VALUE GROUPS OF REAL CLOSED FIELDS AND FRAGMENTS OF PEANO ARITHMETIC

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Abstract. We investigate real closed fields admitting an integer part of which non-negative cone is a model of Peano Arithmetic (or of its fragment $I\Delta_0 + EXP$). We obtain necessary conditions on the value group of such a real closed field. These allow us to construct a class of examples of real closed fields which do not admit such integer parts.

1. Introduction

A real closed field (RCF) is a model of $Th(\mathbb{R})$, the first order theory of $\mathbb{R}$ in the language of ordered rings $\mathcal{L} = \{+, \cdot, 0, 1, <\}$. An integer part of a real closed field R is a subring of R that generalizes in a natural way the relation between the integers and the reals.

Definition 1. An *integer part* (IP) for an ordered field R is a discretely ordered subring Z such that for each $r \in R$, there exists some $z \in Z$ with $z \leq r < z + 1$. This z will be denoted by $\lfloor r \rfloor$.

If R is an ordered field, then R is Archimedean if and only if $\mathbb{Z}$ is (the unique) integer part. We will consider only non-Archimedean real closed fields. Integer parts provide an approach for building connections between real algebra and models of fragments of Peano Arithmetic (PA). Shepherdson [S] showed that a discretely ordered ring Z is an integer part for some real closed field if and only if its non-negative cone, i.e. $Z^{\geq 0}$ is a model of $IOpen$ (see below). Every real closed field has an integer part, see [MR]. However, an integer part for R need not be unique, not even up to elementary equivalence (see [BO], [M]). In [DKS] a characterization of countable real closed fields with an integer part which is a model of PA was obtained in terms of recursive saturation. In [JK] fragments of PA have also been considered in this context. In [MS] the authors show that in the [DKS] characterization the countability condition cannot be dropped.

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We recall that Peano Arithmetic is the first order theory in the language $\mathcal{L} = \{+, \cdot, 0, 1, <\}$ of discretely ordered commutative rings with 1 of which positive cone satisfies the induction axiom for each first order formula. Fragments of PA are obtained by restricting the induction scheme to certain classes of formulas. The following subsystems will be relevant for this paper:

(1) *IOpen* (open induction) denotes induction restricted to open formulas, i.e. formulas that do not contain quantifiers;
(2) $I\Delta_0$ (bounded induction) denotes induction restricted to bounded formulas, i.e. formulas in which all quantifiers are bounded by a term of the language;
(3) $I\Delta_0 + EXP$ is $I\Delta_0$ with an extra axiom stating that the exponential function is total. It is known that $I\Delta_0 + EXP$ is a proper extension of $I\Delta_0$, for details see [HP, Chapter V.3].

In what follows we will not make any distinction between the discrete ordered ring and its non-negative cone as a model of PA or one of its fragments. For convenience we introduce the following definition.

Definition 2. An IPA is an integer part of a non Archimedean real closed field which is a model of PA. A non-Archimedean real closed field which admits an integer part which is a model of Peano Arithmetic is called an IPA real closed field, or $IPA - RCF$.

In this paper we investigate the valuation theoretical properties of any $IPA - RCF$. Our main result (Theorem 9) shows that the value group of a non-Archimedean $IPA - RCF$ must satisfy strong valuation theoretical conditions (i.e. it is an exponential group, see Definition 8). This provides a method for constructing real closed fields with no IPA, see Section 3. Theorem 9 follows from Lemma 12, where we prove that *any* $IPA - RCF$ admits a left exponential function (see Definition 7). This has many implications on the valuation theoretic structure of $IPA - RFC$. In particular, non-Archimedean $IPA - RCF$ cannot be maximally valued (see Corollary 15). Corollary 15 can be interpreted as an arithmetical counterpart to [KKS, Theorem 1] stating that no maximally valued non-Archimedean real closed field admits left exponentiation. Finally, we remark that in the above results PA can be replaced by its proper fragment $I\Delta_0 + EXP$.

