field R together with A-embeddings $R_i \to R$ $(i = 1, 2)$.

1.3. Let A be a ring. By ΣA^2 we denote the set of (finite) sums of squares in A. A subset $M \subseteq A$ is called a quadratic module of A if $1 \in M$, $M + M \subseteq M$, and $a^2M \subseteq M$ for every $a \in A$ hold. If, in addition, $MM \subseteq M$ holds, then M is called a preordering of A.

A quadratic module M is finitely generated if there exist finitely many elements $h_1, \dots, h_r \in M$ such that (putting $h_0 := 1$)

$$M \;=\; (\Sigma A^2)h_0 + \cdots + (\Sigma A^2)h_r \;:=\; \left\{\sum_{i=0}^{r} s_i h_i \colon s_0, \dots, s_r \in \Sigma A^2\right\}.$$

We say in this case that the quadratic module M is generated by $h_1, \dots, h_r$.

A quadratic module M of A is said to be archimedean if $\mathbb{Z} + M = A$, or equivalently if for every $a \in A$ there exists a positive integer n such that $n \pm a \in M$.

Given a quadratic module $M \subseteq A$, one associates with M the closed subset $\mathcal{X}_M := \{\alpha \in \mathrm{Sper}(A)\colon f(\alpha) \ge 0$ for every $f \in M\}$ of Sper(A). The saturation of M is the preordering $\mathrm{Sat}(M) := \{f \in A\colon f \ge 0 \text{ on } \mathcal{X}_M\}$ of A. The quadratic module M is called saturated if $M = \mathrm{Sat}(M)$. Any of [9], [16], or [25] contains more background on quadratic modules or preorderings and their saturations.

The notion of stability for a quadratic module is basic for this paper. It will be recalled in 2.1 below.

1.4. Let R be a real closed field, and let V be an affine R-variety. Given a semialgebraic set $K \subseteq V(R)$, we denote the associated constructible subset of $\mathrm{Sper}\, R[V]$ by $\widetilde{K}$; see [3, section 7.2]. Given any finite system of inequalities that describes K, the set $\widetilde{K}$ is the subset of $\mathrm{Sper}\, R[V]$ that is described by the same system. The saturated preordering associated with K is denoted by $\mathcal{P}(K)$, that is,

$$\mathcal{P}(K) \;=\; \{f \in R[V]\colon f|_K \ge 0\}.$$

Example 1.5. Let $h_1, \dots, h_r \in \mathbb{R}[x_1, \dots, x_n]$, and consider the basic closed set

$$K \;=\; \{\xi \in \mathbb{R}^n\colon h_1(\xi) \ge 0, \dots, h_r(\xi) \ge 0\}$$

in $\mathbb{R}^n$. The quadratic module M generated by $h_1, \dots, h_r$ satisfies $M \subseteq \mathcal{P}(K)$. In general, equality does not hold; i.e., there exist polynomials f with $f|_K \ge 0$ but $f \notin M$. If M is archimedean, then M contains every polynomial f with $f|_K > 0$, by the archimedean

Positivstellensatz (see [9], [16], or [17]). Note that M archimedean implies that K is compact. Conversely, if K is compact, and if M is a preordering, then M is archimedean (Schmüdgen Positivstellensatz; see [29], [9], or [16]).

1.6. The convex hull of a set $S \subseteq \mathbb{R}^n$ is denoted by $\mathrm{conv}(S)$. If $K \subseteq \mathbb{R}^n$ is a closed convex set, a point $a \in K$ is called an extreme point of K if $a = (1-t)b + tc$, where $b, c \in K$ and $0 < t < 1$, implies $b = c = a$. The set of extreme points of K is denoted by $\mathrm{Ex}(K)$. When K is a semialgebraic set, the set $\mathrm{Ex}(K)$ is semialgebraic as well.

2. The relaxation method.

2.1. Let A be a finitely generated $\mathbb{R}$-algebra, and let M be a finitely generated quadratic module in A, say $M = \Sigma_A h_0 + \cdots + \Sigma_A h_r$ with $1 = h_0, h_1, \dots, h_r \in A$, and $\Sigma_A := \Sigma A^2$ (the cone of sums of squares in A). The quadratic module M is said to be *stable* (see [15], [24]) if, given any finite-dimensional linear subspace U of A, there exists a finite-dimensional linear subspace W of A with

$$M \cap U \;\subseteq\; \Sigma_W h_0 + \cdots + \Sigma_W h_r.$$

Here Σ_W denotes the set of sums of squares of elements of W. The property of being stable does not depend on the choice of the generators $h_0, \dots, h_r$ of M. If A is a polynomial ring over $\mathbb{R}$, stability of M means that there exists a map $\varphi\colon \mathbb{N} \to \mathbb{N}$ such that, for every $f \in M$, there exists a representation $f = \sum_{i,j} p_{ij}^2 h_i$ with suitable polynomials p_{ij} such that $\deg(p_{ij}^2 h_i) \le \varphi(\deg(f))$ for all i, j.

2.2. By a *semidefinite representation* of a set $S \subseteq \mathbb{R}^n$ one means a representation

$$S \;=\; \left\{ x \in \mathbb{R}^n \colon \exists\, y \in \mathbb{R}^k \;\; M_0 + \sum_{i=1}^n x_i M_i + \sum_{j=1}^k y_j N_j \succeq 0 \right\}$$

with suitable $k \ge 0$ and real symmetric matrices M_i, N_j of some size. The set S is called a *spectrahedral shadow* if it has a semidefinite representation. Other terms often used in the literature are projected spectrahedron, semidefinitely representable set, or lifted-LMI representable set.

We now recall the method of moment relaxation [7] for constructing semidefinite representations, in a generality adapted to our needs. For more background we refer the reader to [8, Chapter 11] and to [2, Chapters 6–7]. We only outline the basic principle of the construction, ignoring possible refinements.

2.3. Let A be a finitely generated reduced $\mathbb{R}$-algebra. We denote the associated affine $\mathbb{R}$-variety by $V = \mathrm{Spec}(A)$, so $A = \mathbb{R}[V]$, and we always equip the set $V(\mathbb{R}) = \mathrm{Hom}(A, \mathbb{R})$ of real points of V with its natural Euclidean topology. Fix elements $1 = h_0, h_1, \dots, h_r \in A$, write $\Sigma_A := \Sigma A^2$ for the cone of sums of squares in A, let

$$M \;=\; h_0 \Sigma_A + \cdots + h_r \Sigma_A$$

be the quadratic module in A generated by the h_i, and let

$$K \;=\; \{\xi \in V(\mathbb{R}) \colon h_1(\xi) \ge 0, \dots, h_r(\xi) \ge 0\}$$

be the associated basic closed semialgebraic subset of $V(\mathbb{R})$. We assume that K is Zariski dense in V. Fix a finite-dimensional linear subspace $L \subseteq A$ containing 1, and let $1, x_1, \ldots, x_n$ be a basis of L. We consider the morphism $\varphi = \varphi_L = (x_1, \ldots, x_n)$ from V to affine n-space determined by L, and the induced map $\varphi\colon V(\mathbb{R}) \to \mathbb{R}^n$.

Given a linear subspace $B \subseteq A$ we denote by BB the linear subspace of A spanned by all products $b_1 b_2$ with $b_1, b_2 \in B$. Fix a tuple $W = (W_0, \ldots, W_r)$ of finite-dimensional linear subspaces of A, and consider the linear subspace

$$U \ := \ W_0 W_0 + h_1 W_1 W_1 + \cdots + h_r W_r W_r$$

of A. We assume that L is contained in U, and we denote by $\rho\colon U' \to L'$ the restriction map between the dual linear spaces. By U'_1 (resp., L'_1) we denote the set of all linear forms λ in U' (resp., in L') with $\lambda(1) = 1$, and we identify $\mathbb{R}^n$ with L'_1 via the map

$$L'_1 \xrightarrow{\sim} \mathbb{R}^n, \quad \lambda \mapsto \big(\lambda(x_1), \ldots, \lambda(x_n)\big).$$

For $i = 0, \ldots, r$, let $\Sigma_{W_i} \subseteq W_i W_i$ denote the cone of sums of squares of elements of W_i. The set

$$M_W \ := \Sigma_{W_0} + h_1 \Sigma_{W_1} + \cdots + h_r \Sigma_{W_r}$$

is contained in $M \cap U$ and is a convex semialgebraic cone in U. Since K is Zariski dense in V, we have $M \cap (-M) = \{0\}$. This implies that M_W is closed in U [15, Proposition 2.6]. Let $M_W^* \subseteq U'$ be the dual cone of M_W. Then M_W^* can be defined by a (homogeneous) linear matrix inequality; that is, M_W^* is a spectrahedral cone in U'. The subset $M_W^* \cap U'_1$ of M_W^* is therefore a spectrahedron as well. Its image set

$$K_W \ := \ \rho(M_W^* \cap U'_1) \ = \ L'_1 \cap \rho(M_W^*) \ \subseteq \ \mathbb{R}^n$$

under the restriction map $\rho\colon U'_1 \to L'_1 = \mathbb{R}^n$ is therefore a spectrahedral shadow by construction. For every $\xi \in K$, the cone M_W^* contains the evaluation map at ξ (restricted to U). Therefore K_W contains the set $\varphi(K)$, and therefore we have $\operatorname{conv}(\varphi(K)) \subseteq K_W$. Increasing the subspaces $W_0, \ldots, W_r$ of A results in making the set K_W smaller. The main facts are summarized in the following theorem (cf. [7, Theorem 2]).

Theorem 2.4. *Let $L \subseteq A$ be a fixed linear subspace with basis $1, x_1, \ldots, x_n$, and let $\varphi\colon V \to \mathbb{A}^n$ be the associated morphism. With assumptions and notation from 2.3, we have the following:*

(a) $\overline{K_W} \ = \ \{\eta \in \mathbb{R}^n \colon \forall\, f \in L \cap M_W \ f(\eta) \geq 0\}$*;*
(b) *the inclusion $\overline{\operatorname{conv}(\varphi(K))} \subseteq \overline{K_W}$ of closed convex sets is an equality if and only if $L \cap \mathcal{P}(K) \subseteq M_W$;*
(c) *if M is archimedean (see 1.3), then $\operatorname{conv}(\varphi(K)) = \bigcap_W K_W$, the intersection over all systems $W = (W_0, \ldots, W_r)$ of finite-dimensional subspaces of A.*

If K is compact, then $\operatorname{conv}(\varphi(K))$ is again compact by Carathéodory's lemma, and for any fixed tuple W as above we get the following lemma.

Corollary 2.5. *If K is compact, then $\operatorname{conv}(\varphi(K)) = K_W$ holds if and only if $L \cap \mathcal{P}(K) \subseteq M_W$.*

2.6. The moment relaxation for the closed convex hull $\overline{\mathrm{conv}(\varphi(K))}$ is said to become *exact* if the equality $\overline{\mathrm{conv}(\varphi(K))} = \overline{K_W}$ holds for some choice $W = (W_0, \dots, W_r)$ of finite-dimensional subspaces. When K is compact, this is equivalent to $\mathrm{conv}(\varphi(K)) = K_W$.

