Spectral Stability of Small-Amplitude Traveling Waves via Geometric Singular Perturbation Theory

Dissertation zur Erlangung des akademischen Grades
eines Doktors der Naturwissenschaften
vorgelegt von

Johannes Wächtler

an der
Universität Konstanz
Mathematisch-Naturwissenschaftliche Sektion
Fachbereich Mathematik und Statistik

Tag der mündlichen Prüfung: 30. Mai 2012

Referenten: Prof. Dr. Heinrich Freistühler, Universität Konstanz
Prof. Dr. Peter Szmolyan, Technische Universität Wien

Abstract

This thesis is concerned with the spectral stability of small-amplitude traveling waves in two different systems: First, in a system of reaction-diffusion equations where the reaction term undergoes a pitchfork bifurcation; second, in a strictly hyperbolic system of viscous conservation laws with a characteristic family that is not genuinely nonlinear.

In either case, there exist families ϕ_ε, $\varepsilon \in (0, \varepsilon_0]$, $\varepsilon_0 \ll 1$, *of small-amplitude traveling waves. The eigenvalue problem associated with the linearization at* ϕ_ε *is a system of ordinary differential equations depending on two parameters, the amplitude* ε *and the spectral value* κ*. Suitably scaled, the system reveals a slow-fast structure. Using methods from geometric singular perturbation theory, this will be exploited to thoroughly describe the dynamics of the eigenvalue problem in the zero-amplitude limit. I will prove that the eigenvalue problem converges, in the limit* $\varepsilon \to 0$, *to the well-understood eigenvalue problem associated with a traveling wave* ϕ_0 *of a certain scalar equation. The profiles* ϕ_ε *then inherit the spectral stability from the respective limit profile* ϕ_0.

The proofs rely on concepts from dynamical system theory, most notably on invariant manifold theory and geometric singular perturbation theory.

Contents

1 Introduction

Various phenomena in science are mathematically described by systems of reaction-diffusion equations

$$u_t = Du_{xx} + f(u), \quad u \in \mathbb{R}^n, x \in \mathbb{R}, t \geq 0, \tag{1.1}$$

or systems of viscous conservation laws

$$u_t + f(u)_x = \left(B(u)u_x\right)_x, \quad u \in \mathbb{R}^n, x \in \mathbb{R}, t \geq 0. \tag{1.2}$$

Systems as (1.1) and (1.2) are abundant in the mathematical modeling of physical, chemical and biological systems.
Equations of both type frequently admit traveling waves, i.e. solutions $u(x,t)$ having the special form

$$u(x,t) = \phi(x - st), \quad \lim_{\xi \to \pm\infty} \phi(\xi) = u^{\pm} \in \mathbb{R}^n, \tag{1.3}$$

where $\phi \colon \mathbb{R} \to \mathbb{R}^n$, $s \in \mathbb{R}$ and $\xi = x - st$. That is, $u(x,t)$ travels at speed s through the state space without changing its profile ϕ. The ansatz (1.3) turns (1.1) resp. (1.2) into an autonomous ordinary differential equation whose heteroclinic and homoclinic orbits correspond to profiles of the respective system. Traveling waves can thus be constructed by dynamical system techniques; see for instance [34, 49].
A traveling wave ϕ is said to be spectrally stable if the linear operator L associated with the linearization of the equation at ϕ has no spectrum in the closed half-plane $\mathbb{H} = \{\kappa \in \mathbb{C} : \operatorname{Re}\kappa \geq 0\}$, except a simple eigenvalue $\kappa = 0$ which is due to the shift invariance of the profile [1].

This thesis establishes the spectral stability of traveling waves with small amplitude in two different contexts. First, consider a reaction-diffusion system

$$u_t = u_{xx} + f(u, \varepsilon^2), \quad u \in \mathbb{R}^n, x \in \mathbb{R}, t \geq 0, \tag{1.4}$$

where the reaction term f, depending on a positive parameter ε, undergoes a supercritical pitchfork bifurcation at $(u, \varepsilon) = (0, 0)$. For small $\varepsilon > 0$, there are two rest states $u_\varepsilon^{\pm} = \mathcal{O}(\varepsilon)$ of (1.4), i.e. $f(u_\varepsilon^{\pm}, \varepsilon^2) = 0$, and there exist heteroclinic orbits of the profile equation for (1.1),

$$\begin{aligned} u' &= v, \\ v' &= -sv - f(u, \varepsilon^2), \end{aligned} \tag{1.5}$$

connecting these rest states ($' = \frac{d}{d\xi}$). Thus, there are families of traveling fronts with end states u_ε^- and u_ε^+. I will prove that existence and stability of these small fronts can be traced to existence and stability of a traveling front in the scalar reaction-diffusion equation

$$u_t = u_{xx} - u(u+1)(u-1), \quad u \in \mathbb{R}.$$

The second question at issue is the spectral stability of small viscous shock waves associated with a non-convex mode. Consider a system of viscous conservation laws

$$u_t + f(u)_x = u_{xx} \tag{1.6}$$

with $u \in \mathbb{R}^n$, $x \in \mathbb{R}$, $t \geq 0$ and a smooth flux function $f\colon \mathbb{R}^n \to \mathbb{R}^n$. Assume that (1.6) is strictly hyperbolic, i.e. $Df(u)$ has n real and distinct eigenvalues for every $u \in \mathbb{R}$. Let $\lambda(u)$ be an eigenvalue of $Df(u)$. A traveling wave $u(x,t) = \phi(x - st)$ of (1.6) with end states $\phi(\pm\infty) = u^\pm$ is called a non-characteristic viscous shock wave associated with $\lambda(u)$ if it solves

$$u' = f(u) - su - c \tag{1.7}$$

with $c = f(u^-) - su^-$ and $\lambda(u^-) > s > \lambda(u^+)$. Majda and Pego [34] showed that for every reference state $u_0 \in \mathbb{R}^n$ there is a neighbourhood of u_0 such that any two points $u^\pm$ in this neighbourhood are connected by a viscous shock wave $\phi(x - st)$ of (1.6) if and only if both the Rankine-Hugoniot condition $s(u^+ - u^-) = f(u^+) - f(u^-)$ and Liu's strict entropy condition are satisfied [34].
The mode $\lambda(u)$ is said to be genuinely nonlinear if, with $r(u)$ being a right eigenvector of $\lambda(u)$,

$$\nabla\lambda(u) \cdot r(u) \neq 0 \quad \text{for all } u \in \mathbb{R}^n.$$

Freistühler and Szmolyan [14] proved that the spectral stability of small viscous shock waves associated with a genuinely nonlinear mode $\lambda(u)$ can be derived from the stability of a shock wave in the Burgers equation,

$$u_t + (u^2)_x = u_{xx}, \quad u \in \mathbb{R}$$

(see also [40]). Here, I address the spectral stability of small shock waves close to a hypersurface Σ in $\mathbb{R}^n$ with

$$\nabla\lambda(u) \cdot r(u) = 0, \quad (r(u) \cdot \nabla)^2 \lambda(u) \neq 0, \qquad \text{for all } u \in \Sigma.$$

The condition of genuine nonlinearity is thus not satisfied as $\nabla\lambda(u) \cdot r(u)$ changes its sign in points on Σ. Systems with such a non-convex mode frequently appear in applications, see for instance [13].
I will prove that the spectral stability of these small shock waves can be derived from the spectral stability of a fixed profile in

$$u_t + (u^3)_x = u_{xx}, \quad u \in \mathbb{R}.$$

Following the line of thought of Freistühler and Szmolyan [14], the proofs in this thesis rely on dynamical system techniques. In particular, the notion of normally hyperbolic invariant manifolds and geometric singular perturbation theory [12] provide the basis for the argumentation.
The important observation is that both profile equations (1.5) and (1.7) have an intrinsic slow-fast structure which, by suitably scaling the equation, is made accessible to geometric singular perturbation theory (the singular perturbation parameter is the amplitude ε). The heteroclinic orbits corresponding to the traveling waves lie on slow manifolds of the respective profile equation. In either case, the eigenvalue problem for the associated linear operator L_ε will inherit this slow-fast structure. While the eigenvalue problem for the small fronts reveals two time scales, there are three distinct time scales in the eigenvalue problem for the small shock waves.
The eigenvalue problem naturally induces a flow on the Grassmann manifold $\mathcal{G}_n^{2n}(\mathbb{C})$ of n-dimensional subspaces of $\mathbb{C}^{2n}$. Many arguments in the proofs are in fact based on an analysis of this induced flow on $\mathcal{G}_n^{2n}(\mathbb{C})$ which also has different time scales. The key objects are the so-called Evans bundles

$$\mathcal{H}_\varepsilon^\pm \colon \mathbb{H} \to \mathcal{G}_n^{2n}(\mathbb{C}),$$

which are analytic mappings with the property that $\kappa \in \mathbb{H} \setminus \{0\}$ is an eigenvalue of L_ε if and only if $\mathcal{H}_\varepsilon^-(\kappa) \cap \mathcal{H}_\varepsilon^+(\kappa) \neq \{0\}$. The Evans bundles will be constructed explicitly. Using the slow-fast structure of the eigenvalue problem, one finds a splitting of the Evans bundles into slow and fast parts. While the fast parts converge in the zero-amplitude limit to constant, spectrally isolated spaces, the slow parts converge to embeddings of the Evans bundles of the respective limit profile. In the course of the proof, several degeneracies of the eigenvalue problem near $(\varepsilon, \kappa) = (0, 0)$ will arise and different scalings will be required to overcome these.
The idea to use the induced slow-fast structure of the eigenvalue problem to reduce the stability problem to a lower dimensional problem has already been carried out successfully in many contexts, starting with Jones' stability proof for the fast pulses in the FitzHugh-Nagumo equation [26]; see also [17, 18].
Although both questions at issue, that is, the stability of the small fronts in (1.4) and the stability of the small shock waves in (1.6), are studied in the same manner, the proofs differ in many details owing to the different structures of the equations. These differences, however, are instructive as they reveal how properties of the partial differential equation translate into properties of the eigenvalue problem.

The small shock waves near Σ are already known to be nonlinearly stable [16]. Their spectral stability is thus not new. However, it has not yet been established directly. The strength of the geometric techniques proposed in [14] is that they provide good insight into how the shock waves in the system are modeled by the limit shock wave in the scalar equation. It should also be noted here that, in comparison to the situation in [14], the dynamics of the eigenvalue problem associated with the small shock waves for (1.6) has a

richer structure that leads to a more degenerate setting. For instance, the curvature of the slow manifolds has to be taken into account. In contrast to the genuinely nonlinear case, the tangent space approximation does not suffice.
In the context of bistable reaction-diffusion systems, the spectral stability of traveling waves arising through a bifurcation has been studied in different systems by various authors; see for instance [6] and references therein. The investigations often rely on rather specific assumptions on the reaction terms. To the best of my knowledge, the stability of fronts near a pitchfork singularity has not yet been investigated in the systematic and general manner that I propose here. Raugel and Kirchgässner [43] studied the stability of traveling waves in systems modeled by the KPP-equation but their approach differs substantially from the one here. In fact, the approach in this thesis is general enough to be applied to other systems as well.

Applications of systems as (1.1) and (1.2) can be found in the books by Dafermos [4] and by Murray [38] which contain numerous examples of such models in continuum physics and in biology.

The thesis is organized as follows. Chapter 2 outlines the preliminaries needed for the proofs, including a brief introduction to the folllowing: geometric singular perturbation theory; flows on Grassmannian; spectral stability. Chapter 3 is concerned with the spectral stability of small fronts near a pitchfork singularity. Chapter 4 then focuses on the spectral stability of small shock waves associated with a non-convex mode.

Acknowledgements
I wish to thank Heinrich Freistühler for his support and his guidance over the last years. The first time I heard about geometric singular perturbation theory and traveling waves was in his lectures on dynamical systems at Leipzig University in winter 2006/07. I learnt a lot from him in countless discussions in which he generously shared his knowledge. In summer 2011, I spent two very productive months at the Technical University of Vienna. I am deeply grateful to Peter Szmolyan for all the fruitful discussions and the inspiring time.
I would also like to thank the members of the Konstanz Department of Mathematics and Statistics for their support. In particular, thank you to Johannes Höwing and Andreas Klaiber for always being willing to discuss problems and questions.

2 Preliminaries

2.1 Normally hyperbolic invariant manifolds and geometric singular perturbation theory

The proofs in this thesis rely on the theory of invariant manifolds for dynamical systems. The key concept is the persistence of normally hyperbolic invariant manifolds as developed by Fenichel [11] and by Hirsch, Pugh and Shub [24]. Due to the importance of this concept for the argumentation in Chapters 3 and 4, we begin this section with a precise definition of normally hyperbolic invariant manifolds for flows and state the fundamental theorem on the persistence of normally hyperbolic invariant manifolds. Subsequently, we address the basic notions of the closely related geometric singular perturbation theory that Fenichel [12] derived from his persistence results for invariant manifolds [11].

Normally hyperbolic invariant manifolds

Let M be a smooth manifold and let f be a C^r vector field on M, $r \geq 1$. Let $\Phi\colon \mathbb{R}\times M \to M$ denote the flow of f. Then, for each $t \in \mathbb{R}$,

$$\Phi_t\colon M \to M, \quad p \mapsto \Phi(t,p),$$

is a C^r diffeomorphism of M, and $\Phi_t \circ \Phi_s = \Phi_{t+s}$ for all $t, s \in \mathbb{R}$.

Definition 2.1 *A smooth compact submanifold V of M is called a normally hyperbolic invariant manifold for f if it is invariant under the flow, i.e. $\Phi_t(V) = V$ for all $t \in \mathbb{R}$, and if*

(i) The restriction of the tangent bundle of M to V splits into three continuous subbundles,

$$TM\mid_V = N^u \oplus TV \oplus N^s,$$

which are invariant under $D\Phi_t$ for all $t \in \mathbb{R}$.

(ii) There exist $\mu > \rho \geq 0$ and $C > 0$ such that, for all $p \in V$,

$$\begin{aligned}
\forall t \in \mathbb{R}:\ & \|D\Phi_t(p)|_{T_pV}\| \leq Ce^{\rho|t|},\\
\forall t \geq 0:\ & \|D\Phi_t(p)|_{N_p^s}\| \leq Ce^{-\mu t},\\
\forall t \leq 0:\ & \|D\Phi_t(p)|_{N_p^u}\| \leq Ce^{\mu t}.
\end{aligned}$$

In other words, V is a normally hyperbolic invariant manifold if the flow expands and contracts vectors in normal direction at an exponential rate that is higher than the rate of expansion resp. contraction of vectors tangent to the manifold.

Theorem 2.1 *Let V be a normally hyperbolic invariant manifold for f and assume that M is compact. If*

$$\frac{\rho}{\mu} < \frac{1}{r},$$

then the following holds.

(i) V has a locally invariant stable manifold $W^s(V)$ that is tangent at V to $TV \oplus N^s$. $W^s(V)$ is a C^r manifold and invariantly foliated by C^r submanifolds that are tangent at V to the stable subspace N^s.
V has a locally invariant unstable manifold $W^u(V)$ that is tangent at V to $N^u \oplus TV$. $W^u(V)$ is a C^r manifold and invariantly foliated by C^r submanifolds that are tangent at V to the unstable subspace N^u.

(ii) If $\tilde{f}$ is a C^r vector field that is C^1 close to f, $\|f - \tilde{f}\|_{C^1} \ll 1$, then there exists a unique normally hyperbolic invariant manifold $\tilde{V}$ for $\tilde{f}$ that is C^r diffeomorphic to V. The unique manifolds $W^s(\tilde{V})$, $W^u(\tilde{V})$ as well as the fibers of the respective foliation are C^r diffeomorphic to those of V.

Theorem 2.1 states Theorem 4.1 in [24] for flows of vector fields, a proof can be found there. A similar result was obtained by Fenichel [11], see also [52].

Assertion (ii) states that normally hyperbolic invariant manifolds persist as normally hyperbolic invariant manifolds. To prove the persistence of V, one uses the hyperbolic structure in normal direction. The idea is to prove first existence and persistence of the stable and unstable manifolds. Their intersection then gives the perturbed manifold $\tilde{V}$; cf. [52]. To underline the importance of normally hyperbolic invariant manifolds, we note here that Mañé [35] proved the converse of the persistence result of Hirsch, Pugh and Shub [24].

In this thesis, we will only encounter normally hyperbolic invariant manifolds for which $\rho = 0$ holds. The statements of Theorem 2.1 then hold for $r = \infty$.

Geometric singular perturbation theory

Consider a system of the form

$$\begin{aligned} x' &= f(x, y, \varepsilon), \\ y' &= \varepsilon g(x, y, \varepsilon) \end{aligned} \tag{2.1}$$

with $(x, y) \in U$, $U \subset \mathbb{R}^{k+n}$, $' = \frac{d}{d\tau}$ and $\varepsilon \in (-\bar{\varepsilon}, \bar{\varepsilon})$, $\bar{\varepsilon} > 0$ small. For notational convenience, we assume that $f, g \in C^\infty(U \times (-\bar{\varepsilon}, \bar{\varepsilon}), \mathbb{R}^{k+n})$.

The characteristic feature of systems as (2.1) is the presence of two different timescales. The variable x evolves much faster than y whose dynamics is of order ε. Correspondingly, x is called the *fast* variable, and y is referred to as the *slow* variable. Written on the slow timescale $t = \varepsilon\tau$, system (2.1) becomes

$$\begin{aligned} \varepsilon\dot{x} &= f(x, y, \varepsilon), \\ \dot{y} &= g(x, y, \varepsilon) \end{aligned} \tag{2.2}$$

where $\dot{} = \frac{d}{dt}$. For $\varepsilon \neq 0$, the fast system (2.1) and the slow system (2.2) are equivalent. However, setting $\varepsilon = 0$ in (2.1) and (2.2), respectively, yields two completely different systems: The *layer problem*

$$\begin{aligned} x' &= f(x, y, 0), \\ y' &= 0, \end{aligned} \tag{2.3}$$

and the *reduced problem*

$$\begin{aligned} 0 &= f(x, y, 0), \\ \dot{y} &= g(x, y, 0). \end{aligned} \tag{2.4}$$

These two limit systems capture the essence of the dynamics on the respective time scale. In (2.3), the slow variable y only appears as a parameter. In (2.4), the x-equation is degenerated to an algebraic condition and (2.4) thereby defines a dynamical system on the set

$$\mathcal{S} = \{(x, y) \in U \ : \ f(x, y, 0) = 0\}$$

which, in turn, consists of fixed points of (2.3). In [12], Fenichel presented geometric methods of how to combine features of these two limit systems to a complete picture of the dynamics in the perturbed system (2.1). The analysis of (2.1) then reduces to that of the two lower dimensional systems (2.3) and (2.4).
Fenichel's approach is based on his results on the persistence of invariant manifolds [11]. We will describe this theory in some detail, as it provides the technical basis for the proofs in Chapter 3 and 4. There is an excellent exposition of the geometric singular perturbation theory by Jones [27]. For later purposes, however, we need a more general version of the theory. Thus, closely following Fenichel [12] and Szmolyan [49], we will state Fenichel's main theorem for the general case of vector fields and explain subsequently how this theorem applies to (2.1).

Before doing so, let us briefly sketch one of the central ideas. Denote by $\mathcal{S}_H \subset \mathcal{S}$ the open subset where all eigenvalues of $D_x f$ have non-vanishing real parts. That is to say, any point in $\mathcal{S}_H$ is a hyperbolic fixed point of the layer problem (2.3). A compact subset $M_0 \subset \mathcal{S}_H$ is thus a normally hyperbolic critical manifold of (2.3). The idea is now to prove the existence of locally invariant manifolds M_ε for small $\varepsilon \neq 0$ that are diffeomorphic to M_0. Restricted to M_ε, (2.2) becomes a regular perturbation of (2.4). A possible boundary of M_0 is a technical difficulty which has to be overcome; cf. [27, 52].

The general theorem for vector fields on manifolds
In order to state Theorem 9.1 in [12], the following notation is required; cf. section VII in [12].
Let X_ε be a family of vector fields on a manifold $\mathcal{M}$ with $\varepsilon \in (-\bar{\varepsilon}, \bar{\varepsilon})$ and let $\cdot\tau$ denote

the flow that $X_\varepsilon \times 0$ induces on $\mathcal{M} \times (-\bar{\varepsilon}, \bar{\varepsilon})$. A manifold $M \subset \mathcal{M}$ is said to be *locally invariant* under the flow if there exists a neighbourhood $V \subset \mathcal{M}$ of M so that for all $m \in M$, $m \cdot [0,\tau] \subset V$ implies that $m \cdot [0,\tau] \subset M$, and correspondingly $m \cdot [\tau, 0] \subset V$ implies that $m \cdot [\tau, 0] \subset M$ for $\tau < 0$. This says that no trajectory can leave M without also leaving V.

Assume that $\mathcal{S}$ is a submanifold of $\mathcal{M}$ consisting entirely of equilibrium points of X_0. Then $T_m\mathcal{S}$ is in the kernel of the linearization LX_0 and LX_0 induces a linear map on the quotient space,

$$QX_0(m)\colon T_m\mathcal{M}/T_m\mathcal{S} \to T_m\mathcal{M}/T_m\mathcal{S}.$$

The eigenvalues of $QX_0(m)$ are called the *nontrivial* eigenvalues.

Let $\mathcal{S}_R \subset \mathcal{S}$ be the open set where QX_0 is invertible and let N be the LX_0-invariant complement of $T\mathcal{S}_R$. Denote the projection $T\mathcal{M}\mid_{\mathcal{S}_R} \to T\mathcal{S}_R$ by $\pi^{\mathcal{S}}$. Further, let $\mathcal{S}_H \subset \mathcal{S}_R$ be the open subset where QX_0 has no pure imaginary eigenvalues.

Let $M_0 \subset \mathcal{S}$ be a compact subset such that $QX_0(m)$ has k_s eigenvalues in the left half plane, k_c eigenvalues on the imaginary axis, and k_u eigenvalues in the right half plane, for each $m \in K$. Denote the stable, center and unstable eigenspaces of $LX_0 \times \{0\}$ at $(m,0) \in M_0 \times \{0\}$ by E^s_m, E^c_m and E^u_m, respectively. Therefore $\dim E^s_m = k_s$, $\dim E^c_m = n + 1 + k_c$ and $\dim E^u_m = k_u$ where n is the dimension of $\mathcal{S}$.

A locally invariant manifold $\mathcal{C}$ is a center manifold for $X_\varepsilon \times 0$ near M_0 if $M_0 \times \{0\} \subset \mathcal{C}$ and $\mathcal{C}$ is tangent to E^c_m at $(m,0)$ for all $(m,0) \in M_0 \times \{0\}$. The center-stable and center-unstable manifolds $\mathcal{C}^s$ and $\mathcal{C}^u$ are similarly defined as locally invariant manifolds, containing $M_0 \times \{0\}$ whilst being tangent to $E^c_m \oplus E^s_m$ and $E^c_m \oplus E^u_m$, respectively, at $(m,0)$ for all $(m,0) \in M_0 \times \{0\}$.

Definition 2.2 *Let $\mathcal{C}^s$ be a center-stable manifold for $X_\varepsilon \times 0$ near M_0. A family $\{\mathcal{F}^s(p) : p \in \mathcal{C}^s\}$ is called a C^{r_2} family of C^{r_1} stable manifolds for $\mathcal{C}^s$ near M_0 if*

(i) $\mathcal{F}^s(p)$ is a C^{r_1} manifold for each $p \in \mathcal{C}^s$.

(ii) $p \in \mathcal{F}^s(p)$ for each $p \in \mathcal{C}^s$.

(iii) $\mathcal{F}^s(p)$ and $\mathcal{F}^s(q)$ are either disjoint or identical for each $p, q \in \mathcal{C}^s$.

(iv) $\mathcal{F}^s(m,0)$ is tangent to $E^s_{(m,0)}$ at $(m,0)$ for each $m \in M_0$.

(v) $\{\mathcal{F}^s(p) : p \in \mathcal{C}^s\}$ is a positively invariant C^{r_2} family of manifolds, i.e.

$$\mathcal{F}^s(p) \cdot \tau \subset \mathcal{F}^s(p \cdot \tau)$$

for all $p \in \mathcal{C}^s$ and $\tau \geq 0$ with $p \cdot [0,\tau] \subset \mathcal{C}^s$.

A C^{r_1} family of C^{r_2} unstable manifolds $\{\mathcal{F}^u(p) : p \in \mathcal{C}^u\}$ for $\mathcal{C}^u$ is defined analogously.

We now have everything needed at hand to cite the most important parts of Theorem 9.1 in [12], see also [49].

Theorem 2.2 *Let $\mathcal{M}$ be a C^{r+1} manifold, $1 \leq r < \infty$. Let X_ε, $\varepsilon \in (-\bar{\varepsilon}, \bar{\varepsilon})$, be a C^r family of vector fields on $\mathcal{M}$, and let $\mathcal{S}$ be a C^r submanifold of $\mathcal{M}$ consisting entirely of equilibrium points of X_0. Let k_s, k_c and k_u be fixed integers, and let $M_0 \subset \mathcal{S}$ be a compact subset such that $QX_0(m)$ has k_s eigenvalues in the left half plane, k_c eigenvalues on the imaginary axis, and k_u eigenvalues in the right half plane, for all $m \in M_0$. Then*

(i) There is a C^r center-stable manifold $\mathcal{C}^s$ for $X_\varepsilon \times 0$ near M_0. There is a C^r center-unstable manifold $\mathcal{C}^u$ for for $X_\varepsilon \times 0$ near M_0. There is a C^r center manifold $\mathcal{C}$ for for $X_\varepsilon \times 0$ near M_0.

(ii) There is a C^{r-1} family $\{\mathcal{F}^s(p) : p \in \mathcal{C}^s\}$ of C^r stable manifolds for $\mathcal{C}^s$ near M_0. If $p \in \mathcal{M} \times \{\varepsilon\}$, then $\mathcal{F}^s(p) \subset \mathcal{M} \times \{\varepsilon\}$. Each manifold $\mathcal{F}^s(p)$ intersects $\mathcal{C}$ transversally, in exactly one point. There is a C^{r-1} family $\{\mathcal{F}^u(p) : p \in \mathcal{C}^u\}$ of C^r unstable manifolds for $\mathcal{C}^u$ near M_0. If $p \in \mathcal{M} \times \{\varepsilon\}$, then $\mathcal{F}^u(p) \subset \mathcal{M} \times \{\varepsilon\}$. Each manifold $\mathcal{F}^u(p)$ intersects $\mathcal{C}$ transversally, in exactly one point.

(iii) Let $K_s < 0$ be larger than the real parts of the eigenvalues of $QX_0(m)$ in the left half plane, for all $m \in M_0$. Then there is a constant C_s such that if $p \in \mathcal{C}^s$ and $q \in \mathcal{F}^s(p)$, then

$$d(p \cdot \tau, q \cdot \tau) \leq C_s e^{K_s \tau} d(p, q)$$

*for all $\tau \geq 0$ with $p \cdot [0, \tau] \subset \mathcal{C}^s$.
Let $K_u > 0$ be smaller than the real parts of the eigenvalues of QX_0 in the right half plane, for all $m \in M_0$. Then there is a constant C_u such that if $p \in \mathcal{C}^u$ and $q \in \mathcal{F}^u(p)$, then*

$$d(p \cdot \tau, q \cdot \tau) \leq C_u e^{K_u \tau} d(p, q)$$

for all $\tau \leq 0$ with $p \cdot [\tau, 0] \subset \mathcal{C}^u$.

(iv) If $M_0 \subset \mathcal{S}_H$, define for $(m, \varepsilon) \in \mathcal{C}$, $X_R(m) = \pi^{\mathcal{S}}(\partial/\partial\varepsilon) X_\varepsilon(m) \mid_{\varepsilon=0}$ and

$$X_{\mathcal{C}}(m, \varepsilon) = \begin{cases} \varepsilon^{-1} X_\varepsilon(m) \times \{0\} & \text{if } \varepsilon \neq 0, \\ X_R(m) \times \{0\} & \text{if } \varepsilon = 0. \end{cases}$$

Then $X_{\mathcal{C}}$ is a C^{r-1} vector field on $\mathcal{C}$ near $M_0 \times 0$.

The proof of Theorem 2.2 is due to Fenichel [12]. An alternative proof can be found in Jones [27]; see also [44, 52]. The assertion (iv) is the heart of the theorem: If $M_0 \subset \mathcal{S}_H$, the restriction of the slow dynamics to $\mathcal{C}$ lead to a system that is no longer singularly, but now regularly perturbed.

If $\mathcal{M}$, $\mathcal{S}$ and X_ε are C^∞, Theorem 2.2 holds for any $r < \infty$ and $\mathcal{F}^u(p)$, $\mathcal{F}^s(p)$ are C^∞ manifolds for each fixed p. In general, even if $r = \infty$, the manifolds $\mathcal{C}$, $\mathcal{C}^u$ and $\mathcal{C}^s$, and the families $\mathcal{F}^u$ and $\mathcal{F}^s$ will not be C^∞ as one may construct different $\mathcal{C}$, $\mathcal{C}^u$ and $\mathcal{C}^s$ for each r. In particular, $\mathcal{C}$, $\mathcal{C}^u$ and $\mathcal{C}^s$, and $\mathcal{F}^u$ and $\mathcal{F}^s$ will not in general be unique; cf. [12].

Applications to slow-fast systems on $\mathbb{R}^{k+n}$

Let us now address how Theorem 2.2 applies to the system (2.1). In the notation of Theorem 2.2, $X_\varepsilon = (f, \varepsilon g)$, $r = \infty$, $\mathcal{M} = U$ and $\mathcal{S} = \{(x,y) : f(x,y,0) = 0\}$. Let $M_0 \subset \mathcal{S}_H$ be a compact subset such that, for all $(x,y) \in M_0$, $D_x f(x,y,0)$ has k_s eigenvalues in the left half plane and k_u eigenvalues in the right half plane, $k_s + k_u = k$ (these are the nontrivial eigenvalues). By assertion (i) of Theorem 2.2, it follows that the extended system

$$\begin{aligned} x' &= f(x,y,\varepsilon), \\ y' &= \varepsilon g(x,y,\varepsilon), \\ \varepsilon' &= 0 \end{aligned} \tag{2.5}$$

on $U \times (-\bar{\varepsilon}, \bar{\varepsilon})$ possesses an $(n+1)$-dimensional center manifold $\mathcal{C}$, an $(n+1+ks)$-dimensional center-stable manifold $\mathcal{C}^s$ and an $(n+1+ks)$-dimensional center-unstable manifold $\mathcal{C}^u$. As the flow of (2.5) stays in hyperplanes with $\varepsilon =$ const, the existence of $\mathcal{C}$ implies that manifolds $M_\varepsilon \subset U$ exist which are regular $\mathcal{O}(\varepsilon)$-perturbations of M_0 such that

$$\mathcal{C} = \bigcup_{-\varepsilon_0 < \varepsilon < \varepsilon_0} M_\varepsilon \times \{\varepsilon\}$$

wherein $\varepsilon_0 > 0$ is sufficiently small. Due to the fact that $\mathcal{C}$ is locally invariant under the flow of $X_\varepsilon \times 0$, these *slow* manifolds M_ε are locally invariant under the flow of (2.1). Part (iv) of Theorem 2.2 asserts that the restriction of the slow problem (2.2) to M_ε is a regular perturbation of the reduced problem (2.4). We want to explain this important fact in more detail.