2. Valuation-theoretical Preliminaries

Let $(K, +, \cdot, 0, 1, <)$ be a non Archimedean real closed field. Set $|x| := \max\{x, -x\}$. For $x, y \in K^\times$, we write $x \sim y$ iff there exists $n \in \mathbb{N}$ such that $|x| < n|y|$ and $|y| < n|x|$. It is easy to see that $\sim$ is an equivalence relation on $K^\times$. Denote by $[x]_\sim$ the equivalence class of $x \in K$ in $\sim$ and let G be the set of equivalence classes. G carries a group structure via $[x]_\sim + [y]_\sim := [xy]_\sim$. This is the value group of K.

The natural valuation v on K is the map $v : K \to G \cup \{\infty\}$ given by $v(0) = \infty$ and $v(x) = [x]_\sim$.

We recall some basic notions relative to the natural valuation v.

(1) $R_v = \{a \in K : v(a) \geq 0\}$ is the valuation ring, i.e. the finite elements of K (the convex closure of $\mathbb{Z}$ in K).
(2) $\mu_v = \{a \in K : v(a) > 0\}$ the valuation ideal, i.e. the unique maximal ideal of R_v ($x \in \mu_v \leftrightarrow \forall n \in \mathbb{N} |x| < \frac{1}{n}$).
(3) $\mathcal{U}_v^{>0} = \{a \in K^{>0} : v(a) = 0\}$ the group of positive units of R_v ($\mathcal{U}_v^{>0} = R_v^{>0} \setminus \mu_v$).
(4) $1 + \mu_v = \{a \in K : v(a-1) > 0\}$, the group of 1-units. It is a subgroup of $\mathcal{U}_v^{>0}$ ($x \in \mu_v \leftrightarrow 1 + x \in 1 + \mu_v$).
(5) $\mathbf{P}_K := \{x \in K : v(x) < 0\}$ is the set of infinite elements of K
(6) $\overline{K} = R_v/\mu_v$, the residue field of K, which is an Archimedean field, i.e. isomorphic to a subfield of $\mathbb{R}$.

Definition 3. (i) Let Γ be a linearly ordered set and $\{B_\gamma : \gamma \in \Gamma\}$ be a family of (additive) Archimedean groups. The Hahn group (Hahn sum) $G = \oplus_{\gamma \in \Gamma} B_\gamma$ is the group of functions

$$f : \Gamma \to \cup_{\gamma \in \Gamma} B_\gamma$$

where $f(\gamma) \in B_\gamma$, supp $(f) = \{\gamma \in \Gamma : f(\gamma) \neq 0\}$ is finite. G is endowed with componentwise addition and the lexicographical ordering. The Hahn product is defined analogously, replacing finite by well-ordered above.

(ii) Let G be an ordered abelian group and k an Archimedean field. Then $k((G))$ is the field of generalized power series over G with coefficients in k. That is, $k((G))$ consists of formal expressions of the form $\Sigma_{g \in G} c_g t^g$, where $c_g \in k$ and $\{g \in G : c_g \neq 0\}$ is well-ordered. Again, the addition is pointwise, the ordering is lexicographic and the multiplication is given by the convolution formula.

Definition 4. Let $(K, +, \cdot, 0, 1, <)$ be a real closed field. An exponential map is an isomorphism between $(K, +, 0, <)$ and $(K^{>0}, \cdot, 1, <)$.

We recall the decompositions of the abelian groups $(K, +, 0, <)$ and $(K^{>0}, \cdot, 1, <)$, see [K, Theorem 1.4 and Theorem 1.8].

Theorem 5. Let $(K, +, \cdot, 0, 1, <)$ be an ordered field. There is a group complement $\mathbf{A}$ of R_v in $(K, +, 0, <)$ and a group complement $\mathbf{A}'$ of μ_v in R_v such that

$$(K, +, 0, <) = \mathbf{A} \oplus \mathbf{A}' \oplus \mu_v = \mathbf{A} \oplus R_v.$$

$\mathbf{A}$ and $\mathbf{A}'$ are unique up to order preserving isomorphism, and $\mathbf{A}'$ is order isomorphic to the Archimedean group $(\overline{K}, +, 0, <)$. Furthermore, the vaue set of $\mathbf{A}$ is $G^{<0}$, the one of μ_v is $G^{>0}$, and the nonzero components of $\mathbf{A}$ and μ_v are all isomorphic to $(\overline{K}, +, 0, <)$.