If one is aiming at describing the convex hull of $\varphi(K)$ in $\mathbb{R}^n$, approximately or exactly, note that there is a two-fold freedom of modifying the above construction. On the one hand, we may enlarge the subspaces $W_0, \dots, W_r$. We may as well enlarge the quadratic module M by adding finitely many more generators h_i from $\mathcal{P}(K)$. Both steps result in making the approximation tighter. When the saturated preordering $\mathcal{P}(K)$ itself is finitely generated, then choosing $M = \mathcal{P}(K)$ will give the closest approximations for $\mathrm{conv}(\varphi(K))$.

When $K = V(\mathbb{R})$ is a real algebraic set, and when an embedding $V \subseteq \mathbb{A}^n$ is fixed, the closed convex sets $\overline{K_W} \subseteq \mathbb{R}^n$ resulting from taking $M = \Sigma\mathbb{R}[V]^2$ approximate the closed convex hull $\overline{\mathrm{conv}(V(\mathbb{R}))}$. Under the name *theta bodies* of V they have been studied by Gouveia, Parrilo, Thomas, and others (see [4] and [2, Chapter 7]).

Varying the embedding φ, we have the following corollary.

Corollary 2.7. *Let V be an affine $\mathbb{R}$-variety, let $K \subseteq V(\mathbb{R})$ be a basic closed set, Zariski dense in V, and assume that the saturated preordering $\mathcal{P}(K)$ in $\mathbb{R}[V]$ is finitely generated. Then the following two conditions are equivalent:*

(i) *for any $n \in \mathbb{N}$ and any morphism $\varphi\colon V \to \mathbb{A}^n$ of $\mathbb{R}$-varieties, the moment relaxation for the closed convex hull $\overline{\mathrm{conv}(\varphi(K))}$ becomes exact (2.6);*

(ii) *the preordering $\mathcal{P}(K)$ in $\mathbb{R}[V]$ is stable (2.1).*

Proof. After fixing a finite description $\mathcal{P}(K) = h_0\Sigma + \cdots + h_r\Sigma$ (with $\Sigma = \Sigma\mathbb{R}[V]^2$), stability of $\mathcal{P}(K)$ means that, for every finite-dimensional subspace $L \subseteq \mathbb{R}[V]$ containing 1, there exists a tuple $W = (W_0, \dots, W_r)$ of finite-dimensional subspaces such that $L \cap \mathcal{P}(K) \subseteq M_W$. By Theorem 2.4(b), it is equivalent that $\overline{\mathrm{conv}(\varphi(K))} = \overline{K_W}$, where φ is the morphism associated with L. Having L range over all finite-dimensional subspaces means to have φ range over all morphisms from V to affine space of arbitrary dimension. Therefore, (i) and (ii) are equivalent. ■

3. Auxiliary results. Let C be a nonsingular curve over $\mathbb{R}$. Here we collect results that are needed for working in the base extension of C to a real closed valuation ring $B \supseteq \mathbb{R}$. The situation has some resemblance to arithmetic surfaces. The main result that will be needed in the next section is Proposition 3.15.

3.1. The following setup will be fixed for the entire section. Let R be a real closed field containing $\mathbb{R}$, the field of real numbers. The unique ordering of R is denoted by $\le$. Let

$$B \;:=\; \big\{b \in R\colon \exists\, n \in \mathbb{N}\;\; -n < b < n\big\}$$

be the convex hull of $\mathbb{R}$ in R. Then B is a valuation ring with quotient field R, and we denote by $v\colon R \to \Gamma \cup \{\infty\}$ the associated Krull valuation. The maximal ideal of B will be denoted by $\mathfrak{m}$. The residue field is $B/\mathfrak{m} = \mathbb{R}$.

Let A be a finitely generated $\mathbb{R}$-algebra, and write $A_B = A \otimes B$ and $A_R = A \otimes R$ (with $\otimes := \otimes_{\mathbb{R}}$ always). Given $0 \ne f \in A_R$, we can write $f = \sum_{i=1}^r a_i \otimes b_i$ with $a_i \in A$ and $b_i \in R$ in such a way that $a_1, \dots, a_r$ are linearly independent over $\mathbb{R}$. Putting

$$w(f) \;:=\; \min\{v(b_i)\colon i = 1, \dots, r\}$$

and $w(0) := \infty$ gives a well-defined map $w\colon A_R \to \Gamma \cup \{\infty\}$ that extends the valuation v. (To see that w is well-defined, let $f = \sum_{j=1}^{s} a'_j \otimes b'_j$ be a second representation with $a'_1, \dots, a'_s$ $\mathbb{R}$-linearly independent. Then $b_1, \dots, b_r$ and $b'_1, \dots, b'_s$ span the same $\mathbb{R}$-linear subspace of R, so we can write $b'_j = \sum_i c_{ij} b_i$ with $c_{ij} \in \mathbb{R}$. It follows that $\min_j v(b'_j) \geq \min_i v(b_i)$. By symmetry, the opposite inequality holds as well.) For $f, g \in A_R$, it is easy to see that $w(f+g) \geq \min\{w(f), w(g)\}$ and $w(fg) \geq w(f) + w(g)$ hold. For $b \in R$ we have, moreover, $w(bf) = w(f) + v(b)$.

The residue map $B \to B/\mathfrak{m} = \mathbb{R}$ will be denoted by either $b \mapsto \pi(b)$ or $b \mapsto \overline{b}$. Accordingly we often denote the induced homomorphism $A_B \to A$ by $f \mapsto \overline{f}$. We have $A_B = \{f \in A_R \colon w(f) \geq 0\}$, and for $f \in A_B$ we have $\overline{f} = 0$ if and only if $w(f) > 0$.

Lemma 3.2. *Assume that the $\mathbb{R}$-algebra A is an integral domain. Then $w(fg) = w(f) + w(g)$ holds for all $f, g \in A_R$, and so w extends to a valuation of $\operatorname{Quot}(A_R)$, the field of fractions of A_R.*

Clearly, the residue field of the valuation w of $\operatorname{Quot}(A_R)$ is $\operatorname{Quot}(A)$.

Proof. Since A is a domain, and since $\mathbb{R}$ is relatively algebraically closed in R, the tensor product A_R is a domain, too. We can write $f = af_0$ and $g = bg_0$ with $a, b \in R$, where $f_0, g_0 \in A_B$ satisfy $w(f_0) = w(g_0) = 0$. So we can assume $w(f) = w(g) = 0$, which means $\overline{f}, \overline{g} \neq 0$ in A. Since A is a domain, we have $\overline{f} \cdot \overline{g} \neq 0$, which implies that $w(fg) = 0$. The lemma is proved. ■

3.3. Let A be a finitely generated reduced $\mathbb{R}$-algebra, as before, and write $V = \operatorname{Spec}(A)$ for the affine $\mathbb{R}$-variety associated with A. We need to work with the real spectrum of $A_B = A \otimes B$. As a set, $\operatorname{Sper}(A_B)$ can be identified with the disjoint union of the real spectra of the rings $A \otimes R(\mathfrak{q})$, where $\mathfrak{q}$ is a prime ideal of B and $R(\mathfrak{q})$ denotes the residue field of $\mathfrak{q}$ (a real closed field extension of $\mathbb{R}$). Given any point $\xi \in V(\mathbb{C}) = \operatorname{Hom}_{\mathbb{R}}(A, \mathbb{C})$, we consider the homomorphism

$$\xi \otimes \pi \colon A \otimes B \to \mathbb{C}, \quad a \otimes b \ \mapsto \ a(\xi)\overline{b}$$

and denote its kernel by M_ξ. So

$$M_\xi \ := \left\{ \sum_i a_i \otimes b_i \in A \otimes B \colon \ \sum_i a_i(\xi)\overline{b}_i = 0 \text{ in } \mathbb{C} \right\}.$$

Clearly, M_ξ is a maximal ideal of $A \otimes B$ whose residue field is the residue field of ξ (hence $\mathbb{R}$ or $\mathbb{C}$). When ξ is real, i.e., $\xi \in V(\mathbb{R})$, there is a unique point in $\operatorname{Sper}(A \otimes B)$ whose support is M_ξ. This point will be denoted by α_ξ. Conversely, any point $\alpha \in \operatorname{Sper}(A \otimes B)$ with residue field $\mathbb{R}$ has the following form.

Lemma 3.4. *Given $\alpha \in \operatorname{Sper}(A \otimes B)$, there exists $\xi \in V(\mathbb{R})$ with $\alpha = \alpha_\xi$ if and only if $(A \otimes B)/\operatorname{supp}(\alpha) = \mathbb{R}$.*

3.5. We fix a semialgebraic subset K of $V(\mathbb{R})$ and denote by $\widetilde{K}$ the constructible subset of $\operatorname{Sper}(A) = \operatorname{Sper} \mathbb{R}[V]$ corresponding to K; see 1.4. The natural homomorphism $i\colon A \to A_B$ induces a continuous map $i^*\colon \operatorname{Sper}(A_B) \to \operatorname{Sper}(A)$ of the real spectra (see 1.2), and we write $X_K := (i^*)^{-1}(\widetilde{K})$. So X_K is a constructible subset of $\operatorname{Sper}(A_B)$, which is closed in $\operatorname{Sper}(A_B)$

if K is a closed subset of $V(\mathbb{R})$. By K_R we denote the base field extension of K to R (see [3, section 5.1]). So K_R is the semialgebraic subset of $V(R)$ that is defined by the same finite system of inequalities as K (this does not depend on the choice of such a system). Considering $V(R)$ as a subset of $\mathrm{Sper}(A_B)$ in the natural way, we have $K_R = V(R) \cap X_K$ (cf. 1.4).

Recall that a closed point of a topological space T is a point $x \in T$ for which the singleton set $\{x\}$ is closed in T.

