When are Options Overpriced? The Black-Scholes Model and Alternative Characterisations of the Pricing Kernel.

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Abstract

An important determinant of option prices is the elasticity of the pricing kernel used to price all claims in the economy. In this paper, we first show that for a given forward price of the underlying asset, option prices are higher when the elasticity of the pricing kernel is declining than when it is constant. We then investigate the implications of the elasticity of the pricing kernel for the stochastic process followed by the underlying asset. Given that the underlying information process follows a geometric Brownian motion, we demonstrate that constant elasticity of the pricing kernel is equivalent to a Brownian motion for the forward price of the underlying asset, so that the Black-Scholes formula correctly prices options on the asset. In contrast, declining elasticity implies that the forward price process is no longer a Brownian motion: it has higher volatility and exhibits autocorrelation. In this case, the Black-Scholes formula underprices all options.

1 Introduction

Following Black and Scholes (1973), the traditional approach to the pricing of European-style options on an underlying asset assumes that the asset price follows a given, exogenous process and prices the options using an arbitrage-free hedging argument. An alternative equilibrium approach, followed by Rubinstein (1976) and Brennan (1979), assumes that the asset price and the value of the market portfolio at the end of a single period have a given joint probability distribution and that a representative investor exists, with a given utility function for end of period wealth. It has been shown that both these approaches can lead to the same risk-neutral valuation relationship for the option price. A third approach, following Harrison and Kreps (1979), assumes a no-arbitrage economy which in turn implies the existence of a pricing kernel. This pricing kernel variable has the important property that the option forward price equals the expected value of the product of the option payoff and the pricing kernel. This third approach is consistent with the equilibrium approach, since the Brennan-Rubinstein assumptions imply a pricing kernel which equals the relative marginal utility of the representative investor.

In this paper, we adopt the more general pricing kernel framework. Assuming that the asset-specific pricing kernel exhibits constant elasticity, yields the Black-Scholes assumption of a geometric Brownian motion of the asset price. Assuming a representative investor exists with constant relative risk aversion, implies the Brennan-Rubinstein world. However, the general framework permits the pricing of options under less restrictive assumptions. In particular, it turns out that the curvature of the pricing kernel is critical for the pricing of options. Alternative characterizations of the elasticity of the pricing kernel with respect to the price of the underlying asset lead to different option prices. In order to investigate the effect of alternative pricing kernels, we start with an assumption regarding the price of the underlying asset. Option pricing models typically take as given either the price of the underlying asset and the risk-free rate of interest, or alternatively, the asset forward price. In this paper, we assume throughout that the current forward price, for delivery at a fixed terminal date, is given. Thus, when we compare the effect on option prices of different characterizations of the pricing kernel, we do so assuming that the different pricing kernels lead to the same current forward price of the asset.

We first investigate the relative pricing of options in a general setting, where the forward price of the asset is given and the asset-specific pricing kernel exhibits either constant or declining elasticity. We find that the prices of all options are higher in the economy with declining elasticity than in the economy with constant elasticity. These higher prices are the result of the increased convexity of the pricing kernel. In the special case where asset prices on the terminal date are lognormal, all option prices exceed the Black-Scholes prices, if the pricing kernel has declining elasticity.

How can it be, then, that the Black-Scholes model underprices all options, when we know that if the asset forward price follows a geometric Brownian motion, no-arbitrage arguments can be used to establish the Black-Scholes prices? We investigate the answer to this puzzle and find that, in a declining elasticity pricing kernel economy, the asset forward price does *not* follow a geometric Brownian motion, even though the information process does. We first establish the conditions under which the asset forward price follows a geometric Brownian motion. We then investigate the effect on the price process of the alternative assumptions regarding the elasticity of the pricing kernel.

The organization of this paper is as follows. In the following section we review previous related work. Then, in section 3, we establish our principal result: all options have higher prices in the declining elasticity economy than in the constant elasticity economy. In section 4, we consider a Black-Scholes world in which the terminal asset price is lognormal, and we establish the equivalence of two alternative assumptions: constant elasticity of the pricing kernel, and a geometric Brownian motion of the asset forward price. Section 5 then investigates the effect on the stochastic process of the asset forward price, of the alternative assumption of declining elasticity. In section 6, we assume a more traditional, representative agent economy, and establish sufficient conditions for declining elasticity of the pricing kernel in an economy in which the asset price and aggregate consumption are related by a log-linear regression. Section 7 summarizes the main conclusions of our analysis.