Theorem 6. Let $(K, +, \cdot, 0, 1, <)$ be an ordered field, and assume that $(K^{>0}, \cdot, 1, <)$ is divisible. There is a group complement $\mathbf{B}$ of $\mathcal{U}_v^{>0}$ in $(K^{>0}, \cdot, 1, <)$ and a group complement $\mathbf{B}'$ of $1 + \mu_v$ in $\mathcal{U}_v^{>0}$ such that

$$(K^{>0}, \cdot, 1, <) = \mathbf{B} \odot \mathbf{B}' \odot (1 + \mu_v) = \mathbf{B} \odot \mathcal{U}_v^{>0}.$$

Every group complement $\mathbf{B}$ of $\mathcal{U}_v^{>0}$ in $(K^{>0}, \cdot, 1, <)$ is order isomorphic to G through the isomorphism $-v$. Every group complement $\mathbf{B}'$ of $1 + \mu_v$ in $\mathcal{U}_v^{>0}$ is order isomorphic to $(\overline{K}^{>0}, +, 0, <)$.

Definition 7. An ordered field K is said to have left exponentiation iff there is an isomorphism from a group complement $\mathbf{A}$ of R_v in $(K, +, 0, <)$ onto a group complement $\mathbf{B}$ of $\mathcal{U}_v^{>0}$ in $(K^{>0}, \cdot, 1, <)$.

3. Value groups of $IPA - RCF$

In this section we analyze the structure of the value group of an IPA-RCF (or $I\Delta_0 + EXP$). We state and prove (modulo Lemma 12 and Corollary 13 which we postpone to Section 4) the main result Theorem 9 and its consequences. We need the following definition from [K, page 26]

Definition 8. Let G be an ordered abelian group with rank Γ and Archimedean components $\{B_\gamma : \gamma \in \Gamma\}$. Let C be an Archimedean group. We say that G is exponential group in C if Γ is an isomorphic (as linear order) to the negative cone $G^{<0}$, and B_γ is isomorphic (as ordered group) to C.

In [K, Proposition 1.22 and Corollary 1.23], it is shown that for non-Archimedean K, if K admits left exponentiation, then the value group of K is an exponential group in the (Archimedean) additive group $(\overline{K}, +, 0, <)$ of the residue field. Hence, by Lemma 12 we get

Theorem 9. If K is a non-Archimedean real closed field that admits an IPA (or $I\Delta_0 + EXP$), then the value group of K is an exponential group in the additive group $(\overline{K}, +, 0, <)$. In particular, the rank of $v(K)$ is a dense linear order without endpoints.

Remark 10. Note that there are plenty of divisible ordered abelian groups that are not exponential groups in $(\overline{K}, +, 0, <)$. For example, take the Hahn group $G = \oplus_{\gamma \in \Gamma} B_\gamma$ where the Archimedean components B_γ are divisible but not all isomorphic and/or Γ is not a dense linear order without endpoints (say, a finite Γ). Alternatively, we could choose all Archimedean components to be divisible and all isomorphic, say to C, and Γ to be a dense linear order without endpoints, but choose the residue field so that $(\overline{K}, +, 0, <)$ not isomorphic to C.

Notice that the converse of Theorem 9 fails, see Corollary 16. We now present a class of subfields of power series fields that are not IPA.

A class of not IPA (or $I\Delta_0 + EXP$) real closed fields. Let k be any real closed subfield of $\mathbb{R}$. Let $G \neq \{0\}$ be any divisible ordered abelian group which is *not* an exponential group in $(k, +, 0, <)$, see Remark 10. Consider the field of power series $k((G))$ and its subfield $k(G)$ generated by k and $\{t^g : g \in G\}$. Let K be any real closed field satisfying

$$k(G)^{rc} \subseteq K \subseteq k((G))$$

where $k(G)^{rc}$ is the real closure of $k(G)$. Any such K has G as value group and k as residue field. By Theorem 9, K does not admit an IPA.