Proposition 3.6. *Assume that the semialgebraic set $K \subseteq V(\mathbb{R})$ is compact. Then the closed points of X_K are precisely the points α_ξ for $\xi \in K$ (see 3.3).*

Proof. For $\xi \in K$ we have $\alpha_\xi \in X_K$ by construction, and this is a closed point of $\mathrm{Sper}(A_B)$ since $\mathrm{supp}(\alpha_\xi) = M_\xi$ is a maximal ideal of A_B. Conversely, let $\alpha \in X_K$ be a closed point of X_K, and let $\phi\colon A \otimes B \to S$ be a homomorphism that represents α, where S is a real closed field (cf. 1.2). Let $C \subseteq S$ be the convex hull of $\mathbb{R}$ in S; then we have $C/\mathfrak{m}_C = \mathbb{R}$. We claim that $\mathrm{im}(\phi) \subseteq C$ holds. Indeed, let $a \in A$ and $b \in B$. Since K is compact, there is $c \in \mathbb{R}$ with $|a| < c$ on K, and it follows that $|\phi(a \otimes 1)| < c$ in S. On the other hand, there is a real number $c' > 0$ such that $|b| < c'$ holds on $\mathrm{Sper}(B)$, for example $c' = 1 + |\overline{b}|$. So we get $|\phi(a \otimes b)| < cc'$ in S, whence $\phi(a \otimes b) \in C$. Now, since $\mathrm{im}(\phi) \subseteq C$, we can compose $\phi\colon A \otimes B \to C$ with the residue homomorphism $C \to \mathbb{R}$, resulting in a homomorphism $\psi\colon A \otimes B \to \mathbb{R}$. By construction, the point $\beta \in \mathrm{Sper}(A \otimes B)$ represented by ψ is a specialization of α. Since K is closed in $V(\mathbb{R})$, we have $\beta \in X_K$, and so $\beta = \alpha$, which proves the claim by Lemma 3.4. ■

Lemma 3.7. *Let $K \subseteq V(\mathbb{R})$ be a semialgebraic set, and let $f \in A_B$. Then f is nonnegative on the constructible subset X_K of $\mathrm{Sper}(A_B)$ if and only if f is nonnegative on $K_R \subseteq V(R)$.*

Proof. Let $\mathfrak{q}$ be a prime ideal of B. The quotient field $R(\mathfrak{q})$ of $B/\mathfrak{q}$ is real closed. Let $\pi_\mathfrak{q}(f) \in A \otimes R(\mathfrak{q}) = A_{R(\mathfrak{q})}$ be the coefficientwise reduction of f modulo $\mathfrak{q}$. On the other hand, let $K_{R(\mathfrak{q})} \subseteq V(R(\mathfrak{q}))$ be the base field extension of K from $\mathbb{R}$ to $R(\mathfrak{q})$. Then $f \geq 0$ on X_K is equivalent to $\pi_\mathfrak{q}(f) \geq 0$ on $K_{R(\mathfrak{q})}$ for every prime ideal $\mathfrak{q}$ of B. Thus we have to show the following: If $f \geq 0$ on $K_R \subseteq V(R)$, then $\pi_\mathfrak{q}(f) \geq 0$ on $K_{R(\mathfrak{q})}$ for every prime ideal $\mathfrak{q}$ of B. To see this, recall that the residue map $B_\mathfrak{q} \to R(\mathfrak{q})$ has a homomorphic section s. Thus if $\eta \in K_{R(\mathfrak{q})}$ is a given homomorphism $\eta\colon A \to R(\mathfrak{q})$, then $\xi := s \circ \eta$, considered as a homomorphism $A \to B_\mathfrak{q} \subseteq R$, is a point in K_R. Since $f \geq 0$ at ξ, it follows that $\pi_\mathfrak{q}(f) \geq 0$ at η. ■

3.8. Now we specialize to the case where C is an irreducible affine curve over $\mathbb{R}$ and $A = \mathbb{R}[C]$ is the affine coordinate ring of C. We keep fixed the extension $\mathbb{R} \subseteq R$ of real closed fields and the convex hull B of $\mathbb{R}$ in R, and we will write $R[C] := A \otimes R$ and $B[C] := A \otimes B$. The following technical lemma is specific to the curves case.

Lemma 3.9. *Let C be an irreducible affine curve over $\mathbb{R}$, and let $K \subseteq C(\mathbb{R})$ be a compact semialgebraic set. Let M be a maximal ideal of $\mathbb{R}[C] \otimes B = B[C]$, and assume that there exists $\beta \in X_K$ with $\mathrm{supp}(\beta) \subseteq M$ and with $\mathrm{supp}(\beta) \not\subseteq \mathbb{R}[C] \otimes \mathfrak{m}$. Then $M = M_\xi$ for some $\xi \in K$.*

Proof. Write $A = \mathbb{R}[C]$ as before. Let $P = \mathrm{supp}(\beta)$, write $\mathfrak{q} = P \cap B$, and let $k = R(\mathfrak{q}) = B_\mathfrak{q}/\mathfrak{q}B_\mathfrak{q}$ be the residue field of the prime ideal $\mathfrak{q}$ of B. The field k is real closed. The sequence of ring homomorphisms $B \to A \otimes B \to A \otimes k$ induces, by taking preimages, a sequence of maps

$$\mathrm{Spec}(A \otimes k) \xrightarrow{j} \mathrm{Spec}(A \otimes B) \xrightarrow{\pi} \mathrm{Spec}(B)$$

between the Zariski spectra. The map j is a bijection from $\mathrm{Spec}(A \otimes k)$ to the preimage $\pi^{-1}(\mathfrak{q}) = C_k$ of $\mathfrak{q}$ under π, and this bijection preserves residue fields of prime ideals. Since $A \otimes k = k[C]$ is a one-dimensional integral domain, since the zero ideal of $A \otimes k$ corresponds to $A \otimes \mathfrak{q} \in \pi^{-1}(\mathfrak{q})$, and since $P \not\subseteq A \otimes \mathfrak{m}$ by assumption, we see that P corresponds to a maximal ideal of $A \otimes k$ under this bijection. Since, moreover, the residue field of P is real, there exists a point $\eta \in C(k)$ such that P is the kernel of the homomorphism

$$A \otimes B \;=\; \mathbb{R}[C] \otimes B \;\to\; \mathbb{R}[C] \otimes k \;=\; k[C] \xrightarrow{\eta} k.$$

So $(A \otimes B)/P$ is isomorphic to a subring of k that contains the valuation ring $B/\mathfrak{q}$ of k. Therefore $(A \otimes B)/P$ is a valuation ring itself and, in particular, a local ring. Therefore M is the unique maximal ideal of $A \otimes B$ that contains P. On the other hand, by Proposition 3.6, there exists $\xi \in K \subseteq C(\mathbb{R})$ such that β specializes to α_ξ, and hence $P \subseteq M_\xi$. This shows that $M = M_\xi$. ■

3.10. We keep fixing the extension $\mathbb{R} \subseteq R$ and the valuation ring B of R as before. We now assume that C is a nonsingular and geometrically irreducible affine algebraic curve over $\mathbb{R}$, and we consider the affine scheme $C \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(B) = \mathrm{Spec}(\mathbb{R}[C] \otimes B)$. This is a relative affine curve over $\mathrm{Spec}(B)$. If B were a discrete valuation ring, the situation would be a (very particular) instance of a relative curve over a Dedekind scheme and hence an arithmetic surface. However, B has a divisible value group and therefore is not noetherian (as long as $R \neq \mathbb{R}$). Moreover, the Krull dimension of B can be arbitrarily large. Therefore we cannot directly rely on arguments that are well known for arithmetic surfaces, or simply for noetherian rings. Still, the situation and the auxiliary results we are about to prove resemble the case of a relative curve over a discrete valuation ring.

The function field of C (resp., of C_R) is, as usual, denoted by $\mathbb{R}(C) := \mathrm{Quot}\,\mathbb{R}[C]$ (resp., by $R(C) := \mathrm{Quot}\,R[C]$).

3.11. Let $R' = R(\sqrt{-1})$ be the algebraic closure of R, and let $B' = B[\sqrt{-1}]$, a valuation ring of R' that extends the valuation ring B of R. The maximal ideal of B' will be denoted $\mathfrak{m}'$, and we have $B'/\mathfrak{m}' = \mathbb{C}$. The valuation v on R (see 3.1) (resp., w on $R(C)$ (see 3.2)) extends uniquely to a valuation on R' (resp., on $R'(C)$), and we use the same letter v (resp., w) to denote this extension. The residue field of the valuation v on R' is $\mathbb{C}$, and the residue field of the valuation w on $R'(C)$ is $\mathbb{C}(C)$, the complex function field of the curve C. Given $g \in R'(C)$ with $w(g) \geq 0$, we denote the residue class of g in $\mathbb{C}(C)$ by $\overline{g}$. Also, we write $B'[C] = \mathbb{R}[C] \otimes B'$ and $R'[C] = \mathbb{R}[C] \otimes R'$. Again we have $B'[C] = \{f \in R'[C] \colon w(f) = 0\}$.

We consider the natural specialization map

$$C(B') \;\to\; C(\mathbb{C}), \quad \eta \mapsto \overline{\eta}$$

defined by composing a homomorphism $\eta \colon \mathbb{R}[C] \to B'$ with the residue map $B' \to B'/\mathfrak{m}' = \mathbb{C}$. Note that $\eta \in C(B')$ specializes to $\xi \in C(\mathbb{C})$ (that is, $\overline{\eta} = \xi$) if and only if $h(\eta) = 0$ implies that $\overline{h}(\xi) = 0$ for every $h \in B'[C]$. Given $\xi \in C(\mathbb{C})$, we will use the notation

$$U(\xi) \;:=\; \{\eta \in C(B') \colon \overline{\eta} = \xi\},$$

so this is the set of B'-rational points of C that specialize to the $\mathbb{C}$-rational point ξ. The maximal ideal of $B[C]$ associated with $\xi \in C(\mathbb{C})$ is denoted by $M_\xi = \{f \in B[C] \colon \overline{f}(\xi) = 0\}$; see 3.3.

The zero or pole order of a rational function g on a nonsingular curve in a geometric point ξ will be denoted by $\mathrm{ord}_\xi(g)$. Thus, given $f \in B'[C]$ and $\eta \in C(R')$, the symbol $\mathrm{ord}_\eta(f)$ denotes the vanishing order of f in the point η of the generic fibre $C_{R'}$. For $\xi \in C(\mathbb{C})$, on the other hand, the symbol $\mathrm{ord}_\xi(\overline{f})$ denotes the vanishing order in ξ of the restriction $\overline{f}$ of f to the special fibre C. Below (Proposition 3.15) we show how the vanishing orders of f in points of the generic fibre determine the vanishing orders of $\overline{f}$ in the points of the special fibre.

Lemma 3.12. *Let $g \in R(C)^*$ satisfy $w(g) = 0$, let $\xi \in C(\mathbb{C})$ be a geometric point of the special fibre, and assume $\mathrm{ord}_\eta(g) \geq 0$ for every $\eta \in U(\xi)$. Then there exist $0 \neq f, h \in B[C]$ with $g = \frac{f}{h}$ and $\overline{h}(\xi) \neq 0$. In other words, g lies in the localized ring $B[C]_{M_\xi}$.*

Proof. We can write $g = \frac{b}{a}$ with $0 \neq a, b \in R[C]$. By scaling a and b with a nonzero element of R we clearly can assume $w(a) = w(b) = 0$. So, in particular $a, b \in B[C]$.

Let $\eta_1, \dots, \eta_r$ be the zeros of a in $U(\xi)$, and let $\zeta_1, \dots, \zeta_s$ be the remaining zeros of a in $C(R')$. For each $j = 1, \dots, s$ there exists $h_j \in B[C]$ satisfying $\overline{h}_j(\xi) \neq 0$ and $h_j(\zeta_j) = 0$ since $\overline{\zeta}_j \neq \xi$. By taking a product of suitable powers of these h_j, we find $h \in B[C]$ satisfying $\overline{h}(\xi) \neq 0$ and $\mathrm{ord}_{\zeta_j}(h) \geq \mathrm{ord}_{\zeta_j}(a)$ for $j = 1, \dots, s$.