2 Recent Literature on the Mispricing of Options by the Black-Scholes Model

Empirical research in the last few years has suggested that options are underpriced by the Black-Scholes model, i.e., the implied volatility of options typically exceeds the historical volatility of the price of the underlying asset (see, for example Canina and Figlewski (1993)). This evidence is corroborated by studies that estimate the expected value of the implied pricing kernel and the parameters of the risk-neutral distribution, using index options data (for example, see Longstaff (1995), Brenner and Eom (1998), and Buraschi and Jackwerth (1998)). Although many alternative explanations have been proposed for these findings, ranging from jumps in the price process to the existence of "fat tails" in the return distribution of the underlying asset, most of the explanations relate one way or another to the stochastic process followed by the price of the underlying asset. We suggest an alternative explanation. We derive a model in which all options are underpriced by the Black-Scholes model, even though the underlying asset price has a lognormal distribution on the terminal date. Also, the price process exhibits excess volatility, even though the information process for the underlying asset follows a geometric Brownian motion. In our model, it is the characteristics of the pricing kernel, i.e. of the risk adjustment, that produces the excess-pricing of the options.

In a closely related recent paper, Mathur and Ritchken (1995) consider the price of options on the market portfolio, in a single-period, representative agent model. Restricting their analysis to agents with declining absolute risk aversion, they conclude that the price of an option given constant proportional risk aversion (CPRA), is the minimum option price. The implication is that declining proportional risk aversion will produce higher option prices. In the special case of a lognormal market portfolio payoff, the Black-Scholes price, resulting from CPRA, is the minimum option price. Our results, cast in terms of the characteristics of the asset-specific pricing kernel rather than risk attitudes, generalize and explain this conclusion in several ways. First, we consider options on assets in a multi-asset economy. In the special case where we consider options on the market portfolio, our results are consistent with those of Mathur and Ritchken. The second generalization is that we do not assume a representative agent economy. In contrast, we assume, in section 3, that the pricing kernel has declining elasticity. This is consistent with, but does not require, declining proportional risk aversion of the representative agent.[1] Thirdly, our conclusions hold in a general, multi-period economy rather than only in the single-period economy. Our conclusions in section 5, regarding the effect of declining elasticity on the stochastic process

[1]Another set of conditions in which CPRA investors act as if they have declining proportional risk aversion is provided by Franke, Stapleton and Subrahmanyam (1998). They show that if investors face non-hedgeable background risks, they act like investors with declining proportional risk aversion and demand options to hedge the marketable risks that they face.

followed by asset forward prices also help to explain Mathur and Ritchken's results. Although they assume a single period economy, the question arises as to how their results are consistent with the stochastic process followed by prices between the two dates. The answer in our model is that underpricing by the Black-Scholes model is consistent with a process for the forward price that exhibits excess volatility.

Benninga and Mayshar (1997) analyze a model in which heterogeneous investors with different levels of CPRA act like a representative investor with declining proportional risk aversion. They also find that certain options are underpriced by the Black-Scholes model. Our paper is also closely related to the prior work of Bick [(1987) and (1990)], Franke (1984) and Stapleton and Subrahmanyam (1990). These authors investigated the consistency of various asset price processes in a representative investor economy, Bick in a continuous-time setting and Franke and Stapleton and Subrahmanyam in a discrete-time setting.[2] Our analysis, in section 4, on the equivalence of constant elasticity of the pricing kernel and a random walk in the asset forward price, parallels that of Bick. Again, our analysis here is somewhat more general, relying on the existence of a pricing kernel, rather than a representative investor who is limited to puchasing claims on the market portfolio.

3 Contingent Claims Prices Given Declining Elasticity of the Pricing Kernel

In this section, we analyze the prices of contingent claims in a perfect capital market, where arbitrage possibilities do not exist. We do so by examining the properties of the pricing kernel, a variable which can be used to price any claim in this economy.