4. Left exponentiation on $IPA - RCF$

In this section we prove Lemma 12 and its corollaries. We recall the following standard facts about Peano arithmetic that we will frequently use:

The graph of the exponential function over $\mathbb{N}$ is definable by an $\mathcal{L}$-formula $E(x, y, z)$, which stands for $x^y = z$. PA proves the basic properties of exponentiation which we list below. For convenience we fix a basis 2 (this is just a closed term of the language) for exponentiation. We use $E(x, y)$ for $E(2, x, y)$.

(1) $E(0, 1)$, $\forall x, y(E(x, y) \rightarrow E(x + 1, 2y))$ and $\forall x \exists! y E(x, y)$.
(2) $\forall x \forall y((x \geq 1 \wedge E(x, y)) \rightarrow (y \geq (x + 1)))$
(3) $\forall a, b, c, d((E(a, c) \wedge E(b, d)) \rightarrow E(a + b, cd))$
(4) $\forall x \exists! y \exists z(E(y, z) \wedge z \leq x < 2z)$
"each element lies between two successive powers of 2"
(5) $\forall a, b, c, d((E(a, c) \wedge E(b, d) \wedge a < b) \rightarrow c < d)$
"exponentiation is strictly increasing"

The above properties imply that over any model of PA there is a function which *behaves* as the exponential function over $\mathbb{N}$, i.e. satisfies the properties of the exponential function over $\mathbb{N}$. This is the function $Exp(x)$ defined by the formula $E(x, y)$. In what follows when we are working in a model $\mathcal{M}$ of PA the unique element m of $\mathcal{M}$ such that $\mathcal{M} \models E(n, m)$ will be denoted either by 2^n or by $Exp(n)$.

It is known that there is a Δ_0-formula $E_0(x, y, z)$ defining the graph of the exponential function in $\mathbb{N}$ for which the above properties of exponentiation, except the totality, are provable in $I\Delta_0$. These properties give an invariant meaning to exponentiation in every non standard model of $I\Delta_0$ (see [HP, Chapter V.3]). EXP will denote the axiom $\forall x \exists! y E(x, y)$.

Lemma 11. Let K be a RCF with an integer part Z, and let R_v be its valuation ring. Then $R_v \cap Z = \mathbb{Z}$.

Proof. We certainly have $\mathbb{Z} \subseteq R_v \cap Z$, since $1 \in Z$ by definition of an IP and $Z \cap R_v$ is a group.
Let $m \in R_v \cap Z$. Then there is $z \in \mathbb{Z}$ such that $z \leq m < z+1$.
To see that such z indeed exists, note that R_v is the convex closure of $\mathbb{Z}$. So for every $x \in R_v$, there are $k_1, k_2 \in \mathbb{Z}$ such that $k_1 \leq x < k_2$. Since $\mathbb{Z} = \omega^* + \omega$ as a linear ordering, we can pick k_1 maximal and k_2 minimal with this property. Then it follows that $k_2 = k_1 + 1$.
Now $z \leq m < z+1$ implies $0 \leq m-z < 1$. As $m-z \in Z$, we must have $m - z = 0$ since Z is an IP and hence by definition does not contain any positive element smaller than 1. Thus $m = z \in \mathbb{Z}$, as desired. □

Lemma 12. Let $(K, +, \cdot, 0, 1, <)$ be a real closed field and let M be a nonstandard model of PA. Assume that $Z := -M \cup M$ is an integer part of K. Then K has left exponentiation.