For any point $\eta \in C(R')$ we claim that $\mathrm{ord}_\eta(bh) \geq \mathrm{ord}_\eta(a)$ holds. Indeed, this is trivial if $a(\eta) \neq 0$. For $\eta \in \{\zeta_1, \dots, \zeta_s\}$ it is so by the choice of h. For $\eta \in \{\eta_1, \dots, \eta_r\}$ it is true since $\mathrm{ord}_\eta(b) \geq \mathrm{ord}_\eta(a)$ by the assumption on g. So $gh = \frac{bh}{a}$ has no poles in $C(R')$ and therefore lies in $R[C]$. Since $w(gh) = 0$, we have $gh \in B[C]$, so it suffices to take $f := gh$. ∎

The analogue of Lemma 3.12 in algebraic geometry would be the following statement: If V is a nonsingular complex algebraic surface and $\xi \in V(\mathbb{C})$, and if a rational function $g \in \mathbb{C}(V)^*$ has no pole along any curve $C \subseteq V$ through ξ, then $g \in \mathcal{O}_{V,\xi}$. (Indeed, the noetherian local ring $\mathcal{O}_{V,\xi}$, being integrally closed, is the intersection of its localizations at all height one prime ideals.)

Lemma 3.13. *Let $f, g \in B'[x, y]$ be polynomials such that the coefficientwise reduced polynomials $\overline{f}, \overline{g} \in \mathbb{C}[x, y]$ are not identically zero. Assume $f(0, 0) = g(0, 0) = 0$, and assume the curves $\overline{f} = 0$ and $\overline{g} = 0$ in $\mathbb{C}^2$ intersect transversally at $(0, 0)$. Then the curves $f = 0$ and $g = 0$ in R'^2 intersect transversally at $(0, 0)$, and they do not intersect in any point $(a, b) \neq (0, 0)$ in R'^2 with $a, b \in \mathfrak{m}'$.*

Proof. The gradient vectors of f and g at the origin lie in B'^2, and by assumption they are linearly independent modulo $\mathfrak{m}'$. Hence they are linearly independent in R'^2, which is the first assertion. After a linear change of coordinates we can assume

$$f \;=\; x + \sum_{d \geq 2} f_d(x, y), \quad g \;=\; y + \sum_{d \geq 2} g_d(x, y),$$

where $f_d, g_d \in B'[x, y]$ are homogeneous polynomials of degree d for $d \geq 2$. Let $(0, 0) \neq (a, b) \in \mathfrak{m}' \times \mathfrak{m}'$, and assume $v(a) \leq v(b)$. Since $v(a) > 0$, we see that $v(f(a, b) - a) > v(a)$, whence $v(f(a, b)) = v(a)$, and therefore $f(a, b) \neq 0$. Likewise, $v(a) \geq v(b)$ implies that $v(g(a, b)) = v(b)$ and $g(a, b) \neq 0$. ∎

Lemma 3.14. *Let* $\eta \in C(B')$*, and let* $\xi = \overline{\eta} \in C(\mathbb{C})$*.*

(a) *There is* $s \in B'[C]$ *such that* $s(\eta) = 0$ *and* $\mathrm{ord}_\xi(\overline{s}) = 1$*.*

(b) *If* $\eta \in C(B)$*, then an element* s *satisfying* (a) *can be found in* $B[C]$*.*

(c) *For any element* s *satisfying* (a) *one has* $\mathrm{ord}_\eta(s) = 1$ *and* $s(\eta') \neq 0$ *for any* $\eta' \in U(\xi) \smallsetminus \{\eta\}$*.*

Proof. Choose $t \in B'[C]$ such that $\overline{t} \in \mathbb{C}[C]$ is a local uniformizer at $\xi = \overline{\eta}$. Then $t(\eta) \in \mathfrak{m}'$. The element $s := t - t(\eta)$ of $B'[C]$ has $s(\eta) = 0$ and $\overline{s} = \overline{t}$, and hence $\mathrm{ord}_{\overline{\eta}}(\overline{s}) = 1$. If η is real, i.e., $\eta \in C(B)$, then t (and therefore s) can be found in $B[C]$. This proves (a) and (b).

(c) The question is local around the point $\overline{\eta} \in C(\mathbb{C})$. Zariski locally around any given $\mathbb{C}$-point, any nonsingular curve over $\mathbb{C}$ is isomorphic to a Zariski open subset of a plane curve over $\mathbb{C}$. Therefore we can assume that C is a (possibly singular) closed curve in $\mathbb{A}^2_{\mathbb{C}}$ and that $\xi = (0, 0)$ is a nonsingular point of C. Now assertion (c) follows from Lemma 3.13. ∎

Proposition 3.15. *Let* $f \in B'[C]$ *satisfy* $w(f) = 0$*. The vanishing order of* $\overline{f}$ *in a point* $\xi \in C(\mathbb{C})$ *satisfies*

$$\mathrm{ord}_\xi(\overline{f}) \;=\; \sum_{\eta \in U(\xi)} \mathrm{ord}_\eta(f).$$

Proof. Let e denote the right-hand sum in the assertion, and let

$$\{\eta \in U(\xi)\colon\; f(\eta) = 0\} \;=:\; \{\eta_1, \dots, \eta_r\},$$

a finite set of points in $U(\xi) \subseteq C(B') \subseteq C(R')$. For every $i = 1, \dots, r$, choose $s_i \in B'[C]$ with $w(s_i) = 0$, $s_i(\eta_i) = 0$, and $\mathrm{ord}_\xi(\overline{s}_i) = 1$, according to Lemma 3.14(a). Moreover, put $e_i := \mathrm{ord}_{\eta_i}(f)$. Let $s := s_1^{e_1} \cdots s_r^{e_r} \in B'[C]$; then we have $w(s) = 0$ and $\mathrm{ord}_\xi(\overline{s}) = e_1 + \cdots + e_r = e$. Moreover, from Lemma 3.14(c) we see that $\mathrm{ord}_{\eta_i}(s) = e_i = \mathrm{ord}_{\eta_i}(f)$ for $i = 1, \dots, r$ and $s(\eta) \neq 0$ for any $\eta \in U(\xi) \smallsetminus \{\eta_1, \dots, \eta_r\}$. Hence the rational function $g := \frac{f}{s} \in R'(C)^*$ has $\mathrm{ord}_\eta(g) = 0$ for any $\eta \in C(B')$ with $\overline{\eta} = \xi$. Applying Lemma 3.12 to g and g^{-1} shows that g is a unit in the localized ring $B'[C]_{M_\xi}$. Thus $\overline{g}(\xi) \neq 0$, and hence $\mathrm{ord}_\xi(\overline{f}) = \mathrm{ord}_\xi(\overline{s}) = e$. ∎

For an analogue of Proposition 3.15 in algebraic geometry let V be a nonsingular complex surface and $C \subseteq V$ be an irreducible curve. Given a rational function $f \in \mathbb{C}(V)^*$ of order zero along C, the proposition corresponds to the formula for the divisor of the restriction of f to C.

4. Main theorem.

The following fact is well known.

Theorem 4.1. *Let* C *be a nonsingular affine curve over* $\mathbb{R}$*, and let* $K \subseteq C(\mathbb{R})$ *be a compact semialgebraic set. Then the saturated preordering* $\mathcal{P}(K)$ *of* K *in* $\mathbb{R}[C]$ *is finitely generated.*

This is proved in [22, Theorem 5.21]. More precisely (by [22, Theorem 5.22(b)]), $\mathcal{P}(K)$ can be generated by two elements, even as a quadratic module, and can in fact be generated by a single element whenever K has no isolated points. If $K = C(\mathbb{R})$ (assuming this set is compact), we have $\mathcal{P}(K) = \Sigma\mathbb{R}[C]^2$.

4.2. Let us briefly indicate how Theorem 4.1 can be proved. (The proof given in [22] was more complicated since the archimedean local-global principle was not yet available at that time.) When K has no isolated points, $\mathcal{P}(K)$ is generated by any $f \in \mathcal{P}(K)$ which has simple zeros in the boundary points of K and has no other zeros in K (one can show that such f exists). This follows from the archimedean local-global principle (see Theorem 4.6 below). In the general case, let $\xi_1, \dots, \xi_r$ be the isolated points of K. We modify the set K by replacing each isolated point ξ_i with a small closed interval $[\xi_i, \eta_i]$ on $C(\mathbb{R})$, for which $\eta_i \neq \xi_i$ lies on the same connected component of $C(\mathbb{R})$ as ξ_i, and the interval is so small that $[\xi_i, \eta_i] \cap K = \{\xi_i\}$. Let K_1 be the modified set obtained in this way, and note that K_1 has no isolated points. Let K_2 be a second such modification of K in which ξ_i gets replaced by $[\eta_i', \xi_i]$, where $\eta_i' \neq \xi_i$ is again chosen close to ξ_i, but such that η_i and η_i' lie on opposite sides of ξ_i on the local branch of $C(\mathbb{R})$ around ξ_i. Then, by the first part of the argument, there exists a single generator f_j of $\mathcal{P}(K_j)$ for both $j = 1, 2$. Again using the archimedean local-global principle, one concludes that $\mathcal{P}(K)$ is generated by f_1 and f_2.

The following theorem (resp., its corollary) is the main result of this paper. The results on spectrahedral representations in sections 5–6 are derived from it.

Theorem 4.3. *Let C be a nonsingular affine curve over $\mathbb{R}$, let $K \subseteq C(\mathbb{R})$ be a compact semialgebraic set, and let $T = \mathcal{P}(K)$ be the saturated preordering of K in $\mathbb{R}[C]$. For any real closed field R containing $\mathbb{R}$, the preordering T_R generated by T in $R[C]$ is saturated as well.*

Using the notion of stable preordering (see 2.1), we can give the following equivalent formulation.

Corollary 4.4. *For C and K as in Theorem 4.3, the preordering $\mathcal{P}(K)$ in $\mathbb{R}[C]$ is stable.*

Proof. By [24, Corollary 3.8], $T = \mathcal{P}(K)$ is stable if and only if for every real closed field R containing $\mathbb{R}$ the preordering T_R is saturated in $R[C]$. So Corollary 4.4 is equivalent to Theorem 4.3. ■

Remarks 4.5. For the following remarks assume that the nonsingular affine curve C is irreducible.

1. When the curve C is rational, the assertions of Theorem 4.3 and Corollary 4.4 are true regardless of whether K is compact or not. More precisely, assume that C is a nonsingular rational affine curve, and let $K \subseteq C(\mathbb{R})$ be any closed semialgebraic subset. Then the saturated preordering $\mathcal{P}(K)$ of K in $\mathbb{R}[C]$ is finitely generated and is stable. This is well known and essentially elementary.

2. When C has genus one and $C(\mathbb{R})$ is compact, Theorem 4.3 and Corollary 4.4 were proved for $K = C(\mathbb{R})$ in [27]. In all other cases of positive genus, these results are new.

3. When C is nonsingular of genus ≥ 1 and $K \subseteq C(\mathbb{R})$ is a closed semialgebraic set that is not compact, two situations can occur. Either K is virtually compact (see 7.1 below), in which case we will later prove that the above results remain true (Theorem 7.3 below), or else K fails to be virtually compact; then it is known that the preordering $\mathcal{P}(K)$ fails to be finitely generated [22, Theorem 5.21], and so the notion of stability does not even make sense for it. See Example 7.2 below for both examples and nonexamples of virtually compact sets.

