

Nonclassical time correlation functions in continuous quantum measurement

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A continuous projective measurement of a quantum system often leads to a suppression of the dynamics, known as the Zeno effect. Alternatively generalized nonprojective, so called “weak” measurements can be performed. Such a measurement is parametrized by its strength parameter that can interpolate continuously between ideal strong measurement with no dynamics – strict Zeno effect, and weak measurement characterized by almost free dynamics but blurry observations. Here we analyze the stochastic properties of this uncertainty component in the resulting observation trajectory. The observation uncertainty results from intrinsic quantum uncertainty, the effect of measurement on the system (backaction) and detector noise. It is convenient to separate the latter, system independent contribution from the system dependent uncertainty, and this paper shows how to accomplish this separation. The system dependent uncertainty is found in terms of a quasiprobability, which, despite its weaker properties is shown to satisfy a weak positivity condition. We discuss the basic properties of this quasiprobability with special emphasis on its time correlation functions as well as their relationship to the full correlation functions along the observation trajectory, and illustrate our general results with simple examples. We demonstrate a violation of classical macrorealism using the fourth order time correlation functions with respect to the quasiprobability in the two-level system.

I. INTRODUCTION

The continuous projective von Neumann quantum measurement [1] leads to a suppression of the dynamics, known as the quantum Zeno effect (QZE) [2]. To escape this problem, modern quantum measurement theory offers generalizations of the projective measurement to so-called positive operator-valued measures (POVM) [3, 4], where a projection is replaced by a softer operation such as a Kraus operator [5]. Such operators can describe not only projective measurements but also weak measurement, in which case the action of the POVM leaves the state almost unchanged. By virtue of the Naimark theorem [6] POVMs are equivalent to projective measurements in an extended Hilbert space that includes additional detector degrees of freedom. The effect of a continuous application of Kraus operators, which correspond to a time-continuous measurement can be described by stochastic evolution equations such as Lindblad-type equations [7] for the system density matrix or Langevin equations for individual system trajectories, physically describing irreversible effects like decoherence and decay affected by the measurement process.

Weak measurements[8] make it possible to escape the QZE by paying a price in terms of an imperfect detection. In the extreme case the dynamics of the system is (almost) free but the measurement outcome is obscured by a large detection noise. This is similar to the problem of a quantum linear amplifier, which can amplify

both complementary noncommuting observables, like \hat{x} and \hat{p} , but only if accompanied by a large noise[9]. The interpretation of weak measurements of correlation functions is sometimes paradoxical: one must either accept unusually large values of the physical quantity[8] or replace the probability by a quasiprobability [10]. Weak measurements are also very useful in quantum feedback protocols [11].

The QZE lies at the strong limit of a spectrum of measurements whose strong/weak character can be changed continuously [12–18], using e.g. a Gaussian POVM [19, 20]. The Gaussian POVM is also the key element of the continuous collapse interpretation of quantum mechanics [21]. Here, for the first time we use Gaussian POVM for continuous measurement to describe time correlation functions. Such functions are known for special cases in the weak measurement limit [22] (but not in general) and are necessary to explain the recent experiment that shows nonclassical behavior of time correlation functions in a two-level system [23]. The calculations are facilitated by making a deconvolution of the outcome time trace probability into the probability component associated with the white detection noise and a quasiprobability that describes the intrinsic system uncertainty, and discuss their general properties. Such a deconvolution has the advantage that we can make use of basic properties of the quasiprobability, e.g., the weak positivity [24], which states that the second order correlation function matrix is positive definite. Our scheme provides a unified and concise treatment of weak measurements and the QZE, pointing out the general trade-off between measurement and decoherence. By comparing the average signal to the associated noise we also establish limits

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on the uncertainty of the outcome and its dependence on measurement characteristics.

The time correlation functions obtained by our approach provide a convenient route for the analysis of uncertainty properties of systems undergoing weak measurements. Taking a two-level system as an example, a single nondemolishing measurement of an observable not commuting with the Hamiltonian is not possible in both time and frequency domain although the latter gives a better signal-to-noise ratio. Although this is intuitively clear, using our approach, it is possible to establish and compare bounds on the outcome uncertainty. For another simple example, the continuous position measurement of a harmonic oscillator, we show that the same measurement procedure does not lead to the QZE. Instead, the continuous measurement leads to unbounded growth in noise, in agreement with the general observation [20] and in analogy to anti-Zeno effect[25].

The proposed separation has another important consequence. If we assume classical macrorealism in quantum mechanics then the statistics of the outcomes with the detection noise subtracted in the limit of noninvasive measurement should correspond to a positive definite probability. In contrast, we show that macrorealism assumption is violated by demonstrating that our quasiprobability is somewhere negative. Such violation has been recently demonstrated experimentally [23]. In fact, if we additionally assume dichotomy or boundedness of the quantum outcomes, the violation can occur already on the level of second order correlations as shown by Leggett, Garg and others [22, 26]. However, as follows from weak positivity, without these additional assumptions, second order correlations are not sufficient to violate macrorealism. Instead, one needs at least fourth order averages to see this violation. In this paper we demonstrate, that a special fourth order correlation function in the two-level system, reminiscent of the Leggett-Garg proposal[26], can reveal the negativity of the quasiprobability in this case and consequently can be used to violate macrorealism, without having to make any additional assumptions.

The paper is organized as follows. We first define the continuous Gaussian POVM and obtain the probability distribution for the continuously measured observable. We then make the deconvolution of this probability into detection noise and a quasiprobability and introduce a formalism for evaluating time correlation functions. With that we are able to prove the weak positivity. We show that the time evolution associated with the quasiprobability can be formulated either as a quantum Langevin equation driven by a white Gaussian noise, or a Lindblad-type master equation for the non-selective system density matrix. We also show how the required time correlation functions can be calculated from these stochastic equations. Next, we demonstrate the general trade-off between dynamics and measurement, taking a two-level system and a harmonic oscillator as examples, and discuss the behavior of the average signal and the

noise in these prototype system. Then we construct the Leggett-Garg inequality without assuming dichotomy or boundedness of the measurement variable. Finally, for completeness, we show how the Zeno effect emerges within our formalism. Several instructive proofs of formulas are presented in Appendices.

II. QUASIPROBABILITY AND WEAK POSITIVITY

We begin by introducing a general scheme of continuous measurement and describe its properties. For a given system characterized by a Hamiltonian \hat{H} and an initial system state given by a density matrix $\hat{\rho}$, we consider the measurement of one, generally time-dependent observable \hat{A} . A description amenable to continuous interpolation between hard and soft measurement can be formulated in terms of Kraus operators [5, 8]. We assume a Gaussian form of the Kraus operators, whereupon the state of the system following a single instantaneous measurement is given by

$$\hat{\rho}_1(a) = \hat{K}(a)\hat{\rho}\hat{K}(a), \quad (1)$$

$$\hat{K}(a) = (2\bar{\lambda}/\pi)^{1/4}e^{-\bar{\lambda}(a-\hat{A})^2}. \quad (2)$$

Note that in (1) the non-negative definite operators $\hat{\rho}$ and $\hat{\rho}_1$ represent the states of the system just before and just after the measurement. The probability that the measurement of \hat{A} gives the outcome a is given by [3]

$$P(a) = \text{Tr}\hat{\rho}_1(a), \quad (3)$$

which is normalized, $\int da P(a) = 1$. The Kraus operator (2) depends on the parameter $\bar{\lambda}$, which characterizes the weakness of the measurement. For $\bar{\lambda} \rightarrow \infty$, we recover a strong, projective measurement with an exact result but a complete destruction of coherence, while $\bar{\lambda} \rightarrow 0$ corresponds to a weak measurement with almost no influence on the state of the system, $\hat{\rho}_1(a) \sim \hat{\rho}$, but very large measurement uncertainty of the order of $\sim 1/\bar{\lambda}$. The probability distribution (3) is consistent with projective measurement scheme, namely $\langle a \rangle = \text{Tr}\hat{A}\hat{\rho}$.