Note that, since $D_x f(x,y,0)$ is invertible for any $(x,y) \in M_0$, the Implicit Function Theorem implies that M_0 is locally given as the graph of a function. Let us assume that this holds globally with a function h^0,

$$M_0 = \{(x,y) \,:\, x = h^0(y), y \in K\}$$

where K is a compact set. Then the slow manifolds M_ε are as well given as the graph of a function. The following lemma is proved in [27].

Lemma 2.1 *Assume that* M_0 *is given as the graph of a* C^∞ *function* h^0*,*

$$M_0 = \{(x,y) \,:\, x = h^0(y), y \in K\}$$

with K *being a compact, simply connected domain whose boundary is a* $(n-1)$*-dimensional* C^∞ *submanifold. Then there exists* $\varepsilon_0 > 0$ *such that, for every* $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$*, there is a function* h^ε*, defined on* K*, so that*

$$M_\varepsilon = \{(x,y) \,:\, x = h^\varepsilon(y), y \in K\}$$

is locally invariant under (2.1). Moreover, h^ε *is* C^r*, for any* $r < \infty$*, both in* y *and* ε*.*

With these functions h^ε, the restriction of (2.2) to the manifolds M_ε is

$$\dot{y} = g(h^\varepsilon(y), y, \varepsilon). \tag{2.6}$$

This system is regular in ε and the limit $\varepsilon \to 0$ is given as

$$y' = g(h^0(y), y, 0),$$

which is the reduced problem (2.4). Hence, restricted to M_ε, the singular perturbation becomes a regular perturbation. This is the gist of Theorem 2.2 (iv) as any structure on M_0 that persists under regular perturbations can thus be found on M_ε as well. For example, hyperbolic fixed points of (2.4) on M_0 persist as hyperbolic fixed points of the perturbed system (2.2). More generally, normally hyperbolic invariant manifolds of the reduced system (2.4) persist as normally hyperbolic invariant manifolds of the perturbed system (2.6); cf. [49]. The dynamics of (2.1) on M_ε is thereby fully captured by (2.6).

Next, we seek a characterization of the dynamics outside M_ε. For this, the families $\{\mathcal{F}^s(p) : p \in \mathcal{C}^s\}$ and $\{\mathcal{F}^u(p) : p \in \mathcal{C}^u\}$ will be crucial as they form a foliation of $\mathcal{C}^s$ and $\mathcal{C}^u$, respectively,

$$\mathcal{C}^s = \bigcup_{p \in \mathcal{C}} \mathcal{F}^s(p), \quad \mathcal{C}^u = \bigcup_{p \in \mathcal{C}} \mathcal{F}^u(p).$$

Let $\mathcal{F}^{s/u}_\varepsilon(p)$ denote the projection of $\mathcal{F}^{s/u}(p)$ from $U \times (-\varepsilon_0, \varepsilon_0)$ on U. The important observation is now that, for $\varepsilon \geq 0$, we have

$$W^s(M_\varepsilon) = \bigcup_{p \in M_\varepsilon} \mathcal{F}^s_\varepsilon(p), \quad W^u(M_\varepsilon) = \bigcup_{p \in M_\varepsilon} \mathcal{F}^u_\varepsilon(p). \tag{2.7}$$

Thus, the stable and unstable manifolds of M_ε are invariantly foliated by the smooth fibers $\mathcal{F}^s_\varepsilon(x, y, 0)$ and $\mathcal{F}^u_\varepsilon(x, y, 0)$, respectively.

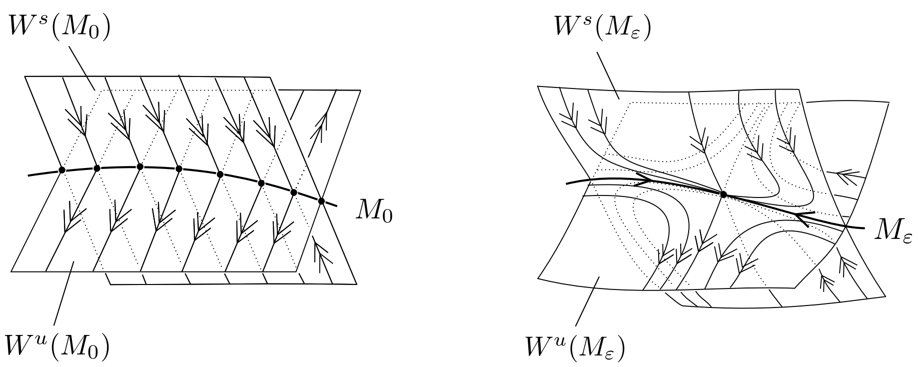

Figure 2.1: Dynamics near M_0 and near M_ε in the case of an attracting fixed point of the reduced problem. Double arrows indicate the fast flow.

Although M_ε is no longer a critical manifold, the terms stable and unstable manifold are justified by the decay rates of points on the fibers as stated in Theorem 2.2 (iii). Note that a single fiber will not in general be invariant under (2.5). However, any fiber $\mathcal{F}_\varepsilon^{s/u}(p)$ whose base point p is a fixed point of (2.5) is itself invariant.

For $\varepsilon = 0$, $\mathcal{F}_0^s(x,y,0)$ and $\mathcal{F}_0^u(x,y,0)$ are the stable and unstable invariant manifolds of the hyperbolic fixed point $(x,y) \in M_0$ of the layer problem (2.3). As the fibers $\mathcal{F}_\varepsilon^s(x,y,0)$ and $\mathcal{F}_\varepsilon^u(x,y,0)$ are regular perturbations of $\mathcal{F}_0^s(x,y,0)$ and $\mathcal{F}_0^u(x,y,0)$, the dynamical situation in the proximity of M_ε can therefore be grasped by studying the layer problem (2.3). Together with (2.6), we receive a complete picture of the perturbed dynamics (2.1) near M_ε for $\varepsilon > 0$ sufficiently small. In particular, one can construct heteroclinic and homoclinic orbits in the perturbed system (2.1) as perturbations of a singular orbit that is pieced together by solutions of (2.3) and (2.4); cf. Szmolyan [49].

2.2 Flows on Grassmann manifolds

Let $\mathcal{G}_d^n(\mathbb{C})$ denote the Grassmann manifold of the d-dimensional linear subspaces of $\mathbb{C}^n$, $d, n \in \mathbb{N}$, $d \le n$. In this section, we compile the elementary facts about $\mathcal{G}_d^n(\mathbb{C})$ that we are going to use later.

We begin with the construction of local charts. To this end, let $(e_1, \dots, e_n)$ be an ordered basis of $\mathbb{C}^n$. For every set $I = \{i_1, \dots, i_d\} \subset \{1, \dots, n\}$ with $i_1 < \dots < i_d$, define $X_0^I = \operatorname{span}\{e_{i_1}, \dots, e_{i_d}\}$ and $Y_0^I = \operatorname{span}\{e_j : j \notin I\}$. Furthermore, set

$$U^I = \{X \in \mathcal{G}_d^n(\mathbb{C}) \, : \, X \oplus Y_0^I = \mathbb{C}^n\}.$$

The set U^I is an open neighbourhood of X_0^I in $\mathcal{G}_d^n(\mathbb{C})$. For each $X \in U^I$, there exists a unique matrix $T = T(X) \in \mathbb{C}^{(n-d)\times d}$ such that $T(X)$ represents X. That is, the rows of

$$\begin{pmatrix} I_d \\ T \end{pmatrix}$$

give a coordinate representation of X with respect to the ordered basis $E^I = (e_{i_1}, \dots, e_{i_d}, e_{j_1}, \dots, e_{j_{n-d}})$, $j_1 < \dots < j_{n-d}$. The bijection

$$\varphi^I : U^I \to \mathbb{C}^{(n-d)\times d}, \quad X \mapsto T(X),$$

defines a local chart of $\mathcal{G}_d^n(\mathbb{C})$. We call φ^I the *canonical chart* with respect to the basis E^I of $\mathbb{C}^n$; cf. [14]. If $I = \{1, \dots, d\}$, we simply write $\varphi^I = \varphi$.
The set of all these charts φ^I, defined on the open sets U^I, defines an atlas and gives $\mathcal{G}_d^n(\mathbb{C})$ the structure of a compact, complex-analytic manifold of dimension $(n-d)d$. For the details, we refer to [21].

Consider now the constant coefficient system

$$W' = AW \quad \text{on } \mathbb{C}^n \tag{2.8}$$

with a matrix $A \in \mathbb{C}^{n\times n}$. As (2.8) is linear, it naturally induces a flow on $\mathcal{G}_d^n(\mathbb{C})$ for each $d \le n$, which we denote by

$$X' = \mathcal{A}^d(X). \tag{2.9}$$

A solution $X\colon \mathbb{R} \to \mathcal{G}_d^n(\mathbb{C})$ of (2.9) is spanned by d linearly independent solutions of (2.8). In some situation, it is of advantage to consider (2.9) instead of the original system (2.8). To describe in a convenient manner how a linear subspace $X \subset \mathbb{C}^n$ evolves under (2.8), one considers the evolution of the point $X \in \mathcal{G}_d^n(\mathbb{C})$ under (2.9).
The fixed points of (2.9) are the d-dimensional invariant subspaces of the coefficient matrix A. The content of the following lemma, taken from [14], is the local behavior of solutions near such a fixed point.

Lemma 2.2 *Fix a basis $\{e_1, \dots, e_n\}$ of $\mathbb{C}^n$ and let $X_0 = \operatorname{span}\{e_1, \dots, e_d\}$. Let φ be the canonical chart for $\mathcal{G}_d^n(\mathbb{C})$ with respect to the basis $(e_1, \dots, e_n)$ of $\mathbb{C}^n$.*
If

$$A = \operatorname{diag}(\mu_1, \dots, \mu_n),$$

then modulo φ the flow of (2.9) near X_0 obeys the linear system

$$t'_{ab} = (\mu_{d+a} - \mu_b)t_{ab} \quad \text{on } \mathbb{C}^{(n-d)\times d}. \tag{2.10}$$

If in addition

$$\operatorname{Re}(\operatorname{spec}(A\,|_{X_0})) > \operatorname{Re}(\operatorname{spec}(A\,|_{Y_0})) \tag{2.11}$$

with $Y_0 = \operatorname{span}\{e_{d+1}, \dots, e_n\}$, then X_0 is a hyperbolic attractor for (2.9) and, via φ^{-1}, any sphere in $\mathbb{C}^{(n-d)\times d}$ defines a positively invariant neighbourhood of X_0.

Proof. (Cf. [14]) Let $X\colon \mathbb{R} \to \mathcal{G}_d^n(\mathbb{C})$ be an orbit of (2.9) near X_0 and represent X by a matrix-valued function $\Xi\colon \mathbb{R} \to \mathbb{C}^{n\times d}$,

$$\Xi = \begin{pmatrix} x_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & x_d \\ y_{11} & \dots & y_{1d} \\ \vdots & & \vdots \\ y_{(n-d)1} & \dots & y_{(n-d)d} \end{pmatrix}$$

with $x_j \neq 0$, $j = 1, \dots, d$. The columns of Ξ are a basis of X as a subspace of $\mathbb{C}^n$ and therefore each column solves (2.8),

$$\begin{pmatrix} x_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & x_d \\ y_{11} & \dots & y_{1d} \\ \vdots & & \vdots \\ y_{(n-d)1} & \dots & y_{(n-d)d} \end{pmatrix}' = \begin{pmatrix} \mu_1 x_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mu_d x_d \\ \mu_{d+1} y_{11} & \dots & \mu_{d+1} y_{1d} \\ \vdots & & \vdots \\ \mu_n y_{(n-d)1} & \dots & \mu_n y_{(n-d)d} \end{pmatrix}.$$

This then yields (2.10) as

$$(t_{ab}) = \varphi(X) = (y_{ab}/x_b)$$

with

$$(y_{ab}/x_b)' = (x_b\mu_{d+a}y_{ab} - \mu_b x_b y_{ab})/x_b^2 = (\mu_{d+a} - \mu_b)(y_{ab}/x_b). \tag{2.12}$$

If (2.11) holds, $\operatorname{Re}\mu_{d+a} - \mu_b < 0$ for all $a = 1, \ldots, n-d$, $b = 1, \ldots, d$. Hence, X_0 is then a hyperbolic attractor. □

Note that in the case $\operatorname{Re}(\operatorname{spec}(A\,|_{X_0})) < \operatorname{Re}(\operatorname{spec}(A\,|_{Y_0}))$, it follows from (2.12) that X_0 is a hyperbolic repeller. In the case that $Y_0 = Y_0^s \oplus Y_0^u$ with

$$\left(\operatorname{spec}(A\,|_{Y_0^s})\right) < \operatorname{Re}(\operatorname{spec}(A\,|_{X_0})) < \operatorname{Re}\left(\operatorname{spec}(A\,|_{Y_0^u})\right),$$

one finds that X_0 is a hyperbolic fixed point of saddle type.

Let $J = [\tau^-, \tau^+]$ be a closed interval and let τ be the solution of

$$\tau' = (\tau - \tau^-)(\tau^+ - \tau)g(\tau), \quad \tau(0) = \frac{\tau^+ - \tau^-}{2},$$

with some smooth function $g\colon J \to (0, \infty)$, that is, $\tau(-\infty) = \tau^-$, $\tau(+\infty) = \tau^+$. We consider the non-autonomous system

$$W' = A(\tau)W \tag{2.13}$$

on $\mathbb{C}^n$ with a smooth matrix-valued function $A\colon J \to \mathbb{C}^{n\times n}$. Assume that $A(\tau^\pm)$ is hyperbolic with an invariant unstable space $U^\pm$ and an invariant stable space $S^\pm$,

$$\dim U^\pm = n - k, \quad \dim S^\pm = k.$$

Due to the hyperbolicity of $A(\tau^\pm)$, a solution $W(\xi)$ of (2.13) decays for $\xi \to -\infty$ if and only if it is asymptotic to U^-, and it decays for $\xi \to +\infty$ if and only if it is asymptotic to S^+. Viewed as subsets of $\mathcal{G}_1^n(\mathbb{C})$, $U^\pm$ and $S^\pm$ are invariant manifolds for the autonomous end systems

$$X' = \mathcal{A}^1(\tau^\pm)(X).$$

Definition 2.3 *A solution* $X : \mathbb{R} \to \mathcal{G}_1^n(\mathbb{C})$ *of*

$$X' = \mathcal{A}^1(\tau)(X)$$

with

$$X(-\infty) \in U^-, \quad X(+\infty) \in S^+$$

is called an unstable-to-stable-bundle connection (USBC) *for* (2.13).

With this definiton, we can say that there is a solution $W(\xi)$ of (2.13) decaying for $|\xi| \to \infty$ if and only if there is an unstable-to-stable-bundle connection. In the course of the stability analysis, we will frequently find that possible eigenfunctions may only arise in a reduced, lower dimensional subsystem of the eigenvalue problem. The notion of an unstable-to-stable-bundle connection, introduced in [15], will then prove to be more expedient than that of an eigenfunction.

Lastly, we state a slightly more general version of Lemma 6 in [14]. The proof in [14] can easily be adopted to the situation here and we thus omit it.

Lemma 2.3 *Let τ and A be as above. For any $n \in \mathbb{N}$ there exists a constant $c > 0$, depending only on n, so that the following holds. Let $R\colon J \to Gl(n, \mathbb{C})$ be a smooth matrix function such that, for every $\tau \in J$,*

$$R^{-1}(\tau)A(\tau)R(\tau) = \operatorname{diag}(\mu_1(\tau), \ldots, \mu_n(\tau))$$

with

$$\begin{aligned} \operatorname{Re}\mu_j(\tau) > 0 &\quad \textit{for } j = 1, \ldots, n-k, \\ \operatorname{Re}\mu_j(\tau) < 0 &\quad \textit{for } j = n-k+1, \ldots, n. \end{aligned}$$

If

$$\left| (R(0))^{-1} \frac{dR}{d\tau}(\tau) \right| \le c$$

for all $\tau \in J$, then the unique solution $X^-\colon J \to \mathcal{G}^n_{n-k}(\mathbb{C})$ of

$$X' = \mathcal{A}^{n-k}(\tau)(X) \quad \textit{on } \mathcal{G}^n_{n-k}(\mathbb{C}) \tag{2.14}$$

with

$$X^-(-\infty) = U^-$$

satisfies as well

$$X^-(+\infty) = U^+.$$

In other words, there are no unstable-to-stable-bundle connections for (2.13).

Lemma 2.3 will in fact only be used at one place in Chapter 4.

2.3 Spectral stability of traveling waves

We recall the basic facts about spectral stability, focusing on traveling waves in either a system of reaction-diffusion equations,

$$u_t = u_{xx} + f(u), \tag{2.15}$$

or a strictly hyperbolic system of viscous conservation laws,

$$u_t + f(u)_x = u_{xx}, \tag{2.16}$$

with $u \in \mathbb{R}^n$, $x \in \mathbb{R}$, $t \geq 0$, and a sufficiently smooth nonlinearity f. For more general settings, we refer to the surveys by Sandstede [45] and Zumbrun [53] as well as to the references therein.

We write both equations in a unified manner as

$$u_t = u_{xx} + \mathcal{F}(u). \tag{2.17}$$

In the case of (2.15), the nonlinear part reads $\mathcal{F}(u) = f(u)$. For (2.16), $\mathcal{F}(u) = -f(u)_x$.

Let $\phi(x - st)$, $\phi(\pm\infty) = u^{\pm}$, be a traveling wave for (2.17). If (2.17) is transformed to co-moving coordinates $(\xi = x - st, t)$, it becomes

$$u_t = u_{\xi\xi} + su_\xi + \mathcal{F}(u). \tag{2.18}$$

The profile ϕ is now a steady state of (2.18), that is

$$0 = \phi_{\xi\xi} + s\phi_\xi + \mathcal{F}(\phi). \tag{2.19}$$

Observe that if ϕ solves (2.19), so does $\phi(\cdot + \delta)$ with $\delta \in \mathbb{R}$. This has to be taken into account by the different notions of stability for ϕ.

The traveling wave ϕ is said to be *nonlinearly stable*, if small initial perturbations of ϕ render solutions that are time-asymptotic to a possibly shifted version of ϕ. In other words, if there exists $\varepsilon > 0$ such that any solution $u(\xi, t)$ of (2.18) with $\|u(\cdot, 0) - \phi\| < \varepsilon$ satisfies

$$\|u(\cdot, t) - \phi(\cdot + \delta)\|' \to 0 \quad \text{for } t \to \infty$$

with some $\delta \in \mathbb{R}$ and norms $\|\cdot\|$, $\|\cdot\|'$. The choice of the norms depends on the specific structure of the equation at study and we refer to [1, 16, 20, 28, 31, 36, 48] for examples.

A standard approach to nonlinear stability is to consider first the linearization of (2.18) about the steady state ϕ,

$$p_t = p_{\xi\xi} + sp_\xi + \frac{\partial \mathcal{F}}{\partial u}(\phi)p =: Lp, \tag{2.20}$$

where L is a densely defined linear operator on a function space $\mathcal{X}$. Here, we take either $\mathcal{X} = \mathbb{C}^0_{\text{unif}}(\mathbb{R}, \mathbb{C}^n)$ or $\mathcal{X} = L_2(\mathbb{R}, \mathbb{C}^n)$. One can conclude stability of ϕ by considering the spectrum $\sigma(L)$. Note that $\sigma(L)$ consists of both point spectrum $\sigma_p(L)$ and essential spectrum $\sigma_{ess}(L) = \sigma(L) \setminus \sigma_p(L)$, where we define $\sigma_p(L)$ as the union of all isolated eigenvalues of finite multiplicity. Due to the shift invariance of ϕ, $\kappa = 0$ is an eigenvalue with eigenfunction ϕ_ξ, i.e. $0 \in \sigma_p(L)$.

Definition 2.4 *The traveling wave ϕ is called spectrally stable if L has no spectrum in the closed right half plane $\mathbb{H} := \{\kappa \in \mathbb{C}\colon\ \mathrm{Re}\,\kappa \geq 0\}$, except for a simple eigenvalue at $\kappa = 0$.*

For the problems that we study, spectral stability implies nonlinear stability. For traveling waves in (2.15) the theory in Henry [23] is applicable. In the context of viscous profiles in (2.16), a corresponding general result of this kind is due to Zumbrun and Howard [54].

Point spectrum and essential spectrum may both cause spectral instability. We recall the well-known fact that the location of $\sigma_{ess}(L)$ is prescribed by the essential spectrum of the constant-coefficient operators $L^\pm$, obtained from (2.20) at $\xi = \pm\infty$; cf. Theorem A.2 in [23]. The operator L is of the form

$$Lp = p_{\xi\xi} + a(\xi)p_\xi + b(\xi)p$$

with bounded real matrix functions $a(\xi)$, $b(\xi)$ which are asympotic to constant matrices $a^\pm$, $b^\pm$ for $\xi \to \pm\infty$. Define

$$s^\pm = \{\kappa \in \mathbb{C} \,:\, \det\left(-\tau^2 + i\tau a^\pm + b^\pm - \kappa I\right) = 0 \text{ for some } \tau \in \mathbb{R}\}. \tag{2.21}$$

The sets $s^\pm$, consisting of curves in the complex plane, define the boundary of $\sigma_{ess}(L)$. Let P be the subset of $\mathbb{C}$ such that $\mathbb{C} \setminus P$ is the component of $\mathbb{C} \setminus (s^- \cup s^+)$ containing the right half plane. Then $\sigma_{ess}(L)$ is contained in P. Moreover, $s^\pm \subset \sigma_{ess}(L)$. For ϕ being a traveling wave for (2.15), the essential spectrum is bounded to the left half plane if all eigenvalues of $Df(u^\pm)$ have strictly negative real part. In the case of a viscous profile for (2.16), the essential spectrum touches the imaginary axis in $\kappa = 0$. See Figure 2.2.

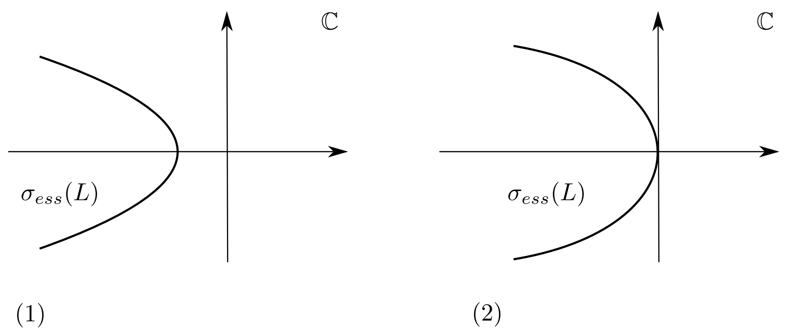

Figure 2.2: Typical configuration of $\sigma_{ess}(L)$ for profiles in (1) reaction-diffusion equation with stable end states $u^\pm$ (bistable nonlinearity) and (2) viscous conservation laws.

In order to locate $\sigma_p(L)$, recast the eigenvalue problem $p' = Lp$ as a non-autonomous, linear system of first order on $\mathbb{C}^{2n}$,

$$W' = A(\phi, \kappa)W \tag{2.22}$$

with $' = \frac{d}{d\xi}$ and a coefficient matrix $A(\phi(\xi), \kappa) \in \mathbb{C}^{2n\times 2n}$ tending to limits $A^{\pm}(\kappa) = A(u^{\pm}, \kappa)$ for $\xi \to \pm\infty$. The matrices $A(\phi(\xi), \kappa)$ and $A^{\pm}(\kappa)$ are analytic functions of κ. If ϕ is a profile for (2.15), the coefficient matrix is given as

$$A(\phi, \kappa) = \begin{pmatrix} 0 & I \\ \kappa I - Df(\phi) & -sI \end{pmatrix}.$$

If ϕ is a viscous profile for (2.16), we take

$$A(\phi, \kappa) = \begin{pmatrix} Df(\phi) - sI & I \\ \kappa I & 0 \end{pmatrix}.$$

From now on, we assume the following.

(A1) ϕ converges to its end states $u^{\pm}$ at exponential rate,

$$|\phi(\xi) - u^{\pm}| \leq Ce^{-\mu|\xi|} \quad \text{for } \xi \to \pm\infty$$

with constants $C > 0$ and $\mu > 0$.

(A2) The end matrices $A^{\pm}$ have *consistent splitting* on $\mathbb{H} \setminus \{0\}$, i.e. $A^{\pm}(\kappa)$ are hyperbolic matrices with an n-dimensional stable space $S^{\pm}(\kappa)$ and an n-dimensional unstable space $U^{\pm}(\kappa)$, for all $\kappa \in \mathbb{H} \setminus \{0\}$.

Assumption (A1) is equivalent to assuming that the end points $u^{\pm}$ are hyperbolic fixed points of the respective profile equation. The assumption (A2) on the consistent splitting asserts that there is no essential spectrum in $\mathbb{H} \setminus \{0\}$. In fact, $s^{\pm}$ in (2.21) are the curves where the end matrices $A^{\pm}(\kappa)$ have pure imaginary eigenvalues; cf. [23].

If the trivial eigenvalue $\kappa = 0$ is simple, spectral instability can only be caused by eigenvalues $\kappa \in \sigma_p(L) \cap \mathbb{H} \setminus \{0\}$ and one needs to track those $\kappa \in \mathbb{H} \setminus \{0\}$ for which a nontrivial solution $W(\xi, \kappa)$ of (2.22) exists. Due to (A1) and (A2), such a solution $W(\xi, \kappa)$ must in fact be asympotic to $U^{-}(\kappa)$ for $\xi \to -\infty$ and asymptotic to $S^{+}(\kappa)$ for $\xi \to +\infty$. That is, $W(\xi, \kappa)$ necessarily decays exponentially for $\xi \to \pm\infty$. Using Definition 2.3, this fact may be stated as follows: $\kappa \in \mathbb{H}\setminus\{0\}$ is an eigenvalue if and only if an unstable-to-stable-bundle connection for (2.22) at κ exists. Therefore, the search for unstable eigenvalues amounts to the search for unstable-to-stable-bundle connections for (2.22). The crucial objects for this are the so-called *unstable* resp. *stable Evans bundles*

$$\mathcal{H}^{-} : \mathbb{H} \setminus \{0\} \to \mathcal{G}_n^{2n}(\mathbb{C}), \quad \mathcal{H}^{+} : \mathbb{H} \setminus \{0\} \to \mathcal{G}_n^{2n}(\mathbb{C}),$$

which are analytic mappings with the property that, for every solution $W(\xi,\kappa)$ of (2.22) and every $\kappa \in \mathbb{H} \setminus \{0\}$,

$$W(0,\kappa) \in \mathcal{H}^-(\kappa) \iff W(\xi,\kappa) \to 0 \text{ for } \xi \to -\infty,$$
$$W(0,\kappa) \in \mathcal{H}^+(\kappa) \iff W(\xi,\kappa) \to 0 \text{ for } \xi \to +\infty.$$

That is to say, $\kappa \in \mathbb{H} \setminus \{0\}$ is an eigenvalue of L if and only if

$$\mathcal{H}^-(\kappa) \cap \mathcal{H}^+(\kappa) \neq \{0\}.$$

In the course of the stability analysis in the subsequent chapters, the Evans bundles will be explicitly constructed. At this point, we will just sketch the basic idea of the construction inside the region of consisting splitting; see also [1, 14].

Consider the non-autonomous system that (2.22) induces on $\mathcal{G}_n^{2n}(\mathbb{C})$,

$$X' = \mathcal{A}^d(\phi,\kappa)(X), \tag{2.23}$$

and let $\kappa \in \mathbb{H} \setminus \{0\}$. Coupling the profile equation (i.e. the first-order version of (2.19)) to (2.23) yields an autonomous system on $\mathbb{R}^m \times \mathcal{G}_n^{2n}(\mathbb{C})$ where $m = n$ or $m = 2n$. For this extended system, $(u^\pm, U^\pm(\kappa))$ and $(u^\pm, S^\pm(\kappa))$ are hyperbolic fixed points. By Lemma 2.2, the spectrally isolated invariant spaces $U^\pm(\kappa)$ and $S^\pm(\kappa)$ are hyperbolic fixed points of the constant-coefficient versions of (2.23) with $\phi = u^\pm$; $U^\pm(\kappa)$ is an attractor and $S^\pm(\kappa)$ is a repeller. Meanwhile, the end states $u^\pm$ are repelling resp. attracting hyperbolic fixed points of the profile equation. We therefore find a unique solution (ϕ, X_κ^-) with α-limit $(u^-, U^-(\kappa))$ and a unique solution (ϕ, X_κ^+) with ω-limit $(u^+, S^+(\kappa))$. The unique intersection points of these solutions with the $\{\phi(0)\}$-section are the spaces $\mathcal{H}^\pm(\kappa)$; see Figure 2.3. As $A(\phi,\kappa)$ is analytic in κ, the unstable and stable manifolds of $(u^-, U^-(\kappa))$ and $(u^+, S^+(\kappa))$ depend analytically on κ. Thus $\mathcal{H}^\pm(\kappa)$ are analytic functions of κ too.

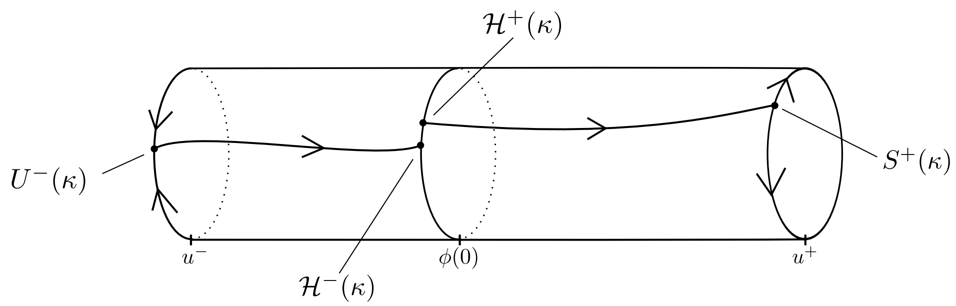

Figure 2.3: The extended phase space $[u^-, u^+] \times \mathcal{G}_1^2(\mathbb{C})$ for a scalar equation ($n = 1$).