Consider a date t in the interval $[0, T]$ where 0 is the current date and T is some terminal date. Let $S_{T,j}$ be the price of the asset j at time T. The forward price at date t, for delivery of the asset at date T is denoted $F_{t,T,j}$.

Based on the absence of arbitrage there exists a pricing kernel, $\psi_{t,t+1}$, such that for any asset or claim on an asset, j,

[2] Franke (1984) and Stapleton and Subrahmanyam (1990) use a somewhat different approach to characterize the preferences that support a geometric random walk. They start with a process for the cash flows, the fundamental exogenous variable, and derive the restrictions required for the process for cash flows to be transformed into a geometric random walk for returns.

$$F_{t,T,j} = E_t[F_{t+1,T,j}\psi_{t,t+1}] \tag{1}$$

where E_t is the expectation operator conditional on the information set at time t. $\psi_{t,t+1}$ is a positive random variable. Since a risk-free claim on a dollar to be paid at date T always has a forward price of one dollar, it follows from the no-arbitrage condition in equation (1) that the pricing kernel has an expectation of unity, i.e. $E_t(\psi_{t,t+1}) = 1$. Now, defining the pricing kernel over the interval from t to T as

$$\psi_{t,T} = \psi_{t,t+1}\psi_{t+1,t+2}...\psi_{T-1,T}$$

it follows by successive substitution and using the unbiased expectations property of conditional expectations, that the asset forward price is

$$F_{t,T,j} = E_t[S_{T,j}\psi_{t,T}], \tag{2}$$

since $F_{T,T,j} = S_{T,j}$. Also, it follows that $E_t(\psi_{t,T}) = 1$.

The pricing kernel, $\psi_{t,T}$, prices any date T claim. If we now consider claims contingent on a single asset, j, with price $S_{T,j}$, we can define and use a pricing kernel unique to asset j. Defining

$$\phi_{t,T,j} = E_t[\psi_{t,T}|S_{T,j}]$$

and using the property of conditional expectations we can re-write equation (2) as

$$F_{t,T,j} = E_t[S_{T,j}\phi_{t,T,j}] \tag{3}$$

where the expectation is over states of $S_{T,j}$ and $\phi_{t,T,j}$ is a time T measurable random variable, unique to asset j. Clearly, $\phi_{t,T,j}$ is a function of $S_{T,j}$. [3]

Since we are concerned here with the pricing of contingent claims on (any) single asset, we drop the subscript j in equation (3) and write the basic pricing equation as simply

$$F_{t,T} = E_t[S_T\phi_{t,T}] \tag{4}$$

We assume that $\phi_{t,T}$ is twice differentiable in S_T. Having described the basic economy, we can now proceed to price contingent claims.

[3] The pricing kernel, $\phi_{t,T}$, can also be derived using the first order condition for the optimal portfolio choice of the investor in a representative agent economy. This is discussed in more detail in section 6 below.

Pricing of European Options: The General Case

A similar no-arbitrage pricing argument can be used to evaluate the forward price of a European-style contingent claim on the risky asset. If the payoff on the contingent claim at time T is $g(S_T)$, then the forward price of the contingent claim at time t, for delivery at T, denoted $C_{t,T}$, is given by

$$C_{t,T} = E_t[g(S_T)\phi_{t,T}] \tag{5}$$

In option pricing, we generally take the price of the underlying asset as given, and consider only the *relative* pricing of the option. We take a similar approach here, $F_{t,T}$ is assumed to be at a given level, $F^*_{t,T}$:

$$F_{t,T} = F^*_{t,T} = E_t[S_T\phi_{t,T}]. \tag{6}$$

We then ask the following question. How does the forward price of the option $C_{t,T}$ depend on the pricing kernel, $\phi_{t,T}$, given that $F_{t,T} = F^*_{t,T}$? Clearly, assuming only that $F_{t,T} = F^*_{t,T}$ leaves room for several alternative shapes of the pricing kernel $\phi_{t,T}$, since there is an infinite number of possible pricing kernels that satisfy the constraint in equation (6). We now establish a result which characterizes the $\phi_{t,T}$ functions which satisfy equation (6).