Before giving the actual proof of Theorem 4.3, we need some preparations. First recall the archimedean local-global principle.

Theorem 4.6 ([23, Corollary 2.10]). *Let A be a ring containing $\frac{1}{2}$, let P be an archimedean* *preordering in A, and let f be an element of the saturation of P. Then f lies in P if (and only if) f lies in $P_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of A.*

Here $P_{\mathfrak{m}}$ is the preordering generated by P in the localized ring $A_{\mathfrak{m}}$. See 1.3 for the notions of archimedean preordering and saturation.

4.7. In the following C let be a nonsingular affine curve over $\mathbb{R}$, let $K \subseteq C(\mathbb{R})$ be a compact semialgebraic subset, and let $T = \mathcal{P}(K) \subseteq \mathbb{R}[C]$. Moreover, let R be a real closed field containing $\mathbb{R}$, and let B be the convex hull of $\mathbb{R}$ in R (see 3.1). We shall work in the ring $B[C] = \mathbb{R}[C] \otimes B$ and use the auxiliary results from section 3. In particular, we use the notation introduced there. Let T_B be the preordering generated by T in $B[C]$. The saturation of T_B consists of all $f \in B[C]$ with $f \geq 0$ on X_K.

Lemma 4.8. *The preordering T_B in $B[C]$ is archimedean.*

Proof. Since $T = \mathcal{P}(K)$ is the saturated preordering in $\mathbb{R}[C]$ associated with the compact set K, it is clear that T is archimedean. Let $f \in B[C] = \mathbb{R}[C] \otimes B$. Since $T - T = \mathbb{R}[C]$, we can write f in the form $f = \sum_{i=1}^r f_i \otimes b_i$ with $f_i \in T$ and $b_i \in B$ $(i = 1, \dots, r)$. Since T is archimedean, there exists $0 < c_1 \in \mathbb{R}$ with $c_1 - f_i \in T$ $(i = 1, \dots, r)$. By the definition of B there exists $0 < c_2 \in \mathbb{R}$ with $b_i \leq c_2$ in R for every i, and hence $c_2 - b_i$ is a square in B for $i = 1, \dots, r$. We conclude that

$$rc_1c_2 - f \;=\; c_2 \sum_{i=1}^r (c_1 - f_i) \otimes 1 + \sum_{i=1}^r f_i \otimes (c_2 - b_i)$$

lies in T_B. ∎

By $\operatorname{int}(K_R)$ we denote the interior, relative to $C(R)$, of the semialgebraic subset K_R of $C(R)$. The following technical lemma is based on Lemma 3.14.

Lemma 4.9. *Let $\xi \in K$, and let $U(\xi) = \{\eta \in C(B') \colon \overline{\eta} = \xi\}$ as in 3.11. For every point $\eta \in U(\xi)$ there exists an element $p_\eta \in T_B$ with $w(p_\eta) = 0$, such that $p_\eta(\eta) = 0$ and*

$$\operatorname{ord}_\xi(\overline{p}_\eta) \;=\; \begin{cases} 1 & \text{if } \eta \in C(R),\ \eta \notin \operatorname{int}(K_R), \\ 2 & \text{if } \eta \in C(R),\ \eta \in \operatorname{int}(K_R), \\ 2 & \text{if } \eta \in C(R') \setminus C(R). \end{cases}$$

Moreover, if $\eta = \xi$ and ξ is an isolated point of K, there exists a second element $p'_\xi \in T_B$ with the same properties as p_ξ and such that $p_\xi p'_\xi \leq 0$ on a neighborhood of ξ in $C(R)$.

Proof. We need to distinguish several cases. First assume $\eta \in C(R') \setminus C(R)$. By Lemma 3.14(a) there exists $s \in B'[C]$ with $s(\eta) = 0$ and $\operatorname{ord}_\xi(\overline{s}) = 1$. Let τ be the R-automorphism of $R'[C]$ of order two that is induced by complex conjugation on R'. Then $p_\eta := s \cdot \tau(s)$ is a sum of two squares in $B[C]$, and hence $p_\eta \in T_B$, and clearly $p_\eta(\eta) = 0$ and $\operatorname{ord}_\xi(\overline{p}_\eta) = 2$.

When $\eta \in \operatorname{int}(K_R)$, choose $s \in B[C]$ with $s(\eta) = 0$ and $\operatorname{ord}_\xi(\overline{s}) = 1$, according to Lemma 3.14(b). Then $p_\eta := s^2$ will do the job.

Now assume $\eta \in C(R)$ and $\eta \notin \operatorname{int}(K_R)$. Then necessarily ξ is a boundary (or isolated) point of K, and either $\eta = \xi$ or $\eta \notin K_R$. Since T is saturated, there exists $t \in T$ with

$\mathrm{ord}_\xi(t) = 1$ and with $t(\eta) \le 0$. (The second condition is automatic if ξ is not an isolated point of K_R.) So $p_\eta := t - t(\eta)$ lies in T_B and has the desired properties.

To prove the additional claim in the case where $\eta = \xi$ is an isolated point of K, fix a local orientation on $C(\mathbb{R})$ around ξ. Since T is saturated, one can find $t_1 \in T$ changing sign from $+$ to $-$ in ξ, as well as $t_2 \in T$ changing sign from $-$ to $+$ in ξ, such that both have vanishing order 1 in ξ. The proof of the lemma is complete. ∎

Proof of Theorem 4.3. We have to show that T_R contains every $g \in R[C]$ with $g \ge 0$ on K_R. It suffices to prove that T_B contains every $f \in B[C]$ with $f \ge 0$ on K_R and with $w(f) = 0$. Indeed, given $g \in R[C]$ with $g \ge 0$ on K_R, we find $0 \ne b \in R$ with $w(g) = v(b^2)$, and hence with $b^{-2}g \in B[C]$ and $w(b^{-2}g) = 0$. Knowing $b^{-2}g \in T_B$ clearly implies that $g \in T_R$.

So fix $f \in B[C]$ with $f \ge 0$ on K_R and with $w(f) = 0$. From Lemma 3.7 we know that $f \ge 0$ on X_K, that is, f lies in the saturation of T_B in $B[C]$. Since T_B is archimedean (Lemma 4.8), we can apply the archimedean local-global principle, Theorem 4.6, to f and T_B. By this theorem, it suffices to prove, for every maximal ideal M of $B[C]$, that f lies in T_M, the preordering generated by T in the local ring $B[C]_M$. To show this, fix M, and let $X_{K,M} := X_K \cap \mathrm{Sper}\, B[C]_M$, where $\mathrm{Sper}\, B[C]_M$ is considered as a subset of $\mathrm{Sper}\, B[C]$ in the natural way. So $X_{K,M}$ is the basic closed constructible subset of $\mathrm{Sper}\, B[C]_M$ associated with T_M.

If $f > 0$ on $X_{K,M}$, then $f \in T_M$ by [26, Proposition 2.1]. So we can assume that there exists $\beta \in X_K$ with $f \in \mathrm{supp}(\beta) \subseteq M$. The hypotheses of Lemma 3.9 apply to M since $w(f) = 0$ implies that $\mathrm{supp}(\beta) \not\subseteq \mathbb{R}[C] \otimes \mathfrak{m}$. By Lemma 3.9, therefore, we have $M = M_\xi$ for some point $\xi \in K$. Recall that

$$U(\xi) \;=\; \{\eta \in C(B')\colon\; \overline{\eta} = \xi\}.$$

We decompose the set of R-zeros of f in $U(\xi)$ as

$$\{\eta \in U(\xi) \cap C(R)\colon\; f(\eta) = 0\} \;=\; \{\eta_1, \dots, \eta_r\} \cup \{\zeta_1, \dots, \zeta_s\}$$

in such a way that $\eta_1, \dots, \eta_r$ are interior points of K_R, while $\zeta_1, \dots, \zeta_s$ are not. Note that f has even order in any of the points η_i. Among the nonreal zeros of f in $U(\xi)$, choose a subset $\{\omega_1, \dots, \omega_t\}$ that contains exactly one representative from each pair of complex conjugate points. Then put

$$p \;:=\; \prod_{i=1}^{r} (p_{\eta_i})^{\frac{1}{2}\mathrm{ord}_{\eta_i}(f)} \cdot \prod_{j=1}^{s} (p_{\zeta_j})^{\mathrm{ord}_{\zeta_j}(f)} \cdot \prod_{k=1}^{t} (p_{\omega_k})^{\mathrm{ord}_{\omega_k}(f)},$$

where the $p_{\eta_i}, p_{\zeta_j}, p_{\omega_k} \in T_B$ are chosen as in Lemma 4.9. Then p, being a product of elements of T_B, lies in T_B. By Proposition 3.15 we have

$$\mathrm{ord}_\xi(\overline{f}) \;=\; \sum_{i=1}^{r} \mathrm{ord}_{\eta_i}(f) + \sum_{j=1}^{s} \mathrm{ord}_{\zeta_j}(f) + 2\sum_{k=1}^{t} \mathrm{ord}_{\omega_k}(f).$$

This number is also equal to $\mathrm{ord}_\xi(\overline{p})$. It follows that $g := \frac{f}{p}$ is a unit in the local ring $B[C]_M$. In particular, g has no zeros or poles in $U(\xi)$.

We would like g to take positive values in all points $\eta \in U(\xi) \cap K_R$. This obviously is the case whenever $p(\eta) \neq 0$. By continuity, it is also true whenever η is not an isolated point of K_R. The remaining case when $\eta \in U(\xi)$ is an isolated point of K_R can occur only for $\eta = \xi \in C(\mathbb{R})$ and when ξ is an isolated point of K. If $g(\xi) < 0$, we replace one of the local factors p_ξ in the definition of p by p'_ξ, where p'_ξ is chosen as in Lemma 4.9. If p' denotes the modification of p obtained in this way, and $g' = f/p'$, we have achieved $g'(\xi) > 0$.

Using Lemma 3.7 we see that the unit g of $B[C]_M$ takes strictly positive values on the set $X_{K,M}$ associated with the preordering T_M. Hence, by another application of [26, Proposition 2.1], we conclude that g lies in T_M. As a consequence, it follows that $f = pg \in T_M$, as desired. The proof of Theorem 4.3 is complete. ∎

5. Semidefinite representations in the compact case.

Now we use moment relaxation to obtain semidefinite representations from the results of the previous section.

Theorem 5.1. *Let $K \subseteq \mathbb{R}^n$ be a compact convex semialgebraic set whose set $\mathrm{Ex}(K)$ of extreme points has (semialgebraic) dimension ≤ 1. Then K is a spectrahedral shadow. A semidefinite representation of K can be obtained from a suitable moment relaxation.*

Proof. The closure $K_0 := \overline{\mathrm{Ex}(K)}$ is a compact semialgebraic set and satisfies $\dim(K_0) \leq 1$. We have $K = \mathrm{conv}(K_0)$ by the Krein–Milman theorem. We may assume that K is not contained in any proper affine-linear subspace of $\mathbb{R}^n$.