The above construction works only inside the region of consisting splitting because it relies on the hyperbolicity of the end matrices $A^\pm(\kappa)$. In order to use the Evans bundles effectively, one possibly needs to extend $\mathcal{H}^\pm$ to a region $\Omega \subset \mathbb{C}$ including the closed right half plane $\mathbb{H}$.

Lemma 2.4 *Under assumptions (A1) and (A2), there exist an open region* $\Omega \subset \mathbb{C}$ *with* $\mathbb{H} = \{\kappa : \operatorname{Re}\kappa \geq 0\} \subset \Omega$ *and unique analytic mappings*

$$\mathcal{H}^- : \Omega \to \mathcal{G}_n^{2n}(\mathbb{C}), \quad \mathcal{H}^+ : \Omega \to \mathcal{G}_n^{2n}(\mathbb{C}),$$

such that, for every solution $W(\xi,\kappa)$ *of* (2.22) *and every* $\kappa \in \mathbb{H} \setminus \{0\}$*, it holds*

$$W(0,\kappa) \in \mathcal{H}^-(\kappa) \iff W(\xi,\kappa) \to 0 \text{ for } \xi \to -\infty,$$
$$W(0,\kappa) \in \mathcal{H}^+(\kappa) \iff W(\xi,\kappa) \to 0 \text{ for } \xi \to +\infty.$$

Instead of giving a proof of this lemma, we refer to the literature, for instance [1, 19], and only remark that if the essential spectrum is bounded away from the imaginary axis, the region of consisting splitting includes a domain $\{\kappa \in \mathbb{C} : \operatorname{Re}\kappa > -\beta\}$ with some $\beta > 0$, cf. [1, 23]. However, if the essential spectrum touches the imaginary axis, one needs to extend $\mathcal{H}^\pm$ into the essential spectrum; cf. [19, 29] and the remarks below.

An effective tool for detecting those κ with $\mathcal{H}^-(\kappa) \cap \mathcal{H}^+(\kappa) \neq \{0\}$ is the *Evans function*. Let $\{\eta_1^-(\kappa), \ldots, \eta_{2n-k(\kappa)}^-\}$ and $\{\eta_1^+(\kappa), \ldots, \eta_k^+(\kappa)\}$ be bases for $\mathcal{H}^-(\kappa)$ and $\mathcal{H}^+(\kappa)$, respectively. It is possible to choose the basis vectors $\eta_j^\pm(\kappa)$ in such a way that they depend analytically on $\kappa \in \Omega$; cf. [30]. An Evans function for the traveling wave ϕ is now defined as

$$\mathcal{E}(\kappa) = \det\left[\eta_1^-(\kappa), \ldots, \eta_{2n-k}^-(\kappa), \eta_1^+(\kappa), \ldots, \eta_k^+(\kappa)\right].$$

Note that, in contrast to the Evans bundles $\mathcal{H}^\pm$, an Evans function is not unique. However, any two Evans functions only differ by a non-vanishing, analytic factor. We state the important properties of $\mathcal{E}$ as a lemma.

Lemma 2.5 *The function* $\mathcal{E} : \Omega \to \mathbb{C}$ *is analytic and has the following properties.*

(i) For $\kappa \in \mathbb{H}$,
$$\mathcal{E}(\kappa) = 0 \iff \kappa \in \sigma_p(L).$$

(ii) The order of κ *as a zero of* $\mathcal{E}$ *equals its multiplicity as an eigenvalue of* L.

A proof of assertion (ii) can be found in [1]. The first assertion follows directly from the definition.

Now, the search for unstable eigenvalues is the search for zeros of an analytic function. In the light of Lemma 2.5, we can say that ϕ is spectrally stable if and only if both

$$\mathcal{E}(\kappa) \neq 0 \quad \text{for } \kappa \in \mathbb{H} \setminus \{0\}$$

and $\mathcal{E}'(0) \neq 0$ hold.

The idea to construct an analytic function whose zeros coincide with the eigenvalues of a wave is due to Evans [7–10] who studied the stability of waves in systems modeling the propagation of a nerve impulse. Following Evans' approach, Jones used the Evans function to prove the stability of the fast pulses in the FitzHugh-Nagumo equation [26]. With Alexander, Gardner and Jones [1] introducing the Evans function for traveling waves in general reaction-diffusion systems, it became the standard device in the stability analysis of traveling waves.

A systematic use of Evans function techniques for profiles in a general system of viscous conservation laws had for a long time been hindered by the presence of the essential spectrum at $\kappa = 0$ as well as by the lack of a general theorem about how spectral stability and nonlinear stability are linked to each other. In some special cases, both problems were successfully solved, see for instance [28]. In the case of a general system, it has been the achievement of Gardner and Zumbrun [19] to show how to extend the Evans function for a viscous profile into the essential spectrum (see also the similar result by Kapitula and Sandstede [29]). The assertion that if a viscous profile is spectrally stable it is even nonlinearly stable, is due to Zumbrun and Howard [54].

3 Small fronts near a pitchfork singularity

3.1 Statement of the results

Consider a system of reaction-diffusion equations that is given as

$$\begin{aligned} u_t &= u_{xx} + f(u, w, \alpha), \\ w_t &= w_{xx} + Bw + g(u, w, \alpha) \end{aligned} \tag{3.1}$$

with $t \geq 0$, $x \in \mathbb{R}$, $(u, w) \in \mathbb{R}^{1+d}$, $d \geq 1$, a real parameter $\alpha \in \mathbb{R}$, and a non-singular matrix $B \in \mathbb{R}^{d \times d}$. Assume that the functions

$$f\colon \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}, \quad g\colon \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$$

are smooth with $f(u, w, \alpha), g(u, w, \alpha) = \mathcal{O}(|u, w, \alpha|^2)$. Furthermore, let $\lambda_1(u, w, \alpha), \ldots, \lambda_{d+1}(u, w, \alpha)$ denote the eigenvalues of the matrix

$$\begin{pmatrix} \frac{\partial f}{\partial u}(u, w, \alpha) & \nabla_w f(u, w, \alpha) \\ \frac{\partial g}{\partial u}(u, w, \alpha) & B + D_w g(u, w, \alpha) \end{pmatrix}$$

such that $\lambda_1^0 = \lambda_1(0, 0, 0) = 0$. Therefore, $\lambda_j^0 = \lambda_j(0, 0, 0)$ with $j = 2, \ldots, 1 + d$ are the eigenvalues of B which we assume to satisfy

$$\lambda_j^0 \notin \mathbb{R}^+ \cup \{0\} \text{ for } j > 1.$$

These assumptions imply the existence of a one-dimensional center manifold for the vector field $(f(u, w, \alpha), Bw + g(u, w, \alpha))$ near $(u, w, \alpha) = (0, 0, 0)$ that is tangent in $(u, w, \alpha) = (0, 0, 0)$ to the u-direction [22].

We are interested in the case where the reduced vector field on this center manifold undergoes a supercritical pitchfork bifurcation at $(u, w, \alpha) = (0, 0, 0)$. To this effect, we assume further that

$$\frac{\partial^2 f}{\partial u^2}(0) = 0, \quad \frac{\partial f}{\partial \alpha}(0) = 0, \quad \frac{\partial^2 f}{\partial u \partial \alpha}(0) > 0, \tag{3.2}$$

$$\frac{\partial^3 f}{\partial u^3}(0) - 3\frac{\partial}{\partial u}\nabla_w f(0) B^{-1} \frac{\partial^2 g}{\partial u^2}(0) < 0. \tag{3.3}$$

In this situation, there exist rest states of (3.1) close to $(u, w, \alpha) = (0, 0, 0)$ and families of small fronts connecting these rest states. Since we assume the bifurcation to be of supercritical type, we may restrict ourselves to the case where $\alpha \geq 0$, and hence use

$$\varepsilon = \sqrt{\alpha}$$

from now on as independent parameter. The following lemma asserts the existence of the small fronts.

Lemma 3.1 *Assuming* (3.2), (3.3), *there exists* $\varepsilon_0 > 0$ *such that there are two regular families* $(u_\varepsilon^\pm, w_\varepsilon^\pm) \in \mathbb{R}^{1+d}$, $\varepsilon \in [0, \varepsilon_0]$, *of rest states for* (3.1),

$$f(u_\varepsilon^\pm, w_\varepsilon^\pm, \varepsilon^2) = 0, \quad Bw_\varepsilon^\pm + g(u_\varepsilon^\pm, w_\varepsilon^\pm, \varepsilon^2) = 0$$

with $(u_\varepsilon^\pm, w_\varepsilon^\pm) = \mathcal{O}(\varepsilon)$. *The following holds.*

(i) The constant states $(u_\varepsilon^\pm, w_\varepsilon^\pm)$ *are stable as solutions of* (3.1) *if and only if* $\operatorname{Re}\lambda_j^0 < 0$ *for* $j > 1$.

(ii) There is a traveling front $\phi_{1,\varepsilon} = (u_{1,\varepsilon}, w_{1,\varepsilon})$ *in* (3.1) *with speed* $s_1(\varepsilon)$ *and end states*

$$\phi_{1,\varepsilon}(\pm\infty) = (u_\varepsilon^\pm, w_\varepsilon^\pm).$$

The speed $s_1(\varepsilon)$ *is a regular function of* ε.

(iii) There is a traveling front $\phi_{2,\varepsilon} = (u_{2,\varepsilon}, w_{2,\varepsilon})$ *in* (3.1) *with speed* $s_2(\varepsilon)$ *and end states*

$$\phi_{2,\varepsilon}(\pm\infty) = (u_\varepsilon^\mp, w_\varepsilon^\mp).$$

The speed $s_2(\varepsilon)$ *is a regular function of* ε.

The proof of Lemma 3.1 will be given in the next section.

The small fronts $\phi_{1,\varepsilon}$ and $\phi_{2,\varepsilon}$ – more precisely, their spectral stability – are the objects which are studied in the following. We shall prove that both existence and spectral stability of $\phi_{1,\varepsilon}$ and $\phi_{2,\varepsilon}$ can be traced to existence and stability of the standing front $\phi_0 \colon \mathbb{R} \to \mathbb{R}$ with $\phi_0(\pm\infty) = \pm 1$ in the scalar equation

$$u_t = u_{xx} - u^3 + u, \quad u \in \mathbb{R}. \tag{3.4}$$

Let

$$\mathcal{H}_\varepsilon^- : \mathbb{H} \to \mathcal{G}_{1+d}^{2+2d}(\mathbb{C}), \quad \mathcal{H}_\varepsilon^+ : \mathbb{H} \to \mathcal{G}_{1+d}^{2+2d}(\mathbb{C})$$

be the Evans bundles for $\phi_{1,\varepsilon}$. By Lemma 3.1 (i), there is an open region $\Omega_\varepsilon \subset \mathbb{C}$ with $\mathbb{H} \subset \Omega_\varepsilon$ for every $\varepsilon > 0$ such that the Evans bundles are well-defined, analytic functions on Ω_ε [17]. Moreover, let

$$\mathcal{H}_0^- : \mathbb{H} \to \mathcal{G}_1^2(\mathbb{C}), \quad \mathcal{H}_\varepsilon^+ : \mathbb{H} \to \mathcal{G}_1^2(\mathbb{C})$$

be the Evans bundles for the standing wave ϕ_0 of (3.4), restricted to $\mathbb{H}$. The main result of this chapter is the following theorem.

Theorem 3.1 *Let all the eigenvalues of B have strictly negative real parts and define $H_\varepsilon^\pm : \mathbb{H} \to \mathcal{G}_{1+d}^{2+2d}$ as*

$$H_\varepsilon^\pm(\kappa) := \mathcal{H}_\varepsilon^\pm(\varepsilon^2\kappa).$$

Then

(i) In the zero-amplitude limit $\varepsilon \to 0$, the scaled Evans bundles $H_\varepsilon^\pm$ converge as analytic functions to suspensions $H_0^\pm$ of $\mathcal{H}_0^\pm$ into $\mathcal{G}_{1+d}^{2+2d}(\mathbb{C})$. With respect to appropriate coordinates of $\mathbb{C}^{2+2d}$, these suspensions $H_0^\pm$ are given as

$$\begin{aligned} H_0^-(\kappa) &= \mathcal{H}_0^-(\kappa) \oplus (\mathbb{C} \times \{0\})^d, \\ H_0^+(\kappa) &= \mathcal{H}_0^+(\kappa) \oplus (\{0\} \times \mathbb{C})^d. \end{aligned}$$

(ii) There exists $\bar{R} > 0$ such that

$$H_\varepsilon^-(\kappa) \cap H_\varepsilon^+(\kappa) = \{0\}$$

for all $\kappa \in \mathbb{H}$ with $|\kappa| > \bar{R}$ and all $\varepsilon \in [0, \varepsilon_0]$.

An analogous statement holds for the Evans bundles of $\phi_{2,\varepsilon}$.

Let $\{\eta_{1,\varepsilon}^\pm(\kappa), \dots, \eta_{1+d,\varepsilon}^\pm(\kappa)\}$ be a globally analytic basis of $H_\varepsilon^\pm(\kappa)$. Assertion (i) of Theorem 3.1 implies that there exists a matrix $T \in Gl(2+2d, \mathbb{C})$ with

$$T^{-1}\left(\eta_{1,0}^-(\kappa), \dots, \eta_{1+d,0}^-(\kappa), \eta_{1,0}^+(\kappa), \dots, \eta_{1+d,0}^+(\kappa)\right) T = \begin{pmatrix} \tilde{\eta}^-(\kappa) & \tilde{\eta}^+(\kappa) & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

where $\tilde{\eta}^\pm(\kappa) \in \mathbb{C}^2$, $\mathcal{H}_0^\pm(\kappa) = \mathbb{C}\tilde{\eta}^\pm(\kappa)$. Therefore, stated in terms of Evans functions, Theorem 3.1 says the following.

Corollary 3.1 *Let all the eigenvalues of B have strictly negative real parts and let $\mathcal{E}_\varepsilon$ be Evans functions for $\phi_{1,\varepsilon}$. Consider then the scaled Evans functions $E_\varepsilon(\kappa) := \mathcal{E}_\varepsilon(\varepsilon^2\kappa)$. They have the following properties.*

(i) The functions E_ε converge for $\varepsilon \to 0$ as analytic functions, $\lim_{\varepsilon\to 0} E_\varepsilon = E_0$, with E_0 being an Evans function for the front ϕ_0 of (3.4).

(ii) There exists $\bar{R} > 0$ such that

$$E_\varepsilon(\kappa) \neq 0$$

for all $\kappa \in \mathbb{H}$ with $|\kappa| > \bar{R}$ and all $\varepsilon \in [0, \varepsilon_0]$.

An analogous statement holds for any family of Evans functions for $\phi_{2,\varepsilon}$.

The set of unstable eigenvalues of ϕ_ε thus converges to the set of unstable eigenvalues of ϕ_0. With ϕ_0 being spectrally stable [23, 47], Theorem 3.1 thereby yields the spectral stability of the small fronts.

Corollary 3.2 *If all eigenvalues of B have negative real parts, the fronts $\phi_{1,\varepsilon}$ and $\phi_{2,\varepsilon}$ are spectrally stable for sufficiently small $\varepsilon > 0$.*

We proceed as follows. In the next section, we discuss the existence of the fronts and prove Lemma 3.1. In section 3.3, we state three lemmas which jointly imply Theorem 3.1. The proof of these lemmas will then be given in the sections 3.3.1 – 3.3.3

3.2 Construction of the fronts

Traveling fronts in (3.1) correspond to heteroclinic orbits of the profile equation

$$\begin{aligned} u' &= v, \\ v' &= -sv - f(u, w, \varepsilon^2), \\ w' &= y, \\ y' &= -sy - Bw - g(u, w, \varepsilon^2). \end{aligned} \tag{3.5}$$

We introduce a scaling to reveal the slow-fast structure of (3.5), induced by the bifurcation. Using geometric singular perturbation theory, we then construct heteroclinic orbits in the re-scaled system.
The scaling

$$u = \varepsilon\bar{u}, \quad v = \varepsilon^2\bar{v}, \quad w = \varepsilon^2\bar{w}, \quad y = \varepsilon^2\bar{y}, \quad s = \varepsilon\bar{s}$$

turns (3.5) into

$$\begin{aligned} \bar{u}' &= \varepsilon\bar{v}, \\ \bar{v}' &= -\varepsilon\bar{s}\bar{v} - \varepsilon^{-2} f(\varepsilon\bar{u}, \varepsilon^2\bar{w}, \varepsilon^2), \\ \bar{w}' &= \bar{y}, \\ \bar{y}' &= -\varepsilon\bar{s}\bar{y} - B\bar{w} - \varepsilon^{-2} g(\varepsilon\bar{u}, \varepsilon^2\bar{w}, \varepsilon^2). \end{aligned} \tag{3.6}$$

This is a slow-fast system with singular perturbation in $\varepsilon = 0$. The slow variables are $\bar{u}$ and $\bar{v}$; the fast variables are $\bar{w}$ and $\bar{y}$. Using the Taylor expansions of f and g,

$$\begin{aligned} f(\varepsilon\bar{u}, \varepsilon^2\bar{w}, \varepsilon^2) &= \varepsilon^3 \left(\frac{\partial}{\partial u}\nabla_w f(0)\bar{w}\bar{u} + \frac{\partial^2 f}{\partial u \partial \alpha}(0)\bar{u} + \frac{1}{6}\frac{\partial^3 f}{\partial u^3}(0)\bar{u}^3 + \mathcal{O}(\varepsilon) \right), \\ g(\varepsilon\bar{u}, \varepsilon^2\bar{w}, \varepsilon^2) &= \frac{\varepsilon^2}{2}\frac{\partial^2 g}{\partial u^2}(0)\bar{u}^2 + \mathcal{O}(\varepsilon^3), \end{aligned} \tag{3.7}$$

system (3.6) becomes

$$\begin{aligned} \bar{u}' &= \varepsilon\bar{v}, \\ \bar{v}' &= \varepsilon\left(-\bar{s}\bar{v} - a_1\bar{w}\bar{u} - a_2\bar{u}^3 - a_3\bar{u} + \mathcal{O}(\varepsilon)\right), \\ \bar{w}' &= \bar{y}, \\ \bar{y}' &= -B\bar{w} - a_4\bar{u}^2 + \mathcal{O}(\varepsilon), \end{aligned} \tag{3.8}$$

wherein

$$a_1 = \frac{\partial}{\partial u}\nabla_w f(0), \quad a_2 = \frac{1}{6}\frac{\partial^3 f}{\partial u^3}(0), \quad a_3 = \frac{\partial^2 f}{\partial u \partial \alpha}(0), \quad a_4 = \frac{1}{2}\frac{\partial^2 g}{\partial u^2}(0).$$

On the slow time scale $\tilde{\xi} = \varepsilon\xi$, (3.8) is given as

$$\begin{aligned} \dot{\bar{u}} &= \bar{v}, \\ \dot{\bar{v}} &= -\bar{s}\bar{v} - a_1\bar{w}\bar{u} - a_2\bar{u}^3 - a_3\bar{u} + \mathcal{O}(\varepsilon), \\ \varepsilon\dot{\bar{w}} &= \bar{y}, \\ \varepsilon\dot{\bar{y}} &= -B\bar{w} - a_4\bar{u}^2 + \mathcal{O}(\varepsilon). \end{aligned} \tag{3.9}$$

By setting $\varepsilon = 0$, we obtain the reduced problem

$$\begin{aligned} \dot{\bar{u}} &= \bar{v}, \\ \dot{\bar{v}} &= -\bar{s}\bar{v} - a_1\bar{w}\bar{u} - a_2\bar{u}^3 - a_3\bar{u}, \end{aligned} \tag{3.10}$$

which is defined on

$$\mathcal{S} = \{(\bar{u}, \bar{v}, \bar{w}, \bar{y}) \in \mathbb{R}^{2+2d} \,:\, \bar{w} = -B^{-1}a_4\bar{u}^2,\, \bar{y} = 0\}.$$

Each point on $\mathcal{S}$ is a fixed point of the layer problem

$$\begin{aligned} \bar{w}' &= \bar{y}, \\ \bar{y}' &= -B\bar{w} - a_4\bar{u}^2. \end{aligned} \tag{3.11}$$

The linearization in such a fixed point,

$$\begin{pmatrix} 0 & I \\ -B & 0 \end{pmatrix},$$

has the eigenvalues $\mu_j^{\pm} = \pm\left(-\lambda_{j+1}^0\right)^{1/2}$, $j = 1, \ldots, d$. Since no eigenvalues of B lie in $\mathbb{R}^+ \cup \{0\}$, we find that $\operatorname{Re}\mu_j^{\pm} \neq 0$ for all $j = 1, \ldots, d$. Therefore, each point in $\mathcal{S} = \mathcal{S}_H$ is a hyperbolic fixed point of the layer problem and, for every compact $K \subset \mathbb{R}^2$,

$$M_0 = \{(\bar{u}, \bar{v}, \bar{w}, \bar{y}) \in \mathcal{S} \,:\, (\bar{u}, \bar{v}) \in K\}$$

is a compact normally hyperbolic critical manifold of (3.8) for $\varepsilon = 0$ with d attracting and d repelling directions. We assume now that K satisfies the conditions of Lemma 2.1 which then implies the following.

Lemma 3.2 *There exists $\varepsilon_0 > 0$ such that there is a family of normally hyperbolic invariant manifolds*

$$M_\varepsilon = \{(\bar{u}, \bar{v}, \bar{w}, \bar{y}) \in \mathbb{R}^{2+2d} \,:\, (\bar{u}, \bar{v}) \in K,\, \bar{w} = h_\varepsilon^1(\bar{u}, \bar{v}), \bar{y} = h_\varepsilon^2(\bar{u}, \bar{v})\}, \quad \varepsilon \in [0, \varepsilon_0],$$

for (3.8)*. The functions h_ε^1, h_ε^2 are regular in $\bar{u}$, $\bar{v}$ and ε with*

$$h_\varepsilon^1(\bar{u}, \bar{v}) = -B^{-1}a_4\bar{u}^2 + \mathcal{O}(\varepsilon), \quad h_\varepsilon^2(\bar{u}, \bar{v}) = \mathcal{O}(\varepsilon).$$

The restriction of (3.9) to M_ε is governed by

$$\begin{aligned}\dot{\bar{u}} &= \bar{v},\\ \dot{\bar{v}} &= -\bar{s}\bar{v} + a\bar{u}^3 - a_3\bar{u} + \mathcal{O}(\varepsilon),\end{aligned} \tag{3.12}$$

where we have set $a = a_1B^{-1}a_4 - a_2 = \frac{1}{6}\left(3\frac{\partial}{\partial u}\nabla_w f(0)B^{-1}\frac{\partial^2 g}{\partial u^2}(0) - \frac{\partial^3 f}{\partial u^3}(0)\right) > 0$. For $\varepsilon = 0$, the fixed points of (3.12) are $(0,0)$ and $(\pm\sqrt{\frac{a_3}{a}}, 0)$. The linearization of (3.12) in $(\pm\sqrt{\frac{a_3}{a}}, 0)$ has the eigenvalues

$$-\frac{\bar{s}}{2} \pm \sqrt{\frac{\bar{s}^2}{4} + 2a_3},$$

hence $(\pm\sqrt{\frac{a_3}{a}}, 0)$ are hyperbolic fixed points of (3.10), for all $\bar{s} \in \mathbb{R}$. Note that, for $\varepsilon = 0$, (3.12) is the profile equation for the bistable scalar reaction-diffusion

$$\bar{u}_t = \bar{u}_{xx} - a\bar{u}^3 + a_3\bar{u}, \quad \bar{u} \in \mathbb{R}.$$

By scaling t, x and $\bar{u}$ with powers of a and a_3, one can achieve

$$a = a_3 = 1,$$

which we will assume from now on.

Lemma 3.3 *There exists $\varepsilon_1 > 0$ such that for $\varepsilon \in (0, \varepsilon_1]$ and all $\bar{s} \in \mathbb{R}$, the scaled profile equation* (3.8) *has two hyperbolic fixed points $p_\varepsilon^\pm = (\bar{u}_\varepsilon^\pm, 0, \bar{w}_\varepsilon^\pm, 0)$ with $\dim W^u(p_\varepsilon^\pm) = \dim W^s(p_\varepsilon^\pm) = 1 + d$. Moreover, there are regular functions $\bar{s}_i : [0, \varepsilon_1] \to \mathbb{R}$, $i = 1, 2$, such that*

(i) For $\varepsilon \in (0, \varepsilon_1]$ and $\bar{s} = \bar{s}_1(\varepsilon)$, there is a heteroclinic orbit in (3.8) *connecting p_ε^- to p_ε^+.*

(ii) For $\varepsilon \in (0, \varepsilon_1]$ and $\bar{s} = \bar{s}_2(\varepsilon)$, there is a heteroclinic orbit in (3.8) *connecting p_ε^+ to p_ε^-.*

Proof. Setting $\varepsilon = 0$ in (3.12) gives the reduced problem on M_0,

$$\begin{aligned}\dot{\bar{u}} &= \bar{v},\\ \dot{\bar{v}} &= -\bar{s}\bar{v} + \bar{u}^3 - \bar{u}.\end{aligned} \tag{3.13}$$

Restricted to the slow manifolds M_ε, the singular perturbation becomes a regular perturbation. Thus, the hyperbolic saddles $(\pm 1, 0)$ of (3.13) persist for small $\varepsilon > 0$ as hyperbolic saddles of (3.12) and consequently as hyperbolic fixed points

$$p_\varepsilon^\pm = (\bar{u}_\varepsilon^\pm, 0, \bar{w}_\varepsilon^\pm, 0) = (\pm 1 + \mathcal{O}(\varepsilon), 0, -B^{-1}a_4 + \mathcal{O}(\varepsilon), 0)$$

of the full system (3.8). Since $\dim W^u(M_0) = \dim W^s(M_0) = d$, we find further that

$$\dim W^u(p_\varepsilon^\pm) = \dim W^s(p_\varepsilon^\pm) = 1 + d,$$

see Theorem 2.2 in [49]. The claim is that these invariant manifolds intersect for small $\varepsilon > 0$ and certain values of $\bar{s}$.
If $\bar{s} = 0$, (3.13) is Hamiltonian with $H(\bar{u}, \bar{v}) = \frac{1}{2}\bar{v}^2 - \int_0^{\bar{u}} u^3 - u\,du$. Solutions of (3.13) are parts of level sets of $H(\bar{u}, \bar{v})$. As $H(-1, 0) = H(1, 0)$, we find two heteroclinic orbits $(\bar{u}_1, \bar{v}_1)$ and $(\bar{u}_2, \bar{v}_2)$ for (3.13) with

$$(\bar{u}_1(\pm\infty), \bar{v}_1(\pm\infty)) = (\pm 1, 0), \quad (\bar{u}_2(\pm\infty), \bar{v}_2(\pm\infty)) = (\mp 1, 0).$$

These two heteroclinic orbits, forming a heteroclinic loop, persist as heteroclinic orbits of the original problem (3.8), if the intersection of the unstable and stable manifolds is transversal in $(\bar{u}, \bar{v}, \bar{s})$-space [49]. This transversality is established by a Melnikov computation. Consider first the heteroclinic orbit $(\bar{u}_1, \bar{v}_1)$. Up to multiplication with a constant,

$$\psi(\tau) = (-\dot{\bar{v}}_1(\tau), \bar{v}_1(\tau))$$

is the unique bounded solution of the associated adjoint problem to (3.13),

$$\dot{\psi} = \begin{pmatrix} 0 & -3\bar{u}_1^2 + 1 \\ -1 & 0 \end{pmatrix} \psi.$$

With $F(\bar{u}, \bar{v}, \bar{s}, \varepsilon)$ denoting the right hand side of (3.12), we find that

$$M_1 = \int_{-\infty}^{+\infty} D_{\bar{s}} F(\bar{u}_1(\tau), \bar{v}_1(\tau), 0, 0)\psi(\tau)d\tau = -\int_{-\infty}^{+\infty} \bar{v}_1^2(\tau)d\tau \neq 0.$$

As the Melnikov integral M_1 does not vanish, the orbit $(\bar{u}_1, \bar{v}_1)$ is transversal, and there exists a regular function $\bar{s}_1(\varepsilon)$ with $\bar{s}_1(0) = 0$ such that, for small $\varepsilon > 0$ and $\bar{s} = \bar{s}_1(\varepsilon)$, this orbit persists as a heteroclinic orbit of (3.12). This implies the existence of a heteroclinic orbit $(\bar{u}_{1,\varepsilon}, \bar{v}_{1,\varepsilon}, \bar{w}_{1,\varepsilon}, \bar{y}_{1,\varepsilon})$ of the perturbed system (3.8) with α-limit p_ε^- and ω-limit p_ε^+ (see Theorem 4.2 in [49]).
Due to the symmetry of (3.13) for $\bar{s} = 0$, an analogous argumentation leads to the heteroclinic orbit $(\bar{u}_{2,\varepsilon}, \bar{v}_{2,\varepsilon}, \bar{w}_{2,\varepsilon}, \bar{y}_{2,\varepsilon})$ of (3.8) going from p_ε^+ to p_ε^- and existing for $\bar{s} = \bar{s}_2(\varepsilon)$. Note that in general $\bar{s}_1(\varepsilon) \neq \bar{s}_2(\varepsilon)$. □

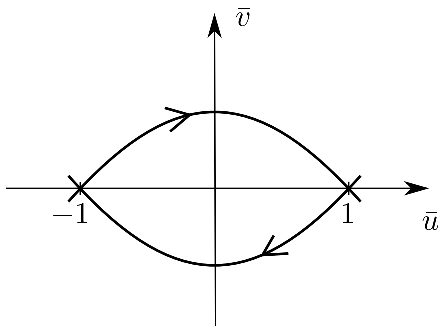

Figure 3.1: The heteroclinic loop on M_0.

As the analysis of (3.8) implies further that $\lambda_1(\varepsilon\bar{u}_\varepsilon^\pm, \varepsilon^2\bar{w}_\varepsilon^\pm, \varepsilon^2) < 0$, setting

$$u_{j,\varepsilon} = \varepsilon\bar{u}_{j,\varepsilon}, \quad w_{j,\varepsilon} = \varepsilon^2\bar{w}_{j,\varepsilon}, \quad s_j(\varepsilon) = \varepsilon\bar{s}_j(\varepsilon), \quad j = 1, 2,$$

yields Lemma 3.1.

