Universität Konstanz

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Konstanzer Schriften in Mathematik

Nr. 277, Januar 2011

ISSN 1430-3558

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Abstract: In correcting a small mistake in [10], we can prove a new result on non-exponential stability for a coupled system arising in resonators. This also gives another (surprising and) simple example for a thermoelastic system changing from exponential stability to non-exponential stability when changing from Fourier's law to Cattaneo's law in the modeling of heat conduction.

1 Introduction

In our paper [10], in particular, the thermoelastic system

$$a\Delta^2 u + \Delta\theta + \ddot{u} = 0, \tag{1.1}$$

$$\Delta\theta - m\theta + d\Delta\dot{\hat{u}} = c\dot{\hat{\theta}}, \tag{1.2}$$

where

$$\hat{f} = f + \tau\dot{f}, \qquad (\dot{f} = f_t = \frac{\partial f}{\partial t}), \tag{1.3}$$

for the functions $(u,\theta) = (u,\theta)(\mathbf{x},t)$ with $\mathbf{x} \in B \subset \mathbb{R}^n$, $t \geq 0$, was studied, where B is (smoothly) bounded, $n \geq 2$, and a, m, d, c are positive constants. Additionally, one has

[0]AMS subject classification: 35 L 35, 74 F 05, 74 G 50

Keywords and phrases: hyperbolic model in thermoelasticity, well-posedness, exponential stability, Fourier law, Cattaneo law

[1]Supported by the project "Partial Differential in Thermomechanics. Theory and Applications (MTM2009-08150)" "

[2]Supported by the DFG-project "Hyperbolic Thermoelasticity" (RA 504/3-3).

the boundary conditions

$$u(\mathbf{x},t) = \Delta u(\mathbf{x},t) = \theta(\mathbf{x},t) = 0, \qquad (\mathbf{x},t) \in \partial B \ \times \ [0,\infty), \tag{1.4}$$

and initial conditions for $(u, u_t, \theta, \theta_t)$ in $t = 0$. (For the background an the interpretation in micro-beam resonators see [10].)

The system is reformulated as a first-order system for

$$\mathbf{V} := (\hat{u}, \hat{u}_t, \theta, \theta_t)',$$

and we obtain

$$\mathbf{V}_t = A\mathbf{V}, \qquad V(0) = V^0 \tag{1.5}$$

with the (yet formal) differential operator A given by the symbol

$$A_f := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -a\Delta^2 & 0 & -\Delta & -\tau\Delta \\ 0 & 0 & 0 & 1 \\ 0 & \frac{d}{c\tau}\Delta & \frac{1}{c\tau}(\Delta - m) & -\frac{1}{\tau} \end{pmatrix}$$

and the initial value

$$\mathbf{V}^0(\mathbf{x}) := (\hat{u}, \hat{u}_t, \theta, \theta_t)'(\mathbf{x}, 0)$$

with its components being given in terms of the originally prescribed initial data by using the differential equations. As underlying Hilbert space we have

$$\mathcal{H} := (H^2(B) \cap H_0^1(B)) \times (L^2(B))^n \times H_0^1(B) \times L^2(B)$$

with inner product

$$\langle V, W \rangle_{\mathcal{H}} := \big(d\langle V^2, W^2\rangle + ad\langle \Delta V^1, \Delta W^1\rangle\big)$$

$$+\tau(\langle \nabla V^3, \nabla W^3\rangle + \tau m\langle V^3, W^3\rangle + c\langle V^3 + \tau V^4, W^3 + \tau W^4\rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the usual $L^2(B)$-inner product. The operator A is now given as

$$A : D(A) \subset \mathcal{H} \mapsto \mathcal{H}, \qquad AV := A_f V,$$

with

$$D(A) := \{V \in \mathcal{H} \mid V^2 \in H^2(B) \cap H_0^1(B),\ V^4 \in H_0^1(B),\ A_f V \in \mathcal{H}\}.$$

A generates a contraction semigroup (cf. [10] for the boundary condition $u \in H_0^2(B)$), and A^{-1} is compact, hence the spectrum $\sigma(A)$ of A equals the point spectrum $\sigma_p(A)$

