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SYMBOL LENGTH AND STABILITY INDEX

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ABSTRACT. We show that a pythagorean field (more generally, a reduced abstract Witt ring) has finite stability index if and only if it has finite 2-symbol length. We give explicit bounds for the two invariants in terms of one another. To approach the question whether those bounds are optimal we consider examples of pythagorean fields.

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1. INTRODUCTION

The aim of this article is to study the relation between two field invariants appearing in the theory of quadratic forms over fields, with a special focus on real pythagorean fields. We shall first recall some facts from the theory of quadratic forms over fields, Milnor K-Theory, Galois cohomology, and real algebra, refering to [8], [11], and [12] for details, and shall then formulate our main results in the context of fields. From Section 2 on, we will mainly work in the abstract theory of quadratic forms, where the field is replaced by an abstract Witt ring, and prove the results in this more general setting.

Let F always denote a field of characteristic different from 2. Let WF denote the Witt ring of quadratic forms over F and IF its fundamental ideal. For $n \in \mathbb{N}$ let $I^nF = (IF)^n$, $\overline{I}^nF = I^nF/I^{n+1}F$, $H^n(F) = H^n(\Gamma_F, \mu_2)$, the nth cohomology group for the trivial action of the absolute Galois group Γ_F of F on $\mu_2 = \{+1, -1\}$, and k_nF the nth group of Milnor K-theory modulo 2 of F defined in [14]. The group $\overline{I}^nF$ is generated by the classes of n-fold Pfister forms $\langle\langle a_1, \dots, a_n\rangle\rangle = \langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$, and similarly k_nF is generated by 'symbols' $\{a_1, \dots, a_n\}$, while $H^n(F)$ contains 'cup products' $(a_1) \cup \cdots \cup (a_n)$, where $a_1, \dots, a_n \in F^\times$. Milnor [14] asked whether for any $n \in \mathbb{N}$ there are natural isomorphisms between the groups $\overline{I}^nF$, H^nF and k_nF making those elements correspond with one another for fixed $a_1, \dots, a_n \in F^\times$. We have $k_0F = \overline{I}^0F = H^0F = \mathbb{Z}/2\mathbb{Z}$, by convention, and $k_1F \cong \overline{I}^1F \cong H^1F \cong F^\times/F^{\times 2}$, via $\{a\} \mapsto \langle\langle a\rangle\rangle + I^2F \mapsto (a) \mapsto aF^{\times 2}$. Moreover, H^2F can be identified with $\mathrm{Br}_2(F)$, the 2-torsion part of the Brauer group of F, by interpreting $(a_1) \cup (a_2)$ as the class of the quaternion algebra $(a_1, a_2)_F$. For any $n \in \mathbb{N}$, Milnor [14] defined a natural homomorphism $s_n : k_nF \longrightarrow \overline{I}^nF$ with $s_n(\{a_1, \dots, a_n\}) = \langle\langle a_1, \dots, a_n\rangle\rangle + I^{n+1}F$,

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which is trivially surjective, and he showed that $s_2 : k_2F \longrightarrow \overline{I}^2F$ is an isomorphism. It was proven in [15] that Milnor's aforementioned question has a positive answer in general and, in particular, that s_n is an isomorphism for any $n \in \mathbb{N}$.

As in [1] we denote by $\lambda_n(F)$ the supremum in $\mathbb{N}\cup\{\infty\}$ over the numbers r such that there exists an element of k_nF that can not be expressed as a sum of less than r symbols, and we call $\lambda_n(F)$ the *n-symbol length of F*. While $\lambda_0(F) = \lambda_1(F) = 1$ independently of F, the 2-symbol length $\lambda_2(F)$ is of particular interest and was studied (with slightly different notation) in [9] and in [10] relative to quadratic forms and the u-invariant of F.

Let $\sum F^2$ denote the subgroup of $F^\times$ consisting of the nonzero sums of squares in F. We say that F is *pythagorean* if $\sum F^2 = F^{\times 2}$. We say that F is *real* if $-1 \notin \sum F^2$, and *nonreal* if $-1 \in \sum F^2$. A *preordering of F* is a subset $T \subseteq F$ that contains all squares in F and is closed under addition and multiplication and such that $-1 \notin T$; if in addition $T \cup -T = F$, then T is called an *ordering of F*. For a preordering T of F we write $T^\times = T \setminus \{0\}$, which is a subgroup of $F^\times$. We denote by X_F the set of all orderings of F and by X_T the set of all orderings containing the preordering T. For any preordering T of F we have $T = \bigcap X_T$. Any $P \in X_F$ yields a map $\mathrm{sign}_P : WF \longrightarrow \mathbb{Z}$ called the *signature at P*. If F is real, then $P \longmapsto \ker(\mathrm{sign}_P)$ gives a one-to-one correspondence between X_F and the set of non-maximal prime ideals of WF.