Let $I \subseteq \mathbb{R}[\mathtt{x}] = \mathbb{R}[x_1, \dots, x_n]$ be the ideal of polynomials vanishing on K_0, and let $C_0 = \mathrm{Spec}(\mathbb{R}[\mathtt{x}]/I)$; then C_0 is the reduced Zariski closure of K_0 in $\mathbb{A}^n$. Then C_0 is an $\mathbb{R}$-variety (possibly reducible) of dimension ≤ 1. Let $\pi\colon C'_0 \to C_0$ be the normalization of C_0. Note that π is a finite morphism and that C'_0 is nonsingular. If C_0 has the irreducible components $X_1, \dots, X_l$, and if we denote by A'_i the integral closure of $\mathbb{R}[X_i]$ in its quotient field, the coordinate ring of C'_0 is therefore $\mathbb{R}[C'_0] = A'_1 \times \cdots \times A'_l$.

The map $\pi\colon C'_0(\mathbb{R}) \to C_0(\mathbb{R})$ on $\mathbb{R}$-points may fail to be surjective. Indeed, when ξ is an isolated point of $C_0(\mathbb{R})$ that lies on a one-dimensional irreducible component of C_0, then $\xi \notin \pi(C'_0(\mathbb{R}))$. To resolve this problem, let $\xi_1, \dots, \xi_k$ be the isolated points of $C_0(\mathbb{R})$ that lie in K_0 and that lie on one-dimensional irreducible components of C_0, and write $P_i = \mathrm{Spec}(\mathbb{R})$ for $i = 1, \dots, k$. Finally let

$$C_1 = C'_0 \amalg P_1 \amalg \cdots \amalg P_k$$

(disjoint sum), and let $\phi\colon C_1 \to C_0$ be the morphism with $\phi|_{C'_0} = \pi$ and with $\phi(P_i) = \xi_i$ $(i = 1, \dots, k)$. Let K_1 be the preimage of K_0 in $C_1(\mathbb{R})$. Since π, and therefore also ϕ, is a finite morphism, the semialgebraic set K_1 is compact. By construction we have $\phi(K_1) = K_0$. Since C_1 is nonsingular with irreducible components of dimension ≤ 1, the saturated preordering $\mathcal{P}(K_1)$ of K_1 in $\mathbb{R}[C_1]$ is finitely generated (see Theorem 4.1). By the main result of the previous section (Corollary 4.4), the preordering $\mathcal{P}(K_1)$ is stable. Note that K_1 is a basic closed set since $\dim(K_1) \leq 1$ (see, for instance, [1, Propositions VI.5.1 and III.3.1]).

The morphism $\phi\colon C_1 \to C_0 \subseteq \mathbb{A}^n$ induces a homomorphism $\varphi\colon \mathbb{R}[\mathtt{x}] \to \mathbb{R}[C_1]$ of the coordinate rings. Since K was assumed not to be contained in a proper affine-linear subspace, the restriction of φ to $L := \mathrm{span}(1, x_1, \dots, x_n)$ is injective. We consider L as a linear subspace of $\mathbb{R}[C_1]$. Let $\Sigma = \Sigma\mathbb{R}[C_1]^2$, and choose $1 = h_0, h_1, \dots, h_r \in \mathbb{R}[C_1]$ with $\mathcal{P}(K_1) = h_0\Sigma + \cdots + h_r\Sigma$. Since $\mathcal{P}(K_1)$ is stable, there exists a tuple $W = (W_0, \dots, W_r)$ of finite-dimensional linear subspaces $W_i \subseteq \mathbb{R}[C_1]$ such that $L \cap \mathcal{P}(K_1)$ is contained in

$$M_W \;=\; \Sigma_{W_0} + h_1\Sigma_{W_1} + \cdots + h_r\Sigma_{W_r};$$

see 2.3. By Corollary 2.5 this implies that we have found a semidefinite representation for $\operatorname{conv}(\phi(K_1)) = K$. ■

Example 5.2. To illustrate the construction in the proof of Theorem 5.1, let us consider the (rational) affine curve C_0 with equation

$$y^2 + x^2(x-1)(x-2) \;=\; 0.$$

The set $C_0(\mathbb{R}) \subseteq \mathbb{R}^2$ is compact and has the origin as an isolated point. To construct a semidefinite representation for the convex hull K of $C_0(\mathbb{R})$, we work in $A_1 = A_0' \times \mathbb{R}$, where A_0' is the integral closure of $A_0 = \mathbb{R}[C_0]$, i.e.,

$$A_0' \;=\; \mathbb{R}[x,z]/\bigl(z^2 + (x-1)(x-2)\bigr)$$

(where $y = xz$). Using the elements $1 = (1,1)$, $u = (x,0)$, $v = (z,0)$, and $e = (1,0)$ of A_1, we let $L = \operatorname{span}(1,u,uv)$, $W = \operatorname{span}(1,e,u,v)$, and $U = WW = \operatorname{span}(1,e,u,v,u^2,uv)$. The relaxation for K obtained from this data is exact. Using the basis $1-e, e, u, v$ for W, we get K as the set of all $(\xi,\eta) \in \mathbb{R}^2$ for which there exist $a,b,c \in \mathbb{R}$ with

$$\begin{pmatrix} 1-c & 0 & 0 & 0 \\ 0 & c & \xi & a \\ 0 & \xi & b & \eta \\ 0 & a & \eta & 3\xi - b - 2c \end{pmatrix} \;\succeq\; 0.$$

For the reader's convenience we include the details of the argument: Since $v^2 = -u^2 + 3u - 2e$, we get, for

$$\mu = \mu_1 + c\mu_e + x\mu_u + a\mu_v + b\mu_{u^2} + y\mu_{uv} \;\in U'$$

a general linear form, the matrix

$$M \;=\; M(x,y,a,b,c) \;=\; \begin{pmatrix} 1-c & 0 & 0 & 0 \\ 0 & c & x & a \\ 0 & x & b & y \\ 0 & a & y & 3x - b - 2c \end{pmatrix}$$

with respect to the basis $1-e$, e, v of W. This matrix represents the pull-back of μ to a symmetric bilinear form on W, via the product map $W \times W \to U$. Exactness of the relaxation is shown as follows. Let $S = \{(\xi,\eta)\colon \exists\, a,b,c\ M(\xi,\eta,a,b,c) \succeq 0\}$, and let K be the convex hull of $C_0(\mathbb{R})$. The inclusion $C_0(\mathbb{R}) \subseteq S$ is obvious. To prove $S \subseteq K$, let $M(\xi,\eta,a,b,c) \succeq 0$. Then $0 \le c \le 1$. Exploiting the 2×2 minors M_{23} and M_{34} we get $\eta^2 + \xi^2(\xi^2 - 3\xi + 2c) \le 0$. This implies that $(\xi,\eta) \in K$ when $c = 0$ or $c = 1$. Let $0 < c < 1$. Since the right bottom 3×3 submatrix of M is homogeneous, we can scale with $\frac{1}{c}$ and get $M(\frac{\xi}{c}, \frac{\eta}{c}, \frac{a}{c}, \frac{b}{c}, 1) \succeq 0$. By what was just remarked we have $(\frac{\xi}{c}, \frac{\eta}{c}) \in K$ and hence $(\xi,\eta) \in K$ as well.

Remark 5.3. It was already mentioned that the dimension hypothesis $\dim(K) \le 1$ in Theorem 5.1 is essential, according to [28]. Similarly, this hypothesis is also essential for the stability result, Theorem 4.4, from which Theorem 5.1 was derived. Indeed, there does not exist any compact semialgebraic set $K \subseteq \mathbb{R}^n$ with $\dim(K) \ge 2$ such that the saturated preordering $\mathcal{P}(K)$ is finitely generated and stable. This follows from the main result of [24].

6. Semidefinite representations in the general case.

Using the compact case, we now establish semidefinite representations for the closed convex hulls of arbitrary one-dimensional semialgebraic sets and will deduce the dimension two case of the Helton–Nie conjecture. I am indebted to Tim Netzer, who showed me how to obtain semidefinite representations for noncompact closed convex sets from such representations for compact sets.

Theorem 6.1. *Let $K \subseteq \mathbb{R}^n$ be the closed convex hull of a semialgebraic set of dimension ≤ 1. Then K is a spectrahedral shadow.*

6.2. Before we start the proof, we need to recall a few notions on convex sets and cones, for which we refer the reader to [20, Theorems 8.1 and 8.2]. Given a nonempty closed convex set $K \subseteq \mathbb{R}^n$, the *recession cone* of K is

$$\mathrm{rc}(K) \;=\; \{x \in \mathbb{R}^n\colon\; K + x \subseteq K\}$$

and is a closed convex cone. Note that $\mathrm{rc}(K)$ can also be described as the set of all existing limits $\lim_{\nu\to\infty} a_\nu x_\nu$ in $\mathbb{R}^n$, where x_ν is a sequence in K and a_ν is a null sequence of positive real numbers. The *homogenization* K^h of K is the closure of the convex cone $K^c = \{(t, tx)\colon t \ge 0, x \in K\}$ in $\mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}$ and is described as

$$K^h \;=\; K^c \cup \{(0, y)\colon y \in \mathrm{rc}(K)\}.$$

The original set K is recovered from its homogenization as $K = \{x \in \mathbb{R}^n\colon (1, x) \in K^h\}$. The extreme rays of K^h are the rays spanned by points $(1, x)$ with $x \in \mathrm{Ex}(K)$, together with the rays spanned by points $(0, y)$ where $\mathbb{R}_+ y$ is an extreme ray of $\mathrm{rc}(K)$.

6.3. Let $S \subseteq \mathbb{R}^n$ be a semialgebraic set. A ray $\mathbb{R}_+ u$ (with $0 \ne u \in \mathbb{R}^n$) will be called an *asymptotic direction of S at infinity* if there exist continuous semialgebraic paths $a(t)$ in $\mathbb{R}$ and $x(t)$ in S (with $0 < t \le 1$) such that $a(t) > 0$, $a(t) \to 0$, and $a(t)x(t) \to u$ for $t \to 0$.

Proposition 6.4. *Let $S \subseteq \mathbb{R}^n$ be a nonempty closed semialgebraic set, and let $K = \overline{\mathrm{conv}(S)}$ be its closed convex hull.*

(a) *Each extreme point of K is contained in S.*
(b) *Each extreme ray of $\mathrm{rc}(K)$ is an asymptotic direction of S at infinity.*

Without the hypothesis that S is semialgebraic, assertion (a) certainly remains true as long as S is bounded, but we are not sure about the general case.