Due to the boundary conditions, the following ansatz for $V = (V^1, V^2, V^3, V^4)^T$ is possible (cf. [10]):

$$V(t,x) = \sum_{j=1}^{\infty} (\alpha_j(t), \gamma_j(t), \delta_j(t), \varepsilon_j(t))^T \, w_j(x),$$

where $(w_j)_j$ denote the eigenfunctions of the Laplace operator under Dirichlet boundary conditions corresponding to the eigenvalue λ_j,

$$-\Delta w_j = \lambda_j w_j, \quad w_j = 0 \text{ on } \partial B, \tag{1.6}$$

with

$$0 < \lambda_1 \leq \cdots \leq \lambda_j \to \infty \quad (\text{as } j \to \infty).$$

Then the coefficients $\alpha_j, \gamma_j, \delta_j, \varepsilon_j$ satisfy the same differential equation,

$$c\tau z'''' + cz''' + (\lambda_j + m + ac\tau\lambda_j^2 + d\tau\lambda_j^2)z'' + (ac\lambda_j^2 + d\lambda_j^2)z' + a\lambda_j^2(\lambda_j + m)z = 0. \tag{1.7}$$

The corresponding characteristic polynomial P_j is given by

$$P_j(\beta) = \beta^4 + \frac{1}{\tau}\beta^3 + \frac{1}{c\tau}(\lambda_j + m + \tau(ac+d)\lambda_j^2)\beta^2 + \frac{1}{c\tau}(ac+d)\lambda_j^2\beta + \frac{a}{c\tau}(\lambda_j^3 + m\lambda_j^2). \tag{1.8}$$

The zeros of P_j are denoted by $\beta_1(j), \ldots, \beta_4(j)$. Let $\mathcal{S}$ denote the spectral set of all zeros,

$$\mathcal{S} := \{\beta_k(j) \mid j = 1, 2, 3 \ldots; \quad k = 1, 2, 3, 4\}.$$

Then it is easy to see ([10]) that

$$\sigma_p(A) \subset \mathcal{S}. \tag{1.9}$$

In [10, Theorem 4] it was claimed that

$$\exists\, \omega > 0 : \quad \sup\{\operatorname{Re}\beta \mid \beta \in \mathcal{S}\} \leq -\omega. \tag{1.10}$$

The final arguments in the proof of (1.10) are based on [10, (5.17)], saying

$$\operatorname{Re}\beta_2(j) = \operatorname{Re}\beta_1(j) \longrightarrow -\frac{1}{2\tau}, \tag{1.11}$$

as $j \to \infty$, which is claimed to follow from [10, (5.8)], saying

$$\operatorname{Re}\beta_1(j) + \operatorname{Re}\beta_3(j) = -\frac{1}{2\tau}, \tag{1.12}$$

and [10, (5.17)], saying

$$\operatorname{Re}\beta_4(j) = \operatorname{Re}\beta_3(j) \longrightarrow -\frac{1}{2\tau}. \tag{1.13}$$

But, (1.12) and (1.13) do *not* imply (1.11), but instead

$$\mathrm{Re}\,\beta_2(j) = \mathrm{Re}\,\beta_1(j) \longrightarrow 0. \tag{1.14}$$

This way, the assertion (1.10) (expressing [10, Theorem 4]) is seen to be wrong. But now, the correct relation (1.14) allows us first to prove that the semigroup is *not* exponentially stable (Theorem 2.1 below).

Since the considerations allow us to take the parameter m in the equation (1.2) for θ to be zero without changing the conclusions, we can conclude another new result that gives a new, simple, somehow surprising example for a thermoelastic system, the corresponding semigroup of which changes from exponentially decaying type to non-exponential decaying type if the model for heat conduction is changed from Fourier's law to Cattaneo's law (Theorem 3.1 below). This kind of behavior was observed for the rather special Timoshenko system with heat conduction in [2]; now we have another, simpler and even more convincing example that the change from Fourier's law to Cattaneo's law — the modification of a partially parabolic system with infinite propagation speed of signals to a hyperbolic model with finite propagagion speed — may change basic properties of the system and hence suggests to think about the modeling character of each system.

2 The non-exponential stability of the semigroup

Theorem 2.1

(i) $\sup\{\mathrm{Re}\,\beta \mid \beta \in \mathcal{S}\} = 0$.

(ii) $\mathcal{S} = \sigma(A) = \sigma_p(A)$.