We recall the definition of fans introduced in [3]. Let T be a preordering of F and $n \in \mathbb{N}$ such that $[F^\times : T^\times] = 2^{n+1}$. Then by [3] we have $n \leq |X_T| \leq 2^n$, and equality $|X_T| = 2^n$ holds if and only if the image of the homomorphism $\mathrm{sign}_T : WF \longrightarrow \mathbb{Z}^{X_T}, \varphi \longmapsto (\mathrm{sign}_P(\varphi))_{P\in X_T}$ is isomorphic to $\mathbb{Z}[(\mathbb{Z}/2\mathbb{Z})^n]$; in this case T is called a *fan of degree n*. A fan of degree 0 is the same as an ordering. A fan of degree 1 is the same as the intersection of two different orderings.

The (*reduced*) *stability index* of a field was introduced in [5]. In [6] this field invariant was characterized in terms of fans. By this characterization the *stability index of F* is given as

$$st(F) \;=\; \sup\{\deg(T) \mid T \text{ fan of } F\} \;\in\; \mathbb{N}\cup\{\infty\},$$

with the convention that $\sup\emptyset = 0$. Hence, $st(F) = 0$ if and only if $|X_F| \leq 1$.

Our aim is to relate the stability index to the symbol lengths, in particular to the 2-symbol length. This will be done in the more general context of abstract Witt rings, introduced in Section 2. For fields (2.3) reads as follows:

> *If $\lambda_i(F) < \infty$, for some $i \geq 2$, then $st(F) < \infty$. In particular, for $i = 2$ we have $st(F) \leq 2\lambda_2(F) - 1$.*

In Section 3 we focus on reduced Witt rings. For fields (3.7) reads as follows:

> *If F is pythagorean, then $\lambda_2(F) < \infty$ if and only if $st(F) < \infty$.*

In order to prove the right-to-left implication, we actually show in (3.6):

> *Let F be pythagorean and $s = st(F)$. If $1 \leq s \leq 2$, then $\lambda_2(F) = s$. If $3 \leq s \leq \infty$, then $[\frac{s}{2}] + 1 \;\leq\; \lambda_2(F) \;\leq\; 2^{s-1}(2^{s-2} - 1)$.*

Here and the sequel, we use the notation $[x] = \max\{z \in \mathbb{Z} \mid z \leq x\}$ for $x \in \mathbb{R}$. We do not know whether the upper bound on λ_2 in the pythagorean case is optimal. To approach this question, we construct in (3.8) for any $r \in \mathbb{N}$ a pythagorean field F with $\lambda_2(F) = st(F) = r$. As for the lower bound, for any $s \in \mathbb{N}$ we have $st(F) = s$ and $\lambda_2(F) = [\frac{s}{2}] + 1$ for the field $F = \mathbb{R}((t_1)) \dots ((t_s))$.

2. Abstract Witt rings

We recall the notion of (abstract) Witt rings from [13]. A *Witt ring* is a triple (W, G, I) where W is a commutative ring, I is the unique ideal of index 2 of W, called the *fundamental ideal*, and $G \subseteq W^\times$ is a group that additively generates W and such that $G \longrightarrow I/I^2, a \longmapsto (1-a) + I^2$ is a group isomorphism.

Let $n \in \mathbb{N}$. The nth power of the fundamental ideal I^n is additively generated by the products $(1+a_1)\cdots(1+a_n)$ where $a_1, \dots, a_n \in G$. We set $\overline{I}^n = I^n/I^{n+1}$.

For $\varphi \in W$ the least number of summands needed to write φ as a sum of elements of G is called the *anisotropic dimension of* φ and denoted by $\operatorname{diman}(\varphi)$. For $\alpha \in \overline{I}^n$ let $l(\alpha)$ denote the least number of summands needed to write α as a sum of classes of elements of the shape $(1+a_1)\cdots(1+a_n)$ with $a_1, \dots, a_n \in G$.

2.1. **Lemma.** *Let $\alpha \in \overline{I}^2$ and $m \geq 1$. Then $l(\alpha) \leq m$ if and only if $\alpha = \varphi + I^3$ for some $\varphi \in I^2$ with* $\operatorname{diman}(\varphi) \leq 2m + 2$.

Proof: The proof is easy, and basically the same as in [1, (3.2)] □

We define the *nth symbol length of* W as

$$\lambda_n(W) \;=\; \sup\{l(\alpha) \mid \alpha \in \overline{I}^n\} \;\in\; \mathbb{N} \cup \{\infty\}\,.$$

It is easy to see that $\lambda_0(W) = \lambda_1(W) = 1$. For the Witt ring $W = WF$ of the field F we then have $\lambda_n(F) = \lambda_n(W)$ in view of the isomorphism $s_n : k_nF \longrightarrow \overline{I}^n F$.