Proof. For the proof of both parts we can assume $\mathrm{rc}(K) \cap (-\mathrm{rc}(K)) = \{0\}$. (Otherwise K contains a line, which implies that $\mathrm{Ex}(K) = \varnothing$ and $\mathrm{rc}(K)$ has no extreme ray.) We are first going to show that for any $\xi \in K$ there exists $u \in \mathrm{rc}(K)$ with $\xi - u \in \mathrm{conv}(S)$; note that this implies $\mathrm{Ex}(K) \subseteq \mathrm{conv}(S)$ and hence (a). Let $\xi \in K$. By the curve selection lemma and by

Carathéodory's lemma, there exist continuous semialgebraic paths $a_i(t)$ in $[0,1]$ and $x_i(t)$ in S, for $i=0,\dots,n$ and $0<t\le 1$, such that $\sum_{i=0}^n a_i(t)\equiv 1$, and such that

$$x(t) \;=\; \sum_{i=0}^n a_i(t)x_i(t) \tag{4}$$

converges to ξ for $t\to 0$. Note that the limit $\alpha_i := \lim_{t\to 0} a_i(t)$ exists in $[0,1]$ for every $0\le i\le n$ since the functions $a_i(t)$ are semialgebraic, and note that $\sum_i \alpha_i \equiv 1$.

We claim that the curves $a_i(t)x_i(t)$ $(i=0,\dots,n)$ are bounded for $t\to 0$. Indeed, assume that $a_i(t)x_i(t)$ is unbounded for at least one index i. Since the $a_i(t)x_i(t)$ have Puiseux–Laurent expansions in t for small $t>0$, we see that there exists a minimal rational number $q>0$ such that, for every $0\le i\le n$, the curve $t^q a_i(t)x_i(t)$ is bounded and therefore the limit $u_i := \lim_{t\to 0} t^q a_i(t)x_i(t)$ exists in $\mathbb{R}^n$. Then $u_i\in \mathrm{rc}(K)$ for every i, and $u_i\ne 0$ for at least one index i. Multiplying (4) with t^q shows that $\sum_{i=0}^n u_i = 0$, contradicting $\mathrm{rc}(K)\cap(-\mathrm{rc}(K))=\{0\}$.

So the curves $a_i(t)x_i(t)$ are all bounded. Hence the limits $u_i := \lim_{t\to 0} a_i(t)x_i(t)$ exist in $\mathbb{R}^n$. If $x_i(t)$ is unbounded, then $\alpha_i=0$ and $u_i\in\mathrm{rc}(K)$. If $x_i(t)$ is bounded, then $\xi_i = \lim_{t\to 0} x_i(t)$ exists in S, and $u_i=\alpha_i\xi_i$. Let y denote the sum of the u_i for those indices i for which $x_i(t)$ is bounded, and let u be the sum of the remaining u_i. Then $y\in\mathrm{conv}(S)$, $u\in\mathrm{rc}(K)$, and $\xi=y+u$. This proves our assertion and hence (a).

The proof of (b) is similar. After making a translation we can assume $0\in K$. Let $0\ne u\in\mathrm{rc}(K)$. Similar to (4) we have

$$\frac{1}{t}u - w(t) \;=\; \sum_{i=0}^n a_i(t)x_i(t)$$

$(0<t\le 1)$ with semialgebraic paths $a_i(t)$ in $[0,1]$ and $x_i(t)$ in S, where $\sum_i a_i(t)\equiv 0$ and $w(t)$ is a correction term with $|w(t)|<1$. Multiplication with t gives

$$u - tw(t) \;=\; \sum_{i=0}^n t a_i(t)x_i(t).$$

For $t\to 0$, the summands on the right remain bounded, as shown above. Therefore the limit $u_i=\lim_{t\to 0} t a_i(t)x_i(t)$ exists in $\mathbb{R}^n$ for $i=0,\dots,n$, and $\mathbb{R}_+u_i$ is an asymptotic direction of S at infinity (see 6.3) if $u_i\ne 0$. From $u=\sum_i u_i$ we see that if $\mathbb{R}_+u$ is an extreme ray of $\mathrm{rc}(K)$, then $\mathbb{R}_+u=\mathbb{R}_+u_i$ for some i, which proves (b). ∎

6.5. We now give the proof of Theorem 6.1. Let $S\subseteq\mathbb{R}^n$ be a nonempty semialgebraic set of dimension at most one, and let $K=\overline{\mathrm{conv}(S)}$ be its closed convex hull. In order to prove that K is a spectrahedral shadow, we may assume $\mathrm{rc}(K)\cap(-\mathrm{rc}(K))=\{0\}$. (Indeed, $U=\mathrm{rc}(K)\cap(-\mathrm{rc}(K))$ is a linear subspace of $\mathbb{R}^n$, and $K+U\subseteq K$. If $\pi\colon\mathbb{R}^n\to\mathbb{R}^n/U$ is the quotient map, then $\pi(K)=\overline{\pi(S)}$, and the recession cone R of $\pi(K)$ satisfies $R\cap(-R)=\{0\}$. A semidefinite representation for $\pi(K)$ immediately gives one for K.) For the homogenization $K^h\subseteq\mathbb{R}^{n+1}$ of K (see 6.2) this implies that $K^h\cap(-K^h)=\{0\}$. So the dual cone $(K^h)^*$ of K^h in $\mathbb{R}^{n+1}$ is full-dimensional, and we can pick an interior point w of $(K^h)^*$. The convex set

$$K_1 \;:=\; \{x\in K^h\colon\ \langle x,w\rangle = 1\}$$

is compact, and K^h is (isomorphic to) the homogenization of K_1. Indeed, since $\langle y, w\rangle > 0$ for $0 \neq y \in K^h$, we have $K^h = \{tx\colon x \in K_1,\ t \geq 0\}$, and the right-hand set is closed and hence equal to K_1^h.

The extreme rays of the convex cone K^h correspond to the extreme points of K and to the extreme rays of $\mathrm{rc}(K)$; see 6.2. By Proposition 6.4(a), $\mathrm{Ex}(K) \subseteq \overline{S}$ has dimension ≤ 1. The set S has only finitely many asymptotic directions at infinity since $\dim(S) \leq 1$, and so $\mathrm{rc}(K)$ has only finitely many extreme rays by Proposition 6.4(b). Considering the set of extreme rays of K^h as a subset of the unit sphere in $\mathbb{R}^{n+1}$, this set therefore has dimension ≤ 1. It follows that the set $\mathrm{Ex}(K_1)$ of extreme points of K_1 has dimension ≤ 1 as well. So we can apply Theorem 5.1 to K_1 and conclude that K_1 is a spectrahedral shadow. By Lemma 6.6 below, this implies that the cone $(K_1)^c = (K_1)^h \cong K^h$ (first equality holds since K_1 is compact) is a spectrahedral shadow as well. This completes the proof of Theorem 6.1 since K, being an affine-linear section of K^h, is again a spectrahedral shadow.

Lemma 6.6. *Let $K \subseteq \mathbb{R}^n$ be a convex set. If K is a spectrahedral shadow, the same is true for the convex cone $K^c \subseteq \mathbb{R} \times \mathbb{R}^n$ (see 6.2).*

Proof. This is certainly well known: Assume

$$K \;=\; \{x \in \mathbb{R}^n\colon \exists\, y \in \mathbb{R}^m \;\; A + M(x) + N(y) \succeq 0\},$$

where $M(x)$, $N(y)$ are linear systems of symmetric matrices. Then K^c is the set of all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ for which there is $(s, y) \in \mathbb{R} \times \mathbb{R}^m$ with

$$tA + M(x) + N(y) \;\succeq\; 0, \quad \begin{pmatrix} t & x_i \\ x_i & s \end{pmatrix} \;\succeq\; 0 \quad (i = 1, \dots, n).$$

The proof of Theorem 6.1 is therefore complete. We can easily extend the theorem to closed conic hulls.

Corollary 6.7. *Let $S \subseteq \mathbb{R}^n$ be a semialgebraic set, and let $S_1 := \{\frac{x}{|x|}\colon 0 \neq x \in S\}$ be its radial projection to the $(n-1)$-sphere. If $\dim(S_1) \leq 1$, then the closed conic hull $\overline{\mathrm{cone}(S)}$ of S is a spectrahedral shadow.*

Proof. Here $\mathrm{cone}(S)$, the convex cone generated by S, consists of all finite linear combinations of elements of S with nonnegative coefficients. For the proof consider $K := \overline{\mathrm{conv}(S_1)}$, a compact convex set in $\mathbb{R}^n$ that is a spectrahedral shadow by Theorem 6.1. By Lemma 6.6, the cone $K^c = K^h \subseteq \mathbb{R} \times \mathbb{R}^n$ of K is a spectrahedral shadow as well. Since $\overline{\mathrm{cone}(S)}$ is the closure of the projection of K^c to $\mathbb{R}^n$, the assertion of the corollary follows. ∎

Now we combine Theorem 6.1 with results of Netzer to show the following theorem.

Theorem 6.8 (Helton–Nie conjecture in dimension two). *Every convex semialgebraic subset of $\mathbb{R}^2$ is a spectrahedral shadow.*

Proof. Let $K \subseteq \mathbb{R}^2$ be a convex semialgebraic set. To prove that K has a semidefinite representation, we first consider the case when K is closed. If K contains a line, the assertion is obvious by reduction to a (closed) convex subset of $\mathbb{R}$. So we assume $\mathrm{rc}(K) \cap (-\mathrm{rc}(K)) = \{0\}$. Then $K = \mathrm{conv}\,\mathrm{Ex}(K) + \mathrm{rc}(K) = \overline{\mathrm{conv}\,\mathrm{Ex}(K)} + \mathrm{rc}(K)$ (Minkowski sum; see [20, Theorem

18.5]), and the set $\mathrm{Ex}(K)$ is semialgebraic of dimension ≤ 1. By Theorem 6.1, $\overline{\mathrm{conv}\,\mathrm{Ex}(K)}$ is a spectrahedral shadow. Since $\mathrm{rc}(K)$ is clearly a spectrahedral shadow, being a closed convex cone in $\mathbb{R}^2$, we see that K is one as well.

Now let $K \subseteq \mathbb{R}^2$ be an arbitrary convex semialgebraic set. We can assume that K has nonempty interior. Let M be the set of points in the boundary $\partial K = \partial\overline{K}$ that do not lie in K. Then M is a semialgebraic set with $\dim(M) \le 1$, and we can decompose M set-theoretically as follows. Let M_0 be the relative topological interior of $M \cap \mathrm{Ex}(\overline{K})$ inside $\partial\overline{K}$, and let $\mathcal{F}$ be the set of one-dimensional faces of $\overline{K}$. The supporting line of every $F \in \mathcal{F}$ is an irreducible component of the Zariski closure of ∂K. Therefore the set $\mathcal{F}$ is finite. For each $F \in \mathcal{F}$, let $M_F = F \cap M$. Moreover, let H_F be the open halfplane with $H_F \cap K \neq \varnothing$ whose boundary line contains F, and let $K_F = H_F \cup (F \cap K) = H_F \cup (\overline{H}_F \cap K)$. Then M is the union of M_0 with finitely many extreme points of $\overline{K}$ and with $\bigcup_{F\in\mathcal{F}} M_F$. Accordingly, K is the intersection of $K_0 := \overline{K} \smallsetminus M_0$ with finitely many sets $K_\xi := \overline{K} \smallsetminus \{\xi\}$ (where $\xi \in \mathrm{Ex}(\overline{K})$) and with the sets K_F $(F \in \mathcal{F})$.