Since option prices are dependent on the joint relationship of the pricing kernel, $\phi_{t,T}$, and the price of the asset on the terminal date, we can analyse option prices by investigating the *elasticity* of the pricing kernel, $\phi_{t,T}$, with respect to the asset price on the terminal date. The elasticity is defined in the conventional manner as

$$\eta(S_T) = -\frac{\partial\phi_{t,T}}{\phi_{t,T}} \Big/ \frac{\partial S_T}{S_T} \tag{7}$$

We define the elasticity of two different pricing kernels, both of which satisfy equation (6) as follows. The first pricing kernel $\phi_{t,T,1}$, written henceforth as ϕ_1, has constant elasticity η_1, i.e., $\eta_1' = 0$. The second pricing kernel $\phi_{t,T,2}$, written as ϕ_2, has declining elasticity η_2, where η_2' is negative for all values of S_T. We first establish the following result about the properties of the two pricing kernels.

Lemma 1 *(Intersections of Pricing Kernels with Different Elasticities)*

Consider two pricing kernels, ϕ_1 and ϕ_2, each of which yields the same forward asset price $F^*_{t,T}$. Suppose that for ϕ_1, the elasticity is constant, i.e. $\eta_1' = 0$, and for ϕ_2, the elasticity is declining, i.e. $\eta_2' < 0$, $\forall S_T$, then the pricing kernels ϕ_1 and ϕ_2 intersect twice.

Proof:
Consider the two pricing kernels ϕ_1 and ϕ_2 with corresponding elasticities η_1 and η_2 for which $\eta_1' = 0$ and $\eta_2' < 0$. This implies that

$$\frac{\partial}{\partial S_T}\left[\frac{\eta_2}{\eta_1}\right] < 0. \tag{8}$$

Suppose that both pricing kernels satisfy equation (6). First, it is necessary that the two pricing kernels (see Figure 1) intersect at least once. Otherwise, it would be impossible for them to have the property $E(\phi_1) = E(\phi_2) = 1$. Second, the two pricing kernels must intersect more than once, since otherwise the forward price of the risky asset, $F^*_{t,T}$, cannot be the same under both pricing kernels. To see this, suppose that the two pricing kernels intersect only once at $S_T = \hat{S}_T$. Suppose that $\phi_1 > [<]\phi_2$ for $S_T < [>]\hat{S}_T$. Then, consider a claim paying $(S_T - \hat{S}_T)$ at date T. Then, $E[(S_T - \hat{S}_T)\phi_2] > E[(S_T - \hat{S}_T)\phi_1]$ follows since $(S_T - \hat{S}_T)(\phi_2 - \phi_1) \geq 0, \forall S_T$. As $E[(S_T - \hat{S}_T)\phi] = E[S_T\phi] - \hat{S}_T$, the forward price of the risky asset would be higher under pricing kernel ϕ_2 than under ϕ_1. Hence, the forward price can be the same only if the pricing kernels intersect at least twice. Finally, we show in Appendix A that more than two intersections contradicts the assumption in equation (8). □

The lemma is illustrated in Figure 1. For prices below S_T^A, $\phi_2 > \phi_1$. This implies that for contingent claims that pay off only in the region $S_T < S_T^A$, contingent claim prices will be higher under ϕ_2 than under ϕ_1. Also, for prices above S_T^B, we have $\phi_2 > \phi_1$. Again, for contingent claims that pay off only in the region $S_T > S_T^B$, contingent claim prices will be higher under ϕ_2 than under ϕ_1. In particular, put options with strike prices at or below S_T^A and call options with strike prices at or above S_T^B have higher prices under the declining elasticity pricing kernel. However, the following Theorem establishes that *all* options have higher prices.

Theorem 1 *(The Pricing of European-Style Options)*
Consider two pricing kernels, ϕ_1 and ϕ_2, both of which yield the same forward price of the risky asset. Suppose that for pricing kernel ϕ_1, the elasticity is constant and for pricing kernel ϕ_2, the elasticity is declining. Then, the price of any European-style option is greater under pricing kernel ϕ_2 than under ϕ_1.