From now on, we focus exclusively on the front $\phi_{1,\varepsilon}$ of (3.1). Omitting the index, we simply write $\phi_\varepsilon = (u_\varepsilon, w_\varepsilon)$ and $s_\varepsilon = s(\varepsilon)$. Before considering the associated eigenvalue problem, we seek a simple and convenient parametrization of ϕ_ε.

The associated heteroclinic orbit $(\bar{u}_\varepsilon, \bar{v}_\varepsilon, \bar{w}_\varepsilon, \bar{y}_\varepsilon)$ of the scaled profile equation (3.8) lies entirely in M_ε and is thus parametrized by its slow $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$-components, being the heteroclinic orbit of (3.12). For $\varepsilon = 0$, the orbit $(\bar{u}_0, \bar{v}_0)$ is part of the level set $H(\bar{u}, \bar{v}) \equiv \frac{1}{4}$. We thus obtain

$$\bar{v}_0(\tilde{\xi}) = \frac{1}{\sqrt{2}}\left(1 - \bar{u}_0^2(\tilde{\xi})\right) =: \hat{\nu}_0(\bar{u}_0(\tilde{\xi})), \quad \tilde{\xi} \in \mathbb{R}.$$

That is, $(\bar{u}_0, \bar{v}_0)(\mathbb{R})$ is the graph $\{(\bar{u}, \hat{\nu}_0(\bar{u})) : \bar{u} \in (\bar{u}_0^-, \bar{u}_0^+)\}$.

For $\varepsilon > 0$, there exist functions $\hat{\nu}_\varepsilon$ that are regular $\mathcal{O}(\varepsilon)$-perturbations of $\hat{\nu}_0$ with $\hat{\nu}_\varepsilon(\bar{u}_\varepsilon^\pm) = 0$ and $\hat{\nu}_\varepsilon(\bar{u}) > 0$ for $\bar{u} \in (\bar{u}_\varepsilon^-, \bar{u}_\varepsilon^+)$ such that the heteroclinic orbit $(\bar{u}_\varepsilon, \bar{v}_\varepsilon) = (\bar{u}_\varepsilon, \nu_\varepsilon(\bar{u}_\varepsilon))$ is governed by the scalar equation

$$\bar{u}' = \varepsilon\bar{v} = \varepsilon\hat{\nu}_\varepsilon(\bar{u}), \quad \bar{u}(0) = 0.$$

The solution $\bar{u}_\varepsilon$ of the above equation parametrizes the front $(\bar{u}_\varepsilon, \bar{w}_\varepsilon)$. However, we will use a parametrization by a shifted and scaled version of $\bar{u}$, namely

$$\tau_\varepsilon = \frac{2\bar{u}_\varepsilon - \bar{u}_\varepsilon^+ - \bar{u}_\varepsilon^-}{\bar{u}_\varepsilon^+ - \bar{u}_\varepsilon^-} = \bar{u}_\varepsilon + \mathcal{O}(\varepsilon),$$

which is governed by

$$\tau' = \varepsilon\left(1 - \tau^2\right)\nu_\varepsilon(\tau), \quad J := [-1, 1], \tag{3.14}$$

with a function $\nu_\varepsilon : J \to (0, +\infty)$. The advantage is now that, for all $\varepsilon > 0$ sufficiently small, $\tau_\varepsilon^\pm = \pm 1$. Let τ_ε be the unique solution of (3.14) with $\tau_\varepsilon(0) = 0$. From now on, we write $\phi_\varepsilon[\tau] = \phi_\varepsilon(\tau_\varepsilon^{-1}(\tau))$, $\tau \in J$. Requiring $\tau_\varepsilon(0) = 0$ then amounts to fix the phase of ϕ_ε. We remark here that (3.14) does not describe the full dynamics of (3.8). Moreover, τ is not the auxiliary variable introduced in [1]. In contrast to the construction there, we do not re-parametrize the independent variable ξ.

3.3 Analysis of the eigenvalue problem

The linearization of (3.1) about ϕ_ε yields the linear operator

$$L_\varepsilon \begin{pmatrix} p \\ r \end{pmatrix} = \begin{pmatrix} p'' + sp' + \frac{\partial f}{\partial u}(u_\varepsilon, w_\varepsilon, \varepsilon^2)p + \nabla_w f(u_\varepsilon, w_\varepsilon, \varepsilon^2)r \\ r'' + sr' + \frac{\partial g}{\partial u}(u_\varepsilon, w_\varepsilon, \varepsilon^2)p + Br + D_w g(u_\varepsilon, w_\varepsilon, \varepsilon^2)r \end{pmatrix},$$

densely defined on $C^0_{unif}(\mathbb{R}, \mathbb{C}^{1+d})$, see [1].

The essential spectrum $\sigma_{ess}(L)$ is determined by the constant states $(u_\varepsilon^\pm, w_\varepsilon^\pm)$. If $(u_\varepsilon^\pm, w_\varepsilon^\pm)$ are unstable as constant solutions of (3.1), there is essential spectrum in the right half plane; cf. (2.21) or [1]. By Lemma 3.1 (i) it thus follows:

Corollary 3.3 *If an eigenvalue of B has strictly positive real part, the fronts ϕ_ε are spectrally unstable.*

We thereby assume from now on

$$\operatorname{Re}\lambda_j^0 < 0 \quad \text{for } j = 2, \dots, 1+d.$$

Let $\kappa \in \mathbb{H}$ and write the eigenvalue problem $(L_\varepsilon - \kappa I)\,(p,r)^T = 0$ as

$$X' = A_\varepsilon(u_\varepsilon, w_\varepsilon, \kappa)X \tag{3.15}$$

with $X = (p, r, q, z)^T$ and

$$A_\varepsilon(u_\varepsilon, w_\varepsilon, \kappa) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & I \\ \kappa - \frac{\partial f}{\partial u}(u_\varepsilon, w_\varepsilon, \varepsilon^2) & -\nabla_w f(u_\varepsilon, w_\varepsilon, \varepsilon^2) & -s_\varepsilon & 0 \\ -\frac{\partial g}{\partial u}(u_\varepsilon, w_\varepsilon, \varepsilon^2) & \kappa I - B - D_w g(u_\varepsilon, w_\varepsilon, \varepsilon^2) & 0 & -s_\varepsilon I \end{pmatrix}.$$

We couple (3.15) with (3.14) to obtain

$$\begin{aligned} \tau' &= \varepsilon\left(1-\tau^2\right)\nu_\varepsilon(\tau), \\ X' &= A_{\varepsilon,\kappa}[\tau]X, \end{aligned} \tag{3.16}$$

which is an autonomous system on $J \times \mathbb{C}^{2+2d}$ with $A_{\varepsilon,\kappa}[\tau] = A_\varepsilon(\varepsilon\bar{u}, \varepsilon^2\bar{w}, \kappa)$. Recall that $\bar{u}$, $\bar{w}$ are functions of τ and that $s_\varepsilon = \varepsilon\bar{s}_\varepsilon$. The eigenvalues of $A_{\varepsilon,\kappa}[\tau]$ are

$$\mu_{j,\varepsilon,\kappa}^\pm[\tau] = -\frac{\varepsilon\bar{s}_\varepsilon}{2} \pm \sqrt{\left(\frac{\varepsilon\bar{s}_\varepsilon}{4}\right)^2 + \kappa - \lambda_j(\varepsilon\bar{u}, \varepsilon^2\bar{w}, \varepsilon^2)}, \quad j = 1, \dots, 1+d.$$

Using the expansions of f and g in (3.7), we write (3.15) as

$$\begin{aligned} \tau' &= \varepsilon\left(1-\tau^2\right)\nu_\varepsilon(\tau), \\ p' &= q, \\ r' &= z, \\ q' &= -\varepsilon\bar{s}q + \kappa p - \varepsilon^2\left((a_1B^{-1}a_4 + 3a_2)\tau^2 + 1 + \mathcal{O}(\varepsilon)\right)p - \varepsilon(a_1\tau + \mathcal{O}(\varepsilon))r, \\ z' &= -\varepsilon\bar{s}z + \kappa r - (B + \mathcal{O}(\varepsilon))r - 2\varepsilon a_4\tau p + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{3.17}$$

which brings the slow-fast structure of (3.16) and the singular perturbation in $\varepsilon = 0$ to light. In order to unravel the dynamics of (3.16) for small $\varepsilon > 0$, we distinguish three different regimes of ε and κ. Theorem 3.1 is a consequence of the following three lemmas.

Lemma 3.4 (Inner regime) *Let* $\mathcal{H}^{\pm}\colon \mathbb{H} \to \mathcal{G}^{2+2d}_{1+d}$ *be the Evans bundles for* ϕ_ε. *Define* $H^{\pm} : \mathbb{H} \to \mathcal{G}^{2+2d}_{1+d}$ *as*

$$H^{\pm}_{\varepsilon}(\kappa) := \mathcal{H}^{\pm}_{\varepsilon^2}(\varepsilon^2\kappa).$$

In the zero-amplitude limit $\varepsilon \to 0$, *the scaled Evans bundles* $H^{\pm}_{\varepsilon}$ *converge as analytic functions to suspensions* $H^{\pm}_0$ *of* $\mathcal{H}^{\pm}_0$ *into* $\mathcal{G}^{2+2d}_{1+d}(\mathbb{C})$. *With respect to appropriate coordinates of* $\mathbb{C}^{2+2d}$, *these suspensions* $H^{\pm}_0$ *are given as*

$$\begin{aligned} H^-_0(\kappa) &= \mathcal{H}^-_0(\kappa) \oplus (\mathbb{C}\times\{0\})^d, \\ H^+_0(\kappa) &= \mathcal{H}^+_0(\kappa) \oplus (\{0\}\times\mathbb{C})^d. \end{aligned}$$

Lemma 3.5 (Outer regime) *There exist* $R_0, R_1 > 0$ *such that there are no unstable-to-stable-bundle connections for* (3.16) *if* $\varepsilon > 0$ *is sufficiently small and* $\kappa \in \mathbb{H}$ *satisfies*

$$\varepsilon^2 R_0 \le |\kappa| \le R_1.$$

Lemma 3.6 (Outmost regime) *For any* $R_1 > 0$ *there exists* $\varepsilon_1 > 0$ *such that for* $\varepsilon \le \varepsilon_1$, *there are no unstable-to-stable bundle connections for* (3.16) *for* $\kappa \in \mathbb{H}$ *with* $|\kappa| \ge R_1$.

The proof of these lemmas will be given in the next three sections.

The nature of the dynamics of (3.16) depends on the ratio of ε and $|\kappa|$. The three regimes reflect a transition from a region where the profile dynamics are dominant (inner regime) to a region where, owing to the large modulus of κ, the X-part of (3.16) is almost autonomous (outmost regime).

3.3.1 Inner regime: Proof of Lemma 3.4

The scaling $\kappa = \varepsilon^2\zeta$, $q = \varepsilon\hat{q}$, $r = \varepsilon\hat{r}$, $z = \varepsilon\hat{z}$ turns (3.17) into

$$\begin{aligned} \tau' &= \varepsilon\left(1-\tau^2\right)\nu_\varepsilon(\tau), \\ p' &= \varepsilon\hat{q}, \\ \hat{r}' &= \hat{z}, \\ \hat{q}' &= -\varepsilon\bar{s}\hat{q} + \varepsilon\zeta p - \varepsilon\left((a_1B^{-1}a_4 + 3a_2)\tau^2 + 1 + \mathcal{O}(\varepsilon)\right)p - \varepsilon(a_1\tau + \mathcal{O}(\varepsilon))\hat{r}, \\ \hat{z}' &= -\varepsilon\bar{s}\hat{z} + \varepsilon\zeta\hat{r} - (B + \mathcal{O}(\varepsilon))\hat{r} - 2a_4\tau p + \mathcal{O}(\varepsilon^2). \end{aligned} \tag{3.18}$$

Obviously, (3.18) is singularly perturbed in $\varepsilon = 0$: $\hat{r}$, $\hat{z}$ are fast variables, while τ, p and $\hat{q}$ are slow.

Let $\hat{A}_{\varepsilon,\zeta}[\tau]$ be the coefficient matrix of the linear $(p, \hat{r}, \hat{q}, \hat{z})$-part of (3.18); the eigenvalues of $\hat{A}_{\varepsilon,\zeta}[\tau]$ are

$$\mu^{\pm}_{j,\varepsilon,\zeta}[\tau] = \frac{\varepsilon\bar{s}}{2} \pm \sqrt{\left(\frac{\varepsilon\bar{s}}{2}\right)^2 + \varepsilon^2\zeta - \lambda_j(\varepsilon\bar{u}, \varepsilon^2\bar{w}, \varepsilon^2)}, \quad j = 1, \dots, 1+d.$$

For $\varepsilon > 0$ and $\zeta \in \mathbb{H}$, $\hat{A}_{\varepsilon,\zeta}[\tau]$ is hyperbolic as

$$\operatorname{Re}\mu^{-}_{j,\varepsilon,\zeta}[\tau] < 0 < \operatorname{Re}\mu^{+}_{j,\varepsilon,\zeta}[\tau], \quad \tau \in J.$$

For $\varepsilon = 0$, the eigenvalues $\mu^{\pm}_{j,0,\zeta}[\tau]$ with $j > 1$ still have non-vanishing real parts, but $\mu^{\pm}_{1,0,\zeta}[\tau] \equiv 0$. Hence, the matrix $\hat{A}_0[\tau] = \hat{A}_{0,\zeta}[\tau]$ possesses a d-dimensional stable space S^f_0, a d-dimensional unstable space U^f_0, and a two-dimensional center space $N_0(\tau)$, spanned by the vectors $(1, -2B^{-1}a_4\tau, 0, 0)$ and $(0, 1, 0, 0)$. Note that S^f_0 and U^f_0 are independent of τ. For all $\tau \in J$, the invariant spaces S^f_0, U^f_0 and $N_0(\tau)$ are spectrally isolated and

$$\mathbb{C}^{2+2d} = U^f_0 \oplus N_0(\tau) \oplus S^f_0.$$

Since, for $j > 1$, the eigenvalues $\mu^{\pm}_{j,0,\zeta}[\tau] \equiv \pm\sqrt{-\lambda^0_j}$ all have non-vanishing real part,

$$\mathcal{M}_0 = \bigcup_{\tau \in J} (\{\tau\} \times N_0(\tau)) \subset J \times \mathbb{C}^{2+2d}$$

is a normally hyperbolic critical manifold of (3.18) for $\varepsilon = 0$. The stable and unstable fibers of $\mathcal{M}_0$ are the linear spaces S^f_0 and U^f_0, respectively.

Now fix an arbitrary $\rho > 0$ and consider (3.18) for $\zeta \in D_\rho := \{\zeta \in \mathbb{C} : |\zeta| \le \rho\}$. Fenichel theory implies that $\mathcal{M}_0$ perturbs to an invariant manifold for (3.18). There exists $\varepsilon_0 = \varepsilon_0(\rho) > 0$ such that for every $\varepsilon \in [0, \varepsilon_0]$ there is a unique invariant manifold $\mathcal{M}_{\varepsilon,\zeta}$ for (3.18) with global Lipschitz constant and fibers $N^\tau_\varepsilon(\zeta)$ that are linear spaces,

$$\mathcal{M}_{\varepsilon,\zeta} = \bigcup_{\tau \in J} (\{\tau\} \times N^\tau_\varepsilon(\zeta)) \subset J \times \mathbb{C}^{2+2d},$$

see also Theorem 4.1 in [3]. The linear spaces $N^\tau_\varepsilon(\zeta)$ are given as

$$N^\tau_\varepsilon(\zeta) = \{(p, \hat{r}, \hat{q}, \hat{z}) : \hat{r} = R_\varepsilon(\tau, p, \hat{q}, \zeta),\ \hat{z} = Z_\varepsilon(\tau, p, \hat{q}, \zeta)\}$$

with smooth functions $R_\varepsilon(\tau, p, \hat{q}, \zeta) = -2B^{-1}a_4\tau p + \mathcal{O}(\varepsilon)$, $Z_\varepsilon(\tau, p, \hat{q}, \zeta) = \mathcal{O}(\varepsilon)$, both analytic in ζ and, for every $\tau \in J$, linear in p, $\hat{q}$.
By dividing out a common factor of ε, the restriction of (3.18) to $\mathcal{M}_{\varepsilon,\zeta}$ becomes

$$\begin{aligned} \dot{\tau} &= \left(1 - \tau^2\right) \nu_\varepsilon(\tau), \\ \dot{p} &= \hat{q}, \\ \dot{\hat{q}} &= \left(\zeta + 3\tau^2 - 1\right) p + \mathcal{O}(\varepsilon). \end{aligned} \tag{3.19}$$

Recall that $a_1B^{-1}a_4 - a_2 = 1$, $a_3 = 1$, and note further that (3.19) is a regular perturbation of the eigenvalue problem for the profile ϕ_0 in the scalar limit problem (3.4).
Next, we show how the slow-fast structure of (3.18) induces a decomposition of the Evans bundles into fast and slow subbundles. Lemma 3.4 is an obvious corollary of the following.

Lemma 3.7 *For any $\rho > 0$, there exists $\varepsilon_1 = \varepsilon_1(\rho) > 0$ such that for each $\varepsilon \in [0, \varepsilon_1]$, there exist unique mappings*

$$S_\varepsilon^{+,f} : D_\rho \to \mathcal{G}_d^{2+2d}(\mathbb{C}), \quad S_\varepsilon^{+,s} : D_\rho \to \mathcal{G}_1^{2+2d}(\mathbb{C}),$$
$$U_\varepsilon^{-,f} : D_\rho \to \mathcal{G}_d^{2+2d}(\mathbb{C}), \quad U_\varepsilon^{-,s} : D_\rho \to \mathcal{G}_1^{2+2d}(\mathbb{C}),$$

and

$$H_\varepsilon^{+,f} : D_\rho \to \mathcal{G}_d^{2+2d}(\mathbb{C}), \quad H_\varepsilon^{+,s} : D_\rho \to \mathcal{G}_1^{2+2d}(\mathbb{C}),$$
$$H_\varepsilon^{-,f} : D_\rho \to \mathcal{G}_d^{2+2d}(\mathbb{C}), \quad H_\varepsilon^{-,s} : D_\rho \to \mathcal{G}_1^{2+2d}(\mathbb{C}),$$

all smooth in ε and analytic in $\zeta \in D_\rho = \{\zeta \in \mathbb{C} : \operatorname{Re}\zeta \geq 0, |\zeta| \leq \rho\}$, such that the following holds.

(i) For every $\varepsilon \in (0, \varepsilon_1]$ and $\zeta \in D_\rho$, $S_\varepsilon^+(\zeta) = S_\varepsilon^{+,f}(\zeta) \oplus S_\varepsilon^{+,s}(\zeta)$ is the stable invariant space of $\hat{A}_{\varepsilon,\zeta}[\tau_\varepsilon^+]$ and $U_\varepsilon^-(\zeta) = U_\varepsilon^{-,f}(\zeta) \oplus U_\varepsilon^{-,s}(\zeta)$ is the unstable invariant space of $\hat{A}_{\varepsilon,\zeta}[\tau_\varepsilon^-]$.

(ii) For every $\varepsilon \in [0, \varepsilon_1]$ and $\zeta \in D_\rho$, the solutions of

$$(X^\pm)' = \hat{\mathcal{A}}_{\varepsilon,\zeta}^d[u_\varepsilon](X^\pm), \quad X^\pm(0) = H_\varepsilon^{\pm,f}(\zeta),$$
$$(Y^\pm)' = \hat{\mathcal{A}}_{\varepsilon,\zeta}^1[u_\varepsilon](y^\pm), \quad Y^\pm(0) = H_\varepsilon^{\pm,s}(\zeta),$$

satisfy

$$X^+(+\infty) = S_\varepsilon^{+,f}(\zeta), \quad X^-(-\infty) = U_\varepsilon^{-,f}(\zeta),$$
$$Y^+(+\infty) = S_\varepsilon^{+,s}(\zeta), \quad Y^-(-\infty) = U_\varepsilon^{-,s}(\zeta).$$

(iii) For $\varepsilon = 0$,

$$H_0^{+,f}(\zeta) \equiv S_0^f, \quad H_0^{-,f}(\zeta) \equiv U_0^f,$$

and

$$H_0^{\pm,s}(\zeta) = \hat{\mathcal{H}}_0^{\pm,s}(\zeta),$$

where $\hat{\mathcal{H}}_0^\pm(\zeta) \in N_0(0)$ is a representation of $\mathcal{H}_0^\pm(\zeta) \subset \mathbb{C}^2$ in a basis of $N_0(0)$.

Proof. Let $S_\varepsilon^{\pm,f}(\zeta)$ and $U_\varepsilon^{\pm,f}(\zeta)$ be the unique invariant spaces associated with the sets of fast eigenvalues

$$\{\mu_{j,\varepsilon,\zeta}^-[\tau^\pm] : j > 1\}, \quad \{\mu_{j,\varepsilon,\zeta}^+[\tau^\pm] : j > 1\},$$

respectively. These spaces depend smoothly on ε with $S_0^{\pm,f}(\zeta) \equiv S_0^f$ and $U_0^{\pm,f}(\zeta) \equiv U_0^f$. Let $S_\varepsilon^{+,s}(\zeta)$ be the unique one-dimensional invariant space associated with the slow stable eigenvalue $\mu_{1,\varepsilon,\zeta}^-[\tau^+]$ and let $U_\varepsilon^{-,s}(\zeta)$ be the unique one-dimensional invariant space associated with the slow unstable eigenvalue $\mu_{1,\varepsilon,\zeta}^+[\tau^-]$. All eigenvalues $\mu_{j,\varepsilon,\zeta}^\pm[\tau^\pm]$, $j = 1, \dots, 1+d$, are analytic functions of ζ, and so are all these spaces [2].
For $\varepsilon > 0$ and $\zeta \in \mathbb{H}$,

$$U_\varepsilon^-(\zeta) = U_\varepsilon^{-,f}(\zeta) \oplus U_\varepsilon^{-,s}(\zeta), \quad S_\varepsilon^+(\zeta) = S_\varepsilon^{+,f}(\zeta) \oplus S_\varepsilon^{+,s}(\zeta)$$

are, by construction, the unstable invariant space of $\hat{A}_{\varepsilon,\zeta}[\tau^-]$ and the stable invariant space of $\hat{A}_{\varepsilon,\zeta}[\tau^+]$, respectively. This proves assertion (i) of the lemma.

Construction of $H_\varepsilon^{\pm,f}$. Consider the induced system

$$\begin{aligned} \tau' &= \varepsilon\left(1-\tau^2\right)\nu_\varepsilon(\tau), \\ X' &= \hat{\mathcal{A}}^d_{\varepsilon,\zeta}[\tau](X) \end{aligned} \tag{3.20}$$

on $J \times \mathcal{G}_d^{2+2d}(\mathbb{C})$ and define

$$C_0^{+,f} := J \times S_0^f \subset J \times \mathcal{G}_d^{2+2d}.$$

For all $\tau \in J$, S_0^f is the stable invariant space of $\hat{A}_0^d[\tau]$. This implies that $S_0^f \in \mathcal{G}_d^{2+2d}(\mathbb{C})$ is a repelling hyperbolic fixed point of

$$X' = \hat{\mathcal{A}}_0^d[\tau](X)$$

for each $\tau \in J$, see Lemma 2.2. Therefore, $C_0^{+,f}$ is a repelling normally hyperbolic critical curve of (3.20) for $\varepsilon = 0$. For small $\varepsilon > 0$, the curve $C_0^{+,f}$ perturbs smoothly to a unique repelling invariant curve $C_{\varepsilon,\zeta}^{+,f}$ of (3.20). The uniqueness now follows directly from Theorem 2.1 which we can invoke as $C_0^{+,f}$ is a compact normally hyperbolic critical submanifold of the compact manifold $J \times \mathcal{G}_d^{2+2d}(\mathbb{C})$. Note that, in the notation of Theorem 2.1, $\rho = 0$ holds as $C_0^{+,f}$ is a critical manifold.
The perturbed curve $C_{\varepsilon,\zeta}^{+,f}$ is the range of a unique solution $(\tau_\varepsilon, X_{\varepsilon,\zeta}^+)$ of (3.20) satisfying

$$X_{\varepsilon,\zeta}^+(\pm\infty) = S_\varepsilon^{\pm,f}(\zeta).$$

The intersection point of the solution $(\tau_\varepsilon, X_{\varepsilon,\zeta}^+)$ with $\{\tau = 0\} \times \mathcal{G}_d^{2+2d}(\mathbb{C})$ defines a mapping

$$H_\varepsilon^{+,f} : D_\rho \to \mathcal{G}_d^{2+2d}(\mathbb{C}).$$

As $H_\varepsilon^{+,f}(\zeta)$ is the point in which the one-dimensional stable manifold of the fixed point $(+1, S_\varepsilon^{+,f}(\zeta))$ hits the section $\{\tau = 0\} \times \mathcal{G}_d^{2+2d}(\mathbb{C})$, the mapping $H_\varepsilon^{+,f}$ is smooth in ε and analytic in $\zeta \in D_\rho$. Moreover, $X_{0,\zeta}^+ \equiv S_0^f$ implies $H_0^{+,f}(\zeta) \equiv S_0^f$. That is, $H_\varepsilon^{+,f}$ enjoys all the properties claimed in the lemma. The uniqueness follows from the uniqueness of the stable manifold.

The mapping

$$H_\varepsilon^{-,f} : D_\rho \to \mathcal{G}_d^{2+2d}(\mathbb{C})$$

is constructed by applying the above reasoning to the curve

$$C_0^{-,f} := J \times U_0^f \subset J \times \mathcal{G}_d^{2+2d},$$

which is an attracting normally hyperbolic critical curve of (3.20) for $\varepsilon = 0$, and which perturbs smoothly to a unique attracting invariant curve $C_{\varepsilon,\zeta}^{-,f}$ being the range of a unique solution $(\tau_\varepsilon, X_{\varepsilon,\zeta}^-)$ of (3.20) with

$$X_{\varepsilon,\zeta}^-(\pm\infty) = U_\varepsilon^{\pm,f}(\zeta).$$

Intersecting the one-dimensional unstable manifold of $(-1, U_\varepsilon^{-,f}(\zeta))$ with the $\{\tau = 0\}$-section yields $H_\varepsilon^{-,f}$ with the claimed properties. The fact that $C_0^{-,f}$ is attracting again follows from Lemma 2.2.

Construction of $H_\varepsilon^{\pm,s}$. Considered as a subset of $J \times \mathcal{G}_2^{2+2d}$,

$$\mathcal{M}_0 = \bigcup_{\tau \in J} (\{\tau\} \times N_0(\tau))$$

is a normally hyperbolic critical manifold of the induced system

$$\begin{aligned} \tau' &= \varepsilon \left(1 - \tau^2\right) \nu_\varepsilon(\tau), \\ X' &= \hat{\mathcal{A}}^2_{\varepsilon,\zeta}[\tau](X) \end{aligned} \tag{3.21}$$

on $J \times \mathcal{G}_2^{2+2d}$. Thus, $\mathcal{M}_0$ perturbs smoothly to a unique invariant curve $\mathcal{M}_{\varepsilon,\zeta}$ of (3.21). Here, the uniqueness of $\mathcal{M}_{\varepsilon,\zeta}$ is asserted by Theorem 2.1. Since $N_0(\tau)$ is a hyperbolic saddle of the X-dynamics in (3.21), for $\varepsilon = 0$ and every $\tau \in J$ (see Lemma 2.2), $\mathcal{M}_{\varepsilon,\zeta}$ has both stable and unstable manifolds.

The slow flow on $\mathcal{M}_{\varepsilon,\zeta}$ is given by

$$\begin{aligned} \dot{\tau} &= \left(1 - \tau^2\right) \nu_\varepsilon(\tau), \\ \dot{Y} &= \varepsilon^{-1} \hat{\mathcal{A}}^2_{\varepsilon,\zeta}[\tau](Y). \end{aligned} \tag{3.22}$$

By virtue of Theorem 2.2 (iv), the slow system (3.22) is regular in $\varepsilon \in [0, \varepsilon_0]$. For all $\varepsilon \in [0, \varepsilon_0]$, the point $(-1, U_\varepsilon^{-,s}(\zeta))$ is a hyperbolic fixed point of (3.22) with a one-dimensional unstable manifold. Therefore, there is a unique solution $(\tau_\varepsilon, Y_{\varepsilon,\zeta}^-)$ with α-limit $(-1, U_\varepsilon^{-,s}(\zeta))$. The intersection point of this solution with the $\{\tau = 0\}$-section defines the mapping $H_\varepsilon^{-,s}(\zeta)$. Since the unstable manifold of $(-1, U_\varepsilon^{-,s}(\zeta))$ depends smoothly on ε and analytically on ζ, so does $H_\varepsilon^{-,s}(\zeta)$. Applied to the hyperbolic fixed point $(+1, S_\varepsilon^{-,s}(\zeta))$ whose stable manifold is of dimension one, this argumentation gives the slow stable bundle $H_\varepsilon^{+,s}(\zeta)$.

Due to the uniqueness of $\mathcal{M}_{\varepsilon,\zeta}$, we find that, in linear coordinates, the slow system (3.22) is given by (3.19) which, for $\varepsilon = 0$, is the eigenvalue problem for ϕ_0. In consequence, $H_0^{\pm,s}(\zeta)$ are embeddings of the Evans bundles $\mathcal{H}_0^\pm(\zeta) \subset \mathbb{C}^2$ for ϕ_0 into the two-dimensional space $N_0(0)$. □

3.3.2 Outer regime: Proof of Lemma 3.5

Set $\varepsilon = \gamma\delta$ and $\kappa = \gamma^2 e^{i\varphi}$ with $|\varphi| \le \frac{\pi}{2}$. Together with the scaling

$$q = \gamma\check{q}, \quad r = \gamma\check{r}, \quad z = \gamma\check{z}$$

system (3.17) then becomes

$$\begin{aligned}
\tau' &= \gamma\delta\left(1-\tau^2\right)\nu_{\gamma\delta}(\tau),\\
p' &= \gamma\check{q},\\
\check{r}' &= \check{z},\\
\check{q}' &= \gamma e^{i\varphi}p + \mathcal{O}(\gamma\delta),\\
\check{z}' &= \gamma e^{i\varphi}\check{r} - (B+\mathcal{O}(\gamma\delta))\check{r} - 2\delta a_4\tau p + \mathcal{O}(\gamma^2\delta^2).
\end{aligned} \tag{3.23}$$

We are interested in the case where both γ and δ are small, $\gamma, \delta \ll 1$.