(iii) $\{e^{tA}\}_{t\geq 0}$ *is not exponentially stable.*

Proof: Since (i) follows from (1.14), and since (iii) is an easy consequence of (i) and (ii), it remains to show, in view of (1.9),

$$\mathcal{S} \subset \sigma_p(A). \tag{2.1}$$

To prove (2.1) let $\beta \in \mathcal{S}$, i.e. $\beta = \beta_k(j)$ for some fixed $j \in \mathbb{N}$ and some fixed $k \in \{1, 2, 3, 4\}$, in particular

$$P_j(\beta) = 0. \tag{2.2}$$

Let $-\Delta w_j = \lambda w_j$ as given above in (1.6). Let $V^1 \in H^2(B) \cap H_0^1(B)$ be the solution to

$$(\frac{d\beta}{c\tau}\Delta)V^1 = (\beta^2 + \frac{\beta}{\tau})w_j - \frac{1}{c\tau}(\Delta - m)w_j, \tag{2.3}$$

and let

$$V := (V^1, \beta V^1, w_j, \beta w_j)^T.$$

Then we have $V \in D(A)$, and we claim:

$$AV = \beta V, \tag{2.4}$$

which would finish the proof. Now, the following equivalences hold: In view of the definition of V^1 in (2.3), the claim (2.4) is equivalent to

$$-a\Delta^2 V^1 - \Delta w_j - \tau\beta\Delta w_j = \beta^2 V^1. \tag{2.5}$$

Applying $(\frac{d\beta}{c\tau}\Delta)$ to both sides of (2.5), this is equivalent to

$$-a\Delta^2(\frac{d\beta}{c\tau}\Delta)V^1 - (\frac{d\beta}{c\tau}\Delta)\Delta w_j - \tau\beta(\frac{d\beta}{c\tau}\Delta)\Delta w_j = \beta^2(\frac{d\beta}{c\tau}\Delta)V^1, \tag{2.6}$$

which, using (2.3) and (1.6), is equivalent to

$$\Big(\beta^4 + \frac{1}{\tau}\beta^3 + \frac{1}{c\tau}(\lambda_j + m + \tau(ac+d)\lambda_j^2)\beta^2 + \frac{1}{c\tau}(ac+d)\lambda_j^2\beta + \frac{a}{c\tau}(\lambda_j^3 + m\lambda_j^2)\Big)w_j = 0,$$

which is equivalent to $P_j(\beta) = 0$ and hence finishes the proof.

3 Fourier versus Cattaneo law — from exponential to non-exponential stability

As a corollary to the previous section, we are able to present a new example, where the change from the Fourier law to the Cattaneo law in the modeling of the heat conduction changes the system from an exponentially stable system to a non-exponentially stable one.

This effect was known for a special Timoshenko type system with heat conduction and presented in [2]. Here, we give a simpler and even more convincing example for this partially surprising effect that asks for a discussion of the models and their range of validity — even for very simple systems.

In the resonator system discussed above, we may assume $m = 0$ and still are led to the same conclusions. The system — for $m = 0$ from now on — can be rewritten as follows: Introducing a heat flux vector q, a solution (u, θ, q) to

$$u_{tt} + a\Delta^2 u + \Delta\theta = 0, \tag{3.1}$$

$$c\theta_t + \operatorname{div} q - d\Delta u_t = 0, \tag{3.2}$$

$$\tau q_t + q + \nabla\theta = 0 \tag{3.3}$$

(plus initial and boundary conditions) gives a solution (u, θ) to the resonator equations (1.1), (1.2) by eliminating q again. The equation (3.3) represents the Cattaneo law for heat conduction. As we have seen in the previous section, the solutions do not tend to zero exponentially (and uniformly in the data).

On the other hand, if we take $\tau = 0$ in (3.3) — corresponding to Fourier's law for heat conduction —, we obtain, after eliminating the heat flux q, the classical system for the thermoelastic plate equation,

$$u_{tt} + a\Delta^2 u + \Delta\theta = 0, \tag{3.4}$$

$$c\theta_t - \Delta\theta - d\Delta u_t = 0, \tag{3.5}$$

which is known to be represented by an exponentially stable semigroup, see [3, 9, 8, 1, 4, 5, 6, 7]. Hence we have the following

Theorem 3.1 *The systems (3.1)–(3.3) and (3.4), (3.5), respectively, give an example where the change from Fourier's law to Cattaneo's law in the modeling of the heat conduction part changes the system from an exponentially stable one to a non-exponentially stable one.*

The just presented example for the effect described in the last theorem is simpler than the one presented in [2] which relied on special relations of coefficients, the advantages of which are destroyed by the change from Fourier to Cattaneo.

Acknowledgment: We thank Bugra Kabil for pointing out the mistake in [10] that led to the new results presented here.

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