Since any finite intersection of spectrahedral shadows is a spectrahedral shadow, it suffices to show that each of K_0, K_ξ, and K_F as above has a semidefinite representation. Each of the sets K_F is a union of an open halfplane H with a convex subset of the line ∂H. Using the result of Netzer and Sinn [12], such a K_F has a semidefinite representation. (Due to the elementary nature of this situation, one can easily find an explicit such representation directly.) The sets K_ξ $(\xi \in \mathrm{Ex}(\overline{K}))$ have semidefinite representations by [11, Proposition 3.1]. For K_0 we employ Netzer's construction from [11]. Let $N = \partial\overline{K} \smallsetminus M_0$, a closed subset of $\partial\overline{K}$ with $K_0 = \mathrm{int}(K) \cup N$, and let $T = \overline{\mathrm{conv}(N)}$. Then T is a closed convex subset of $\overline{K}$ and is a spectrahedral shadow by Theorem 6.1. By construction, and by Proposition 6.4, $T \cap \partial\overline{K} = N = \partial\overline{K} \smallsetminus M_0$. Using the notation introduced in [11], let $(T \looparrowleft \overline{K})$ denote the union of the relative interiors of all the faces of $\overline{K}$ that meet T. We see that $(T \looparrowleft \overline{K}) = \mathrm{int}(K) \cup N = K_0$. By [11, Theorem 3.8], $(T \looparrowleft \overline{K})$ is a spectrahedral shadow, which proves our theorem. ■

Homogenizing, we see that the Helton–Nie conjecture holds also for convex cones in $\mathbb{R}^3$.

Corollary 6.9. *Every semialgebraic convex cone $C \subseteq \mathbb{R}^3$ is a spectrahedral shadow.*

Proof. We may assume $C \cap (-C) = \{0\}$. In fact we easily reduce to the case where $C \neq \{0\}$ and there exists $u \in \mathbb{R}^3$ with $\langle u, x\rangle > 0$ for every $0 \neq x \in C$. Let $L := \{x \in \mathbb{R}^3 \colon \langle u, x\rangle = 1\}$. Then $K := C \cap L$ has a semidefinite representation by Theorem 6.8. Since $C = \{tx \colon x \in K, t \ge 0\}$ is a linear image of the cone K^c and K^c has a semidefinite representation by Lemma 6.6, we are done. ■

7. Stability in the virtually compact case.

7.1. Let C be an irreducible affine curve over $\mathbb{R}$, and let $K \subseteq C(\mathbb{R})$ be a closed semialgebraic subset. Adopting the terminology of [22], [25], we say that K is *virtually compact* if there exists a nonconstant regular function $f \in \mathbb{R}[C]$ that is bounded on K. Equivalently, K is virtually compact if and only if there exists an irreducible affine curve C_1 containing C as a Zariski open subset, in such a way that the points in $C_1 \smallsetminus C$ are nonsingular on C_1 and the closure K_1 of K in $C_1(\mathbb{R})$ is compact.

When the affine curve C is not necessarily irreducible, a closed semialgebraic set $K \subseteq C(\mathbb{R})$ is called virtually compact if $K \cap C'(\mathbb{R})$ is virtually compact on C' for every irreducible component C' of C. A closed semialgebraic set $K \subseteq \mathbb{R}^n$ of dimension ≤ 1 is called virtually compact if it has this property with respect to its Zariski closure C.

Examples 7.2. A closed semialgebraic set $K \subseteq \mathbb{R}$ is virtually compact only if it is compact. For more interesting examples let C be an irreducible plane curve with equation $f(x, y) = 0$. If the highest degree homogeneous part of f has a nonreal linear factor, then every closed semialgebraic set $K \subseteq C(\mathbb{R})$ is virtually compact. For yet another class of examples consider plane curves C with equation $y^2 = p(x)$, where $p \in \mathbb{R}[x]$ is monic and separable with $\deg(p) = d$. If $d = 2$ or d is odd, only compact sets $K \subseteq C(\mathbb{R})$ are virtually compact. If $d \equiv 0 \pmod 4$, then $K \subseteq C(\mathbb{R})$ is virtually compact if and only if K is contained in the union of a bounded set with either the upper or the lower halfplane. If $d \equiv 2 \pmod 4$, $d \geq 6$, a similar characterization holds with upper or lower halfplanes replaced by the unions of diagonally opposite quadrants.

We show that the analogues of the stability results from section 4 remain true for virtually compact sets K.

Theorem 7.3. *Let C be an irreducible nonsingular affine curve over $\mathbb{R}$, and let $K \subseteq C(\mathbb{R})$ be a closed semialgebraic set that is virtually compact. Then the saturated preordering $\mathcal{P}(K)$ in $\mathbb{R}[C]$ is finitely generated and stable.*

Proof. That $\mathcal{P}(K)$ is finitely generated was already proved (in greater generality) in [22, Theorem 5.21]. We are going to reprove this fact here using a different reasoning because we will need the same argument to prove stability. Since Theorem 7.3 has already been proved when K is compact, we can assume that K is not compact. In particular, the set K is infinite.

Let C_1 and K_1 be as in 7.1. Then C_1 is a nonsingular irreducible affine curve containing C as a Zariski open set, and the closure K_1 of K in $C_1(\mathbb{R})$ is compact. Note that $\mathbb{R}[C_1]$ is a subring of $\mathbb{R}[C]$. Let $T = \mathcal{P}_C(K)$, the saturated preordering of K in $\mathbb{R}[C]$, and let $T_1 = \mathcal{P}_{C_1}(K_1)$, the saturated preordering of K_1 in $\mathbb{R}[C_1]$. Since K_1 is compact, the preordering T_1 in $\mathbb{R}[C_1]$ is finitely generated according to Theorem 4.1. So there are nonzero elements $1 = h_0, h_1, \ldots, h_r \in \mathbb{R}[C_1]$ that generate T_1 as a quadratic module in $\mathbb{R}[C_1]$. (We can even do this with $r \leq 2$; see the remarks after Theorem 4.1.) We will prove that $T = \mathcal{P}_C(K)$ is generated by $h_0, \ldots, h_r$ as a quadratic module in $\mathbb{R}[C]$.

Let $\widetilde{C}$ be the nonsingular projective curve over $\mathbb{R}$ that contains C_1 as an open dense subscheme. We consider Weil divisors on $\widetilde{C}$ and regard them as conjugation-invariant Weil divisors on the complexified curve $\widetilde{C}_{\mathbb{C}}$. Since $\widetilde{C}(\mathbb{R}) \neq \varnothing$, we have $\mathrm{Pic}(\widetilde{C}) = \mathrm{Pic}(\widetilde{C}_{\mathbb{C}})^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}$. Let J be the Jacobian variety of $\widetilde{C}$, an abelian variety over $\mathbb{R}$.

Let $C_1(\mathbb{C}) \setminus C(\mathbb{C}) = \{Q_1, \ldots, Q_s\}$, and let $0 \neq f \in \mathbb{R}[C]$ with $f|_K \geq 0$. For $i = 1, \ldots, s$ let $m_i \geq 0$ be an integer satisfying $2m_i + \mathrm{ord}_{Q_i}(f) \geq 0$. Consider the divisor

$$D = \sum_{i=1}^{s} m_i Q_i$$

on $\widetilde{C}$. Choose a point $Q \in \widetilde{C}(\mathbb{C}) \setminus C_1(\mathbb{C})$, and let $E = Q + \overline{Q}$ (again a divisor on $\widetilde{C}$, the case $Q = \overline{Q}$ is allowed, the bar denoting complex conjugation). There exist integers l, $n \geq 1$ such that the divisor $nE - lD$ has degree zero and such that the divisor class $[nE - lD] \in J(\mathbb{R})$ lies in the identity connected component $J(\mathbb{R})_0$ of the compact real Lie group $J(\mathbb{R})$. Fix an arbitrary $\mathbb{R}$-point P_0 in the interior $\mathrm{int}(K)$ of K relative to $C(\mathbb{R})$. By the argument in [21, Lemmas 2.11 and 2.12], there is an integer $k \geq 1$ such that, for every $\alpha \in J(\mathbb{R})_0$, there exist $2k$ points $P_1, \dots, P_{2k} \in \mathrm{int}(K)$ with

$$\alpha \;=\; \sum_{j=1}^{2k} [P_j - P_0].$$

Applying this to the divisor class $\alpha := [nE - lD - k(2P_0 - E)]$ (which lies in $J(\mathbb{R})_0$; cf. [21, Lemma 2.6]), we conclude that there exist $P_1, \dots, P_{2k} \in \mathrm{int}(K)$ such that

$$lD + \sum_{j=1}^{2k} P_j \;\sim\; (n+k)E$$

on $\widetilde{C}$. Since $\mathrm{supp}(E)$ is disjoint to C_1, there exists $0 \neq h \in \mathbb{R}[C_1]$ such that the divisor of h on C_1 is $lD + \sum_{j=1}^{2k} P_j$. Since $\mathrm{ord}_{Q_i}(h^2 f) \geq 2lm_i + \mathrm{ord}_{Q_i}(f) \geq 0$, we see that $h^2 f$ lies in $\mathbb{R}[C_1]$ as well. Moreover, every zero of h on C is real and is an interior point of K. In addition, we can ensure that h has no common zero with any of $h_0, \dots, h_r$.

Since $f \geq 0$ on K, and since K is dense in K_1, it follows that $h^2 f \geq 0$ on K_1. So $h^2 f \in T_1$, which means that there is an identity

$$h^2 f \;=\; \sum_{i=0}^{r} \sum_j p_{ij}^2 h_i$$

with suitable $p_{ij} \in \mathbb{R}[C_1]$. Since any zero of h on C is real and is an interior point of K, it follows that each summand $p_{ij}^2 h_i$ of the right-hand sum is divisible (inside $\mathbb{R}[C]$) by h^2; see [21, Lemma 0.1]. By the choice of h, none of the h_i vanishes in any of the zeros of h. Hence we even have $h \mid p_{ij}$ inside $\mathbb{R}[C]$ for all indices i, j. Dividing we conclude that f lies in the quadratic module generated by $h_0, \dots, h_r$ in $\mathbb{R}[C]$.

We have thus proved that $T = \mathcal{P}_C(K)$ is finitely generated in $\mathbb{R}[C]$. To prove that T is stable is equivalent to proving the following assertion (cf. [24, Corollary 3.8]): Let R be any real closed extension field of $\mathbb{R}$. Then the preordering T_R generated by T in $R[C]$ is saturated.

To prove this, let $0 \neq f \in R[C]$ be nonnegative on K_R, where K_R denotes the extension of the semialgebraic set $K \subseteq C(\mathbb{R})$ to a semialgebraic subset of $C(R)$. Arguing literally as in the first part of the proof, we find $h \in \mathbb{R}[C_1]$ (sic) such that $h^2 f \in R[C_1]$, and such that any zero of h on C is real and is an interior point of K in which $h_0 \cdots h_r$ does not vanish. Completing the argument exactly as before, we see that f lies in the quadratic module of $R[C]$ generated by $h_0, \dots, h_r$. In other words, $f \in T_R$, as desired. The theorem is proved. ■

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