Repeated measurements of this type can be described by applying such Kraus operators sequentially, separated by time steps Δt . In what follows we make the reasonable assumption that for a given measuring device (“meter”) the weakness parameter $\bar{\lambda}$ is inversely proportional to the measurements frequency, i.e.,

$$\bar{\lambda} = \lambda\Delta t \quad (4)$$

with constant λ . In the continuum limit, $\bar{\lambda}, \Delta t \rightarrow 0$, we obtain (Appendix A) the Kraus operator as a functional of $a(t)$

$$\hat{K}_h[a(t)] \equiv e^{(i/\hbar)\hat{H}t}\hat{K}[a(t)] = C\mathcal{T}e^{-\int \lambda(a(t)-\hat{A}(t))^2 dt} \quad (5)$$

where $a(t)$ is the measurement outcome, $\hat{A}(t)$ is the operator \hat{A} in the Heisenberg representation with respect to the Hamiltonian \hat{H} , $\hat{A}(t) = \exp(i\hat{H}t/\hbar)\hat{A}\exp(-i\hat{H}t/\hbar)$, \mathcal{T} denotes time ordering (later times on the left) and C is a normalization factor. Note that $\hat{K}_h[a]$ is the Heisenberg representation of $\hat{K}[a]$. The analog of (3) is the functional probability

$$P[a] = \text{Tr} \hat{K}^\dagger[a] \hat{K}[a] \hat{\rho} \quad (6)$$

which satisfies the normalization $\int Da P[a] = 1$. Whenever some functional measure D is introduced here, we tacitly include all proper normalization factors in it.

It is convenient to write (5) as a Fourier transform

$$\hat{K}_h[a] = \int D\phi \mathcal{T} e^{\int dt [i\phi(t)(\hat{A}(t) - a(t)) - \phi^2(t)/4\lambda]} \quad (7)$$

so that

$$\begin{aligned} P[a] &= \text{Tr} \left(\hat{K}[a] \hat{\rho} \hat{K}[a] \right) \\ &= \text{Tr} \left(\int D\phi_+ \mathcal{T} e^{\int dt [i\phi_+(t)(\hat{A}(t) - a(t)) - \phi_+^2(t)/4\lambda]} \hat{\rho} \times \right. \\ &\quad \left. \int D\phi_- \tilde{\mathcal{T}} e^{\int dt [i\phi_-(t)(\hat{A}(t) - a(t)) - \phi_-^2(t)/4\lambda]} \right) \quad (8) \end{aligned}$$

where $\tilde{\mathcal{T}}$ denotes inverse time ordering (later times on the right). Changing integration variables according to $\chi = \phi_+ + \phi_-$ and $\phi = (\phi_+ - \phi_-)/2$ we can write

$$\begin{aligned} P[a] &= \int D\phi e^{-\int dt \phi^2(t)/2\lambda} \times \\ &\int D\chi e^{-\int dt \chi^2(t)/8\lambda} e^{-\int i\chi(t)a(t)dt} \\ &\times \text{Tr} \mathcal{T} e^{\int i\chi(t)/2 + \phi(t)\hat{A}(t)dt} \hat{\rho} \tilde{\mathcal{T}} e^{\int i\chi(t)/2 - \phi(t)\hat{A}(t)dt} \quad (9) \end{aligned}$$

The last line can be written alternatively as (see Appendix B)

$$\text{Tr} \mathcal{T} e^{\int i\chi(t)\hat{A}_\phi(t)dt/2} \hat{\rho} \tilde{\mathcal{T}} e^{\int i\chi(t)\hat{A}_\phi(t)dt/2} \quad (10)$$

where $\hat{A}_\phi(t)$ denotes the operator \hat{A} in a modified Heisenberg picture, namely with respect to the Hamiltonian $\hat{H} - \hbar\phi(t)\hat{A}$.

Eqs. (9) and (10) describe the outcome of the continuous measuring process in terms of the probability distribution functional $P[a(t)]$ of the observation function $a(t)$. This distribution reflects the quantum uncertainty, the modified system time evolution caused by the measurement (the backaction effect) and the uncertainty associated with the weak measurement that can be thought of as reflecting detector noise. A more transparent view of these contributions is obtained by separating the latter, system independent contribution from the system dependent effects. This is achieved by considering the moment generating functional (MGF) $M[\chi] = e^{S[\chi]}$, where

$S[\chi]$ is the cumulant generating functional (CGF), given by

$$\begin{aligned} M[\chi] &= e^{S[\chi]} = \int Da e^{i\int \chi(t)a(t)} P[a] = \quad (11) \\ &e^{-\int dt \chi^2(t)/8\lambda} \int D\phi e^{-\int dt \phi^2(t)/2\lambda} \times \\ &\text{Tr} \mathcal{T} e^{\int i\chi(t)\hat{A}_\phi(t)dt/2} \hat{\rho} \tilde{\mathcal{T}} e^{\int i\chi(t)\hat{A}_\phi(t)dt/2}, \end{aligned}$$

where $S[0] = 0$. The CGF can be divided into two parts $S[\chi] = S_d[\chi] + S_q[\chi]$ with

$$S_d[\chi] = -\int dt \chi^2(t)/8\lambda \quad (12)$$

and

$$\begin{aligned} e^{S_q[\chi]} &= \int D\phi e^{-\int dt \phi^2(t)/2\lambda} \times \\ &\text{Tr} \mathcal{T} e^{\int i\chi(t)\hat{A}_\phi(t)dt/2} \hat{\rho} \tilde{\mathcal{T}} e^{\int i\chi(t)\hat{A}_\phi(t)dt/2}. \quad (13) \end{aligned}$$

Note that $S[0] = S_d[0] = S_q[0] = 0$. On the level of probabilities this decomposition corresponds to the convolution

$$P[a] = \int Da' P_d[a - a'] P_q[a'], \quad (14)$$

where

$$P_d[a] = \int D\chi e^{\int dt (\chi(t)a(t)/i - \chi^2(t)/8\lambda)} \propto e^{-2\lambda \int a^2(t)dt} \quad (15)$$

corresponds to a Gaussian noise with zero average and correlation $\langle a(t)a(t') \rangle_d = \delta(t - t')/4\lambda$ that may be interpreted as the noise associated with the detector, and where

$$P_q[a] = \int D\chi e^{-\int i\chi(t)a(t)dt} e^{S_q[\chi]} \quad (16)$$

is a distribution associated with the intrinsic system uncertainty as well as the measurement backaction. It is normalized, $\int Da P_q[a] = 1$, but not necessarily positive, and will be referred to as a quasi-probability [10, 30]. In the limit of weak, noninvasive measurement, $\lambda \rightarrow 0$, P_d diverges while P_q has a well defined limit

$$\begin{aligned} e^{S_q[\chi]} &\xrightarrow{\lambda \rightarrow 0} \quad (17) \\ &\text{Tr} \mathcal{T} e^{\int i\chi(t)\hat{A}(t)dt/2} \hat{\rho} \tilde{\mathcal{T}} e^{\int i\chi(t)\hat{A}(t)dt/2}. \end{aligned}$$

Consider now this distribution P_q . First note that while it is not a real probability functional, it is possible to calculate moments $\langle \rangle$ and cumulants $\langle \langle \rangle \rangle$ with respect to this measure as partial derivatives of the quasi-CGF, respectively

$$\begin{aligned} \langle a(t_1) \cdots a(t_n) \rangle_q &= \frac{\delta^n \exp S_q[\chi]}{\delta i\chi(t_1) \cdots \delta i\chi(t_n)} \Big|_{\chi=0}, \quad (18) \\ \langle \langle a(t_1) \cdots a(t_n) \rangle \rangle_q &= \frac{\delta^n S_q[\chi]}{\delta i\chi(t_1) \cdots \delta i\chi(t_n)} \Big|_{\chi=0}. \end{aligned}$$

In particular, for $t_n \geq \dots \geq t_2 \geq t_1$,

$$\langle a(t) \rangle_q = \int D_\lambda \phi \text{Tr} \left[\hat{A}_\phi(t) \hat{\rho} \right], \quad (19)$$

$$\langle a(t_1) a(t_2) \rangle_q = \int D_\lambda \phi \text{Tr} \left[\{ \hat{A}_\phi(t_1), \hat{A}_\phi(t_2) \} \hat{\rho} \right] / 2, \quad (20)$$

$$\langle a(t_1) a(t_2) a(t_3) \rangle_q = \int D_\lambda \phi \times \quad (21)$$

$$\text{Tr} \left[\{ \hat{A}_\phi(t_1), \{ \hat{A}_\phi(t_2), \hat{A}_\phi(t_3) \} \} \hat{\rho} \right] / 4,$$

$$\langle a(t_1) \dots a(t_n) \rangle_q = \int D_\lambda \phi \times \text{Tr} \left[\{ \hat{A}_\phi(t_1), \{ \hat{A}_\phi(t_2), \dots \hat{A}_\phi(t_n) \} \dots \} \hat{\rho} \right] / 2^{n-1},$$

(see Appendix C), where we have denoted $D_\lambda \phi = D\phi e^{-\int dt \phi^2(t)/2\lambda}$. Here and below we use standard notation $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$ and $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$.