Consider the linear spaces

$$N^\tau_{0,\delta} := \{(p,\check{q},\check{r},\check{z}) \,:\, \check{r} = -2\delta B^{-1}a_4\tau p,\, \check{z} = 0\} \subset \mathbb{C}^{2+2d}, \quad \tau \in J.$$

With $\check{A}_{\gamma,\delta}[\tau]$ denoting the coefficient matrix for the linear part of (3.23), the space $N^\tau_{0,\delta}$ is, for each $\tau \in J$, the two-dimensional invariant center space of $\check{A}_{0,\delta}[\tau]$ and therefore

$$\mathcal{M}_{0,\delta} = \bigcup_{\tau\in J}\left(\{\tau\}\times N^\tau_{0,\delta}\right)\right) \subset J \times \mathbb{C}^{2+2d}$$

is a normally hyperbolic critical manifold of (3.23) for $\gamma = 0$. Fenichel theory implies the following lemma.

Lemma 3.8 *Fix $\delta_0 > 0$ and let $\delta \in [0,\delta_0]$. Then there exists $\gamma_0 = \gamma_0(\delta_0) > 0$ such that for every $\gamma \in [0,\gamma_0]$ there is a unique invariant slow manifold*

$$\mathcal{M}_{\gamma,\delta} = \bigcup_{\tau\in J}\left(\{\tau\}\times N^\tau_{\gamma,\delta}\right)\right) \subset J \times \mathbb{C}^{2+2d}$$

with linear subspaces $N^\tau_{\gamma,\delta}$ of $\mathbb{C}^{2+2d}$ that depend smoothly on τ, γ and δ.

The uniqueness of the slow manifolds $\mathcal{M}_{\gamma,\delta}$ is established as in the previous section.

In coordinates $(\tau, p, \check{q}) \in J \times \mathbb{C}^2$, the slow flow on $\mathcal{M}_{\gamma,\delta}$ is given by the system

$$\begin{aligned}
\dot{\tau} &= \delta\left(1-\tau^2\right)\nu_{\gamma\delta}(\tau),\\
\dot{p} &= \check{q},\\
\dot{\check{q}} &= e^{i\varphi}p + \mathcal{O}(\delta).
\end{aligned} \tag{3.24}$$

The slow-fast structure of (3.23) induces a decomposition of the unstable space $U^-_{\gamma,\delta}$ of $\check{A}_{\gamma,\delta}[\tau^-]$ and the stable space $S^+_{\gamma,\delta}$ of $\check{A}_{\gamma,\delta}[\tau^+]$ into fast and slow components,

$$U^-_{\gamma,\delta} = U^{-,f}_{\gamma,\delta} \oplus U^{-,s}_{\gamma,\delta}, \quad S^+_{\gamma,\delta} = S^{+,f}_{\gamma,\delta} \oplus S^{+,s}_{\gamma,\delta}.$$

The d-dimensional fast spaces $U^{-,f}_{\gamma,\delta}$ and $S^{+,f}_{\gamma,\delta}$ are spectrally isolated and any unstable-to-stable-bundle connection in (3.23) must lie inside the slow manifold $\mathcal{M}_{\gamma,\delta}$. That is, it must connect $U^{-,s}_{\gamma,\delta}$ to $S^{+,s}_{\gamma,\delta}$. The reason for this can easily be seen as in the proof of Lemma 3.7.

The slow system (3.24) decouples for $\delta = 0$ and the $(p, \check{q})$-equations become autonomous with the coefficient matrix

$$\begin{pmatrix} 0 & 1 \\ e^{i \arg \kappa} & 0 \end{pmatrix}$$

that is hyperbolic for all $\kappa \in \mathbb{H}$. This implies that, for sufficiently small $\delta > 0$, the flow of (3.24) connects the slow unstable bundle $U^{-,s}_{\gamma,\delta}$ at τ^- to the slow unstable bundle $U^{+,s}_{\gamma,\delta}$ at τ^+. Hence, there are no unstable-to-stable-bundle connections in the original system (3.23) for sufficiently small $\gamma, \delta > 0$. This proves Lemma 3.5.

3.3.3 Outmost regime: Proof of Lemma 3.6

The scaling

$$\xi = |\kappa|^{-\frac{1}{2}} \tilde{\xi}, \quad q = |\kappa|^{\frac{1}{2}} \tilde{q}, \quad z = |\kappa|^{\frac{1}{2}} \tilde{z}$$

turns (3.16) into

$$\begin{aligned} \dot{\tau} &= \varepsilon |\kappa|^{-\frac{1}{2}} \left(1 - \tau^2\right) \nu_\varepsilon(\tau), \\ \dot{X} &= \tilde{A}_{\varepsilon,\kappa}[\tau] X \end{aligned} \tag{3.25}$$

with coefficient matrix

$$\tilde{A}_{\varepsilon,\kappa}[\tau] = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & I \\ e^{i \arg \kappa} & 0 & 0 & 0 \\ 0 & e^{i \arg \kappa} I - |\kappa|^{-1} B & 0 & 0 \end{pmatrix} + \mathcal{O}(\varepsilon |\kappa|^{-\frac{1}{2}}).$$

Let $R_1 > 0$ be arbitrary and consider (3.25) for κ with $|\kappa| \geq R_1$. The matrix $\tilde{A}_{0,\kappa}$ is then uniformly hyperbolic with a $(1 + d)$-dimensional invariant unstable space $U_{0,\kappa}$. The curve

$$C_{0,\kappa} := J \times U_{0,\kappa} \subset J \times \mathcal{G}^{2+2d}_{1+d}(\mathbb{C})$$

is thus an attracting normally hyperbolic critical curve of the induced system on $J \times \mathcal{G}^{2+2d}_{1+d}(\mathbb{C})$ with $\varepsilon = 0$. By Theorem 2.1, there exists $\varepsilon_1 > 0$ such that, for every $\varepsilon \in [0, \varepsilon_1]$, one finds a unique attracting invariant curve $C_{\varepsilon,\kappa}$ which, as a smooth $\mathcal{O}(\varepsilon)$-perturbation of $C_{0,\kappa}$, is the range of a unique solution of the induced system, connecting the unstable bundle at τ^- to the unstable bundle at τ^+. In particular, there are no unstable-to-stable-bundle connections for $\varepsilon \in [0, \varepsilon_1]$. This proves Lemma 3.6. Note that the scaling that we used above is the well-known one from [1].

3.4 Remarks

The coefficient a in (3.12),

$$a = \frac{1}{6}\left(3\frac{\partial}{\partial u}\nabla_w f(0)B^{-1}\frac{\partial^2 g}{\partial u^2}(0) - \frac{\partial^3 f}{\partial u^3}(0)\right),$$

is the third derivative of the restriction of f to the center manifold, evaluated at $(u, w, \varepsilon^2) = (0, 0, 0)$. The term

$$\frac{\partial}{\partial u}\nabla_w f(0)B^{-1}\frac{\partial^2 g}{\partial u^2}(0)$$

is due to the curvature of the center manifold. Note in particular that we do not assume f to have a leading cubic term as the curvature alone can lead to $a \neq 0$. The Lorentz system [22] is a simple example for such a case. For $d = 1$, (3.3) is one of the conditions for a cusp singularity as stated by Whitney [51].

A general unfolding of the heteroclinic loop in the reduced problem (3.12) is beyond the scope of this thesis. If, for example, certain transversality conditions are satisfied, homoclinic orbits bifurcate from the heteroclinic loop, i.e. a gluing bifurcation occurs [32]. However, due to the singularity of (3.1) in $\varepsilon = 0$, existing results on the stability of pulses generated by a gluing bifurcation, see for instance [39, 46], cannot be applied directly. In general, the spectral stability of such bifurcating pulses is – at least from the uniform perturbation angle – difficult to control as the "time of flight" near the fixed point that is not part of the orbit gets infinite; see Fig. 3.2.
A gluing bifurcation can, for example, be observed in the Nagumo equation,

$$u_t = u_{xx} + u(1-u)(u-\tilde{a}), \quad u \in \mathbb{R},\ \tilde{a} \in [0, 1/2],$$

at $\tilde{a} = \frac{1}{2}$; cf. [37]. While the fronts forming the heteroclinic loop are stable, the bifurcating pulse is unstable [23].

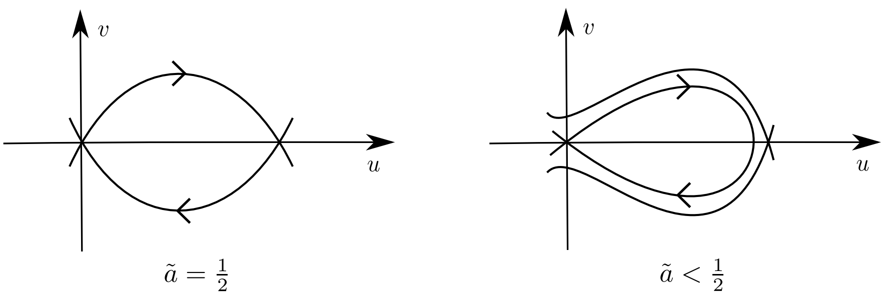

Figure 3.2: The gluing bifurcation in the Nagumo equation.

A gluing bifurcation in (3.12) can be regarded as a model for the more complicated situation in in the FitzHugh-Nagumo equation where both stable and unstable pulses bifurcate from a heteroclinic loop in the layer problem [33]. Good insights into stability of pulses bifurcating from a singular heteroclinic loop would thus lead to a better understanding of the exchange of stability in the FitzHugh-Nagumo equation; see also [50].

4 Small shock waves beyond genuine nonlinearity

4.1 Statement of the results

Consider a strictly hyperbolic system of viscous conservation laws

$$u_t + f(u)_x = u_{xx} \tag{4.1}$$

with $x \in \mathbb{R}$, $t \geq 0$, $u \in \mathbb{R}^n$ and a smooth flux function $f : \mathbb{R}^n \to \mathbb{R}^n$. Let

$$\lambda_1(u) < \cdots < \lambda_n(u), \quad u \in \mathbb{R}^n,$$

denote the eigenvalues of $Df(u)$ and assume that $r_j(u)$, $l_j(u)$ are associated smooth right, respectively left, eigenvectors,

$$Df(u)r_j(u) = \lambda_j(u)r_j(u), \quad l_j(u)Df(u) = \lambda_j(u)l_j(u), \quad l_i(u)r_j(u) = \delta_{ij}$$

for all $u \in \mathbb{R}^n$ and $i, j = 1, \dots, n$.

Fix $k \in \{1, \dots, n\}$ and let $\Sigma \subset \mathbb{R}^n$ be a hypersurface such that for each point $u \in \Sigma$

$$\nabla\lambda_k(u) \cdot r_k(u) = 0, \tag{4.2}$$

$$\left(r_k(u) \cdot \nabla\right)^2 \lambda_k(u) > 0. \tag{4.3}$$

We consider a family of small-amplitude shock waves

$$\phi_\varepsilon(x - s(\varepsilon)t), \quad 0 < \varepsilon \leq \varepsilon_0, \tag{4.4}$$

for (4.1) with end states $\phi_\varepsilon(\pm\infty) = u_\varepsilon^\pm$. Exclusively studying non-characteristic shock waves,

$$\lambda_k(u_\varepsilon^-) > s(\varepsilon) > \lambda_k(u_\varepsilon^+),$$

we correspondingly assume that

$$\begin{aligned} u_\varepsilon^- &= u_* - \varepsilon r_k(u_*), \\ u_\varepsilon^+ &= u_* + \varepsilon(\alpha r_k(u_*) + \varepsilon w(u_*, \varepsilon, \alpha)) \end{aligned} \tag{4.5}$$

with $u_* \in \Sigma$, $\alpha \in (-1, \frac{1}{2})$, and a unique vector $w(u_*, \varepsilon, \alpha)$ perpendicular to $r_k(u_*)$. The analysis in the next section will reveal that (4.5) describes the end states of small non-characteristic profiles in the vicinity of Σ; see also the remarks in section 4.4. The existence of these profiles is asserted by Majda and Pego [34, Corollary 2]. However, the existence will as well follow from the results of the subsequent section.
Note that there exist also characteristic shock waves near Σ, i.e. shock waves with speed $s \in \{\lambda_k(u^-), \lambda_k(u^+)\}$.

The main result of this chapter is the following theorem.

Theorem 4.1 *Fix $u_* \in \Sigma$ and $\alpha \in (-1, \frac{1}{2})$. Let $\mathcal{H}_\varepsilon^\pm : \mathbb{H} \to \mathcal{G}_n^{2n}(\mathbb{C})$ be the Evans bundles of ϕ_ε and let $\mathcal{H}_0^\pm : \mathbb{H} \to \mathcal{G}_1^2(\mathbb{C})$ be the Evans bundles of the shock wave*

$$\phi_0 \colon \mathbb{R} \to \mathbb{R}, \quad \phi_0(-\infty) = -1,\ \phi_0(+\infty) = \alpha,$$

of the scalar viscous conservation law

$$u_t + (u^3)_x = u_{xx}, \quad u \in \mathbb{R}.$$

Define $H_\varepsilon^\pm : \mathbb{H} \to \mathcal{G}_n^{2n}(\mathbb{C})$ by $H_\varepsilon^\pm(\kappa) = \mathcal{H}_\varepsilon^\pm(\varepsilon^4 \kappa)$. It holds:

(i) The scaled Evans bundles $H_\varepsilon^\pm$ converge for $\varepsilon \to 0$ as analytic functions with

$$\lim_{\varepsilon \to 0} H_\varepsilon^\pm = H_0^\pm$$

where $H_0^\pm$ are suspensions of $\mathcal{H}_0^\pm$ in $\mathcal{G}_n^{2n}(\mathbb{C})$; in suitable coordinates of $\mathbb{C}^{2n}$,

$$H_0^-(\kappa) = \mathcal{H}_0^-(\kappa) \oplus (\mathbb{C} \times \{0\})^{n-1},$$
$$H_0^+(\kappa) = \mathcal{H}_0^+(\kappa) \oplus (\{0\} \times \mathbb{C})^{n-1}.$$

(ii) There exist $R > 0$ and $\varepsilon_0 > 0$ such that

$$H_\varepsilon^-(\kappa) \cap H_\varepsilon^+(\kappa) = \{0\}$$

for all $\varepsilon \in [0, \varepsilon_0]$ and all $\kappa \in \mathbb{H}$ with $|\kappa| \geq R$.

As in Chapter 3, we state the result in terms of Evans functions.

Corollary 4.1 *Let $\mathcal{E}_\varepsilon : \mathbb{H} \to \mathbb{C}$ be Evans functions for the shock waves ϕ_ε and define $E_\varepsilon : \mathbb{H} \to \mathbb{C}$ by $E_\varepsilon(\kappa) = \mathcal{E}_\varepsilon(\varepsilon^4 \kappa)$. These scaled Evans functions E_ε then have the following properties.*

(i) For $\varepsilon \to 0$, they converge as analytic functions,

$$\lim_{\varepsilon \to 0} E_\varepsilon = E_0,$$

where the limit E_0 is an Evans function for the shock wave

$$\phi_0 \colon \mathbb{R} \to \mathbb{R}, \quad \phi_0(-\infty) = -1,\ \phi_0(+\infty) = \alpha,$$

of the scalar viscous conservation law

$$u_t + (u^3)_x = u_{xx}, \quad u \in \mathbb{R}.$$

(ii) There exist $R > 0$ and $\varepsilon_0 > 0$ such that

$$E_\varepsilon(\kappa) \neq 0$$

for all $\varepsilon \in [0, \varepsilon_0]$ and all $\kappa \in \mathbb{H}$ with $|\kappa| > R$. That is, for all $\varepsilon \in [0, \varepsilon_0]$, any possible zero of E_ε lies inside $D_R = \{\kappa \in \mathbb{H} : |\kappa| \leq R\}$.

The profile ϕ_0 of the limit equation is known to be spectrally stable [28]. Therefore, E_0 has only a single zero in $\kappa = 0$. By Theorem 4.1, this holds for $\mathcal{E}_\varepsilon$ too. An immediate consequence is thus the spectral stability of ϕ_ε for sufficiently small $\varepsilon > 0$.

Corollary 4.2 *For sufficiently small $\varepsilon > 0$, the shock wave ϕ_ε is spectrally stable.*

Before entering the proof, we express (4.2) and (4.3) in terms of derivatives of f.
By differentiating $(Df - \lambda_k I) r_k = 0$, we obtain

$$(Df(u) - \lambda_k(u)I) Dr_k(u) r_k(u) = -D^2 f(u)(r_k(u), r_k(u)) + r_k(u) \nabla\lambda_k(u) r_k(u). \tag{4.6}$$

Multiplying this equality with the k-th left eigenvector $l_k(u)$ gives

$$\nabla\lambda_k(u) r_k(u) = l_k(u) D^2 f(u)(r_k(u), r_k(u)).$$

Hence, (4.2) is equivalent to assuming that

$$l_k(u) D^2 f(u)(r_k(u), r_k(u)) = 0 \tag{4.7}$$

for all $u \in \Sigma$.
As $(r_k(u) \cdot \nabla)^2 \lambda_k(u) = r_k(u)^T D^2 \lambda_k(u) r_k(u) + \nabla\lambda_k(u) Dr_k(u) r_k(u)$, it follows from (4.6) and (4.7) that condition (4.3) can equivalently be stated as

$$l_k(u) \left(D^3 f(u)(r_k(u), r_k(u), r_k(u)) + 3 D^2 f(u)(r_k(u), Dr_k(u) r_k(u)) \right) > 0 \tag{4.8}$$

for all $u \in \Sigma$. Moreover, multiplying (4.6) with $l_j(u)$, $j \neq k$, yields

$$(\lambda_j(u) - \lambda_k(u)) l_j(u) Dr_k(u) r_k(u) = -l_j(u) D^2 f(u)(r_k(u), r_k(u)),$$

therefore

$$Dr_k(u) r_k(u) = \sum_{j \neq k} -\frac{l_j(u) D^2 f(u)(r_k(u), r_k(u))}{\lambda_j(u) - \lambda_k(u)} r_j(u). \tag{4.9}$$

Without loss of generality, we assume from now on

$$u_* = 0, \quad f(0) = 0, \quad Df(0) = \operatorname{diag}(\lambda_1^0, \ldots, \lambda_n^0) \text{ with } \lambda_j^0 = \lambda_j(0),$$
$$\lambda_k(0) = 0, \quad r_k(0) = e_k.$$

By (4.7), assumption (4.2) then becomes

$$\frac{\partial^2 f_k}{\partial u_k^2}(0) = 0, \tag{4.10}$$

while, using (4.8) and (4.9), assumption (4.3) reads

$$\frac{\partial^3 f_k}{\partial u_k^3}(0) - 3 \sum_{j \neq k} \frac{1}{\lambda_j^0} \frac{\partial^2 f_k}{\partial u_k \partial u_j}(0) \frac{\partial^2 f_j}{\partial u_k^2}(0) > 0. \tag{4.11}$$

The end points $u_\varepsilon^\pm$ of ϕ_ε now satisfy $(u_\varepsilon^-)_k = -\varepsilon$ and $(u_\varepsilon^+)_k = \varepsilon\alpha$.

4.2 Reduction of the profile equation

A viscous profile for (4.1) corresponds to a heteroclinic orbit, connecting u^- to u^+, of the system

$$u' = f(u) - su - c. \tag{4.12}$$

Recall that

$$s\left(u^+ - u^-\right) = f(u^+) - f(u^-), \quad c = f(u^-) - su^-.$$

We introduce the scaling

$$u = \varepsilon\bar{u}, \; s = \varepsilon^2\bar{s}, \; c = \varepsilon^3\bar{c}$$

which turns (4.12) into

$$\bar{u}' = \varepsilon^{-1} f(\varepsilon\bar{u}) - \varepsilon^2\bar{s}\bar{u} - \varepsilon^2\bar{c}. \tag{4.13}$$

A Taylor expansion of $f(\varepsilon\bar{u})$ gives

$$\bar{u}' = Df(0)\bar{u} + \mathcal{O}(\varepsilon),$$

that is, (4.13) is a slow-fast system with singular perturbation in $\varepsilon = 0$. By writing (4.13) as

$$\bar{u}_j' = \lambda_j^0 \bar{u}_j + \mathcal{O}(\varepsilon), \quad j \neq k,$$
$$\bar{u}_k' = \mathcal{O}(\varepsilon),$$

we find that the fast variables are $\bar{u}_j$, $j \neq k$; $\bar{u}_k$ is slow. It now follows from the strict hyperbolicity,

$$\lambda_1^0 < \cdots < \lambda_{k-1}^0 < 0 < \lambda_{k+1}^0 < \cdots < \lambda_n^0,$$

that

$$M_0 = \{\bar{u} \in \mathbb{R}^n \; : \; \bar{u}_j = 0, \, j \neq k\}$$

is a normally hyperbolic critical manifold of (4.13) for $\varepsilon = 0$ with $k - 1$ stable and $n - k$ unstable directions.

Lemma 4.1 *For any closed interval $J \subset \mathbb{R}$, there exist $\varepsilon_0 > 0$ and $n-1$ functions $h_j : J \times [0, \varepsilon_0] \to \mathbb{R}$, $j \neq k$, such that, for every $\varepsilon \in [0, \varepsilon_0]$,*

$$M_\varepsilon = \{\bar{u} \in \mathbb{R}^n \ : \ \bar{u}_k \in J, \bar{u}_j = \varepsilon h_j(\bar{u}_k, \varepsilon), \, j \neq k\}$$

is a normally hyperbolic invariant manifold for (4.13). The functions $h_j(\bar{u}_k, \varepsilon)$ are regular functions of $\bar{u}_k$ and ε,

$$h_j(\bar{u}_k, \varepsilon) = -\frac{1}{2\lambda_j^0} \frac{\partial^2 f_j}{\partial u_k^2}(0)\bar{u}_k^2 + \mathcal{O}(\varepsilon), \quad j \neq k. \tag{4.14}$$

Proof. The existence of the slow manifolds M_ε and of the functions h_j is guaranteed by Fenichel theory as M_0 is a normally hyperbolic critical manifold of (4.13). By inserting $\bar{u}_j = \varepsilon h_j(\bar{u}_k, \varepsilon)$ into (4.13), one finds the expansions (4.14). Note that $\varepsilon h_j(\bar{u}_k, \varepsilon)'$ has to be calculated using the chain rule. □

In order to describe the restriction of (4.13) to M_ε, we need to expand $\varepsilon^{-1} f_k(\varepsilon \bar{u})$ for $\bar{u} \in M_\varepsilon$. Using (4.10) and (4.14), we find that the flow on M_ε is governed by

$$\bar{u}_k' = \varepsilon^2 \left(a\bar{u}_k^3 - \bar{s}(0)\bar{u}_k - \bar{c}_k(0) + \mathcal{O}(\varepsilon) \right) \tag{4.15}$$

with

$$a = \frac{1}{6}\frac{\partial^3 f_k}{\partial u_k^3}(0) - \frac{1}{2}\sum_{j \neq k} \frac{1}{\lambda_j^0} \frac{\partial^2 f_k}{\partial u_k \partial u_j}(0) \frac{\partial^2 f_j}{\partial u_k^2}(0).$$

Due to (4.11), $a > 0$ holds. Transformed to the slow time scale $\varepsilon^2 \xi$, one recovers, for $\varepsilon = 0$, the profile equation of the scalar viscous conservation law

$$u_t + (au^3)_x = u_{xx}, \quad u \in \mathbb{R},$$

which, by rescaling the independent variables x and t, can be transformed to the case $a = 1$. Without loss of generality, we thus assume from now on

$$a = \frac{1}{6}\frac{\partial^3 f_k}{\partial u_k^3}(0) - \frac{1}{2}\sum_{j \neq k} \frac{1}{\lambda_j^0} \frac{\partial^2 f_k}{\partial u_k \partial u_j}(0) \frac{\partial^2 f_j}{\partial u_k^2}(0) = 1. \tag{4.16}$$

Moreover, we consider only that part of M_ε in which the profiles ϕ_ε lie, i.e. the part with $\bar{u}_k \in J_\alpha := [-1, \alpha]$. Setting $\tau = \bar{u}_k$, the reduced profile equation (4.15) therefore is

$$\tau' = \varepsilon^2 \left(\tau^3 - \bar{s}^0 \tau - \bar{c}_k^0 + \mathcal{O}(\varepsilon) \right), \quad \tau \in J_\alpha. \tag{4.17}$$

As $\bar{u}_j = \varepsilon h_j(\tau, \varepsilon)$, we can use τ to parametrize the profile $\bar{u}_\varepsilon$. Since by (4.5) the end points of the scaled profiles, $(\bar{u}_\varepsilon^-)_k = -1$ and $(\bar{u}_\varepsilon^+)_k = \alpha$, are independent of ε, we find that the fixed points of (4.17) for $\varepsilon \in [0, \varepsilon_1]$ are

$$\tau^- = -1 \quad \text{(repelling)}, \qquad \tau^+ = \alpha \quad \text{(attracting)}.$$

Therefore, (4.17) may as well be written as

$$\tau' = \varepsilon^2 (\tau - 1)(\tau - \alpha)(\tau - 1 + \alpha + \mathcal{O}(\varepsilon))$$

wherein $\bar{s}^0 = \frac{\alpha^3-1}{\alpha+1} = \alpha^2 - \alpha + 1$ and $\bar{c}_k^0 = -1 + \bar{s}^0 = \alpha^2 - \alpha$. Lastly, we fix the phase of the profiles ϕ_ε by setting

$$\tau(0) = \frac{\alpha - 1}{2}.$$

Remarks.

1. The tangency of M_0 to the integral curve of $r_k(u)$ at $u = 0$ is of second order; their curvatures agree. For this reason, (4.8) translates to (4.11). Note that (4.11) is identical to the assumption (3.3) in Chapter 3 and that we do not assume $f_k(\varepsilon\bar{u})$ to have a leading term of cubic order (see the remark at the end of Chapter 3).

2. The dynamics of $\bar{u}_k$ in (4.13) is of order ε^2. In the genuinely nonlinear case [14], the corresponding dynamics is just of order ε and assumptions on the dynamics in the other directions, i.e. on f_j, $j \neq k$, are not necessary. The tangent space approximation of the slow manifold proves sufficient in that context; the curvature can be neglected.

3. In contrast to the construction in Chapter 3, τ is not an auxiliary variable but (4.17) is indeed the reduced profile equation. For any α with $-1 < \alpha \leq \frac{1}{2}$, there exists a heteroclinic orbit in

   $$\dot{\tau} = \tau^3 - \bar{s}^0\tau - \bar{c}_k^0$$

   that connects $\tau^- = -1$ to $\tau^+ = \alpha$. However, $\tau^+ = \frac{1}{2}$ is not a hyperbolic fixed point, the convergence to τ^+ is merely algebraic and the shock wave is characteristic. Note further that $0 = \tau^3 - \tilde{s}\tau - \tilde{c}$ describes the cusp surface in $(\tau, \tilde{s}, \tilde{c})$-space. While the assumption of genuine nonlinearity leads to a fold singularity in the profile equation [14], the right-hand side of (4.17) is the general unfolding of the cusp singularity; cf. Figure 4.1.

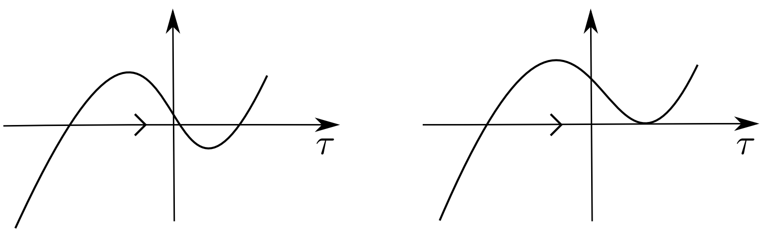

Figure 4.1: The phase portrait of (4.17). For $\alpha < \frac{1}{2}$, $\tau^+ = \alpha$ is a hyperbolic fixed point (non-characteristic case, left). For $\alpha = \frac{1}{2}$, a profile exists but $\tau^+ = \alpha$ is not a hyperbolic fixed point (characteristic case, right).

4.3 Analysis of the eigenvalue problem

The eigenvalue problem for ϕ_ε is

$$\kappa p + ((Df(\phi_\varepsilon) - sI)p)' = p'', \quad \kappa \in \mathbb{H},$$

which we recast as a linear non-autonomous system of first order,

$$W' = A(u_\varepsilon, \kappa, s(\varepsilon))W$$

with

$$W = \begin{pmatrix} p \\ q \end{pmatrix}, \quad A(u, s, \kappa) = \begin{pmatrix} Df(u) - sI & I \\ \kappa I & 0 \end{pmatrix}.$$

The eigenvalues of $A(u, s, \kappa)$ are given by

$$\mu_j^\pm(u, s, \kappa) = \frac{\lambda_j(u) - s}{2} \pm \sqrt{\left(\frac{\lambda_j(u) - s}{2}\right)^2 + \kappa}, \quad j = 1, \dots, n.$$

Since $\alpha < \frac{1}{2}$, the assumptions (A1) and (A2) in section 2.3 are satisfied.
In order to work in an autonomous setting, consider the eigenvalue problem coupled to the reduced profile equation (4.17),

$$\begin{aligned} \tau' &= \varepsilon^2 \left(\tau^3 - \bar{s}^0 \tau - \bar{c}_k^0 + \mathcal{O}(\varepsilon)\right), \\ W' &= A_{\varepsilon,\kappa}[\tau] W \end{aligned} \tag{4.18}$$

with $A_{\varepsilon,\kappa}[\tau] := A(\varepsilon \bar{u}(\tau), \varepsilon^2 \bar{s}, \kappa)$. By the assumptions made on f, we may alternatively write (4.18) in the more explicit form

$$\begin{aligned} \tau' &= \varepsilon^2 \left(\tau^3 - \bar{s}_0 \tau - \bar{c}_k^0 + \mathcal{O}(\varepsilon)\right), \\ p_j' &= \lambda_j^0 p_j + \varepsilon \sum_{l=1}^n a_{jl} \tau p_l + q_j + \mathcal{O}(\varepsilon^2), && j \neq k, \\ p_k' &= \varepsilon \sum_{l=1}^n (a_{kl}\tau + \varepsilon b_l \tau^2) p_l - \varepsilon^2 \bar{s}^0 p_k + q_k + \mathcal{O}(\varepsilon^3), \\ q_j' &= \kappa p_j, && j \neq k, \\ q_k' &= \kappa p_k, \end{aligned} \tag{4.19}$$

where the coefficients are

$$a_{jl} = \frac{\partial^2 f_j}{\partial u_k \partial u_l}(0), \quad b_l = \frac{1}{2}\left(\frac{\partial^3 f_k}{\partial u_k^2 \partial u_l}(0) - \sum_{i \neq k} \frac{1}{\lambda_i^0} \frac{\partial^2 f_k}{\partial u_l \partial u_i}(0) \frac{\partial^2 f_i}{\partial u_k^2}(0)\right).$$

Note that $a_{kk} = 0$ and that the normalization $a = 1$ yields

$$b_k - \sum_{j \neq k} \frac{a_{kj} a_{jk}}{\lambda_j^0} = -3.$$

In a first step, we shall show that, provided $\varepsilon > 0$ is sufficiently small, unstable-to-stable-bundle connections in (4.18) can only occur for $\kappa \in \mathbb{H}$ in a small neighborhood of $\kappa = 0$; the unstable spectrum of ϕ_ε is then restricted to a region $D_R = \{\kappa \in \mathbb{H} : |\kappa| \le R\}$ with arbitrary $R > 0$.