For $i \in \mathbb{N}$ we define

$$\Lambda_i : \mathbb{N} \longrightarrow \mathbb{N},\; n \longmapsto \lambda_i(\mathbb{Z}\,[(\mathbb{Z}/2\mathbb{Z})^n])\,.$$

Note that $\mathbb{Z}\,[(\mathbb{Z}/2\mathbb{Z})^n]$ is the Witt ring of the pythagorean field $\mathbb{R}((t_1)) \dots ((t_n))$. So Λ_i yields the values of the ith symbol length for a particular sequence of fields. By [1] we have $\Lambda_2(n) = [\frac{n}{2}] + 1$, but no formula is known for Λ_i when $i > 2$.

Let X_W be the set of non-maximal prime ideals in W. By [13, Corollary 4.18], elements of X_W are in one-to-one correspondence with ring homomorphisms $W \longrightarrow \mathbb{Z}$, called *signatures of* W. The signature corresponding to $P \in X_W$ is denoted by sign_P. We say that W is *real* if $X_W \neq \emptyset$, and *nonreal* otherwise.

For $d \in \mathbb{N}$, a subset $\mathcal{F} \subseteq X_W$ is called a *fan of degree d on* W if $|\mathcal{F}| = 2^d$ and $W/\bigcap \mathcal{F} \cong \mathbb{Z}\,[(\mathbb{Z}/2\mathbb{Z})^d]$. The *stability index of* W is then defined as

$$st(W) \;=\; \sup\{n \in \mathbb{N} \mid \text{there exists a fan of degree } n \text{ on } W\} \;\in\; \mathbb{N} \cup \{\infty\}\,.$$

Given the Witt ring $W = WF$ of a field F, associating to a preordering T of F the set of prime ideals $\{\ker(\operatorname{sign}_P) \mid P \in X_T\}$ gives a degree preserving one-to-one correspondence between the two concepts of fans, so that $st(F) = st(W)$.

2.2. **Theorem.** *For $n \leq st(W)$ we have $\lambda_i(W) \geq \Lambda_i(n)$ for any $i \in \mathbb{N}$.*

Proof: As $\mathbb{Z}\left[(\mathbb{Z}/2\mathbb{Z})^n\right]$ is a quotient of W, this is obvious. □

2.3. **Corollary.** *We have $st(W) \leq 2\lambda_2(W) - 1$. Moreover, if $\lambda_i(W) < \infty$ for some $i \geq 2$, then $st(W) < \infty$.*

Proof: For any $n \leq st(W)$ one has $\lambda_2(W) \geq \Lambda_2(n) = [\frac{n}{2}] + 1 \geq \frac{n+1}{2}$, which shows the first statement. For fixed i, one has $\Lambda_i(n) \longrightarrow \infty$ for $n \longrightarrow \infty$, and the second statement thus follows using (2.2). □

3. Reduced Witt rings

A commutative ring is *reduced* if it contains no nonzero nilpotent elements. By [13, Corollary 4.22], if the Witt ring W is reduced, then $G = W^\times$.

3.1. **Question.** *Assume that W is reduced. If $st(W) < \infty$, does it follow that $\lambda_i(W) < \infty$ for every $i \in \mathbb{N}$?*

We are going to give a positive answer to this question for $i = 2$.

3.2. **Lemma.** *Let $r \geq st(W)$. For every $\varphi \in W$, there exists $\varphi' \in W$ such that $\varphi \equiv \varphi' \bmod I^r$ and $0 \leq \operatorname{sign}_P(\varphi') < 2^r$ for all $P \in X_W$; if W is reduced, then φ' is uniquely determined by φ.*

Proof: The proof is essentially given in [2, (2.2)]. □

Given $\varphi \in W$ we put

$$\Delta(\varphi) = \max\{|\operatorname{sign}_P(\varphi)| \mid P \in X_W\}$$

and call this number the *amplitude of* φ. In the reduced case, the anisotropic dimension is bounded in terms of the amplitude and the stability index.

3.3. **Theorem** (Bonnard). *Assume that W is reduced of stability index $s \geq 1$. Then $\operatorname{diman}(\varphi) \leq 2^{s-1}\Delta(\varphi)$ for any $\varphi \in W$.*

Proof: See [4, Proposition 4] or [16, Theorem 1]. □

3.4. **Theorem.** *Assume that W is reduced with $st(W) \geq 2$. Let $s = st(W)$ and $r = \max\{s, 3\}$. Any element of $\overline{I}^2$ is of the shape $(\psi + 2) + I^3$ with some $\psi \in I$ of discriminant -1 and with $\operatorname{diman}(\psi) \leq 2^s(2^{r-2} - 1)$.*