Secondly, from Eq. (20) follows the important so-called *weak positivity* property of second order correlations [24]

$$\langle F^2[a] \rangle_q = \int D\phi e^{-\int dt \phi^2(t)/2\lambda} \text{Tr} F^2[\hat{A}_\phi] \hat{\rho} \geq 0 \quad (22)$$

for $F[a] = \int dt (f(t)a(t) + g(t))$ and arbitrary functions f and g . This property has an important implication that no test based solely on maximally second order correlations can reveal the negativity of the quasiprobability. First and second order correlations can be represented by a completely classical, positive Gaussian probability distribution

$$P'_q[a] \propto \exp \left(- \int dt dt' \delta a(t) f^{-1}(t, t') \delta a(t') / 2 \right), \quad (23)$$

where $\delta a(t) = a(t) - \langle a(t) \rangle_q$, $f(t, t') = \langle \delta a(t) \delta a(t') \rangle_q$ and f^{-1} is its inverse defined by $\int dt f(t', t) f^{-1}(t, t'') = \delta(t' - t'')$. The weak positivity guarantees that both f and f^{-1} are positive definite and consequently P'_q is a correct real probability distribution. To check that P_q differs from P'_q and demonstrate its negativity one needs higher order correlations or additional assumptions (e.g. boundedness or dichotomy of a as it happens in Leggett-Garg inequality [26]).

To end this Section, we consider the special case in which the Hamiltonian commutes with \hat{A} (or the noncommuting part is negligible during the interesting timescale). Furthermore, let us take the initial state of the system to be an eigenstate $|a\rangle$ of \hat{A} , i.e. $\hat{\rho}(t=0) = |a\rangle\langle a|$, $\hat{A}|a\rangle = a|a\rangle$. Consider a measurement performed during the time interval t_0 ,

$$\bar{a} = (1/t_0) \int_0^{t_0} dt a(t) \quad (24)$$

In this case we find (Appendix D) that $\langle \bar{a} \rangle = \langle \bar{a} \rangle_q = a$ and $\langle (\delta \bar{a})^2 \rangle = \langle (\delta \bar{a})^2 \rangle_d = 1/4\lambda t_0$ with $\delta X = X - \langle X \rangle$. We can see the intuitively expected effect of an increasing

measurement duration to lead to an improved signal to noise ratio with time, which goes as

$$\frac{\langle \bar{a} \rangle}{\sqrt{\langle (\delta \bar{a})^2 \rangle}} = 2a \sqrt{\lambda t_0}. \quad (25)$$

Thus, even the weakest measurement (small λ) turns into strong a one, if performed often enough for sufficiently long time.

III. REPRESENTATION BY STOCHASTIC EVOLUTION EQUATIONS

Turning back to the general case, we note first that the correlation functions associated with the quasiprobability $P_q[a(t)]$, given by Eqs. (19)-(21), can be calculated from the Heisenberg equations

$$d\hat{B}_\phi(t)/dt = (i/\hbar)[\hat{H}_\phi(t) - \phi(t)\hat{A}_\phi(t), \hat{B}_\phi(t)], \quad (26)$$

where \hat{A} represents the measured variable while \hat{B} is any system operator. In particular

$$\begin{aligned} d\hat{A}_\phi(t)/dt &= (i/\hbar)[\hat{H}_\phi(t), \hat{A}_\phi(t)], \\ d\hat{H}_\phi(t)/dt &= \phi(t)d\hat{A}_\phi(t)/dt. \end{aligned} \quad (27)$$

We can solve these equations for a general stochastic trajectory $\phi(t)$, then take the averages as defined by Eqs. (19)-(21), over a Gaussian distribution of such trajectories. The correlation functions obtained in this way coincide with the ones derived directly from the CGF Eq. (16). If $\hat{H} = \hat{p}^2/2m + V(\hat{x})$, with $[\hat{x}, \hat{p}] = i\hbar\hat{1}$, and $\hat{A} = \hat{x}$ is the position operator, the Heisenberg equations for \hat{x}_ϕ and \hat{p}_ϕ are

$$\begin{aligned} \hat{H}_\phi &= \hat{p}_\phi^2/2m + V(\hat{x}_\phi), \\ \frac{d\hat{x}_\phi}{dt} &= \hat{p}_\phi/m, \\ \frac{d\hat{p}_\phi}{dt} &= (i/\hbar)[V(\hat{x}_\phi), \hat{p}_\phi] + \hbar\phi(t) = -\frac{\partial V(\hat{x}_\phi)}{\partial \hat{x}_\phi} + \hbar\phi(t). \end{aligned} \quad (28)$$

Eq. (28) is a quantum Langevin equation in which the quantum dynamics is augmented by a zero centered white Gaussian noise, $\langle \phi(t) \rangle = 0$, $\langle \phi(t)\phi(t') \rangle = \lambda\delta(t-t')$. Closed form solutions of this equation can be obtained for the harmonic oscillator, a case we discuss below.

Alternatively, the stochastic dynamics affected by the continuous measurement process may be described by a Lindblad-type master equation[7] for the non-selective system density matrix. The latter is defined by

$$\hat{\rho}(t) = \int_{a(0)}^{a(t)} Da \hat{\rho}[a] = \int_{a(0)}^{a(t)} Da \hat{K}[a] \hat{\rho} \hat{K}^\dagger[a] \quad (29)$$

where the integral is over all observation trajectories between times 0 and t . It is shown (appendix E) to evolve according to (using a Liouville-superoperator \hat{L})

$$\frac{d\hat{\rho}}{dt} = \hat{L}\hat{\rho} := [\hat{H}, \hat{\rho}]/i\hbar - \lambda[\hat{A}, [\hat{A}, \hat{\rho}]]/2. \quad (30)$$

In the representation of eigenstates of \hat{A} ,

$$\hat{\rho} = \sum_{a,a'} \tilde{\rho}_{aa'} |a\rangle\langle a'|, \quad (31)$$

$$\check{L}\tilde{\rho}_{a,a'} = \frac{1}{i\hbar} \sum_b (H_{ab}\tilde{\rho}_{ba'} - \tilde{\rho}_{ab}H_{ba'}) - \lambda(a - a')^2 \tilde{\rho}_{aa'}, \quad (32)$$

showing, as is well known [4] and as may be intuitively expected, that the measurement damps the off-diagonal terms ($a \neq a'$) with the rate proportional to the measurement strength. Note that if some eigenvalues a are degenerate, then the corresponding off-diagonal elements of $\tilde{\rho}$ are not damped.