Proof:

We show in Appendix A that the two pricing kernels ϕ_1 and ϕ_2 intersect twice, at points which we denote as S_T^A and S_T^B. That is

$$\begin{array}{lll} \phi_2 & > & \phi_1 \quad \text{for } S_T < S_T^A, \\ \phi_2 & < & \phi_1 \quad \text{for } S_T^A < S_T < S_T^B, \\ \phi_2 & > & \phi_1 \quad \text{for } S_T^B < S_T. \end{array} \tag{9}$$

Now let $L_k(S_T) = a_k + b_k S_T$, where a_k and b_k are chosen so that

$$L_k(S_T) = (S_T - k)^+ \text{ ,for } S_T = S_T^A, \text{and } S_T = S_T^B. \tag{10}$$

The forward price of a call option with strike price k is

$$C_{k,j} = E[(S_T - k)^+ \phi_j], j = 1, 2 \tag{11}$$

which can be written

$$C_{k,j} = E[((S_T - k)^+ - L_k(S_T))\phi_j] + E[L_k(S_T)\phi_j] \text{ }, j = 1, 2 \tag{12}$$

Since the forward price of a linear payoff is the same under both pricing kernels, i.e.,

$$E[L_k(S_T)\phi_1] = E[L_k(S_T)\phi_2], \tag{13}$$

it follows that

$$C_{k,2} - C_{k,1} = E[((S_T - k)^+ - L_k(S_T))(\phi_2 - \phi_1)]. \tag{14}$$

It follows from the definition of $L_k(S_T)$ that $(S_T - k)^+ - L_k(S_T) \geq [=][\leq]0$, when $\phi_2 - \phi_1 > [=][<]0$, and hence $C_{k,2} > C_{k,1}$.

Also, by put-call parity, all puts must have higher forward prices under ϕ_2 than under ϕ_1. □

Theorem 1 shows that given the same forward price for the underlying asset, all options, both puts and calls at *any* strike price are more highly priced by the declining elasticity pricing kernel, ϕ_2, compared to the constant elasticity pricing kernel, ϕ_1.[4] The intuitive reason for this "mispricing" is that the declining elasticity pricing kernel is more convex than the one

[4] We exclude cases where there is a zero probability of finishing out-of-the-money. For example, a call option at a strike price of zero, always finishes in-the-money. By definition, its forward price is the same as the forward price of the underlying asset, and hence equal under the two pricing kernels.

with constant elasticity. This convexity implies that convex claims, such as options, are valued more highly by the declining elasticity pricing kernel, all else being the same. In other words, extreme payoffs on either side of the mean are priced more highly by the declining elasticity pricing kernel. However, linear claims such as the forward contract on the asset are priced the same, by assumption. Although the payoffs close to the mean are priced lower by the declining pricing kernel, this is not sufficient to outweigh the higher pricing of the more extreme payoffs.

Theorem 1 is a general result for the pricing of European-style options: it holds for any probability distribution of S_T. An important implication of the result is that option pricing models that implicitly assume a constant elasticity for the pricing kernel yield lower option prices than those that assume declining elasticity. If the true pricing kernel has declining elasticity, the use of such models leads to mispricing.

4 Constant Elasticity of The Pricing Kernel: The Black-Scholes Economy

We have shown above that if the pricing kernel exhibits declining elasticity, then European options are underpriced by any model that assumes, either explicitly or implicitly, that the pricing kernel has constant elasticity. Hence, the question arises as to what pricing kernel property would yield the same option prices as the Black-Scholes model. Since the Black-Scholes model follows from the assumption that the forward price of the underlying asset follows a geometric Brownian motion, we need to investigate the relationship between the properties of the pricing kernel and the asset price process. In this section, we first examine the relationship between the two assumptions: the elasticity of the pricing kernel is constant, and the asset forward price follows a geometric Brownian motion. We then illustrate the case of constant elasticity using an example, where the forward price follows a stationary geometric binomial process. In the following section, we relax the assumption of constant elasticity and investigate the effects on the price process.

4.1 The General Case

We assume here that the conditional expectation of the underlying asset price at time T, S_T, evolves as a geometric Brownian motion. We show in the following theorem that two properties: A) the pricing kernel has constant, non-state dependent elasticity, and B) the forward price of the asset follows a geometric Brownian motion, are equivalent.[5] In the following section we then proceed to derive the implications of declining elasticity for the forward price process.