Proof. Let $H := \{qm : q \in \mathbb{Q} \wedge m \in Z\}$. It is easy to see that H is a subgroup of $(K, +, 0, <)$. (Obviously, $\mathbb{Q} \subset K$ and $Z \subset H$.) Clearly, H is also a $\mathbb{Q}$-vector space. There is a basis $\bar{\mathbb{B}}$ of H such that $1 \in \bar{\mathbb{B}}$. If $\bar{\mathbb{B}} \ni b = \frac{m}{n}$ with $m \in Z$ and $0 < n \in \mathbb{N}$, then $span_V(b) = span_V(nb)$, where $nb \in Z$. So from now on we will assume without loss of generality that $\bar{\mathbb{B}} \subset Z$. Now let $\mathbb{B} := \bar{\mathbb{B}} \setminus \{1\}$ and let $A := span_V(\mathbb{B})$.

Claim 1: A is a group complement of R_v in $(K, +, 0, <)$.
Proof: We proceed in two steps, showing that each element of K can be decomposed as a sum of an element of A and an element of R_v and that the decomposition is unique. Obviously, $0 \in A$.
Subclaim (i): (Existence of decompositions) For all $x \in K$, there are $y \in A$, $r \in R_v$ such that $x = y + r$.
Proof: Let $x \in K$. If $x \in R_v$, let $y = 0$ and $r = x$.
Assume now that $x \in K^\times \setminus R_v$. Since Z is an IP of K, there is $\lfloor x \rfloor \in Z$ such that $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$. Then $\lfloor x \rfloor \in span_V(\bar{\mathbb{B}})$, e.g. $\lfloor x \rfloor = \Sigma_{b \in \bar{\mathbb{B}}} c_b b$, where $c_b \in \mathbb{Q}$ and $c_b = 0$ for all but finitely many $b \in \bar{\mathbb{B}}$. We split off the summand corresponding to 1 and rewrite this as $\lfloor x \rfloor = c \cdot 1 + \Sigma_{b \in \mathbb{B}} c_b b$, where $c \in \mathbb{Q}$. Obviously $\Sigma_{b \in \mathbb{B}} c_b b \in A$ and $c \cdot 1 \in R_v$, so $\lfloor x \rfloor \in A + R_v$. Furthermore, $0 \leq x - \lfloor x \rfloor < 1$ and, so $x - \lfloor x \rfloor \in [0, 1[\subset R_v$. Hence, $x = \lfloor x \rfloor + (x - \lfloor x \rfloor) \in A + R_v$.
In the rest of the proof, sums of the form $\Sigma_{b \in \bar{\mathbb{B}}} c_b b$ with $c_b \in \mathbb{Q}$ will always be sums where all but finitely many coefficients are 0.
Subclaim (ii): $R_v \cap A = \{0\}$.
Proof: $0 \in R_v \cap A$ is obvious. We show that every linear combination over $\mathbb{B}$ with coefficients from $\mathbb{Q}$ will be either 0 or infinite.
First, let $b \in \mathbb{B}$, $q = \frac{q_1}{q_2} \in \mathbb{Q}^\times$, where $q_1 \in \mathbb{Z}$, $0 < q_2 \in \mathbb{N}$. Then $q_1 \cdot b \in \mathbf{P}_K (= K^\times \setminus R_v)$. Now, if $\frac{1}{q_2} q_1 \cdot b =: c \in R_v$, then also $q_1 \cdot b = c q_2 \in R_v$, a contradiction. So if $qb \neq 0$, then $qb \notin R_v$. Now, suppose $x = \Sigma_{i=1}^n c_i b_i \in \mathbf{P}_K$, where $c_i \in \mathbb{Q}$ and $b_i \in \mathbb{B}$. Let $c_i = \frac{q_i}{p_i}$, $q_i \in \mathbb{Z}$, $0 < p_i \in \mathbb{N}$. So $x = \Sigma_{i=1}^n \frac{q_i}{p_i} b_i = \frac{\Sigma_{i=1}^n m_i b_i}{\Pi_{i=1}^n p_i}$ for some $m_i \in \mathbb{Z}$. Then