Together with the Liouville-Lindblad superoperator \check{L} we define the corresponding evolution superoperator $\check{U}(a,b) = \mathcal{T} \exp \int_b^a \check{L} dt$. It can be then shown (Appendix F) that the correlation functions (19)-(21) are given by

$$\langle a(t) \rangle_q = \text{Tr} \left[\check{A} \check{U}(t,0) \hat{\rho} \right], \quad (33)$$

$$\langle a(t_1)a(t_2) \rangle_q = \text{Tr} \left[\check{A} \check{U}(t_2, t_1) \check{A} \check{U}(t_1, 0) \hat{\rho} \right], \quad (34)$$

$$\langle a(t_1)a(t_2)a(t_3) \rangle_q = \quad (35)$$

$$\text{Tr} \left[\check{A} \check{U}(t_3, t_2) \check{A} \check{U}(t_2, t_1) \check{A} \check{U}(t_1, 0) \hat{\rho} \right],$$

$$\langle a(t_1) \cdots a(t_n) \rangle_q =$$

$$\text{Tr} \left[\check{A} \check{U}(t_n, t_{n-1}) \cdots \check{A} \check{U}(t_2, t_1) \check{A} \check{U}(t_1, 0) \hat{\rho} \right],$$

where $\check{A}\hat{B} = \{\hat{A}, \hat{B}\}/2$. Note that in (33)-(35), $\hat{\rho} = \hat{\rho}(t=0) = \hat{\rho}$. Eqs. (33)-(35) provide a more convenient route for the evaluation of these correlation functions.

In the following Sections we apply this general formalism to the two simplest quantum systems, the two-level system and the harmonic oscillator.

IV. THE TWO-LEVEL SYSTEM

Consider a two-level system defined by the Hamiltonian

$$\hat{H} = \hbar\omega\hat{\sigma}_x/2 \quad (36)$$

and suppose that the system is in the initial state

$$\hat{\rho}(t=0) = (\hat{1} + \hat{\sigma}_z)/2, \quad (37)$$

where $\hat{\sigma}$ denotes Pauli matrices and $\hat{1}$ is the corresponding unit operator. Left uninterrupted, the system will oscillate between the two eigenstates of $\hat{\sigma}_z$, a process analogous to Rabi oscillations in a harmonically driven system. We focus on the measurement of $\hat{A} = \hat{\sigma}_z$ and denote the measurement outcome by $a(t) = \sigma_z(t)$. We pose the following questions: Can the oscillatory time trace of σ_z be observed? How does the measurement process affect this oscillation? Is the oscillation visible in a single run

of an experiment or only as a statistical effect – average over many runs or many copies of the same experiment? The latter question is particularly relevant in light of the growth of activity in single molecule spectroscopy.

To answer these questions we start by writing the action of \check{L} , Eq. (30), in the basis of Hermitian operators $(\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$. In a compact notation it reads,

$$\check{L}(x\hat{\sigma}_x + y\hat{\sigma}_y + z\hat{\sigma}_z) = \omega(y\hat{\sigma}_z - z\hat{\sigma}_y) - 2\lambda(x\hat{\sigma}_x + y\hat{\sigma}_y) \quad (38)$$

and $\check{L}\hat{1} = 0$. Next, expressing the operation of $\check{U}(t,0)$ on $\hat{\sigma}_z$ by

$$\hat{\sigma}_z(t) = \check{U}(t,0)\hat{\sigma}_z = x(t)\hat{\sigma}_x + y(t)\hat{\sigma}_y + z(t)\hat{\sigma}_z \quad (39)$$

and using (38) and (39) in (30) we find $dx/dt = -2\lambda x$; $dy/dt = -(\omega z + 2\lambda y)$, and $dz/dt = \omega y$, which, for $z(t=0) = 1$, $x(0) = y(0) = 0$ yields

$$z(t) = e^{-\lambda t} [\cos(\Omega t) + \lambda \sin(\Omega t)/\Omega], \quad (40)$$

$$y(t) = dz/dt, \quad x(t) = 0,$$

where $\Omega = \sqrt{\omega^2 - \lambda^2}$. This allows to write down the relevant averages (see Appendix G), namely

$$\langle \sigma_z(t) \rangle_q = z(t), \quad (41)$$

$$\langle \sigma_z(t)\sigma_z(t') \rangle_q = z(|t - t'|).$$

The last line is known in the existing literature only for the stationary case ($t, t' \rightarrow \infty$) in the weak measurement limit $\lambda \rightarrow 0$ [17, 22]. It is interesting to note that although the system under consideration is not in a stationary state and in fact evolves irreversibly, this correlation function depends only on the time difference $t' - t$ and remains finite when this difference is constant while both t and t' increase.

Recall that Eqs. (D3) and (D4) imply that $\langle \sigma_z(t) \rangle = \langle \sigma_z(t) \rangle_q$, while $\langle \sigma_z(t)\sigma_z(t') \rangle = \langle \sigma_z(t)\sigma_z(t') \rangle_d + \langle \sigma_z(t)\sigma_z(t') \rangle_q$, which implies $\langle \delta\sigma_z(t)\delta\sigma_z(t') \rangle = \langle \delta\sigma_z(t)\delta\sigma_z(t') \rangle_d + \langle \delta\sigma_z(t)\delta\sigma_z(t') \rangle_q$. In the limit $\lambda \ll \omega$ we see (cf. Eqs. (40) and (41)) clear oscillation of $\langle \sigma_z(t) \rangle$. However, in a single run this signal cannot be distinguished from the noise. Indeed, defining as in (24) $\bar{\sigma}_z = (1/t_0) \int_0^{t_0} dt \sigma_z(t)$, we obviously need to take $t_0 \ll \omega^{-1}$. Therefore

$$\langle (\delta\bar{\sigma}_z)^2 \rangle > \langle (\delta\bar{\sigma}_z)^2 \rangle_d = 1/t_0\lambda \gg 1 \quad (42)$$

The large detection noise covers the signal. This implies that Rabi oscillations cannot be seen in a single run.

The above result was obtained in the time domain. We can also ask whether the Rabi oscillation is visible in the frequency domain. This would imply seeing the peak in the Fourier transform

$$\tilde{\sigma}_z(\nu) = (2/t_0) \int_0^{t_0} dt \cos(\nu t) \sigma_z(t), \quad (43)$$

where t_0 is a time much larger than the oscillation period, but obviously much smaller than the damping time:

$\omega^{-1} \ll t_0 \ll \lambda^{-1}$. From (40) and (41), the peak intensity is $\langle \tilde{\sigma}_z(\nu = \Omega) \rangle \simeq 1$. On the other hand, under the measurement conditions the white detector noise satisfies

$$\langle (\delta \tilde{\sigma}_z(\Omega))^2 \rangle \geq \langle (\delta \tilde{\sigma}_z(\Omega))^2 \rangle_d = 1/2t_0\lambda \gg 1 \quad (44)$$

implying that, again, the noise exceeds the signal and a peak in the frequency domain will not be seen. This time, however, the signal-noise ratio is not as bad as in the time domain because t_0 can be longer.

We conclude that Rabi oscillation cannot be seen in a single run/copy of the experiment but only in a statistical average. The sample size, that is the number of runs/copies needed for this average is of the order $(t_0\lambda)^{-1}$, where $\lambda^{-1} \gg t_0 \gg \omega^{-1}$ in the frequency domain and $t_0 \ll \omega^{-1} \ll \lambda^{-1}$ in the time domain. In the overdamped regime, $\lambda \gg \omega$, one can see the quantum Zeno effect, discussed below in Section VII.