First, let B_τ be a Brownian motion on the probability space (Ω, F, P). We define the information process for the price S_T as the conditional expectation process of S_T, $I_\tau = E_\tau(S_T)$, $\tau \epsilon (t, T)$. We assume that the behaviour of I_τ is governed by the stochastic differential equation:

$$\frac{dI_\tau}{I_\tau} = \alpha d\tau + \sigma dB_\tau \tag{15}$$

where σ is a constant and α, the mean of the process, is zero, simply because it is an information process. It follows that S_T is lognormally distributed. We now investigate conditions under which the forward price $F_{\tau,T}$ follows a geometric Brownian motion process of the form

$$dF_{\tau,T} = F_{\tau,T}\mu_\tau d\tau + F_{\tau,T}\sigma dB_\tau, \; t \leq \tau \leq T, \tag{16}$$

where the drift, μ_τ is non-stochastic, but possibly time dependent. It is known that, if the forward price is governed by (16), then the Black-Scholes prices for European-style options must obtain. Hence, we are also looking at conditions for the Black-Scholes theorem to hold. We establish:

Theorem 2 *(Constant Elasticity of the Pricing Kernel)*
Given that the information process for the underlying asset is

$$\frac{dI_\tau}{I_\tau} = \alpha d\tau + \sigma dB_\tau$$

with $\alpha = 0$, then the following statements are equivalent:
A) The pricing kernel, $\phi_{t,T}$ has constant elasticity,

$$\eta_{t,T} = \frac{\int_t^T \eta_\tau d\tau}{(T-t)} = \frac{\int_t^T \mu_\tau d\tau}{\sigma^2(T-t)}$$

[5] Note that in the multi-period world A) includes the condition of non-state dependency of the elasticity of the pricing kernel. In principle, it is possible for the pricing kernel elasticity to be state dependent, i.e. for the elasticity of $\phi_{t,T}$ to depend on the state at t, for $t < T$.

in each state and at each date, where $\eta_\tau = \mu_\tau/\sigma^2$*.*

B) The asset forward price, $F_{\tau,T}$ *follows a geometric Brownian motion, with drift,* μ_τ *and standard deviation,* σ*.*
Also, if A) or B) holds, then the Black-Scholes formula for the price of a European-style option on S_T *holds, at each date and in each state.*

Proof:
$\underline{B \Rightarrow A}$ Assume that the forward price follows the geometric Brownian motion

$$dF_{\tau,T} = F_{\tau,T}\mu_\tau d\tau + F_{\tau,T}\sigma dB_\tau,\ t \leq \tau < T.$$

For notational covenience, since T is fixed, we write this as

$$dF_\tau = F_\tau \mu_\tau d\tau + F_\tau \sigma dB_\tau. \tag{17}$$

We now consider the process for the conditional expectation of the pricing kernel, $\phi_{t,T}$. $\phi_{t,T}$ is a time T measureable random variable, and its conditional expectation is $E_\tau(\phi_{t,T})$. For simplicity, we denote

$$E_\tau(\phi_{t,T}) \equiv \theta_\tau = \theta_\tau(F_\tau, \tau)$$

where, by assumption, θ_τ is a twice continuously differentiable function of the forward price F_τ and of time τ. By Ito's lemma,

$$d\theta_\tau = (\frac{\partial \theta_\tau}{\partial \tau} + \frac{1}{2}\frac{\partial^2 \theta_\tau}{\partial F_\tau^2} F_\tau^2 \sigma^2) d\tau + \frac{\partial \theta_\tau}{\partial F_\tau} F_\tau dF_\tau$$

Since θ_τ is the conditional expectation of the pricing kernel, it is a P martingale. It follows that the terms in $d\tau$ must add to zero. Hence, we have

$$d\theta_\tau = \frac{\partial \theta_\tau}{\partial F_\tau} F_\tau \sigma dB_\tau. \tag{18}$$

In appendix B, we show that it follows from the definition of the forward price that $E_\tau(dF_\tau \theta_{\tau+d\tau}) = 0$. Since $\theta_{\tau+d\tau} = \theta_\tau + d\theta_\tau$, we have, using the expressions for $d\theta_\tau$ and dF_τ,

$$\mu_\tau d\tau \theta_\tau + \sigma^2 F_\tau \frac{\partial \theta_\tau}{\partial F_\tau} d\tau = 0$$

which implies that, for the elasticity η_τ,

$$\eta_\tau \equiv -\frac{\partial \theta_\tau}{\partial F_\tau}\frac{F_\tau}{\theta_\tau} = \frac{\mu_\tau}{\sigma^2},\ \forall \tau. \tag{19}$$