Lemma 4.2 (Outmost regime) *For any $R_0 > 0$ there exists $\varepsilon_0 > 0$ such that there are no unstable-to-stable-bundle connections for (4.18) if*

$$\varepsilon \in (0, \varepsilon_0], \quad |\kappa| > R_0.$$

Proof. We distinguish two subregimes:
Subregime 1: $R_0 \le |\kappa| \le R_1$ with arbitrary $0 < R_0 < R_1$.
Subregime 2: $|\kappa| > R_1$ with sufficiently large $R_1 > 0$.

Subregime 1. Let $0 < R_0 < R_1$ be arbitrary and consider (4.18) for $\kappa \in \mathbb{H}$ with $R_0 \le |\kappa| \le R_1$. For $\varepsilon = 0$, the coefficient matrix

$$A_{0,\kappa}[\tau] = \begin{pmatrix} Df(0) & I \\ \kappa I & 0 \end{pmatrix}$$

is hyperbolic with an n-dimensional unstable space U_0 and an n-dimensional stable space S_0. As the unstable space U_0 is an attracting hyperbolic fixed point of the constant-coefficient system

$$X' = \mathcal{A}^n_{0,\kappa}[\tau](X)$$

on $\mathcal{G}^{2n}_n(\mathbb{C})$, we find that

$$C_0^- = J_\alpha \times U_0$$

is, for $\varepsilon = 0$, a normally hyperbolic critical manifold of

$$\begin{aligned} \tau' &= \varepsilon^2 \left(\tau^3 - \bar{s}^0 \tau - \bar{c}^0_k + \mathcal{O}(\varepsilon)\right), \\ X' &= \mathcal{A}^n_{\varepsilon,\kappa}[\tau](X) \end{aligned} \tag{4.20}$$

on $J_\alpha \times \mathcal{G}^{2n}_n(\mathbb{C})$. For small $\varepsilon > 0$, C_0^- perturbs smoothly to a unique invariant attracting curve $C^-_{\varepsilon,\kappa}$ of (4.20). This perturbed curve is the range of a unique solution $(\tau, X^-_{\varepsilon,\kappa})$ of (4.20) with

$$(\tau(\pm\infty), X^-_{\varepsilon,\kappa}(\pm\infty)) = (\tau^\pm, U^\pm_\varepsilon(\kappa)),$$

where $U^\pm_\varepsilon(\kappa)$ denotes the invariant unstable space of $A_{\varepsilon,\kappa}[\tau^\pm]$. Hence, there are no USBC in (4.18) for sufficiently small $\varepsilon > 0$ and $\kappa \in \mathbb{H}$ with $R_0 \le |\kappa| \le R_1$.

Subregime 2. The scaling

$$\hat{q} = |\kappa|^{-\frac{1}{2}} q, \quad \hat{\xi} = |\kappa|^{\frac{1}{2}} \xi$$

turns (4.18) into a system where the coefficient matrix of the $(p, \hat{q})$-equations is

$$\begin{pmatrix} 0 & I \\ e^{i \arg \kappa} I & 0 \end{pmatrix} + \mathcal{O}(|\kappa|^{-\frac{1}{2}}).$$

For every $\varepsilon_0 > 0$, there exists $R_1 > 0$ such that, for $\varepsilon \in [0, \varepsilon_0]$ and $\kappa \in \mathbb{H}$ with $|\kappa| > R_1$, the $(p, \hat{q})$-part of the scaled system is uniformly close to a hyperbolic constant-coefficient system and the flow connects the unstable bundle at τ^- with the unstable bundle at τ^+. □

The argumentation in Subregime 1 breaks down in $\kappa = 0$; $A_{0,0}$ is not hyperbolic and (4.18) has a multiple degeneracy in $(\varepsilon, \kappa) = (0, 0)$. To overcome this degeneracy, we blow up the point $(\varepsilon, \kappa) = (0, 0)$ by setting

$$\varepsilon = r\bar{\varepsilon}, \quad \kappa = r^2 \bar{\rho} e^{i\varphi}, \quad |\varphi| \leq \frac{\pi}{2}, \tag{4.21}$$

where $\bar{\varepsilon}^2 + \bar{\rho}^2 = 1$, $\bar{\varepsilon}, \bar{\rho} \geq 0$, and $r \in [0, r_0]$ for $r_0 > 0$ suffciently small.

Let S^1_+ denote the quarter of the unit sphere lying in the first quadrant. The transformation (4.21) then defines a mapping

$$S^1_+ \times [0, r_0] \to \mathbb{R}^2, \quad (\bar{\varepsilon}, \bar{\rho}, r) \mapsto (\varepsilon, |\kappa|) = (r\bar{\varepsilon}, r^2\bar{\rho}).$$

Therefore, one can write (4.18) in terms of $(\bar{\varepsilon}, \bar{\rho}, r) \in S^1_+ \times [0, r_0]$. This transformed system will be analyzed by considering two charts of $S^1_+ \times [0, r_0]$, namely chart K_1, given by $\bar{\rho} = 1$, and chart K_2, given by $\bar{\varepsilon} = 1$. Written in these charts, the transformation (4.21) reads

$$\varepsilon = r_1 \varepsilon_1, \quad \kappa = r_1^2 e^{i\varphi},$$
$$\varepsilon = r_2, \quad \kappa = r_2^2 \rho_2 e^{i\varphi},$$

respectively. In chart K_2, the point $(r_2, \rho_2) = (0, 0)$ will turn out to be still degenerate and we use a second blow-up inside K_2, given by

$$r_2 = \delta\bar{r}, \ \rho_2 = \delta^2\bar{\eta}, \quad (\bar{r}, \bar{\eta}) \in S^1_+, \ \delta \in [0, \delta_0].$$

The analysis of the blown-up system will again be made in two charts, K_3 and K_4, obtained by setting $\bar{\eta} = 1$ and $\bar{r} = 1$, respectively.

Written in the charts $K_1, \ldots, K_4$, the dynamics of (4.18) have three distinct time scales and geometric singular perturbation theory is applicable. In each chart, the dynamics will reveal a particular nature that is correlated to the ratio of ε and $|\kappa|$. Somehow the heart of the matter are the dynamics in chart K_4, as it is in this chart that the eigenvalue problem for the scalar viscous shock wave ϕ_0 dominates the dynamics.

As a result of the blow-up analysis, we obtain the following lemmas which, jointly with Lemma 4.2, then prove Theorem 4.1.

Lemma 4.3 (Outer regime I) *There exist $R_0, R_1 > 0$ such that there are no unstable-to-stable-bundle connections for (4.18) if $\varepsilon > 0$ is sufficiently small and κ satisfies*

$$\varepsilon^2 R_1 \leq |\kappa| \leq R_0.$$

Lemma 4.4 (Outer regime II) *For any $R_1, R_2 > 0$ with $R_2 < R_1$, there are no unstable-to-stable-bundle connections for* (4.18) *if $\varepsilon > 0$ is sufficiently small and κ satisfies*

$$\varepsilon^2 R_2 \leq |\kappa| \leq \varepsilon^2 R_1.$$

Lemma 4.5 (Outer regime III) *There exist $R_3, R_2 > 0$ such that there are no unstable-to-stable-bundle connections for* (4.18) *if $\varepsilon > 0$ is sufficiently small and κ satisfies*

$$\varepsilon^4 R_3 \leq |\kappa| \leq \varepsilon^2 R_2.$$

Lemma 4.6 (Inner regime) *For any $R > 0$, the scaled Evans bundles*

$$H_\varepsilon^\pm(\kappa) = \mathcal{H}_\varepsilon^\pm(\varepsilon^4\kappa)$$

converge uniformly on $D_R = \{\kappa \in \mathbb{H} : |\kappa| \leq R\}$ to suspensions of $\mathcal{H}_0^\pm$ in $\mathcal{G}_n^{2n}(\mathbb{C})$; in suitable coordinates of $\mathbb{C}^{2n}$,

$$\begin{aligned} H_0^-(\kappa) &= \mathcal{H}_0^-(\kappa) \oplus (\mathbb{C} \times \{0\})^{n-1}, \\ H_0^+(\kappa) &= \mathcal{H}_0^+(\kappa) \oplus (\{0\} \times \mathbb{C})^{n-1}. \end{aligned}$$

The blow-up does does not involve the dependent variables and one could thereby as well just be talking of scaling regimes for ε and κ. However, the notion of blow-up is most helpful because it gives a systematic approach to overcome the degeneracy of (4.18) in $(\varepsilon, \kappa) = (0, 0)$. For the general theory of blow-up techniques, we refer to [5].

As only parameters are blown up, we omit to label objects such as coefficient matrices, eigenvalues or invariant manifolds with an index for the respective chart. This notational imprecision will be of no consequence since we do not have to match, say, a solution found in one chart with a solution in another chart. What we do need to ensure, however, is that the findings in the charts overlap so that a whole neighbourhood of $(\varepsilon, \kappa) = (0, 0)$ is actually covered.

4.3.1 Outer regime I: Dynamics in chart K_1

In chart K_1, the blow-up transformation (4.21) reads

$$\varepsilon = r_1 \varepsilon_1, \quad \kappa = r_1^2 e^{i\varphi}, \quad q = r_1 \tilde{q}. \tag{4.22}$$

The main purpose of K_1 is to connect chart K_2 with the outmost regime. Therefore, we are interested only in the case where $\varepsilon_1 > 0$ is small.

Plugging (4.22) into (4.18) yields

$$\begin{aligned} \tau' &= r_1^2 \varepsilon_1^2 \left(\tau^3 - \bar{s}^0 \tau - \bar{c}_k^0 + \mathcal{O}(r_1 \varepsilon_1)\right), \\ W' &= A_{r_1, \varepsilon_1}[\tau] W \end{aligned} \tag{4.23}$$

with

$$W = \begin{pmatrix} p \\ \tilde{q} \end{pmatrix}, \quad A_{r_1,\varepsilon_1}[\tau] = \begin{pmatrix} Df(r_1\varepsilon_1\bar{u}[\tau]) - r_1^2\varepsilon_1^2\bar{s}I & r_1 I \\ r_1 e^{i\varphi} I & 0 \end{pmatrix}.$$

The eigenvalues of $A_{r_1,\varepsilon_1}[\tau]$ are

$$\mu^{\pm}_{j,r_1,\varepsilon_1}[\tau] = \frac{\lambda_j(r_1\varepsilon_1\bar{u}) - r_1^2\varepsilon_1^2\bar{s}}{2} \pm \sqrt{\left(\frac{\lambda_j(r_1\varepsilon_1\bar{u}) - r_1^2\varepsilon_1^2\bar{s}}{2}\right)^2 + r_1^2 e^{i\varphi}}, \quad j = 1, \ldots, n.$$

By writing (4.23) as

$$\begin{aligned}
\tau' &= r_1^2\varepsilon_1^2\left(\tau^3 - \bar{s}^0\tau - \bar{c}_k^0 + \mathcal{O}(r_1\varepsilon_1)\right), \\
p_j' &= \lambda_j^0 p_j + r_1\varepsilon_1 \sum_{l=1}^n a_{jl}\tau p_l + r_1\tilde{q}_j + \mathcal{O}(r_1^2\varepsilon_1^2), && j \neq k, \\
p_k' &= r_1\varepsilon_1 \sum_{l=1}^n (a_{kl}\tau + r_1\varepsilon_1 b_l\tau^2)p_l - r_1^2\varepsilon_1^2\bar{s}^0 p_k + r_1\tilde{q}_k + \mathcal{O}(r_1^3\varepsilon_1^3), \\
\tilde{q}_j' &= r_1 e^{i\varphi} p_j, && j \neq k, \\
\tilde{q}_k' &= r_1 e^{i\varphi} p_k,
\end{aligned}$$

we find that (4.23) is of slow-fast type with singular perturbation in $r_1 = 0$. The slow variables are τ, p_k, $\tilde{q}$. The fast variables are p_j, $j \neq k$. The associated layer problem for $r_1 = 0$ is

$$p_j' = \lambda_j^0 p_j, \quad j \neq k.$$

With $\operatorname{Re}\lambda_j^0 \neq 0$ for $j \neq k$, it then follows that $J_\alpha \times N_0$ with $N_0 = \{(p,\tilde{q}) \in \mathbb{C}^{2n} : p_j = 0,\ j \neq k\}$ is a normally hyperbolic critical manifold of (4.23) for $r_1 = 0$.

Lemma 4.7 *Fix $\varepsilon_1^0 > 0$. There exists $r_1^0 > 0$ such that for $\varepsilon_1 \in [0,\varepsilon_1^0]$ and $r_1 \in [0, r_1^0]$ there is a unique invariant slow manifold*

$$\mathcal{M}_{r_1,\varepsilon_1} = \bigcup_{\tau \in J_\alpha} \left(\{\tau\} \times N_{r_1,\varepsilon_1}(\tau)\right)$$

for (4.23). For each $\tau \in J_\alpha$,

$$N_{r_1,\varepsilon_1}(\tau) = \{(p,\tilde{q}) : p_j = r_1 P_j(\tau, p_k, \tilde{q}, r_1, \varepsilon_1), j \neq k\} \subset \mathbb{C}^{2n}$$

is a linear space. The functions P_j are smooth and linear in $p_k, \tilde{q}_1, \ldots, \tilde{q}_n$ for every $\tau \in J_\alpha$. After dividing out a factor of r_1, the restriction of (4.23) to $\mathcal{M}_{r_1,\varepsilon_1}$ is governed by

$$\begin{aligned}
\dot{\tau} &= r_1\varepsilon_1^2\left(\tau^3 - \bar{s}^0\tau - \bar{c}_k^0 + \mathcal{O}(r_1^2\varepsilon_1^3)\right), \\
\dot{\tilde{q}}_j &= \frac{r_1 e^{i\varphi}}{\lambda_j^0}\left(-\tilde{q}_j - \varepsilon_1 a_{jk}\tau p_k + \mathcal{O}(r_1\varepsilon_1) + \mathcal{O}(r_1^2)\right), \\
\dot{p}_k &= r_1\varepsilon_1^2(3\tau^2 - \bar{s}^0)p_k - \varepsilon_1 \sum_{l \neq k} \frac{a_{kl}}{\lambda_l^0}\tau\tilde{q}_l + \tilde{q}_k + \mathcal{O}(r_1^2\varepsilon_1^3), \\
\dot{\tilde{q}}_k &= e^{i\varphi} p_k,
\end{aligned} \tag{4.24}$$

where $\dot{}$ denotes differentiation with respect to the slow time $r_1\xi$.

Proof. For $r_1 = 0$, $\mathcal{M}_{0,\varepsilon_1} = J_\alpha \times N_0$ is a normally hyperbolic critical manifold for (4.23). As such, it perturbs smoothly to an invariant manifold $\mathcal{M}_{r_1,\varepsilon_1}$ for small $r_1 > 0$ with linear fibers $N_{r_1,\varepsilon_1}(\tau)$.
Differentiating $P_j(\tau, p_k, \tilde{q}, r_1, \varepsilon_1)$ with respect to ξ (by use of the chain rule) and plugging the result into the equation for p_j' in (4.23) yield the expansion

$$\begin{aligned} &P_j(r_1, \tau, p_k, \tilde{q}, \varepsilon_1) \\ &= \frac{1}{\lambda_j^0}\left(-\tilde{q}_j - \varepsilon_1 a_{jk}\tau p_k\right) + \frac{r_1\varepsilon_1}{\lambda_j^0}\left(-\frac{a_{jk}\tau}{\lambda_j^0}\tilde{q}_k + \varepsilon_1\sum_{l\neq k}\frac{a_{jl}\tau}{\lambda_l^0}(\tilde{q}_l + \varepsilon_1 a_{lk}\tau p_k) + \mathcal{O}(\varepsilon_1^2)\right) + \mathcal{O}(r_1^2) \\ &= \frac{1}{\lambda_j^0}\left[-\tilde{q}_j - \varepsilon_1 a_{jk}\tau p_k + \mathcal{O}(r_1\varepsilon_1) + \mathcal{O}(r_1^2)\right]. \end{aligned}$$

The slow flow on $\mathcal{M}_{r_1,\varepsilon_1}$ is thus given by (4.24). □

Let $S^f_{r_1,\varepsilon_1}[\tau]$ and $U^f_{r_1,\varepsilon_1}[\tau]$ denote the invariant stable, respectively unstable subspace of $A_{r_1,\varepsilon_1}[\tau]$ associated with the respective set of "fast" eigenvalues

$$\{\mu^-_{j,r_1,\varepsilon_1}[\tau] \,:\, j < k\}, \quad \{\mu^+_{j,r_1,\varepsilon_1}[\tau] \,:\, j > k\}.$$

Their constant values for $r_1 = 0$, S^f_0 and U^f_0, are the stable, respectively unstable fast fibers of $\mathcal{M}_0$. Next, we show that any unstable-to-stable-bundle connection for (4.23) must lie inside $\mathcal{M}_{r_1,\varepsilon_1}$.

As S^f_0 and U^f_0 are spectrally isolated invariant spaces,

$$J_\alpha \times S^f_0 \subset J_\alpha \times \mathcal{G}^{2n}_{k-1}(\mathbb{C}), \quad J_\alpha \times U^f_0 \subset J_\alpha \times \mathcal{G}^{2n}_{n-k}(\mathbb{C})$$

are normally hyperbolic critical curves of the respective Grassmann version of (4.23) for $r_1 = 0$,

$$\tau' = r_1^2\varepsilon_1^2\left(\tau^3 - \bar{s}^0\tau - \bar{c}^0_k + \mathcal{O}(r_1\varepsilon_1)\right), \quad X' = \mathcal{A}^{r_1,k-1}_{1,\varepsilon_1}[\tau](X), \tag{4.25}$$
$$\tau' = r_1^2\varepsilon_1^2\left(\tau^3 - \bar{s}^0\tau - \bar{c}^0_k + \mathcal{O}(r_1\varepsilon_1)\right), \quad X' = \mathcal{A}^{r_1,n-k}_{1,\varepsilon_1}[\tau](X). \tag{4.26}$$

Applying the argumentation in the proof of Lemma 4.2 to these two curves, we find for small $r_1 > 0$ unique solutions $(\tau, X^+_{r_1,\varepsilon_1})$ of (4.25) and $(\tau, X^-_{r_1,\varepsilon_1})$ of (4.26) which depend smoothly on r_1 and ε_1. Further, they satisfy

$$X^+_{r_1,\varepsilon_1}(\pm\infty) = S^f_{r_1,\varepsilon_1}[\tau^\pm], \quad X^-_{r_1,\varepsilon_1}(\pm\infty) = U^f_{r_1,\varepsilon_1}[\tau^\pm].$$

But then any USBC in (4.23) with $r_1 > 0$ sufficiently small must lie inside the invariant slow manifold $\mathcal{M}_{r_1,\varepsilon_1}$.

Analysis of the reduced system. The reduced system (4.24) is again singularly perturbed in $r_1 = 0$. The fast variables are now p_k, $\tilde{q}_k$, while τ and $\tilde{q}_j$, $j \neq k$, are slow. As $|\varphi| \leq \frac{\pi}{2}$,

$$\mathcal{M}^s_0 = \{(\tau, p_k, \tilde{q}) : p_k = \tilde{q}_k = 0\}$$

is a normally hyperbolic critical manifold for (4.24). For small $r_1 > 0$, it perturbs smoothly to a slow invariant manifold $\mathcal{M}^s_{r_1,\varepsilon_1}$ on which $p_k = \tilde{q}_k = \mathcal{O}(r_1)$. By requiring that the fibers of $\mathcal{M}^s_{r_1,\varepsilon_1}$ are linear spaces, $\mathcal{M}^s_{r_1,\varepsilon_1}$ is uniquely determined. After dividing out a common factor of r_1, the restriction of (4.24) to $\mathcal{M}^s_{r_1,\varepsilon_1}$ is governed by

$$\begin{aligned} \mathring{\tau} &= \varepsilon_1^2(\tau^3 - \bar{s}^0\tau - \bar{c}_k^0) + \mathcal{O}(r_1\varepsilon_1^3), \\ \mathring{\tilde{q}}_j &= \frac{e^{i\varphi}}{\lambda_j^0}\left(-\tilde{q}_j + \mathcal{O}(r_1\varepsilon_1) + \mathcal{O}(r_1^2)\right), \end{aligned} \tag{4.27}$$

where $^\circ$ denotes differentiation with respect to the super slow time $r_1^2\xi$. Note that the part for $\tilde{q}_j$, $j \neq k$, is linear.

By reasoning as above, one finds that any USBC of (4.24) – and thus of (4.23) – must now lie inside this super slow manifold $\mathcal{M}^s_{r_1,\varepsilon_1}$. Therefore, we now need to analyze (4.27).

System (4.27) is a little peculiar. Setting $\varepsilon_1 = 0$, the system decouples and the τ-flow becomes trivial. However, for $r_1 = 0$ and $|\varphi| = \frac{\pi}{2}$, the linear $\tilde{q}_k$-dynamics are governed by purely imaginary eigenvalues. In contrast to the situation of the Gap Lemma, we cannot use the exponential dynamics of τ to overcome this loss of hyperbolicity. If we consider (4.27) for φ with modulus uniformly less than $\frac{\pi}{2}$, i.e. if we stay away from the imaginary axis, we could easily apply Fenichel theory. The problem that we encounter here at $|\varphi| = \frac{\pi}{2}$ is due to the fact that the essential spectrum touches the imaginary axis in $\kappa = 0$. Recall that such an obstacle did not arise in the analysis of the corresponding system in section 3.3.3.

As there seems to be no suitable uniform perturbation argument applicable to the whole r_1-range, we will use Lemma 2.3, thus a pointwise argumentation, to preclude unstable-to-stable-bundle connections in (4.27).

Lemma 4.8 *There exists $R > 0$ such that system (4.27) possesses no unstable-to-stable-bundle connections for all $r_1 > 0$, $\varepsilon_1 > 0$ with*

$$r_1\varepsilon_1 \leq R.$$

Proof. The equations for $\tilde{q}_j$ in (4.27) are linear in $\tilde{q}_j$, thus given by a matrix

$$B_{r_1,\varepsilon_1}[\tau] \in \mathbb{C}^{(n-1)\times(n-1)}$$

with

$$B_{0,\varepsilon_1}[\tau] = e^{i\varphi}\,\mathrm{diag}\left(-\frac{1}{\lambda_1^0}, \dots, -\frac{1}{\lambda_{k-1}^0}, -\frac{1}{\lambda_{k+1}^0}, \dots - \frac{1}{\lambda_n^0}\right).$$

Obviously, all eigenvalues of $B_{0,0}[\tau]$ are simple and distinct which implies that for small $r_1 > 0$ the eigenvalues of $B_{r_1,\varepsilon_1}[\tau]$ are simple too. In fact, they are given as

$$\frac{1}{r_1^2}\mu^+_{j,r_1,\varepsilon_1}[\tau] = -\frac{e^{i\varphi}}{\frac{\lambda_j(r_1\varepsilon_1\bar{u}) - r_1^2\varepsilon_1^2 s}{2} - \sqrt{\left(\frac{\lambda_j(r_1\varepsilon_1\bar{u}) - r_1^2\varepsilon_1^2 s}{2}\right)^2 + r_1^2 e^{i\varphi}}}, \quad j < k,$$

and

$$\frac{1}{r_1^2}\mu^-_{j,r_1,\varepsilon_1}[\tau] = -\frac{e^{i\varphi}}{\frac{\lambda_j(r_1\varepsilon_1\bar{u})-r_1^2\varepsilon_1^2 s}{2} + \sqrt{\left(\frac{\lambda_j(r_1\varepsilon_1\bar{u})-r_1^2\varepsilon_1^2 s}{2}\right)^2 + r_1^2 e^{i\varphi}}}, \; j > k.$$

Note that all eigenvalues have non-vanishing real part as soon as $r_1 > 0$. Their associated smooth eigenvectors are given by

$$e_l + \mathcal{O}(r_1^2) + \mathcal{O}(r_1\varepsilon_1) \in \mathbb{C}^{n-1}, \quad l = 1, \ldots, n-1,$$

and we find a basis transformation

$$T[\tau, r_1, \varepsilon_1, \varphi] = I + \mathcal{O}(r_1^2) + \mathcal{O}(r_1\varepsilon_1)$$

such that $T^{-1}B_{r_1,\varepsilon_1}[\tau]T$ is diagonal.

By virtue of Lemma 2.3, there exists a constant $C > 0$, depending only on the dimension $n-1$, such that there are no USBC for (4.27) if

$$\|T[0, r_1, \varepsilon_1, \varphi]^{-1}\frac{\partial}{\partial\tau}T[\tau, r_1, \varepsilon_1, \varphi]\| \le C, \quad \tau \in J_\alpha. \tag{4.28}$$

We will now identify those values of ε_1 and r_1 for which (4.28) holds. To this end, define the matrix $\tilde{T}$ by

$$T = I - \tilde{T}.$$

As $\|\tilde{T}[0, r_1, \varepsilon_1, \varphi]\| = \mathcal{O}(r_1^2) + \mathcal{O}(r_1\varepsilon_1)$, we may assume $\|\tilde{T}\| < 1$. By the standard estimate for the Neumann series it follows

$$\|T[0, r_1, \varepsilon_1, \varphi]^{-1}\| \le \frac{1}{1 - \|\tilde{T}[0, r_1, \varepsilon_1, \varphi]\|},$$

i.e. the norm of $T[0, r_1, \varepsilon_1, \varphi]^{-1}$ is bounded from above.

By construction, τ appears in $B_{r_1,\varepsilon_1}[\tau]$ only multiplied with a factor of $r_1\varepsilon_1$. Therefore,

$$\|\frac{\partial}{\partial\tau}T[\tau, r_1, \varepsilon_1, \varphi]\| = \mathcal{O}(r_1\varepsilon_1).$$

Combining both estimates, we obtain

$$\|T[0, r_1, \varepsilon_1, \varphi]^{-1}\frac{\partial}{\partial\tau}T[\tau, r_1, \varepsilon_1, \varphi]\| = \mathcal{O}(r_1\varepsilon_1)$$

which proves (4.28). □

In summary, we have proved the following lemma.

Lemma 4.9 *There exist a uniform constant $R > 0$ such that for sufficiently small $\varepsilon_1 > 0$ and $r_1 > 0$ with*

$$r_1\varepsilon_1 \le R,$$

there are no unstable-to-stable-bundle connection for (4.23).

Note that the analysis in K_1 covers a region that overlaps with the outmost regime.

4.3.2 Outer regime II: Dynamics in chart K_2

Chart K_2 is obtained by setting $\bar{\varepsilon} = 1$ in (4.21) which gives

$$\varepsilon = r_2, \quad \kappa = r_2^2 \kappa_2,$$

so that system (4.18) becomes

$$\begin{aligned} \tau' &= r_2^2 \left(\tau^3 - \bar{s}^0 \tau - \bar{c}_k^0 + \mathcal{O}(r_2) \right), \\ W' &= A_{r_2,\kappa_2}[\tau] W \end{aligned} \tag{4.29}$$

with

$$A_{r_2,\kappa_2}[\tau] = \begin{pmatrix} Df(r_2 \bar{u}(\tau)) - r_2^2 sI & r_2 I \\ r_2 \kappa_2 I & 0 \end{pmatrix}.$$

The eigenvalues of $A_{r_2,\kappa_2}[\tau]$ are

$$\mu^{\pm}_{j,r_2,\kappa_2}[\tau] = \frac{\lambda_j(r_2 \bar{u}) - r_2^2 s}{2} \pm \sqrt{\left(\frac{\lambda_j(r_2 \bar{u}) - r_2^2 s}{2} \right)^2 + r_2^2 \kappa_2}, \quad j = 1, \ldots, n.$$

For $r_2 > 0$, the matrix $A_{r_2,\kappa_2}[\tau]$ is hyperbolic with n stable and n unstable eigenvalues. As before, (4.29) may alternatively be written as

$$\begin{aligned} \tau' &= r_2^2 \left(\tau^3 - \bar{s}^0 \tau - \bar{c}_k^0 + \mathcal{O}(r_2) \right), \\ p_j' &= \lambda_j^0 p_j + r_2 \sum_{l=1}^{n} a_{jl} \tau p_l + r_2 \tilde{q}_j + \mathcal{O}(r_2^2), && j \neq k, \\ \tilde{q}_j' &= r_2 \kappa_2 p_j, && j \neq k, \\ p_k' &= r_2 \sum_{l=1}^{n} (a_{kl} \tau + r_2 b_l \tau^2) p_l - r_2^2 \bar{s}^0 p_k + r_2 \tilde{q}_k + \mathcal{O}(r_2^3), \\ \tilde{q}_k' &= r_2 \kappa_2 p_k. \end{aligned}$$

System (4.29) has three distinct time scales too. By applying geometric singular perturbation theory we will be able to show that there are no USBC for small $r_2 > 0$ and $\kappa \in \mathbb{H}$ uniformly bounded away from zero. The point $(r_2, \kappa_2) = (0, 0)$, however, will turn out to be degenerate.

For $R > 0$, we set $D_R = \{\kappa \in \mathbb{H} : |\kappa| \leq R\}$. The analysis of (4.29) will give the following lemma.

Lemma 4.10 *For every $R_0, R_1 > 0$ with $R_0 < R_1$ there exists $\bar{r}_2 > 0$ such that, for $0 < r_2 \leq \bar{r}_2$ and $\kappa_2 \in D_{R_1} \setminus D_{R_0}$, there are no unstable-to-stable-bundle connections for* (4.29).