V. LEGGETT-GARG-TYPE INEQUALITY

The analysis above has used second order correlations that, as stated in (22), are not sensitive to the quasiprobabilistic nature of the distribution. The well known Leggett-Garg inequality [26] needs only second order correlations but requires the additional assumption of bounded observable which is effectively equivalent to higher order correlations (e.g. the dichotomy $A = \pm 1$ is equivalent to measuring $\langle (A^2 - 1)^2 \rangle = 0$ which requires the fourth-order correlator $\langle A^4 \rangle$). Without this assumption, the quasiprobabilistic nature is however revealed in higher order correlations. To see this we take $\hat{\rho} = \hat{1}/2$ and consider the following quantity

$$X[\sigma] = \sigma_z(0)\sigma_z(\pi/\omega) + \sigma_z(-\pi/2\omega)\sigma_z(\pi/2\omega) + 2. \quad (45)$$

The fourth order correlation $\langle X^2 \rangle_q$ is given by

$$\begin{aligned} \langle X^2 \rangle_q &= 6 + \\ e^{-\lambda\pi/\omega} [1/r^2 + (10 - 1/r^2) \cos(\pi r) + 10\lambda \sin(\pi r)/\omega r], \end{aligned} \quad (46)$$

where $r = \sqrt{1 - (\lambda/\omega)^2}$. The behavior of $\langle X^2 \rangle_q$ is shown in Fig. 1. In the limit of strong measurement $\langle X^2 \rangle_q = 16$. The origin is the QZE – the evolution is frozen by the measurement and so $\sigma_z(t)$ does not depend on time, which results in $X = 4$. In the opposite limit of non-invasive measurement $\langle X^2 \rangle_q = -2$ and it crosses zero at $\omega/\lambda \approx 11$. This implies that for sufficiently small λ the classical inequality $\langle X^2 \rangle_q \geq 0$ is violated so the function P_q is not positive definite and as such cannot describe a usual probability. Note, however, that (a) it contains the relevant physical information, discarding the irrelevant detection noise; (b) by itself, it cannot be directly measured, namely correlations such as $\langle X^2 \rangle_q$ are not directly measurable since the real probability is the convolution (14); and (c) the actual detected observable certainly satisfies $\langle X^2 \rangle > 0$. However, an independent determination of the detection noise should be experimentally feasible and allows to find the negativity of $\langle X^2 \rangle_q$ after the noise has been subtracted.

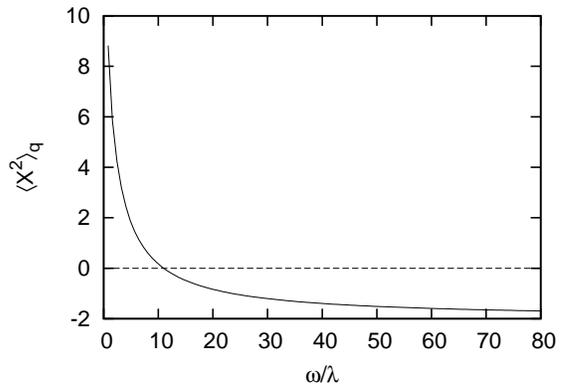


FIG. 1: Demonstration of the violation of the Leggett-Garg-type inequality $\langle X^2 \rangle_q > 0$ as a function of measurement strength λ . The function starts from the value 16 for $\lambda \gg \omega$ as expected for the QZE, crosses the classical bound 0 and tends to -2 in the limit of weak measurement $\lambda \ll \omega$.

VI. THE HARMONIC OSCILLATOR

For completeness, we also consider another much studied simple problem – continuous position measurement, $\hat{A} = \hat{x}$, in a system comprising one harmonic oscillator, described by the Hamiltonian $\hat{H} = \hat{p}^2/2m + m\omega^2\hat{x}^2/2$. Eqs. (28) become

$$\begin{aligned} d\hat{x}_\phi/dt &= \hat{p}_\phi/m, \\ d\hat{p}_\phi/dt &= -m\omega^2\hat{x}_\phi + \hbar\phi(t), \end{aligned} \quad (47)$$

where ϕ represents the zero-centered white Gaussian noise, $\langle \phi(t)\phi(t') \rangle = \lambda\delta(t-t')$. We note in passing that this quantum Langevin equation yields the Fokker-Planck equation for the Wigner function [27]

$$W(x, p) = \int \frac{d\chi d\xi}{2\pi^2} e^{-i\xi x - \xi p} \text{Tr} \hat{\rho} e^{\xi x + i\xi p} \quad (48)$$

in the form [28]

$$\frac{\partial W(x, p, t)}{\partial t} = m\omega^2 x \frac{\partial W}{\partial p} - \frac{p}{m} \frac{\partial W}{\partial x} + \frac{\lambda \hbar^2}{2} \frac{\partial^2 W}{\partial p^2}. \quad (49)$$

However, in what follows we calculate directly the required correlation functions. Solving Eq. (47) we get

$$\begin{aligned} \hat{x}_\phi(t) &= \hat{x}(0) \cos \omega t + (m\omega)^{-1} \hat{p}(0) \sin \omega t \\ &+ \int_0^t dt' \sin \omega(t-t') \hbar \phi(t)/m\omega, \\ \hat{p}_\phi(t) &= \hat{p}(0) \cos \omega t - m\omega \hat{x}(0) \sin \omega t \\ &+ \int_0^t dt' \cos \omega(t-t') \hbar \phi(t'), \end{aligned} \quad (50)$$

This implies that the oscillations of the average position are undamped,

$$\langle x(t) \rangle = \langle x(0) \rangle \cos(\omega t) + (m\omega)^{-1} \langle p(0) \rangle \sin(\omega t) \quad (51)$$

independently of the detection strength.

Turning to the noise term we first note that, as before, the detector noise combines additively with the correlation functions obtained from Eqs. (50). The latter take the form

$$\langle \delta x(t) \delta x(t') \rangle_q = \langle \delta x(t) \delta x(t') \rangle_0 + f_\lambda(t, t') \quad (52)$$

where $\langle \delta x(t) \delta x(t') \rangle_0$ is the free correlation function obtained in the limit $\lambda \rightarrow 0$ or equivalently $\phi \rightarrow 0$, that is, by ignoring the last (noise) terms on the RHS of Eqs. (50),

$$\begin{aligned} \langle \delta x(t) \delta x(t') \rangle_0 &= \langle \delta x(0) \delta x(0) \rangle \cos \omega t \cos \omega t' \\ &+ \langle \delta x(0) \delta p(0) \rangle_W (m\omega)^{-1} \sin \omega(t+t') \\ &+ \langle \delta p(0) \delta p(0) \rangle (m\omega)^{-2} \sin \omega t \sin \omega t' \end{aligned} \quad (53)$$

with the Wigner-ordered average $\langle 2xp \rangle_W = \text{Tr}[\hat{\rho}\{\hat{x}, \hat{p}\}]$, and where $f_\lambda(t, t')$ is the correlation function associated with the noise terms in (50),

$$\begin{aligned} f_\lambda(t, t') &= \frac{\lambda \hbar^2}{2(m\omega)^2} [\min(t, t') \cos \omega(t-t') + \\ &(\sin \omega|t-t'| - \sin \omega(t+t'))/2\omega]. \end{aligned} \quad (54)$$

This measurement-induced correlation function represents the backaction effect of the measuring process. It depends on the detector strength and the parameters of the dynamics but not on the initial state of the oscillator. Moreover, because of the Gaussian nature of ϕ , it contributes solely to the second cumulant $\langle\langle x(t)x(t') \rangle\rangle$, leaving all the other unaffected. As expected, it vanishes in the limit $\lambda \rightarrow 0$. However, the most striking feature in (54) is the growth of noise with time, as expressed by the first term in (54).

In analogy to the two-level system we discuss the behavior of the short time ($t \ll \omega^{-1}$) average \bar{x} and the long time ($t \gg \omega^{-1}$) Fourier transform \tilde{x} , defined by the analogs of Eqs. (24) and (43), respectively. In both limits we are now free to choose $t_0\lambda$ because, in contrast to the 2-level case, the averaged oscillations, Eq. (51), are not damped.

Consider first the time-domain observation. For $t_0\lambda \ll 1$ the uncertainty of \bar{x} is determined by the detection noise as in (42), while in the opposite limit it will be dominated by the backaction (54). In either case the noise exceeds the signal.

In the frequency domain, for $\tilde{x}(\omega) = 2 \int_0^{t_0} dt \cos(\omega t) x(t)/t_0$, we get using Eqs. (51)-(54), the peak signal

$$\langle \tilde{x}(\omega) \rangle = \langle x(0) \rangle, \quad (55)$$

and the intrinsic and backaction noise components

$$\langle (\delta \tilde{x}(\omega))^2 \rangle_q \simeq \langle (\delta x(0))^2 \rangle + \frac{\lambda t_0 \hbar^2}{6(m\omega)^2} \quad (56)$$

to which we need to add the detector noise $(2t_0\lambda)^{-1}$. The total uncertainty originated from the detector satisfies

$$\frac{\lambda t_0 \hbar^2}{6(m\omega)^2} + \frac{1}{2t_0\lambda} \geq \frac{\hbar}{\sqrt{3}m\omega} \quad (57)$$

with the lower bound (obtained as minimum of the LHS with respect to $t_0\lambda$) independent of λ and t_0 . Obviously $\langle x(0) \rangle$ can be chosen large enough for the signal to dominate the noise at intermediate times, but the noise will always exceed the signal at long enough times. As always, the signal-to-noise ratio can be improved by repeated measurements.