$\Sigma_{i=1}^{n} m_i b_i \in Z$ and $\Pi_{i=1}^{n} p_i \in \mathbb{Z}$. Now, $\Sigma_{i=1}^{n} m_i b_i$ is infinite. If not let $\Sigma_{i=1}^{n} m_i b_i =: k \in \mathbb{Z}$. Then $\Sigma_{i=1}^{n} m_i b_i + (-k)\cdot 1 = 0$ is a non-trivial linear combination equal to 0 over $\bar{\mathbb{B}}$, which is a contradiction since $\bar{\mathbb{B}}$ is a basis. Hence, we must have $\Sigma_{i=1}^{n} m_i b_i \in (Z \backslash R_v) \cup \{0\}$. But then, also $\frac{1}{\Pi_{i=1}^{n} p_i} \Sigma_{i=1}^{n} m_i b_i \notin R_v^{\times}$, since $tr \in R_v^{\times}$ for all $t \in \mathbb{Z}^{\times}$, $r \in R_v^{\times}$. Thus $\Sigma_{i=1}^{n} c_i b_i \notin R_v^{\times}$, so $A \cap R_v = \{0\}$.

Hence, decompositions are unique: If $x \in K$, $y_1, y_2 \in A$, $r_1, r_2 \in R_v$ are such that $x = y_1 + r_1 = y_2 + r_2$, then $y_1 = y_2$ and $r_1 = r_2$. Therefore, A is indeed a group complement of R_v in $(K, +, 0, <)$.

The exponential function $Exp(m)$ defined on M can be extended to an exponential function exp' on the ring $Z := -M \cup M$ with values in the fraction field of Z by defining $exp'(m) = Exp(m)$ for $m \in M$ and $exp'(m) = \frac{1}{Exp(-m)}$ for $m \in -M$. Then it is easy to show that exp' satisfies the recursion laws on Z. Since K is real closed, $(K^{>0}, \cdot, 1, <)$ is divisible, hence $x^{\frac{1}{n}}$ is defined for all $x \in K^{>0}$, $0 < n \in \mathbb{N}$. So we can define $\exp : H \to K^{>0}$ by $\exp(\frac{m}{n}) := exp'(m)^{\frac{1}{n}}$ for $m \in Z$, $0 < n \in \mathbb{N}$. Let $B := \exp[A]$.

Claim 2: exp is an isomorphism between A and B (where the operation on B is the multiplication of K).
Proof: We show first that exp is bijective (in fact, strictly increasing) and then that it is a group homomorphism.
Subclaim (i): exp is bijective.
Proof: By definition, exp is surjective. We now prove that exp is strictly increasing, hence it is injective. Suppose that $\frac{n_1}{m_1} < \frac{n_2}{m_2}$, where $n_1, n_2 \in Z$ and $0 < m_1, m_2 \in \mathbb{N}$. Then $exp'(n_1 m_2) < exp'(n_2 m_1)$, and since the root functions are strictly increasing we have $[exp'(n_1)]^{\frac{1}{m_1}} < [exp'(n_2)]^{\frac{1}{m_2}}$, and so $\exp(\frac{n_1}{m_1}) < \exp(\frac{n_2}{m_2})$.
Subclaim (ii): exp is a group homomorphism.
Proof: For $n_1, n_2 \in Z$ and $0 < m_1, m_2 \in \mathbb{N}$ the following equalities hold: $\exp(\frac{n_1}{m_1} + \frac{n_2}{m_2}) = \exp(\frac{n_1 m_2 + n_2 m_1}{m_1 m_2}) = [exp'(n_1 m_2 + n_2 m_1)]^{\frac{1}{m_1 m_2}} = [exp'(n_1 m_2) exp'(n_2 m_1)]^{\frac{1}{m_1 m_2}} = exp'(n_1 m_2)^{\frac{1}{m_1 m_2}} exp'(n_2 m_1)^{\frac{1}{m_1 m_2}}$
$= \exp(\frac{n_1 m_2}{m_1 m_2}) \exp(\frac{n_2 m_1}{m_1 m_2}) = \exp(\frac{n_1}{m_1}) \exp(\frac{n_2}{m_2})$. Hence exp is a group homomorphism.