System (4.29) is singularly perturbed in $r_2 = 0$: While τ, p_k, and $\tilde{q}_1 \ldots, \tilde{q}_n$ are slow variables, p_j, $j \neq k$, are fast. Again,

$$J_\alpha \times N_0$$

with $N_0 = \{(p, \tilde{q}) \,:\, p_j = 0, j \neq k\}$ is a normally hyperbolic critical manifold of (4.29) for $r_2 = 0$. In order to apply Fenichel theory, we fix some $R_1 > 0$ and study (4.29) for $\kappa_2 \in D_{R_1}$. By transferring the proof of Lemma 4.7 to the situation here, we have

Lemma 4.11 *For every $R_1 > 0$, there exists $r_2^0 > 0$ such that for every $r_2 \in [0, r_2^0]$ and $\kappa_2 \in D_{R_1}$ there is a unique invariant manifold*

$$\mathcal{M}_{r_2,\kappa_2} = \bigcup_{\tau \in J_\alpha} (\{\tau\} \times N_{r_2,\kappa_2}(\tau)) \subset J_\alpha \times \mathbb{C}^{2n}$$

for (4.29) with linear spaces

$$N_{r_2,\kappa_2}(\tau) = \{(p, \tilde{q}) \,:\, p_j = r_2 P_j(\tau, p_k, \tilde{q}, r_2, \kappa_2), j \neq k\},$$

where P_j are smooth functions that are linear in $p_k, \tilde{q}_1, \ldots, \tilde{q}_n$ and analytic in κ_2.

The slow flow on $\mathcal{M}_{r_2,\kappa_2}$ is given as

$$\begin{aligned}
\dot{\tau} &= r_2(\tau^3 - \bar{s}^0\tau - \bar{c}_k^0 + \mathcal{O}(r_2)),\\
\dot{\tilde{q}}_j &= r_2\kappa_2\left(-\frac{\tilde{q}_j}{\lambda_j^0} - \frac{a_{jk}\tau p_k}{\lambda_j^0} + \mathcal{O}(r_2)\right), \quad j \neq k,\\
\dot{p}_k &= r_2(3\tau^2 - \bar{s}^0)p_k - r_2\sum_{l\neq k}\frac{a_{kl}}{\lambda_l^0}\tau\tilde{q}_l + \tilde{q}_k + \mathcal{O}(r_2^2),\\
\dot{\tilde{q}}_k &= \kappa_2 p_k,
\end{aligned} \tag{4.30}$$

where a factor of r_2 has been divided out.

The fast stable and unstable fibers S_0^f and U_0^f of $\mathcal{M}_0$ are again spectrally isolated invariant spaces and for small $r_1 > 0$ any USBC of (4.29) must lie within $\mathcal{M}_{r_2,\kappa_2}$.

The slow flow on $\mathcal{M}_{r_2,\kappa_2}$ has itself two different time scales with fast variables p_k, $\tilde{q}_k$, and slow variables $\tau, \tilde{q}_j$, $j \neq k$. The coefficient matrix of the layer problem of (4.30) for $r_2 = 0$,

$$\begin{pmatrix} 0 & 1 \\ \kappa_2 & 0 \end{pmatrix},$$

has eigenvalues $\pm\sqrt{\kappa_2}$, that is, $(p_k, \tilde{q}_k) = (0, 0)$ is a hyperbolic fixed point of the layer problem if $\kappa_2 \neq 0$. We take some $R_0 > 0$ and consider (4.30) for $\kappa_2 \in D_{R_1} \setminus D_{R_0}$. In this case, Fenichel theory implies that for small $r_2 > 0$ there exist slow (or better: super slow) invariant manifolds $\mathcal{M}_{r_2,\kappa_2}^s$ for (4.30) on which $p_k = \tilde{q}_k = \mathcal{O}(r_2)$. The flow of (4.30) restricted to $\mathcal{M}_{r_2,\kappa_2}^s$ is governed by the equations

$$\begin{aligned}
\mathring{\tau} &= \tau^3 - \bar{s}^0\tau - \bar{c}_k^0 + \mathcal{O}(r_2),\\
\mathring{\tilde{q}}_j &= \kappa_2\left(-\frac{\tilde{q}_j}{\lambda_j^0} + \mathcal{O}(r_2)\right), \quad j \neq k,
\end{aligned} \tag{4.31}$$

where $^\circ$ denotes differentiation with respect to the rescaled time $r_2^2\xi$.

Note that both (4.30) and (4.31) are linear in $p_k, \tilde{q}$, and $\tilde{q}_j$, respectively, and that the reductions made above correspond to a decomposition of the stable and unstable bundles of the matrix $A_{r_2,\kappa_2}[\tau^\pm]$ into fast, fast-slow and super slow subbundles. Similar to the situation in K_1, only the super slow unstable and stable subbundles may be connected by the flow of (4.29), i.e. there is a USBC for (4.29) if and only if there is one inside $\mathcal{M}^s_{r_2,\kappa_2}$. Lemma 4.10 thereby follows from

Lemma 4.12 *For small* $r_2 > 0$ *and* $\kappa_2 \in D_{R_1} \setminus D_{R_0}$*, there are no unstable-to-stable-bundle connections for* (4.31)*.*

Proof. Write (4.31) as

$$\begin{aligned} \mathring{\tau} &= \tau^3 - \bar{s}^0 \tau - \bar{c}_k^0 + \mathcal{O}(r_2), \\ \mathring{W} &= B_{r_2,\kappa_2}[\tau] W, \end{aligned} \tag{4.32}$$

where $W = (\tilde{q}_1, \ldots, \tilde{q}_{k-1}, \tilde{q}_{k+1}, \ldots, \tilde{q}_n)^T$ and $B_{r_2,\kappa_2}[\tau] \in \mathbb{C}^{(n-1)\times(n-1)}$ with eigenvalues

$$\frac{1}{r_2^2} \mu^-_{j,r_2,\kappa_2}[\tau] = -\frac{\kappa_2}{\frac{\lambda_j(r_2\bar{u}) - r_2^2 s}{2} + \sqrt{(\frac{\lambda_j(r_2\bar{u}) - r_2^2 s}{2})^2 + r_2^2 \kappa_2}}, \; j > k,$$

and

$$\frac{1}{r_2^2} \mu^+_{j,r_2,\kappa_2}[\tau] = -\frac{\kappa_2}{\frac{\lambda_j(r_2\bar{u}) - r_2^2 s}{2} - \sqrt{(\frac{\lambda_j(r_2\bar{u}) - r_2^2 s}{2})^2 + r_2^2 \kappa_2}}, \; j < k.$$

Let

$$\hat{U}^\pm_{r_2}(\kappa_2), \quad \hat{S}^\pm_{r_2}(\kappa_2)$$

denote the invariant spaces of $B_{r_2,\kappa_2}[\tau^\pm]$ associated with the sets of eigenvalues

$$\left\{ \frac{1}{r_2^2} \mu^+_{j,r_2,\kappa_2}[\tau^\pm] \,:\, j < k \right\}, \quad \left\{ \frac{1}{r_2^2} \mu^-_{j,r_2,\kappa_2}[\tau^\pm] \,:\, j > k \right\},$$

respectively, and consider the systems

$$\mathring{\tau} = \tau^3 - \bar{s}^0 \tau - \bar{c}_k^0 + \mathcal{O}(r_2), \; \mathring{X} = \mathcal{B}^{n-k}_{r_2,\kappa_2}[\tau](X) \quad \text{on } J_\alpha \times \mathcal{G}^{n-1}_{n-k}(\mathbb{C}), \tag{4.33}$$

$$\mathring{\tau} = \tau^3 - \bar{s}^0 \tau - \bar{c}_k^0 + \mathcal{O}(r_2), \; \mathring{X} = \mathcal{B}^{k-1}_{r_2,\kappa_2}[\tau](X) \quad \text{on } J_\alpha \times \mathcal{G}^{n-1}_{k-1}(\mathbb{C}), \tag{4.34}$$

induced by (4.32). We distinguish two cases.

Case 1: $|\arg \kappa_2| \in [\frac{\pi}{2} - \eta, \frac{\pi}{2}]$ *with small* $\eta > 0$ *and* $r_2 > 0$ *sufficiently small.*
If $|\arg \kappa_2| = \frac{\pi}{2}$ and $r_2 = 0$, all eigenvalues of the coefficient matrices $B_{0,\kappa_2}[\tau^\pm]$ are purely imaginary. The τ-flow, however, is repelling from τ^- and attracting towards τ^+. Therefore,

$$G^- := \{\tau^-\} \times \mathcal{G}^{n-1}_{n-k}(\mathbb{C}), \quad G^+ := \{\tau^+\} \times \mathcal{G}^{n-1}_{k-1}(\mathbb{C})$$

are normally hyperbolic invariant manifolds of (4.33) and (4.34), respectively (note that they are not *critical* manifolds). By virtue of Theorem 2.1, there exist an unstable invariant manifold $W^u(G^-)$ of G^- and a stable invariant manifold $W^s(G^+)$ of G^+. Furthermore, both possess an invariant foliation. As τ^- and τ^+ are hyperbolic fixed points, each fiber is a

curve based at a point of G^- and G^+, respectively. If the base point of such a fiber is a fixed point of the flow, the fiber itself is invariant under the flow.
By taking the fiber with base point $(\tau^-, \hat{U}^-_{r_2}(\kappa_2))$, we thus find a unique solution $(\tau, X^-_{r_2,\kappa_2})$ of (4.33) with α-limit $(\tau^-, \hat{U}^-_{r_2}(\kappa_2))$. Similarly, we obtain a unique solution $(\tau, X^+_{r_2,\kappa_2})$ of (4.34) with ω-limit $(\tau^+, \hat{S}^+_{r_2}(\kappa_2))$ by taking the fiber with base point $(\tau^+, \hat{S}^+_{r_2}(\kappa_2))$. The intersection points of these two solutions with $\{\tau = \tau(0)\}$ depend analytically on κ_2 and smoothly on r_1. Both (4.33) and (4.34) decouple for $r_2 = 0$,

$$(\tau(0), X^-_{0,\kappa_2}(0)) = (\tau(0), \hat{U}^-_0(\kappa_2)), \quad (\tau(0), X^+_{0,\kappa_2}(0)) = (\tau(0), \hat{S}^+_0(\kappa_2)).$$

Since $\hat{U}^-_0(\kappa_2)$ and $\hat{S}^+_0(\kappa_2)$ are spectrally isolated invariant subspaces, this implies that there are no unstable-to-stable-bundle connections for (4.31) if $r_2 > 0$ is sufficiently small.

Case 2: $|\arg \kappa_2| \in [0, \frac{\pi}{2} - \eta]$ *with small* $\eta > 0$ *and* $r_2 > 0$ *sufficiently small.*
In this case, $B_{r_2,\kappa_2}[\tau^\pm]$ is uniformly hyperbolic and $\hat{U}^\pm_{r_2}(\kappa_2)$, $\hat{S}^\pm_{r_2}(\kappa_2)$ are the unique unstable and stable subspaces, respectively. The fixed point $(\tau^-, \hat{U}^-_{r_2}(\kappa_2))$ of (4.33) has a one-dimensional unstable manifold, while $(\tau^+, \hat{U}^+_{r_2}(\kappa_2))$ is purely attracting. For $r_2 = 0$, the flow of (4.33) connects $(\tau^-, \hat{U}^-_0(\kappa_2))$ to $(\tau^+, \hat{U}^+_0(\kappa_2))$. This connection persists under small perturbation and, for small $r_2 > 0$, the flow connects $(\tau^-, \hat{U}^-_{r_2}(\kappa_2))$ to $(\tau^+, \hat{U}^+_{r_2}(\kappa_2))$. □

Remark. The proof relies on the fact that the τ-dynamics in (4.31) are $\mathcal{O}(1)$. This is not the case in (4.27) where the τ-dynamics are of order ε_1^2 and where Lemma 2.3 is used. The situation here closely resembles that of the Gap Lemma [19].

The above argumentation is only applicable if one stays uniformly away from $\kappa_2 = 0$; the point $(r_2, \kappa_2) = (0, 0)$ is not accessible. For this reason, we consider next a blow-up of $(r_2, \kappa_2) = (0, 0)$,

$$r_2 = \delta\bar{r}, \ \rho_2 = \delta^2\bar{\eta}, \quad (\bar{r}, \bar{\eta}) \in S^1_+, \ \delta \in [0, \delta_0].$$

The analysis will be made in the two charts K_3, defined by $\bar{\eta} = 1$, and K_4, defined by $\bar{r} = 1$.

4.3.3 Outer regime III: Dynamics in chart K_3

Chart K_3 is given by $\bar{\eta} = 1$, therefore

$$r_2 = \delta_3 r_3, \quad \kappa_2 = \delta_3^2 e^{i\varphi}.$$

Setting $\tilde{q} = \delta_3 \hat{q}$, system (4.29) then becomes

$$\begin{aligned}
\tau' &= \delta_3^2 r_3^2 \left(\tau^3 - \bar{s}^0 \tau - \bar{c}_k^0 + \mathcal{O}(\delta_3 r_3)\right), \\
p_j' &= \lambda_j^0 p_j + \delta_3 r_3 \sum_{l=1}^{n} a_{jl} \tau p_l + \delta_3^2 r_3 \hat{q}_j + \mathcal{O}(\delta_3^2 r_3^2), && j \neq k, \\
p_k' &= \delta_3 r_3 \sum_{l=1}^{n} (a_{kl} \tau + \delta_3 r_3 b_l \tau^2) p_l - \delta_3^2 r_3^2 \bar{s}^0 p_k + \delta_3^2 r_3 \hat{q}_k + \mathcal{O}(\delta_3^3 r_3^3), \\
\hat{q}_j' &= \delta_3^2 r_3 e^{i\varphi} p_j, && j \neq k, \\
\hat{q}_k' &= \delta_3^2 r_3 e^{i\varphi} p_k.
\end{aligned} \tag{4.35}$$

We will treat (4.35) as a slow-fast system with singular perturbation parameter δ_3 and consider $r_3 \in [0, r_3^0]$ with small $r_3^0 > 0$ and fixed $|\varphi| \leq \pi/2$ (note that (4.35) is singularly perturbed in $r_3 = 0$ as well).

The fast variables are again p_j, $j \neq k$, and $J_\alpha \times \{(p, \hat{q}) \in \mathbb{C}^{2n} : p_j = 0,\ j \neq k\}$ is a normally hyperbolic critical manifold of (4.35) for $\delta_3 = 0$. As in the preceding sections, we obtain the following.

Lemma 4.13 *For every $r_3^0 > 0$, there exists $\delta_3^0 > 0$ such that for $\delta_3 \in [0, \delta_3^0]$ and $r_3 \in [0, r_3^0]$ there is a unique invariant slow manifold*

$$\mathcal{M}_{\delta_3, r_3} = \bigcup_{\tau \in J_\alpha} \left(\{\tau\} \times N_{\delta_3, r_3}(\tau)\right)$$

for (4.35). For each $\tau \in J_\alpha$,

$$N_{\delta_3, r_3}(\tau) = \{(p, \tilde{q}) : p_j = \delta_3 P_j(\tau, p_k, \hat{q}, \delta_3, r_3), j \neq k\} \subset \mathbb{C}^{2n},$$

where P_j are smooth functions that are linear in $p_k, \hat{q}_1, \ldots, \hat{q}_n$ with

$$P_j(\tau, p_k, \hat{q}, \delta_3, r_3) = -\frac{r_3 a_{jk}}{\lambda_j^0} \tau p_k - \frac{\delta_3 r_3}{\lambda_j^0} \hat{q}_j + \mathcal{O}(\delta_3 r_3^2), \quad j \neq k.$$

After desingularizing the vector field by rescaling the independent time variable with $\delta_3^2 r_3$, the restriction of (4.35) to $\mathcal{M}_{\delta_3, r_3}$ reads

$$\begin{aligned}
\dot{\tau} &= r_3 \left(\tau^3 - \bar{s}^0 \tau - \bar{c}_k^0 + \mathcal{O}(\delta_3 r_3)\right), \\
\dot{\hat{q}}_j &= r_3 e^{i\varphi} \left(-\frac{\delta_3 a_{jk}}{\lambda_j^0} \tau p_k - \frac{\delta_3^2}{\lambda_j^0} \hat{q}_j + \mathcal{O}(\delta_3^2 r_3)\right), \quad j \neq k, \\
\dot{p}_k &= r_3 (3\tau^2 - \bar{s}^0) p_k - r_3 \sum_{l \neq k} \frac{a_{kl}}{\lambda_l^0} \tau \tilde{q}_l + \hat{q}_k + \mathcal{O}(\delta_3 r_3^2), \\
\dot{\hat{q}}_k &= e^{i\varphi} p_k.
\end{aligned} \tag{4.36}$$

Obviously, (4.36) has a slow-fast structure again and is singularly perturbed in $r_3 = 0$. The eigenvalues of the associated layer problem

$$\dot{p}_k = \hat{q}_k,$$
$$\dot{\hat{q}}_k = e^{i\varphi} p_k$$

have non-vanishing real parts since $|\varphi| \le \pi/2$. Hence, $\mathcal{M}_0^s = \{(\tau, p_k, \hat{q}) \, : \, p_k = \hat{q}_k = 0\}$ is a normally hyperbolic critical manifold of (4.36) for $r_3 = 0$. For small $r_3 > 0$, it perturbs smoothly to normally hyperbolic invariant manifolds (with linear fibers) on which $p_k = \hat{q}_k = \mathcal{O}(r_3)$. The flow of (4.36) restricted to such a slow manifold is thus given by

$$\mathring{\tau} = \tau^3 - \bar{s}^0 \tau - \bar{c}_k^0 + \mathcal{O}(\delta_3 r_3),$$
$$\mathring{\hat{q}}_j = \delta_3 e^{i\varphi} \left(-\delta_3 \frac{\hat{q}_j}{\lambda_j^0} + \mathcal{O}(\delta_3 r_3) \right), \quad j \ne k,$$

where a factor of r_3 has been divided out.
This, however, is system (4.31) written in chart K_3 and we can translate Lemma 4.10 to K_3.

Lemma 4.14 *There exist $\delta_3^0 > 0$ and $r_3^0 > 0$ such that there are no USBC in* (4.35) *if*

$$\delta_3 \in [\bar{\delta}_3, \delta_3^0], \quad r_3 \in [0, r_3^0]$$

with arbitrary $0 < \bar{\delta}_3 < \delta_3^0$.

4.3.4 Inner regime: Dynamics in chart K_4

Chart K_4 is given by $\bar{r} = 1$,

$$r_2 = \delta_4, \; \kappa_2 = \delta_4^2 \kappa_4 = \delta_4^2 \eta_4 e^{i\varphi}. \tag{4.37}$$

Plugging (4.37) into (4.29) and setting $\tilde{q} = \delta_4 \hat{q}$ yields

$$\begin{aligned}
\tau' &= \delta_4^2 \left(\tau^3 - \bar{s}^0 \tau - \bar{c}_k^0 + \mathcal{O}(\delta_4) \right), \\
p_j' &= \lambda_j^0 p_j + \delta_4 \sum_{l=1}^n a_{jl} \tau p_l + \delta_4^2 \hat{q}_j + \mathcal{O}(\delta_4^2), && j \ne k, \\
\hat{q}_j' &= \delta_4^2 \kappa_4 p_j, && j \ne k, \\
p_k' &= \delta_4 \sum_{l=1}^n (a_{kl} \tau + \delta_4 b_l \tau^2) p_l - \delta_4^2 \bar{s}^0 p_k + \delta_4^2 \hat{q}_k + \mathcal{O}(\delta_4^3), \\
\hat{q}_k' &= \delta_4^2 \kappa_4 p_k.
\end{aligned} \tag{4.38}$$

System (4.38) is singularly perturbed in $\delta_4 = 0$. One readily checks that $J_\alpha \times N_0$ with $N_0 = \{(p, \hat{q}) \, : \, p_j = 0, j \ne k\}$ is a normally hyperbolic critical manifold of (4.38) for $\delta_4 = 0$. With an argumentation as in the preceding sections, we obtain the following lemma.

Lemma 4.15 *For every $R > 0$, there exists $\delta_4^0 > 0$ such that for all $\delta_4 \in [0, \delta_4^0]$ and all $\kappa_4 \in D_R$ there exists a unique invariant manifold*

$$\mathcal{M}_{\delta_4,\kappa_4} = \bigcup_{\tau \in J_\alpha} (\{\tau\} \times N_{\delta_4,\kappa_4}(\tau))$$

of (4.38) *where $N_{\delta_4,\kappa_4}(\tau)$ are linear subspaces of $\mathbb{C}^{2n}$,*

$$N_{\delta_4,\kappa_4}(\tau) = \{(p, \hat{q}) \,:\, p_j = \delta_4 P_j(\tau, p_k, \hat{q}, \delta_4, \kappa_4), j \neq k\}$$

with smooth functions

$$P_j(\tau, p_k, \hat{q}, \delta_4, \kappa_4) = -\frac{a_{jk}}{\lambda_j^0}\tau p_k + \mathcal{O}(\delta_4)\,, j \neq k,$$

that are analytic in κ_4 and, for each $\tau \in J_\alpha$, linear in p_k, $\hat{q}_1, \ldots, \hat{q}_n$.
After dividing out a factor of δ_4^2, the restriction of (4.38) *to $\mathcal{M}^4_{\delta_4,\kappa_4}$ reads*

$$\begin{aligned}
\dot{\tau} &= \tau^3 - \bar{s}^0\tau - \bar{c}_k^0 + \mathcal{O}(\delta_4),\\
\dot{\hat{q}}_j &= \delta_4\kappa_4\left(-\frac{a_{jk}}{\lambda_j^0}\tau p_k + \mathcal{O}(\delta_4)\right), \quad j \neq k,\\
\dot{p}_k &= (3\tau^2 - \bar{s}^0)p_k + \hat{q}_k + \mathcal{O}(\delta_4),\\
\dot{\hat{q}}_k &= \kappa_4 p_k.
\end{aligned} \tag{4.39}$$

The reduced system (4.39) is again of slow-fast type with singular perturbation in $\delta_4 = 0$. Note that, in contrast to the situations before, τ is a fast-slow variable here. The layer problem of (4.39),

$$\begin{aligned}
\dot{\tau} &= \tau^3 - \bar{s}^0\tau - \bar{c}_k^0,\\
\dot{p}_k &= 3\tau^2 p_k - \bar{s}^0 p_k + \hat{q}_k,\\
\dot{\hat{q}}_k &= \kappa_4 p_k,
\end{aligned}$$

is the augmented eigenvalue problem for ϕ_0.

Write (4.38) as

$$\begin{aligned}
\tau' &= \delta_4^2\left(\tau^3 - \bar{s}^0\tau - \bar{c}_k^0 + \mathcal{O}(\delta_4)\right),\\
W' &= A_{\delta_4,\kappa_4}[\tau]W
\end{aligned} \tag{4.40}$$

with

$$A_{\delta_4,\kappa_4}[\tau] = \begin{pmatrix} Df(\delta_4\bar{u}) - \delta_4^2\bar{s}I & \delta_4^2 I \\ \delta_4^2\kappa_4 I & 0 \end{pmatrix},$$

and recall that

$$\begin{aligned}
\tau' &= \delta_4^2\left(\tau^3 - \bar{s}^0\tau - \bar{c}_k^0 + \mathcal{O}(\delta_4)\right),\\
W' &= \mathcal{A}^d_{\delta_4,\kappa_4}[\tau]W
\end{aligned}$$

denotes the system that (4.40) induces on $J_\alpha \times \mathcal{G}_d^{2n}(\mathbb{C})$ for any $d \leq n$.
We will show that the slow-fast structure of (4.40) leads to a decomposition of the scaled Evans bundles $H^+_{\delta_4}$, $H^-_{\delta_4}$ into sums of fast and slow parts,

$$H^+_{\delta_4} = H^{+,f}_{\delta_4} \oplus H^{+,s}_{\delta_4}, \quad H^-_{\delta_4} = H^{-,f}_{\delta_4} \oplus H^{-,s}_{\delta_4}$$

corresponding to the splittings into fast and slow parts of the bundles

$$S^+_{\delta_4} = S^{+,f}_{\delta_4} \oplus S^{+,s}_{\delta_4}, \quad U^-_{\delta_4} = U^{-,f}_{\delta_4} \oplus U^{-,s}_{\delta_4}$$

which are the n-dimensional stable and unstable invariant subspaces of the hyperbolic end matrices $A_{\delta_4,\kappa_4}[\tau^\pm]$ if $\delta_4 > 0$ and $\kappa_4 \neq 0$. The proof of the following lemma adapts the geometric techniques proposed in [14] to our situation; see in particular Lemma 4 in [14].

Lemma 4.16 *For every $R > 0$ there exists $\bar{\delta}_4 > 0$ such that for all $\delta_4 \in [0, \bar{\delta}_4]$ there are unique mappings*

$$\begin{aligned} S^{+,f}_{\delta_4} &: D_R \to \mathcal{G}^{2n}_{k-1}(\mathbb{C}), & S^{+,s}_{\delta_4} &: D_R \to \mathcal{G}^{2n}_{n-k+1}(\mathbb{C}), \\ U^{-,f}_{\delta_4} &: D_R \to \mathcal{G}^{2n}_{n-k}(\mathbb{C}), & U^{-,s}_{\delta_4} &: D_R \to \mathcal{G}^{2n}_{k}(\mathbb{C}), \\ H^{+,f}_{\delta_4} &: D_R \to \mathcal{G}^{2n}_{k-1}(\mathbb{C}), & H^{+,s}_{\delta_4} &: D_R \to \mathcal{G}^{2n}_{n-k+1}(\mathbb{C}), \\ H^{-,f}_{\delta_4} &: D_R \to \mathcal{G}^{2n}_{n-k}(\mathbb{C}), & H^{-,s}_{\delta_4} &: D_R \to \mathcal{G}^{2n}_{k}(\mathbb{C}) \end{aligned}$$

which are smooth in $\delta_4 \in [0, \bar{\delta}_4]$ and analytic in $\kappa_4 \in D_R$. They have the following properties.

(i) For $\delta_4 \in (0, \bar{\delta}_4]$ and $\kappa_4 \in D_R \setminus \{0\}$,

$$S^+_{\delta_4}(\kappa_4) = S^{+,f}_{\delta_4}(\kappa_4) \oplus S^{+,s}_{\delta_4}(\kappa_4)$$

is the stable invariant subspace of $A_{\delta_4,\kappa_4}[\tau^+]$, and

$$U^-_{\delta_4}(\kappa_4) = U^{-,f}_{\delta_4}(\kappa_4) \oplus U^{-,s}_{\delta_4}(\kappa_4)$$

is the unstable invariant subspace of $A_{\delta_4,\kappa_4}[\tau^-]$.

(ii) For $\delta_4 \in (0, \bar{\delta}_4]$ and $\kappa_4 \in D_R \setminus \{0\}$, the solutions X^+ and Y^+ of the systems

$$(X^+)' = \mathcal{A}^{k-1}_{\delta_4,\kappa_4}[\tau](X^+), \quad (Y^+)' = \mathcal{A}^{n-k+1}_{\delta_4,\kappa_4}[\tau](Y^+)$$

with initial data

$$X^+(0) = H^{+,f}_{\delta_4}(\kappa_4), \quad Y^+(0) = H^{+,s}_{\delta_4}(\kappa_4)$$

converge at $+\infty$ with limits

$$X^+(+\infty) = S^{+,f}_{\delta_4}(\kappa_4), \quad Y^+(+\infty) = S^{+,s}_{\delta_4}(\kappa_4).$$

(iii) For $\delta_4 \in (0, \bar{\delta}_4]$ and $\kappa_4 \in D_R \setminus \{0\}$, the solutions X^- and Y^- of the systems

$$(X^-)' = \mathcal{A}^{n-k}_{\delta_4,\kappa_4}[\tau](X^-), \quad (Y^-)' = \mathcal{A}^{k}_{\delta_4,\kappa_4}[\tau](Y^-)$$

with initial data

$$X^-(0) = H^{-,f}_{\delta_4}(\kappa_4), \quad Y^-(0) = H^{-,s}_{\delta_4}(\kappa_4)$$

converge at $-\infty$ with limits

$$X^-(-\infty) = U^{-,f}_{\delta_4}(\kappa_4), \quad Y^-(-\infty) = U^{-,s}_{\delta_4}(\kappa_4).$$

(iv) There exist complementary subspaces N_0^{sf}, S_0^{ss} and U_0^{ss} of $\mathbb{C}^{2n}$ with $\dim N_0^{sf} = 2$ such that $\mathbb{C}^{2n}$ may be written as the direct sum

$$\mathbb{C}^{2n} = S_0^f \oplus U_0^f \oplus N_0^{sf} \oplus S_0^{ss} \oplus U_0^{ss}.$$

For each $\kappa_4 \in D_R$,

$$H_0^{+,f}(\kappa_4) = S_0^f, \quad H_0^{-,f}(\kappa_4) = U_0^f$$

and

$$H_0^{+,s}(\kappa_4) = \hat{H}_0^+(\kappa_4) \oplus S_0^{ss}, \quad H_0^{-,f}(\kappa_4) = \hat{H}_0^-(\kappa_4) \oplus U_0^{ss},$$

where $\hat{H}_0^{\pm}(\kappa_4)$ denotes an embedding of $\mathcal{H}_0^{\pm}(\kappa_4) \subset \mathbb{C}^2$ into the two-dimensional subspace $N_0^{sf} \subset \mathbb{C}^{2n}$.