The fact that the backaction contribution (54) to the noise grows with time reflects the continuous pumping of energy to the system affected by the measurement process [31]. This does not happen in the two-level system because of its bounded spectrum, still also in that system the temperature grow to infinity ($\hat{\rho}(t) \rightarrow (1/2)\hat{1}$) as implied by Eqs. (39), (40) and (41). This unlimited growth can be avoided by assuming that the measurement process also involves some friction [20, 28]. Indeed, measurement, even classical, means extraction of information out of the system, so that without compensating friction its entropy must increase and so does the temperature.

VII. THE QUANTUM ZENO EFFECT (QZE)

Also for completeness, we show now how the QZE emerges within the present formalism. So far we have focused on weak measurements, represented by small λ . The opposite limit, $\lambda \rightarrow \infty$, represents the strong measurement case. In systems characterized by a single timescale ω^{-1} , strong and weak measurements are quantified by the inequalities $\lambda \gg \omega$ and $\lambda \ll \omega$, respectively.

Consider the two-level system discussed in Section 4. For $\lambda > \omega$ its dynamics is given by the overdamped analog of Eq. (40), $\Omega = i\sqrt{\lambda^2 - \omega^2}$. In the extreme strong measurement case, $\lambda \gg \omega$, $z(t) \sim e^{-\omega^2 t/2\lambda}$ and the decay slows down as $\lambda \rightarrow \infty$ [14, 32]. This corresponds to the QZE where the system is almost frozen by the measurement, reaching its equilibrium state $z = 0$ only on the timescale $t \sim \lambda/\omega^2$.

For a position measurement in the harmonic oscillator case, we have seen, Eq. (51), that the average position oscillates regardless the strength of the measurement. This implies that the Zeno affect is absent in this system, as is well known [14]. On the other hand, for any measurement strength, the detector-induced backaction noise, Eq. (54), increases without bound at long times at a rate that increases with λ . Already for short times we get $f_\lambda(t, t) \simeq \lambda \hbar^2/3m^2$, and backaction adds fast diffusion in the phase space. This is somewhat analogous to the anti-Zeno effect [25].

VIII. CONCLUSIONS

Gaussian POVMs, here represented by Kraus operators, were used in this paper to formalize the description of weak measurements. A path integral representation of continuous weak measurement described in this way leads directly to an analysis of backaction noise in terms of stochastic evolution equations. The average signal and the associated noise were obtained in terms of moments and time correlation functions of the measured quantity.

In particular, the noise was shown to be an additive combination of a term characteristic of the measurement alone (detector noise) and terms associated with the system, which in turn include contributions from the intrinsic quantum mechanical uncertainty in the system, and from backaction effects from the measurement process. A transparent representation of this stochastic evolution was obtained by separating it into a process characteristic only of the weak measurement, and another, representing the quantum uncertainty intrinsic to the system as well as that arising from the measurement backaction. This yields the noise as an additive combination of the corresponding contributions, while the total probability is found to be convolution of white Gaussian detections noise and intrinsic system's quasiprobability. The quasiprobability can be negative although the negativity is not visible at the level of second order correlations due to weak positivity. The general formalism was applied to two simple problems: continuous monitoring of the level population in a 2-level system and continuous measurement of the position of a harmonic oscillator. For

both systems we have established limits on the possibility to observe oscillatory motion in a single run of an experiment. The negativity property of the quasiprobability can be demonstrated on in the 2-level system, using fourth order correlations. In this way, we have constructed a Leggett-Garg-type inequality without the assumption of dichotomy or boundedness of the variable.

We observe that the QZE occurs when both the Hamiltonian and the observable can be represented in finite-dimensional Hilbert space. When the space is infinite or continuous and both the Hamiltonian and the observable have no finite-dimensional representation, the dynamics will not always be able to 'pin down' the state and consequently the dynamics may get diffusive. Establishing criteria for the occurrence or absence of the QZE in realistic systems continues to be an intriguing and challenging issue.

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Appendix A

Here we derive Eq. (5). Using Eq. (2), a succession of time evolutions and measurements in the interval $(0, t)$ reads

$$\hat{K}(\{a_j\}) = (2\bar{\lambda}/\pi)^{N/4} e^{-(i/\hbar)\hat{H}(t_{N+1}-t_N)} e^{-\bar{\lambda}(a_N-\hat{A})^2} \dots e^{-(i/\hbar)\hat{H}(t_3-t_2)} e^{-\bar{\lambda}(a_2-\hat{A})^2} e^{-(i/\hbar)\hat{H}(t_2-t_1)} e^{-\bar{\lambda}(a_1-\hat{A})^2} e^{-(i/\hbar)\hat{H}t_1}. \quad (\text{A1})$$

Putting $\bar{\lambda} = \lambda\Delta t$ and $t_j - t_{j-1} = t_1 = \Delta t$ and using $\Delta t \rightarrow 0$ leads to

$$\hat{K}(\{a_j\}) = (2\lambda\Delta t/\pi)^{N/4} \prod_{j=1}^N e^{[-(i/\hbar)\hat{H} - \lambda(a_j - \hat{A})^2]\Delta t} e^{-(i/\hbar)H\Delta t} \quad (\text{A2})$$

and, for $\Delta t \rightarrow 0$

$$\hat{K}[a(t)] = C\mathcal{T}e^{\int_0^t [-(i/\hbar)\hat{H} - \lambda(a(t) - \hat{A})^2] dt}. \quad (\text{A3})$$

Alternatively, using $\hat{A}(t_j) = e^{(i/\hbar)\hat{H}t_j} \hat{A} e^{-(i/\hbar)\hat{H}t_j}$ yields

$$\hat{K}(\{a_j\}) = (2\lambda\Delta t/\pi)^{N/4} e^{-(i/\hbar)\hat{H}t_{N+1}} \prod_{j=1}^N e^{[-\lambda(a_j - \hat{A}(t_j))^2]\Delta t} \quad (\text{A4})$$

and in the continuum limit

$$K[a(t)] = C e^{-(i/\hbar)Ht} \mathcal{T}e^{-\lambda \int_0^t (a(t) - \hat{A}(t))^2 dt} \quad (\text{A5})$$

In Eqs. (A3) and (A5) C are normalization factors.

Appendix B

Here we prove Eq. (10). Start from $\mathcal{T} \exp \int i(\chi(t)/2 + \phi(t))\hat{A}(t)dt$ and discretize to get

$$\begin{aligned} \mathcal{T} e^{i \int (\chi(t)/2 + \phi(t))\hat{A}(t)dt} &= \mathcal{T} e^{i\Delta t \sum_j (\chi(t_j)/2 + \phi(t_j))\hat{A}(t_j)} = \{t_j = j\Delta t; j = 1, \dots, N\} \\ &= e^{i\Delta t (\chi(t_N)/2 + \phi(t_N))\hat{A}(t_N)} e^{i\Delta t (\chi(t_{N-1})/2 + \phi(t_{N-1}))\hat{A}(t_{N-1})} \dots e^{i\Delta t (\chi(t_1)/2 + \phi(t_1))\hat{A}(t_1)} = \\ &= e^{(i/\hbar)Ht_N} e^{i\Delta t (\chi(t_N)/2 + \phi(t_N))\hat{A}} e^{-(i/\hbar)Ht_{N-1}} e^{(i/\hbar)Ht_{N-1}} e^{i\Delta t (\chi(t_{N-1})/2 + \phi(t_{N-1}))\hat{A}} e^{-(i/\hbar)Ht_{N-1}} \dots \\ &\dots e^{(i/\hbar)Ht_1} e^{i\Delta t (\chi(t_1)/2 + \phi(t_1))\hat{A}} e^{-(i/\hbar)Ht_1} \end{aligned} \quad (\text{B1})$$