We are almost done. Since B is an isomorphic image of a subgroup, B is a group. It remains to show that it is a group complement of the positive units in $(K^{>0}, \cdot, 1, <)$.

Claim 3: B is a group complement of $\mathcal{U}_v^{>0}$ in $(K^{>0}, \cdot, 1, <)$.
Proof: We proceed as in Claim 1. Obviously, $1 = \exp(0) \in B$.
Subclaim (i): (Existence of decompositions) For all $x \in K^{>0}$, there are $y \in B$ and $u \in \mathcal{U}_v^{>0}$, such that $x = y \cdot u$.
Proof: Let $x \in K^{>0}$. If $x \in \mathcal{U}_v^{>0}$, then we are done since $x = 1 \cdot x$, where

$1 \in B$ and $x \in \mathcal{U}_v^{>0}$. Now suppose that $x \notin \mathcal{U}_v^{>0}$. Assume that $x > 1$. Then, since $M \models PA$, there exists $m \in M$ such that $\mathrm{Exp}(m) \leq \lfloor x \rfloor \leq x < \mathrm{Exp}(m)+1 \leq \mathrm{Exp}(m+1)$. Since $\mathrm{Exp}(m+1) = 2\mathrm{Exp}(m)$, we have $\mathrm{Exp}(m) \sim \mathrm{Exp}(m+1)$, so $\mathrm{Exp}(m) \sim \lfloor x \rfloor \sim x$. In particular, we have $1 \leq \frac{\lfloor x \rfloor}{\mathrm{Exp}(m)}, \frac{x}{\lfloor x \rfloor} < \frac{\mathrm{Exp}(m+1)}{\mathrm{Exp}(m)} = 2$. As $m \in M$, let $m = \Sigma_{b \in \mathbb{B}} c_b b + c \cdot 1$, where $c_b \in \mathbb{Q}$, $c \in \mathbb{Q}$. Then $\mathrm{Exp}(m) = \exp(\Sigma_{b \in \mathbb{B}} c_b b)\exp(c \cdot 1)$. Now $c \cdot 1 \in \mathbb{Q}$, so $\exp(c \cdot 1) > \mu_v$ and $\exp(c \cdot 1)$ is certainly finite, hence $\exp(c \cdot 1) \in \mathcal{U}_v^{>0}$. Al so, $\exp(\Sigma_{b \in \mathbb{B}} c_b b) \in B$, since $\Sigma_{b \in \mathbb{B}} c_b b \in A$. Hence $\mathrm{Exp}(m) \in B \cdot \mathcal{U}_v^{>0}$. Now we have $1 \leq \frac{x}{\mathrm{Exp}(m)} < 2$, which implies $\frac{x}{\mathrm{Exp}(m)} \in K \cap [1, 2[\subset \mathcal{U}_v^{>0}$. Therefore $x = \mathrm{Exp}(m) \cdot \frac{x}{\mathrm{Exp}(m)} \in B \cdot \mathcal{U}_v^{>0}$. If, on the other hand, $0 < x < 1$ and $x \notin \mathcal{U}_v^{>0}$, then $x \in \mu_v^{>0}$, and $\frac{1}{x} \in K \setminus \mathcal{U}_v^{>0} = \mathbf{P}_K$. We use the same argument as before for $\frac{1}{x}$, so there is $m \in M$ such that $2^m \leq \frac{1}{x} < 2^{m+1}$. This implies $\frac{1}{2} < xExp(m) \leq 1$, i.e. $xExp(m) \in \mathcal{U}_v^{>0}$. We can then write $x = \frac{1}{Exp(m)} \cdot xExp(m)$, where $\frac{1}{Exp(m)} \in B$ and $xExp(m) \in U_v^{>0}$.