Proof: Let $\alpha \in \overline{I}^2$. By (3.2), there exists $\varphi \in I^2$ such that $\alpha = \varphi + I^3$ and $0 \leq \operatorname{sign}_P(\varphi) \leq 2^r - 4$ for all $P \in X_W$. Put $\psi = \varphi - 2$ if $s \leq 3$ and $\psi = 2(2^{r-2} - 1) - \varphi$ if $s > 3$. Then $\alpha = \psi + 2 + I^3$ and $\Delta(\psi) \leq 2(2^{r-2} - 1)$. Hence, $\operatorname{diman}(\psi) \leq 2^s(2^{r-2} - 1)$ by (3.3). □

The Witt ring W is said to be *linked* if $\lambda_2(W) \leq 1$; in this case $\lambda_n(W) \leq 1$ for all $n \geq 1$. If W is real, then $\lambda_n(W) \geq 1$ for any $n \in \mathbb{N}$, so that W is linked if and only if $\lambda_n(W) = 1$ for all $n \geq 1$.

3.5. **Proposition.** *Assume that W is reduced. Then W is linked if and only if $I^2 = 2I$, if and only if $st(W) \leq 1$.*

Proof: If $st(W) \geq 2$, then we have $\lambda_2(W) \geq \Lambda_2(2) = 2$ by (2.2). Assume now that $st(W) \leq 1$. Then to any $a, b \in W^\times$ there exists $c \in W^\times$ such that $(1+a)\cdot(1+b) - 2(1+c) \in \bigcap_{\mathfrak{p}\in X_W} \mathfrak{p} = 0$. This shows that $I^2 = 2I$, which in turn implies that W is linked. □

3.6. **Theorem.** *Assume that W is reduced with $s = st(W) < \infty$.*

(*a*) *If $s \leq 1$ then $\lambda_2(W) = 1$.*
(*b*) *If $s = 2$ then $\lambda_2(W) = 2$.*
(*c*) *If $s \geq 3$, then $[\frac{s}{2}] + 1 \leq \lambda_2(W) \leq 2^{s-1}(2^{s-2} - 1)$.*

Proof: Part (*a*) follows from (3.5). If $s \geq 2$, then (2.2) and (3.4) yield that $[\frac{s}{2}] + 1 \leq \lambda_2(W) \leq 2^{s-1}(2^{r-2} - 1)$ with $r = \max\{s, 3\}$. This shows (*b*) and (*c*). □

3.7. **Corollary.** *If W is reduced, then $s(W) < \infty$ if and only if $\lambda_2(W) < \infty$.*

Proof: This is clear from (2.3) and (3.6). □

3.8. **Example.** Let r be a positive integer and let K be a pythagorean SAP-field having exactly r different orderings. For example, such a field K is obtained as the intersection of any r different real closures of $\mathbb{Q}$. It follows from the assumption on K that $|K^\times/K^{\times 2}| = 2^r$. Let P be an ordering of K. There exist $a_1, \dots, a_{r-1} \in P^\times$ such that the square classes $a_1K^{\times 2}, \dots, a_{r-1}K^{\times 2}$ form an $\mathbb{F}_2$-basis of $P^\times/K^{\times 2}$. Let $F = K(\!(t_1)\!) \dots (\!(t_{r-1})\!)$. Then F is pythagorean, $st(F) = r$, and $|F^\times/F^{\times 2}| = 2^{2r-1}$. By [1, (1.1)] the latter implies that $\lambda_2(F) \leq r$. As the ordering P extends to $K(\sqrt{a_1}, \dots, \sqrt{a_{r-1}})$, the quaternion algebra $(-1,-1)_{K(\sqrt{a_1},\dots,\sqrt{a_{r-1}})}$ is not split. Using the results in [17, Sect. 2], it follows that the product of quaternion algebras

$$(-1,-1)_F \otimes_F (a_1, t_1)_F \otimes_F \dots \otimes_F (a_{r-1}, t_{r-1})_F$$

is a division algebra and thus not Brauer equivalent to a product of less than r quaternion algebras, whence $\lambda_2(F) \geq r$. Therefore $\lambda_2(F) = r = st(F)$. Hence, $W = WF$ is a reduced Witt ring with $\lambda_2(W) = r = st(W)$.

If W is reduced with $st(W) = 3$, then we have $2 \leq \lambda_2(W) \leq 4$ by (3.4) and (3.5). The Witt ring of $\mathbb{R}(\!(t_1)\!)(\!(t_2)\!)(\!(t_3)\!)$ and the one obtained for $r = 4$ in (3.8) show that the values 2 and 3 are both possible for $\lambda_2(W)$ in this situation, but this is open for the value 4. More generally, we are left with the following question.

3.9. **Question.** *Is there a reduced Witt ring W with $\lambda_2(W) > st(W)$?*

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