Next replace

$$e^{i\Delta t (\chi(t_j)/2 + \phi(t_j))\hat{A}} \rightarrow e^{i\Delta t \hat{A} \chi(t_j)/2} e^{i\hat{A} \phi(t_j)(t_j - t_{j-1})} \quad (\text{B2})$$

for $j = 1, \dots, N$ and define $\hat{H}_\phi(t) = \hat{H} - \hbar\phi(t)\hat{A}$, $\hat{A}_\phi(t) = \tilde{\mathcal{T}} e^{(i/\hbar) \int_0^t \hat{H}_\phi(t') dt'} \hat{A} \mathcal{T} e^{-(i/\hbar) \int_0^t \hat{H}_\phi(t') dt'}$ and discretize it again,

$$\hat{A}_\phi(t_k) = \tilde{\mathcal{T}} \prod_{j \leq k} e^{(i/\hbar) \hat{H}_\phi(t_j) \Delta t} \hat{A} \mathcal{T} \prod_{j \leq k} e^{-(i/\hbar) \hat{H}_\phi(t_j) \Delta t} \quad (\text{B3})$$

to get

$$e^{it_N \phi(t_N)} \mathcal{T} e^{i\Delta t \sum_{j=1}^N \chi(t_j) \hat{A}_\phi(t_j)/2} \rightarrow e^{it \phi(t)} \mathcal{T} e^{i \int_0^t (\chi(t')/2) \hat{A}_\phi(t') dt'}, \quad (\text{B4})$$

from which (10) follows.

Appendix C

Here we derive Eq.(19)-(21). Start from Eq. (13) and take its functional derivatives

$$\begin{aligned} \langle a(t) \rangle_b &= \int Da a(t) P_b[a] = \frac{1}{i} \left(\frac{\delta e^{S_b[\chi]}}{\delta \chi(t)} \right)_{\chi(t)=0} \\ &= \int D\phi e^{-\int dt \phi^2(t)/2\lambda} \text{Tr} \mathcal{T} \left(\hat{A}_\phi(t) \hat{\rho} + \hat{\rho} \hat{A}_\phi(t) \right) / 2 = \int D\phi e^{-\int dt \phi^2(t)/2\lambda} \text{Tr} \left(\hat{A}_\phi(t) \hat{\rho} \right) \end{aligned} \quad (\text{C1})$$

which is (19).

$$\langle a(t) a(t') \rangle_b = \int Da a(t) a(t') P_b[a] = - \left(\frac{\delta^2 e^{S_b[\chi]}}{\delta \chi(t) \delta \chi(t')} \right)_{\chi(t)=0} = \frac{1}{4} \int D\phi e^{-\int dt \phi^2(t)/2\lambda} \text{Tr} \left\{ \hat{A}_\phi(t'), \left\{ \hat{A}_\phi(t), \hat{\rho} \right\} \right\} \quad (\text{C2})$$

Time ordering implies that for $t' > t$, t' will be placed in the outer commutator, however last expression is equal to

$$\frac{1}{2} \int D\phi e^{-\int dt \phi^2(t)/2\lambda} \text{Tr} \left\{ \hat{A}_\phi(t), \hat{A}_\phi(t') \right\} \hat{\rho}, \quad (\text{C3})$$

which does not depend on the operator ordering. Higher moments are obtained in the same way.

Appendix D

Here we consider the case $[\hat{A}, \hat{H}] = 0$ and $[\hat{\rho}, \hat{A}] = 0$. When the observable \hat{A} commutes with the Hamiltonian, $\hat{A}_\phi(t) = \hat{A}$ so the trace in Eq. (13) becomes independent of ϕ . Using $\int D\phi e^{-\int dt \phi^2(t)/2\lambda} = 1$ it follows that $e^{S_b[\chi]}$ and $P_b[a]$, Eq. (16), do not depend on λ . This implies that the evolution associated with the backaction effect is deterministic and the only source of noise is the detector. To see the implication of this on the moments consider the moment generating function (c.f. Eqs. (12), (13))

$$e^{S[\chi]} = e^{S_a[\chi]} e^{S_b[\chi]} = e^{-\int dt \chi^2(t)/8\lambda} \int D\phi e^{-\int dt \phi^2(t)/2\lambda} \text{Tr} \mathcal{T} e^{i \int \chi(t) \hat{A}_\phi(t) dt/2} \hat{\rho} \tilde{\mathcal{T}} e^{i \int \chi(t) \hat{A}_\phi(t) dt/2} \quad (\text{D1})$$

For the imposed initial conditions this becomes

$$e^{S[\chi]} = e^{-\int dt \chi^2(t)/8\lambda} \left(e^{ia \int \chi(t) dt/2} \right)^2 \quad (\text{D2})$$

The first moment satisfies

$$\langle a \rangle = \left(\frac{\delta}{i\delta\chi(t)} e^{S_d[\chi]+S_b[\chi]} \right)_{\chi(t)=0} = \left(\frac{\delta}{i\delta\chi(t)} e^{S_b[\chi]} \right)_{\chi(t)=0} = a_0 \quad (\text{D3})$$

which implies also $\langle \bar{a} \rangle = a$ when used in (24). It is easy to realize that the second moment satisfies

$$\langle a(t) a(t') \rangle = - \left(\frac{\delta^2}{\delta\chi(t) \delta\chi(t')} e^{S_d[\chi]+S_b[\chi]} \right)_{\chi(t)=0} = - \left(\frac{\delta^2}{\delta\chi(t) \delta\chi(t')} e^{S_d[\chi]} \right)_{\chi(t)=0} - \left(\frac{\delta^2}{\delta\chi(t) \delta\chi(t')} e^{S_b[\chi]} \right)_{\chi(t)=0} \quad (\text{D4})$$

The second term yields a^2 , so $\langle \delta a(t) \delta a(t') \rangle = \langle \delta a(t) \delta a(t') \rangle - a^2$ is determined just by the Gaussian detector noise that in the case of (24) results in

$$\langle (\delta \bar{a})^2 \rangle = 1/(4\lambda t_0). \quad (\text{D5})$$

Appendix E

Here we derive the master equation (30). Following the steps that lead to (9) but without the trace, we find

$$\begin{aligned} \hat{\rho}[a] &= e^{-(i/\hbar)\hat{H}t} \int D\phi e^{-\int dt \phi^2(t)/2\lambda} \int D\chi e^{-\int dt \chi^2(t)/8\lambda} e^{-\int i\chi(t)a(t)dt} \times \\ &\mathcal{T} e^{\int i(\chi(t)/2+\phi(t))\hat{A}(t)dt} \hat{\rho} \tilde{\mathcal{T}} e^{\int i(\chi(t)/2-\phi(t))\hat{A}(t)dt} e^{(i/\hbar)\hat{H}t} \end{aligned} \quad (\text{E1})$$

and

$$\hat{\rho}(t) \equiv \int Da \hat{\rho}[a] = e^{-(i/\hbar)\hat{H}t} \int D\phi e^{-\int dt \phi^2(t)/2\lambda} \mathcal{T} e^{\int \phi(t)\hat{A}(t)dt} \hat{\rho} \tilde{\mathcal{T}} e^{-\int \phi(t)\hat{A}(t)dt} e^{(i/\hbar)\hat{H}t} \quad (\text{E2})$$

In what follows we will use the incremental propagation version of this equation:

$$\hat{\rho}(t+\Delta t) = e^{-(i/\hbar)\hat{H}(t+\Delta t)} \int D\phi e^{-\int_t^{t+\Delta t} dt \phi^2(t')/2\lambda} \mathcal{T} e^{i \int_t^{t+\Delta t} \phi(t')\hat{A}(t')dt} \hat{\rho}(t) \tilde{\mathcal{T}} e^{-i \int_t^{t+\Delta t} \phi(t')\hat{A}(t')dt} e^{(i/\hbar)\hat{H}(t+\Delta t)} \quad (\text{E3})$$