Proof. Fix $R > 0$ and let $\kappa \in D_R$. Consider then the eigenvalues of $A_{\delta_4,\kappa_4}[\tau]$,

$$\mu^{\pm}_{j,\delta_4,\kappa_4}[\tau] = \frac{\lambda_j(\delta_4\bar{u}) - \delta_4^2\bar{s}}{2} \pm \sqrt{\left(\frac{\lambda_j(\delta_4\bar{u}) - \delta_4^2\bar{s}}{2}\right)^2 + \delta_4^4\kappa_4}, \quad j = 1, \ldots, n.$$

For small $\delta_4 > 0$ and $\kappa_4 \neq 0$, the matrix $A_{\delta_4,\kappa_4}[\tau]$ is hyperbolic for all $\tau \in J_\alpha$ with

$$\operatorname{Re} \mu^-_{j,\delta_4,\kappa_4}[\tau] < 0 < \operatorname{Re} \mu^+_{j,\delta_4,\kappa_4}[\tau].$$

For $\delta_4 = 0$, however, there are $n+1$ eigenvalues that vanish, namely the eigenvalues

$$\mu^-_{j,\delta_4,\kappa_4}[\tau],\ j \geq k, \quad \mu^+_{j,\delta_4,\kappa_4}[\tau],\ j \leq k,$$

which we will refer to as the slow eigenvalues. The remaining $n-1$ eigenvalues are uniformly hyperbolic and give the fast dynamics.
Let $E^{\pm}_{j,\delta_4,\kappa_4}[\tau]$ denote the the one-dimensional eigenspace of $\mu^{\pm}_{j,\delta_4,\kappa_4}[\tau]$, $j = 1, \ldots, n$. A short calculation proves that

$$E^{\pm}_{j,\delta_4,\kappa_4}[\tau] = \mathbb{C}\begin{pmatrix} r_j(\delta_4\bar{u}) \\ -\delta_4^{-2}\mu^{\mp}_{j,\delta_4,\kappa_4}[\tau] r_j(\delta_4\bar{u}) \end{pmatrix}, \quad j = 1, \ldots, n.$$

In a first step, we distinguish the following invariant spaces of the end matrices $A_{\delta_4,\kappa_4}[\tau^{\pm}]$,

$$\begin{aligned} S^{+,f}_{\delta_4}(\kappa_4) &= \bigoplus_{j<k} E^-_{j,\delta_4,\kappa_4}[\tau^+], \quad S^{+,s}_{\delta_4}(\kappa_4) = \bigoplus_{j\geq k} E^-_{j,\delta_4,\kappa_4}[\tau^+], \\ U^{-,f}_{\delta_4}(\kappa_4) &= \bigoplus_{j>k} E^+_{j,\delta_4,\kappa_4}[\tau^-], \quad U^{-,s}_{\delta_4}(\kappa_4) = \bigoplus_{j\leq k} E^+_{j,\delta_4,\kappa_4}[\tau^-]. \end{aligned}$$

All the eigenspaces $E^{\pm}_{j,\delta_4,\kappa_4}[\tau^{\mp}]$, $j = 1, \ldots, n$, are smooth in δ_4 and analytic in $\kappa_4 \in D_R$. Therefore, the four invariant spaces defined above are as well. By construction, these spaces are unique and enjoy the properties claimed in (i). See also Lemma 3.7.

Next, let $N_{\delta_4}(\tau,\kappa_4)$ denote the $(n+1)$-dimensional invariant space of $A_{\delta_4,\kappa_4}[\tau]$ associated with the slow eigenvalues $\{\mu^-_{j,\delta_4,\kappa_4}[\tau], j \geq k\} \cup \{\mu^+_{j,\delta_4,\kappa_4}[\tau], j \leq k\}$. In order to construct the mappings $H^{\pm,f}_\varepsilon$ and $H^{\pm,s}_\varepsilon$, we consider the systems

$$\tau' = \delta_4^2\left(\tau^3 - \bar{s}^0\tau - \bar{c}^0_k + \mathcal{O}(\delta_4)\right), \quad X' = \mathcal{A}^{k-1}_{\delta_4,\kappa_4}[\tau](X), \tag{4.41}$$

$$\tau' = \delta_4^2\left(\tau^3 - \bar{s}^0\tau - \bar{c}^0_k + \mathcal{O}(\delta_4)\right), \quad X' = \mathcal{A}^{n+1}_{\delta_4,\kappa_4}[\tau](X), \tag{4.42}$$

$$\tau' = \delta_4^2\left(\tau^3 - \bar{s}^0\tau - \bar{c}^0_k + \mathcal{O}(\delta_4)\right), \quad X' = \mathcal{A}^{n-k}_{\delta_4,\kappa_4}[\tau](X) \tag{4.43}$$

with $\kappa_4 \in D_R$. For $\delta_4 = 0$, the spaces $S^f_0 \equiv S^{+,f}_0(\kappa_4)$, $N_0 \equiv N_0(\tau,\kappa_4)$ and $U^f_0 \equiv U^{-,f}_0(\kappa_4)$ are all invariant, spectrally isolated spaces of the matrix

$$A_0 := A_{0,\kappa_4}[\tau] = \operatorname{diag}(Df(0), 0),$$

i.e. fixed points of the corresponding X-equation. The stable space S^f_0 is a repelling hyperbolic equilibrium, the unstable space U^f_0 is an attracting hyperbolic equilibrium, and the center space N_0 is a hyperbolic equilibrium of saddle-type; cf. Lemma 2.2.
Thus, for $\delta_4 = 0$, the curves

$$\begin{aligned} C^{+,f}_0 &= J_\alpha \times S^f_0 \subset J_\alpha \times \mathcal{G}^{2n}_{k-1}(\mathbb{C}), \\ \mathcal{M}_0 &= J_\alpha \times N_0 \subset J_\alpha \times \mathcal{G}^{2n}_{n+1}(\mathbb{C}), \\ C^{-,f}_0 &= J_\alpha \times U^f_0 \subset J_\alpha \times \mathcal{G}^{2n}_{n-k}(\mathbb{C}) \end{aligned}$$

are normally hyperbolic critical manifolds of (4.41), (4.42) and (4.43), respectively.

For small $\delta_4 > 0$, the two curves $C^{+,f}_0$ and $C^{-,f}_0$ perturb smoothly to unique invariant curves of (4.41) and (4.43), respectively. By arguing just as in the proof of Lemma 3.7, one finds the fast bundles $H^{\pm,f}_{\delta_4}$ with the claimed properties.

It remains to construct the slow bundles $H^{\pm,s}_{\delta_4}$. Theorem 2.1 asserts that the critical curve $\mathcal{M}_0 = J_\alpha \times N_0$ perturbs smoothly to a unique invariant curve $\mathcal{M}_{\delta_4,\kappa_4}$ for system (4.42). With respect to the linear coordinates $(p_k, \hat{q}_1, \ldots, \hat{q}_n)$-coordinates, the slow flow of (4.42) restricted to $\mathcal{M}_{\delta_4,\kappa_4}$ is given by (4.39). This system is again singularly perturbed in $\delta_4 = 0$, and we seek splittings of $N_{\delta_4}(\tau^\pm,\kappa_4)$ into fast-slow and super slow parts.

First, notice that, using $\bar{s}^0 = \alpha^2 - \alpha + 1$ and $\tau^- = -1$, $\tau^+ = \alpha$, we have

$$\mu^+_{k,\delta_4,\kappa_4}[\tau^-] = \delta_4^2\left(1 + \mathcal{O}(\delta_4)\right)\left(\frac{2-\alpha^2+\alpha}{2} + \sqrt{\left(\frac{2-\alpha^2+\alpha}{2}\right)^2 + \kappa_4}\right),$$

$$\mu^-_{k,\delta_4,\kappa_4}[\tau^+] = \delta_4^2\left(1 + \mathcal{O}(\delta_4)\right)\left(\frac{2\alpha^2+\alpha-1}{2} - \sqrt{\left(\frac{2\alpha^2+\alpha-1}{2}\right)^2 + \kappa_4}\right).$$

Recalling that $2-\alpha^2+\alpha > 0$ and $2\alpha^2+\alpha-1 < 0$, we find that $\mu^+_{k,\delta_4,\kappa_4}[\tau^-]$ and $\mu^-_{k,\delta_4,\kappa_4}[\tau^+]$ are the fast-slow eigenvalues. On the other hand,

$$\mu^-_{j,\delta_4,\kappa_4}[\tau^\pm] = \mathcal{O}(\delta_4^4\kappa_4),\ j < k, \quad \mu^+_{j,\delta_4,\kappa_4}[\tau^\pm] = \mathcal{O}(\delta_4^4\kappa_4),\ j > k.$$

We thereby define the spaces

$$S_{\delta_4}^{+,sf}(\kappa_4) := E_{k,\delta_4.\kappa_4}^{-}[\tau^+], \quad S_{\delta_4}^{+,ss}(\kappa_4) := \bigoplus_{j>k} E_{j,\delta_4.\kappa_4}^{-}[\tau^+],$$
$$U_{\delta_4}^{-,sf}(\kappa_4) := E_{k,\delta_4.\kappa_4}^{+}[\tau^-], \quad U_{\delta_4}^{-,ss}(\kappa_4) := \bigoplus_{j<k} E_{j,\delta_4.\kappa_4}^{+}[\tau^-],$$

and denote their constant values for $\delta_4 = 0$ by $S_0^{+,sf}$, $S_0^{+,ss}$, $U_0^{-,sf}$ and $U_0^{-,ss}$, respectively. With these definitions,

$$S_{\delta_4}^{+,s} = S_{\delta_4}^{+,sf} \oplus S_{\delta_4}^{+,ss}, \quad U_{\delta_4}^{-,s} = U_{\delta_4}^{-,sf} \oplus U_{\delta_4}^{-,ss}.$$

Moreover,

$$\mathbb{C}^{2n} = S_0^f \oplus U_0^f \oplus N_0^{sf} \oplus S_0^{ss} \oplus U_0^{ss}$$

where $N_0^{sf} = U_0^{-,sf} \oplus S_0^{+,sf}$ and $\dim N_0^{sf} = 2$.

Consider now the systems

$$\dot{\tau} = \left(\tau^3 - \bar{s}^0\tau - \bar{c}_k^0 + \mathcal{O}(\delta_4)\right), \quad \dot{Y} = (\delta_4^{-2})\mathcal{A}_{\delta_4,\kappa_4}^{k}[\tau](Y), \tag{4.44}$$
$$\dot{\tau} = \left(\tau^3 - \bar{s}^0\tau - \bar{c}_k^0 + \mathcal{O}(\delta_4)\right), \quad \dot{Y} = (\delta_4^{-2})\mathcal{A}_{\delta_4,\kappa_4}^{n-k+1}[\tau](Y) \tag{4.45}$$

on $\mathcal{M}_{\delta_4,\kappa_4}$, which are Grassmann versions of (4.39) and, by virtue of Theorem 2.2 (iv), regular in $\delta_4 \geq 0$. We distinguish two cases.

Case 1: $\kappa_4 \in D_{R_1}$ with $R_1 > 0$ sufficiently small.
Define

$$\mathcal{N}_{\delta_4}^{-}(\kappa_4) = \{Y \in \mathcal{G}_k^{2n}(\mathbb{C}) \,:\, U_{\delta_4}^{-,sf}(\kappa_4) \subset Y \subset N_{\delta_4}(\tau^-, \kappa_4)\},$$
$$\mathcal{N}_{\delta_4}^{+}(\kappa_4) = \{Y \in \mathcal{G}_{n-k+1}^{2n}(\mathbb{C}) \,:\, S_{\delta_4}^{+,sf}(\kappa_4) \subset Y \subset N_{\delta_4}(\tau^+, \kappa_4)\}.$$

With these definitions,

$$C_{\delta_4,\kappa_4}^{-,s} = \{\tau^-\} \times \mathcal{N}_{\delta_4}^{-}(\kappa_4), \quad C_{\delta_4,\kappa_4}^{+,s} = \{\tau^+\} \times \mathcal{N}_{\delta_4}^{+}(\kappa_4)$$

are invariant manifolds for (4.44) and (4.45), respectively. For $\kappa_4 = 0$, they are in fact normally hyperbolic invariant manifolds of the respective systems and hence possess stable and unstable manifolds that are invariantly fibered; cf. Theorem 2.1.
For $\tau = \tau^-$ fixed, $\mathcal{N}_{\delta_4}^{-}(0)$ is a non-repelling invariant manifold of the Y-dynamics of (4.44). On the other hand, τ^- is a repelling fixed point of the τ-equation. The persistence of normally hyperbolic invariant manifolds thus implies that the invariant foliation of the unstable manifold $W^u(C_{\delta_4,\kappa_4}^{-,s})$ has one-dimensional fibers. The unique fiber with base point $(\tau^-, U_{\delta_4}^{-,s}(\kappa_4)) \in C_{\delta_4,\kappa_4}^{-,s}$ is itself invariant under the flow of (4.44). This fiber is the trace of a unique solution $(\tau, Y_{\delta_4,\kappa_4}^{-})$ of (4.44) with α-limit $(\tau^-, U_{\delta_4,\kappa_4}^{-,s})$. Setting

$$H_{\delta_4}^{-,s}(\kappa_4) := Y_{\delta_4,\kappa_4}^{-}(0)$$

gives the slow unstable bundle. The claim that $H_{\delta_4}^{-,s}(\kappa_4)$ is smooth in δ_4 and analytic in κ_4 follows from the respective dependence of $W^u(C_{\delta_4,\kappa_4}^{-,s})$ on δ_4 and κ_4.

Since $\mathcal{N}_{\delta_4}^+(0)$ is non-attracting for the Y-dynamics of (4.45) with $\tau = \tau^+$ fixed, we similarly find that the invariant foliation of the stable manifold $W^s(C^{+,s}_{\delta_4,\kappa_4})$ has one-dimensional fibers as well. By taking the invariant fiber with base point $(\tau^+, S^{+,s}_{\delta_4}(\kappa_4)) \in C^{+,s}_{\delta_4,\kappa_4}$, we find a unique solution $(\tau, Y^+_{\delta_4,\kappa_4})$ of (4.45) whose intersection point with $\{\tau = \tau_0\}$,

$$H^{+,s}_{\delta_4}(\kappa_4) := Y^+_{\delta_4,\kappa_4}(0),$$

defines the slow stable bundle.

Case 2: $\kappa \in D_{R_1} \setminus D_{R_0}$ with $0 < R_0 < R_1$.
Consider

$$\hat{\mathcal{N}}^-_{\delta_4}(\kappa_4) := \{Y \in \mathcal{G}^{2n}_k(\mathbb{C}) \,:\, U^{-,sf}_{\delta_4}(\kappa_4) \subset Y \subset \hat{N}_{\delta_4}(\tau^-, \kappa_4)\},$$
$$\hat{\mathcal{N}}^+_{\delta_4}(\kappa_4) := \{Y \in \mathcal{G}^{2n}_{n-k+1}(\mathbb{C}) \,:\, S^{+,sf}_{\delta_4}(\kappa_4) \subset Y \subset \hat{N}_{\delta_4}(\tau^+, \kappa_4)\}$$

with

$$\hat{N}_{\delta_4}(\tau^\pm, \kappa_4) := \left(\bigoplus_{j>k} E^-_{j,\delta_4,\kappa_4}[\tau^\pm]\right) \oplus \left(\bigoplus_{j<k} E^+_{j,\delta_4,\kappa_4}[\tau^\pm]\right).$$

We can now apply the argumentation from Case 1 to the invariant manifolds

$$\hat{C}^{-,s}_{\delta_4,\kappa_4} = \{\tau^-\} \times \hat{\mathcal{N}}^-_{\delta_4}(\kappa_4), \quad \hat{C}^{+,s}_{\delta_4,\kappa_4} = \{\tau^+\} \times \hat{\mathcal{N}}^+_{\delta_4}(\kappa_4)$$

which, for $\delta_4 = 0$, are normally hyperbolic invariant manifolds of (4.44) and (4.45), respectively. Again, we find that $W^u(\hat{C}^{-,s}_{\delta_4,\kappa_4})$ and $W^s(\hat{C}^{+,s}_{\delta_4,\kappa_4})$ are invariantly foliated by one-dimensional fibers. The fibers whose base points are $(\tau^-, U^{-,s}_{\delta_4}(\kappa_4)) \in \hat{C}^{-,s}_{\delta_4,\kappa_4}$ and $(\tau^+, S^{+,s}_{\delta_4}(\kappa_4)) \in \hat{C}^{+,s}_{\delta_4,\kappa_4}$, respectively, lead to the slow bundles $H^{-,s}_{\delta_4}(\kappa_4)$ and $H^{+,s}_{\delta_4}(\kappa_4)$. This completes the construction of $H^{\pm,s}_{\delta_4}$.

Note that the above reasoning is only valid for the systems restricted to $\mathcal{M}_{\delta_4,\kappa_4}$, in the sense of Theorem 2.2 (iv).

Since the layer problem of (4.39) is the eigenvalue problem for ϕ_0, we find further that

$$H^{-,f}_0(\kappa_4) = \hat{H}^-_0(\kappa_4) \oplus U^{ss}_0, \quad H^{+,s}_0(\kappa_4) = \hat{H}^+_0(\kappa_4) \oplus S^{ss}_0$$

where $\hat{H}^\pm_0(\kappa_4)$ is an embedding of $\mathcal{H}^\pm_0(\kappa_4) \subset \mathbb{C}^2$ into the two-dimensional subspace $N^{sf}_0 \subset \mathbb{C}^{2n}$. This then completes the proof. □

4.4 Completion of the proof and remarks

We summarize now the results of the last three sections. The analysis of the dynamics in chart K_1 shows the existence of a uniform constant $R > 0$ and of $r^0_1, \varepsilon^0_1 > 0$ sufficiently small such that there are no USBC in (4.23) for $r_1 \in (0, r^0_1]$ and $\varepsilon_1 \in (0, \varepsilon^0_1]$ if

$$r_1\varepsilon_1 < R.$$

As $\varepsilon = r_1\varepsilon_1$ and $\kappa = r_1^2 e^{i\varphi}$, we find that there are no USBC for the original problem (4.18) if $\varepsilon > 0$ is sufficiently small and

$$\varepsilon^2 R_1 \le |\kappa| \le R_0 \tag{4.46}$$

where $R_0 = (r_1^0)^2$ and $R_1 = (\varepsilon_1^0)^{-2}$. This proves Lemma 4.3.
Stated in terms of ε and κ, Lemma 4.10 asserts that there are no USBC in (4.18) if $\varepsilon \in (0, \varepsilon^0)$ with a sufficiently small $\varepsilon^0 > 0$ and

$$\varepsilon^2 R_2 \le |\kappa| \le \varepsilon^2 R_1$$

where the constants $0 < R_2 < R_1$ can be chosen arbitrarily. This proves Lemma 4.4.
The analysis in K_3 shows that there are no USBC in (4.18) if

$$\varepsilon^4 R_3 \le |\kappa| \le \varepsilon^2 R_2$$

where $R_3 = (r_3^0)^{-2}$ and $R_2 = (\delta_3^0)^2$. This proves Lemma 4.5.
In chart K_4, $\varepsilon = \delta_4$, $|\kappa| = \delta_4^4|\kappa_4|$. Hence, Lemma 4.16 implies Lemma 4.6. Furthermore, we find that the results in the sections 4.3.1 – 4.3.4 cover a range $0 \le |\kappa| \le R_0$ with a certain $R_0 > 0$. If we thus choose $R_0 > 0$ from Lemma 4.2 to be sufficiently small, we receive a full picture of the dynamics of (4.18) for small $\varepsilon > 0$ and all $\kappa \in \mathbb{H}$. In particular, the charts K_1, K_2, K_3 and K_4 overlap.
Theorem 4.1 is an obvious consequence of the Lemmas 4.2 – 4.6, the proof is thus complete.

Remarks

There is an alternative proof of spectral stability, based on the transversal breaking of the trivial USBC at $\kappa_4 = 0$; cf. [14]. The idea is to show, by a Melnikov computation, that the heteroclinic orbit of (4.39) corresponding to the trivial USBC at $(\delta_4, \kappa_4) = (0, 0)$ is transversal. Therefore, there exists a unique USBC for any sufficiently small $\delta_4 > 0$ with a unique $\kappa_4 = \kappa_4(\delta_4)$. This, however, is then the trivial USBC associated with the translation invariance, i.e. $\kappa_4(\delta_4) = 0$. The same argument can also be applied to (3.19) in Chapter 3. That is, one does not necessarily need to study the Evans bundles.

For $\alpha = \frac{1}{2}$, the argumentation in section 4.3.2 and 4.3.4 is no longer possible as τ^+ is then not hyperbolic; see Figure 4.1. In particular, the argumentation in the analysis of the slow systems (4.31) and (4.39) fails. It seems, however, possible to overcome this loss of hyperbolicity by adapting the techniques of Popovic and Szmolyan [42] who, by means of a blow-up, construct an Evans function for a characteristic profile, see also [41]. Note that the Evans function will not be analytic in the origin. The dispersion relation (2.21) that determines the boundary of $\sigma_{ess}(L)$ has a branch point at $\kappa = 0$; see also [25].

Bibliography

[1] J. Alexander, R. Gardner and C. Jones, *A topological invariant arising in the stability analysis of travelling waves*, J. Reine Angew. Math. **410** (1990), 167–212.

[2] H. Baumgärtel, *Analytic perturbation theory for matrices and operators*, Operator Theory: Advances and Applications, vol. 15, Birkhäuser Verlag, Basel, 1985.

[3] C. Chicone, *Ordinary differential equations with applications*, Texts in Applied Mathematics, vol. 34, Springer, New York, 2006.

[4] C. M. Dafermos, *Hyperbolic conservation laws in continuum physics*, Grundlehren der Mathematischen Wissenschaften, vol. 325, Springer-Verlag, Berlin, 2010.

[5] F. Dumortier, *Techniques in the theory of local bifurcations: blow-up, normal forms, nilpotent bifurcations, singular perturbations*, Bifurcations and periodic orbits of vector fields (Montreal, PQ, 1992), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 408, Kluwer Acad. Publ., Dordrecht, 1993, pp. 19–73.

[6] S.-I. Ei, H. Ikeda and T. Kawana, *Dynamics of front solutions in a specific reaction-diffusion system in one dimension*, Japan J. Indust. Appl. Math. **25** (2008), 117–147.

[7] J. W. Evans, *Nerve axon equations: I Linear approximations*, Indiana Univ. Math. J. **21** (1972), 877–885.

[8] J. W. Evans, *Nerve axon equations: II Stability at rest*, Indiana Univ. Math. J. **22** (1972), 75–90.

[9] J. W. Evans, *Nerve axon equations: III Stability of the nerve impulse*, Indiana Univ. Math. J. **22** (1972), 577–593.

[10] J. W. Evans, *Nerve axon equations: IV The stable and the unstable impulse*, Indiana Univ. Math. J. **24** (1975), 1169–1190.

[11] N. Fenichel, *Persistence and smoothness of invariant manifolds for flows*, Indiana Univ. Math. J. **21** (1971), 193–226.

[12] N. Fenichel, *Geometric singular perturbation theory for ordinary differential equations*, J. Differential Equations **31** (1979), 53–98.

[13] H. Freistühler, *Rotational degeneracy of hyperbolic systems of conservation laws*, Arch. Ration. Mech. Anal. **113** (1990), 39–64.

[14] H. Freistühler and P. Szmolyan, *Spectral stability of small shock waves*, Arch. Ration. Mech. Anal. **164** (2002), 287–309.

[15] H. Freistühler and P. Szmolyan, *Spectral stability of small-amplitude viscous shock waves in several space dimensions*, Arch. Ration. Mech. Anal. **195** (2010), 353–373.

[16] C. Fries, *Stability of viscous shock waves associated with non-convex modes*, Arch. Ration. Mech. Anal. **152** (2000), 141–186.

[17] R. Gardner and C. K. R. T. Jones, *Stability of travelling wave solutions of diffusive predator-prey systems*, Trans. Amer. Math. Soc. **327** (1991), 465–524.

[18] R. A. Gardner and C. K. R. T. Jones, *Stability of one-dimensional waves in weak and singular limits*, Viscous profiles and numerical methods for shock waves (Raleigh, NC, 1990), SIAM, Philadelphia, PA, 1991, pp. 32–48.

[19] R. A. Gardner and K. Zumbrun, *The gap lemma and geometric criteria for instability of viscous shock profiles*, Comm. Pure Appl. Math. **51** (1998), 797–855.

[20] J. Goodman, *Nonlinear asymptotic stability of viscous shock profiles for conservation laws*, Arch. Ration. Mech. Anal. **95** (1986), 325–344.

[21] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1994.

[22] J. Guckenheimer and P. Holmes, *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*, Applied Mathematical Sciences, vol. 42, Springer-Verlag, New York, 1990.

[23] D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics, vol. 840, Springer-Verlag, Berlin, 1981.

[24] M. W. Hirsch, C. C. Pugh and M. Shub, *Invariant manifolds*, Lecture Notes in Mathematics, vol. 583, Springer-Verlag, Berlin, 1977.

[25] P. Howard and K. Zumbrun, *The Evans function and stability criteria for degenerate viscous shock waves*, Discrete Contin. Dyn. Syst. **10** (2004), 837–855.

[26] C. K. R. T. Jones, *Stability of the travelling wave solution of the FitzHugh-Nagumo system*, Trans. Amer. Math. Soc. **286** (1984), 431–469.

[27] C. K. R. T. Jones, *Geometric singular perturbation theory*, Dynamical systems (Montecatini Terme, 1994), Lecture Notes in Mathematics, vol. 1609, Springer-Verlag, Berlin, 1995, pp. 44–118.

[28] C. K. R. T. Jones, R. Gardner and T. Kapitula, *Stability of travelling waves for nonconvex scalar viscous conservation laws*, Comm. Pure Appl. Math. **46** (1993), 505–526.

[29] T. Kapitula and B. Sandstede, *Stability of bright solitary-wave solutions to perturbed nonlinear Schrödinger equations*, Phys. D **124** (1998), 58–103.

[30] T. Kato, *Perturbation theory for linear operators*, Grundlehren der Mathematischen Wissenschaften, Band 132, Springer-Verlag, Berlin, 1976.

[31] S. Kawashima and A. Matsumura, *Asymptotic stability of traveling wave solutions of systems for one-dimensional gas motion*, Comm. Math. Phys. **101** (1985), 97–127.

[32] H. Kokubu, *Homoclinic and heteroclinic bifurcations of vector fields*, Japan J. Appl. Math. **5** (1988), 455–501.

[33] M. Krupa, B. Sandstede and P. Szmolyan, *Fast and slow waves in the FitzHugh-Nagumo equation*, J. Differential Equations **133** (1997), 49–97.

[34] A. Majda and R. L. Pego, *Stable viscosity matrices for systems of conservation laws*, J. Differential Equations **56** (1985), 229–262.

[35] R. Mañé, *Persistent manifolds are normally hyperbolic*, Trans. Amer. Math. Soc. **246** (1978), 261–283.

[36] A. Matsumura and K. Nishihara, *Asymptotic stability of traveling waves for scalar viscous conservation laws with non-convex nonlinearity*, Comm. Math. Phys. **165** (1994), 83–96.

[37] H. P. McKean, *Nagumo's equation*, Advances in Math. **4** (1970), 209–223.

[38] J. D. Murray, *Mathematical biology. II*, Interdisciplinary Applied Mathematics, vol. 18, Springer-Verlag, New York, 2003.

[39] S. Nii, *An extension of the stability index for travelling-wave solutions and its application to bifurcations*, SIAM J. Math. Anal. **28** (1997), 402–433.

[40] R. Plaza and K. Zumbrun, *An Evans function approach to spectral stability of small-amplitude shock profiles*, Discrete Contin. Dyn. Syst. **10** (2004), 885–924.

[41] N. Popovic, *A geometric analysis of logarithmic switchback phenomena*, HAMSA 2004: Proceedings of the International Workshop on Hysteresis and Multi-Scale Asymptotics, Cork 2004, J. Phys. Conference Series, 2005, pp. 164–173.

[42] N. Popovic and P. Szmolyan, *Evans functions and blow-up for degenerate shock waves*, in preparation.

[43] G. Raugel and K. Kirchgässner, *Stability of fronts for a KPP-system II. The critical*

case, J. Differential Equations **146** (1998), 399–456.

[44] K. Sakamoto, *Invariant manifolds in singular perturbation problems for ordinary differential equations*, Proc. Roy. Soc. Edinburgh Sect. A **116** (1990), 45–78.

[45] B. Sandstede, *Stability of travelling waves*, Handbook of dynamical systems II, North-Holland, Amsterdam, 2002, pp. 983–1055.

[46] B. Sandstede and A. Scheel, *Gluing unstable fronts and backs together can produce stable pulses*, Nonlinearity **13** (2000), 1465–1482.

[47] D. H. Sattinger, *On the stability of waves of nonlinear parabolic systems*, Advances in Math. **22** (1976), 312–355.

[48] A. Szepessy and Z. P. Xin, *Nonlinear stability of viscous shock waves*, Arch. Rational Mech. Anal. **122** (1993), 53–103.

[49] P. Szmolyan, *Transversal heteroclinic and homoclinic orbits in singular perturbation problems*, J. Differential Equations **92** (1991), 252–281.

[50] J. Wächtler, *Die Puls-Lösungen der FitzHugh-Nagumo-Gleichungen*, Diploma thesis, Universität Leipzig, 2009.

[51] H. Whitney, *On singularities of mappings of euclidean spaces. I. Mappings of the plane into the plane*, Ann. of Math. (2) **62** (1955), 374–410.

[52] S. Wiggins, *Normally hyperbolic invariant manifolds in dynamical systems*, Applied Mathematical Sciences, vol. 105, Springer-Verlag, New York, 1994.

[53] K. Zumbrun, *Stability and dynamics of viscous shock waves*, Nonlinear conservation laws and applications, IMA Vol. Math. Appl., vol. 153, Springer-Verlag, New York, 2011, pp. 123–167.

[54] K. Zumbrun and P. Howard, *Pointwise semigroup methods and stability of viscous shock waves*, Indiana Univ. Math. J. **47** (1998), 741–871.

All figures were produced by the author.

Zusammenfassung in deutscher Sprache

In dieser Arbeit wird die spektrale Stabilität von Traveling Waves mit kleiner Amplitude in zwei unterschiedlichen Systemen untersucht: in einem Reaktions-Diffusions-System, in dessen Nichtlinearität eine Pitchfork-Bifurkation auftritt, und in einem strikt hyperbolischen System viskoser Erhaltungsgleichungen, in dessen k-tem charakteristischen Feld die Bedingung echter Nichtlinearität nicht erfüllt ist.
In beiden untersuchten Fällen existiert eine Schar ϕ_ε, $\varepsilon \in (0, \varepsilon_0]$, $\varepsilon_0 \ll 1$, von Traveling Waves kleiner Amplitude. Das Eigenwertproblem bezüglich der Linearisierung in ϕ_ε ist ein System gewöhnlicher Differentialgleichungen, das von der Amplitude ε und dem spektralen Parameter κ abhängt und dessen Dynamik mehrere Zeitskalen besitzt. Diese schnell-langsame Struktur wird genutzt, um mittels geometrischer Methoden der singulären Störungstheorie die Dynamik im Limes verschwindender Amplitude genau zu beschreiben. Es wird gezeigt, dass das Eigenwertproblem für $\varepsilon \to 0$ gegen das bereits gut untersuchte Eigenwertproblem einer festen Traveling Wave ϕ_0 in einer skalaren Gleichung konvergiert. Die spektrale Stabilität der Profile ϕ_ε folgt damit aus der spektralen Stabilität des Grenzprofils ϕ_0.
Argumentation und Beweise in dieser Arbeit basieren auf Konzepten der Theorie dynamischer Systeme, insbesondere auf Konzepten der Theorie invarianter Mannigfaltigkeiten und der Theorie singulär gestörter Systeme.