Subclaim (ii): $\mathcal{U}_v^{>0} \cap B = \{1\}$
Proof: $1 \in \mathcal{U}_v^{>0} \cap B$ is obvious. Suppose there exists $x \in \mathcal{U}_v^{>0} = R_v^{>0} \setminus \mu_v$ and $x \neq 1$. Hence there exists $\overline{x} \in A \setminus \{0\}$ such that $x = \exp(\overline{x})$. If $\overline{x} < 0$ then $x \in \mu_v^{>0}$ and if $\overline{x} > 0$ then x is infinite, so in both cases $x \notin \mathcal{U}_v^{>0}$.

Therefore, we know that the decompositions are unique, i.e. if $x \in K^{>0}$, $y_1, y_2 \in B$, $u_1, u_2 \in \mathcal{U}_v^{>0}$ such that $x = y_1 u_1 = y_2 u_2$, then $y_1 = y_2$ and $u_1 = u_2$. Hence, B is indeed a group complement of $\mathcal{U}_v^{>0}$ in $(K^{>0}, \cdot, 1, <)$. □

Note that in the above proof we may replace the hypothesis that K is real closed by the assumption that $(K^{>0}, \cdot)$ is divisible, that is K is root closed for positive elements. The reader will also have noticed that the full power of PA is by far not used in the proof of Lemma 12. All that is needed is a theory T of arithmetic strong enough to prove arithmetical facts (1)-(5) from the beginning of the section. Hence

Corollary 13. Let $(R, +, \cdot, 0, 1, <)$ be a non-archimedean real closed field. Assume that R admits an integer part which is a model of $I\Delta_0 + EXP$. Then R admits a left exponential function.

We also obtain the following exponential analogue to one direction of Shepherdson's theorem [S] cited in the introduction. For the notion of exponential integer part appearing below see [DKKL].

Corollary 14. If $M \models PA$, then M is an exponential integer part of a left exponential real closed field.

Proof. Let $Z = -M \cup M$ and consider $K := ff(Z)^{rc}$, the real closure of the fraction field of Z. As $M \models PA$, $M \models IOpen$, hence Z is an integer part of K by a result of Shepherdson. As remarked above since

M is a model of PA then M has a total exponential function. By Lemma 12, K has a left exponentiation.

□

In [KKS] it is shown that for no ordered abelian group $G \neq \{0\}$ the field $\mathbb{R}((G))$ admits a left exponential function. Therefore, we deduce

Corollary 15. For any divisible ordered abelian group $G \neq \{0\}$ the field $\mathbb{R}((G))$ is not an IPA - RCF.

Proof. We know that $\mathbb{R}((G))$ is a real closed field, see [EP]. By the result from [KKS] just mentioned, it cannot have a left exponential function. Therefore, by Lemma 12, it cannot have an IPA. □

We can now show that the converse to Theorem 9 does not hold.

Corollary 16. Let G be a divisible exponential group. There is a real closed field $(K, +, \cdot, 0, 1, <)$ such that $v(K^{\times}) = G$ but K does not have an IPA.

Proof. Let $K = \mathbb{R}((G))$. By Corollary 15 K cannot have an IPA. □

It is a well known fact that $\mathbb{Z}$ cannot be a direct summand of the additive group of a non standard model of Peano Arithmetic ([Me, Corollary 2]). Our final observation is the following corollary.

Corollary 17. Let K be an $IPA - RCF$ and Z an IPA of K. Then $\mathbb{Q}$ is a direct summand of $\mathbb{Q}Z$, i.e. there is a group complement of $\mathbb{Q}$ in $(\mathbb{Q}Z, +)$.

Proof. Consider $\mathbb{Q}Z$ as a $\mathbb{Q}$-vector space. As in the proof of subclaim (i) of claim 1, pick a basis $\bar{\mathbb{B}}$ of $\mathbb{Q}Z$ containing 1 and consider $C := span_{\mathbb{Q}}(\mathbb{B} \setminus \{1\})$. Obviously, we have $span_{\mathbb{Q}}(\{1\}) = \mathbb{Q}$. Now proceed exactly as in the proof of subclaim (i) of claim 1 of the proof of Lemma 12 to conclude that C is a group complement of $\mathbb{Q}$ in $(\mathbb{Q}Z, +)$. □

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