Next use

$$\begin{aligned} \mathcal{T} e^{i \int_t^{t+\Delta t} \phi(t')\hat{A}(t')dt} &= \mathcal{T} \exp \left(i\Delta t \phi(t) e^{(i/\hbar)\hat{H}t} \hat{A} e^{-(i/\hbar)\hat{H}} \right) = \\ &\prod_j e^{(i/\hbar)\hat{H}t_j} e^{i\Delta t \phi(t)\hat{A}} e^{-(i/\hbar)\hat{H}t_j} = e^{(i/\hbar)\hat{H}(t+\Delta t)} e^{\int_t^{t+\Delta t} (i/\hbar)[\phi(t')\hat{A}-\hat{H}]dt'} e^{-(i/\hbar)\hat{H}t} \end{aligned} \quad (\text{E4})$$

to rewrite Eq. (E3) in the form

$$\hat{\rho}(t+\Delta t) = \int D\phi e^{-\int_t^{t+\Delta t} dt \phi^2(t)/2\lambda} e^{i \int_t^{t+\Delta t} (\phi(t')\hat{A}-\hat{H})dt} \hat{\rho}(t) e^{-i \int_t^{t+\Delta t} (\phi(t')\hat{A}-\hat{H})dt}. \quad (\text{E5})$$

We next expand the RHS of (E5), keeping only terms that can contribute to order $O(\Delta t)$. To this end, we use

$$e^{\pm i \int_t^{t+\Delta t} dt (\phi(t)\hat{A}-\hat{H}/\hbar)} = 1 \pm i\hat{A} \int_t^{t+\Delta t} \phi(t)dt \mp i \left(\hat{H}/\hbar \right) \Delta t - \hat{A}^2 \int_t^{t+\Delta t} \int_t^{t+\Delta t} dt dt' \phi(t)\phi(t')/2 \quad (\text{E6})$$

This leads, using $\langle \phi \rangle = 0$ and $\langle \phi(t)\phi(t') \rangle = \lambda\delta(t-t')$, to

$$\begin{aligned} \hat{\rho}(t+\Delta t) &= \hat{\rho}(t) - \Delta t [\hat{H}, \hat{\rho}(t)] (i/\hbar) - i [\hat{A}, \hat{\rho}(t)] \int_t^{t+\Delta t} \langle \phi(t) \rangle dt \\ &+ \int_t^{t+\Delta t} \langle \phi(t)\phi(t') \rangle (\hat{A}\hat{\rho}(t)\hat{A} - \{\hat{A}^2, \hat{\rho}(t)\}/2) = \hat{\rho}(t) - \Delta t [\hat{H}, \hat{\rho}(t)] (i/\hbar) - \lambda\Delta t \{\hat{A}^2, \hat{\rho}(t)\}/2 + \lambda\Delta t \hat{A}\hat{\rho}(t)\hat{A}, \end{aligned} \quad (\text{E7})$$

which yields

$$\frac{d\hat{\rho}(t)}{dt} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}(t)] - \lambda \{\hat{A}^2, \hat{\rho}(t)\}/2 + \lambda \hat{A}\hat{\rho}(t)\hat{A} = [\hat{H}, \hat{\rho}(t)]/i\hbar - \lambda [\hat{A}, [\hat{A}, \hat{\rho}(t)]]/2. \quad (\text{E8})$$

Appendix F

Here we derive Eqs. (33)-(35). We will demonstrate the derivation of the two-time correlation function, Eq. (34). We start from Eq. (20) in the form (C2) and use the cyclic permutation property of the trace together with identities such as

$$\mathcal{T}e^{(i/\hbar)\int_0^{t'} d\tau \hat{H}_\phi(\tau)} \tilde{\mathcal{T}}e^{(i/\hbar)\int_0^t d\tau \hat{H}_\phi(\tau)} = \mathcal{T}e^{(i/\hbar)\int_{t'}^{t'} d\tau \hat{H}_\phi(\tau)} \quad (t' > t) \quad (\text{F1})$$

to get

$$\langle a(t)a(t') \rangle_b = \frac{1}{2} \int D_\lambda \phi \text{Tr} \left[\hat{A} \tilde{\mathcal{T}}e^{\int_{t'}^{t'} \hat{H}_\phi(s) ds / i\hbar} \left\{ \hat{A}, \tilde{\mathcal{T}}e^{\int_0^t \hat{H}_\phi(s) ds / i\hbar} \hat{\rho} \mathcal{T}e^{\int_0^{t'} i\hat{H}_\phi(s) ds / \hbar} \right\} \mathcal{T}e^{\int_{t'}^{t'} i\hat{H}_\phi(s) ds / \hbar} \right], \quad (\text{F2})$$

where $\hat{H}_\phi(t) = \hat{H} - \hbar\phi(t)\hat{A}$. The functional integral $\int D_\lambda \phi$ can be divided into a product of integrals performed over trajectories $\phi(t)$ between 0 and t and between t and t' . The former operates only on the ϕ dependent expression in the anticommutator brackets, yielding $\int D_\lambda \phi \tilde{\mathcal{T}}e^{-\int_0^t i\hat{H}_\phi(s) ds / \hbar} \hat{\rho} \mathcal{T}e^{\int_0^t i\hat{H}_\phi(s) ds / \hbar} = \check{U}(t, 0) \hat{\rho} = \hat{\rho}(t)$ which satisfies

$$\int D_\lambda \phi \tilde{\mathcal{T}}e^{-\int_0^t i\hat{H}_\phi(s) ds / \hbar} \hat{\rho} \mathcal{T}e^{\int_0^t i\hat{H}_\phi(s) ds / \hbar} = \check{U}(t, 0) \hat{\rho} = \hat{\rho}(t) \quad (\text{F3})$$

(Eq. (F3) is equivalent to (E5), generalized for finite time evolution). Eq. (F2) becomes

$$\langle a(t)a(t') \rangle_b = \int D_\lambda \phi \text{Tr} \left[\hat{A} \tilde{\mathcal{T}}e^{-i\int_{t'}^{t'} \hat{H}_\phi(s) ds / \hbar} \check{A}\check{U}(t, 0) \hat{\rho} \mathcal{T}e^{i\int_{t'}^{t'} \hat{H}_\phi(s) ds / \hbar} \right] \quad (\text{F4})$$

Using (F3) again, now in the form $\int D_\lambda \phi \tilde{\mathcal{T}}e^{-\int_{t'}^{t'} i\hat{H}_\phi(s) ds / \hbar} \hat{x}(t) \mathcal{T}e^{\int_{t'}^{t'} i\hat{H}_\phi(s) ds / \hbar} = \check{U}(t', t) \hat{x}(t)$, leads to (34). Eq. (35) is verified analogously.

Appendix G

Here we prove Eq. (41). From Eq. ((33)), (37) and (39), using also $\check{U}\hat{1} = 1$ (since $\check{L}\hat{1} = 0$) and $\{\hat{\sigma}_j, \hat{\sigma}_k\} = 2\delta_{jk}\hat{1}$ we get

$$\langle \sigma_z(t) \rangle_b = \frac{1}{2} \text{Tr} [\hat{\sigma}_z (1 + \hat{\sigma}_z(t))] = \frac{z(t)}{2} \text{Tr} (\hat{\sigma}_z^2) = z(t). \quad (\text{G1})$$

Next, from Eq. (34) and (37)

$$\langle \sigma_z(t)\sigma_z(t') \rangle_b = \frac{1}{4} \text{Tr} \left[\hat{\sigma}_z \check{U}(t', t) \{ \hat{\sigma}_z, (\hat{1} + \hat{\sigma}_z(t)) \} \right] \quad (\text{G2})$$

for $t' > t$. Using (39) gives $\{ \hat{\sigma}_z, (\hat{1} + \sigma_z(t)) / 2 \} = \hat{\sigma}_z + z(t)\hat{1}$. Finally,

$$\langle \sigma_z(t)\sigma_z(t') \rangle_b = \frac{1}{2} \text{Tr} \left[\hat{\sigma}_z \check{U}(t', t) (\hat{\sigma}_z + z(t)\hat{1}) \right] = \frac{1}{2} \text{Tr} [\hat{\sigma}_z (\hat{\sigma}_z(t' - t) + z(t)\hat{1})] = z(t' - t) \quad (\text{G3})$$

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