From (18) and (19) it follows that

$$d\theta_\tau = -\theta_\tau \eta_\tau \sigma dB_\tau,\ \forall \tau.$$

θ_τ follows a geometric Brownian motion. Hence, $\theta_T = \phi_{t,T}$ is lognormal. Since $\phi_{t,T} = \phi_{t,T}(S_T)$, where S_T is also lognormal, then $\phi_{t,T}$ has constant elasticity with respect to S_T. From (19), $\mu_\tau = \eta_\tau \sigma^2$ so that

$$\int_t^T \mu_\tau d\tau = \sigma^2 \int_t^T \eta_\tau d\tau \equiv \sigma^2 (T-t)\eta_{t,T}$$

This establishes that the pricing kernel has constant, non-state-dependent elasticity, $\eta_{t,T}$, with respect to the terminal spot price, when the forward price follows a geometric Brownian motion.□

<u>$A \Rightarrow B$</u> Assume that the pricing kernel $\phi_{t,T}$ has constant, non-state-dependent elasticity, $\eta_{t,T}$. Constant elasticity with respect to S_T implies that we can write the pricing kernel as

$$\phi_{t,T} = \lambda_{t,T} S_T^{-\eta_{t,T}}.$$

Hence, from the condition $E_t(\phi_{t,T}) = 1$,

$$\lambda_{t,T}^{-1} F_t^{\eta_{t,T}} = E_t\left[\left(\frac{S_T}{F_t}\right)^{-\eta_{t,T}}\right].$$

Also, from $F_t = E_t(S_T \phi_{t,T})$

$$\lambda_{t,T}^{-1} F_t^{\eta_{t,T}} = E_t\left[\left(\frac{S_T}{F_t}\right)^{-\eta_{t,T}+1}\right].$$

Equating these expressions, defining $[\mu(F_t) - \sigma^2/2](T-t)$ as the mean of the logarithm of S_T/F_t, given the forward price F_t, and using the properties of lognormal variables, yields[6]

$$\mu(F_t) = \sigma^2 \eta_{t,T}$$

[6] If X is lognormally distributed with $E(\ln X) = \mu - \sigma^2/2$, then $E(X^a) = exp[a(\mu + (a-1)\sigma^2/2)]$

But, by assumption $\eta_{t,T}$ and hence $\mu(F_\tau)$ is state independent. Hence, since $\eta_{t,T} = \frac{\int_t^T \mu_\tau d\tau}{\sigma^2(T-t)}$ and S_T/F_t is lognormal, for all $t\epsilon[0,T)$,

$$E_t(\ln S_T) - \ln F_t = \int_t^T \mu_\tau d\tau - \frac{\sigma^2}{2}(T-t). \tag{20}$$

Now, consider the information process, I_t. It has zero drift, i.e.

$$\frac{dI_t}{I_t} = \sigma dB_t.$$

This implies that S_T is lognormal with

$$E_t(\ln S_T) = \ln E_t(S_T) - \frac{\sigma^2}{2}(T-t).$$

From this equation and (20) it follows that

$$\ln F_t = \ln E_t(S_T) - \int_t^T \mu_\tau d\tau$$

and since $\ln E_t(S_T)$ is a Brownian motion, so is $\ln F_t$.□

Finally, it is well known that condition B above implies that the Black-Scholes model holds. The proof is similar to the original Black-Scholes proof, with the forward price process substituted for the spot price process.□

Theorem 2 shows that the assumption of the Black-Scholes model, that the asset (forward) price follows a Brownian motion, is equivalent to constant elasticity of the pricing kernel. It follows, using Theorem 1 that the Black-Scholes model underprices options in a declining elasticity economy. We have the following:

Corollary 1 *(Declining Elasticity and Black-Scholes Underpricing)*

Suppose that the information process of S_T follows a standard geometric Brownian motion and that the forward price $F_{t,T}$ is given. Then, if the pricing kernel has the property of declining elasticity, all options on S_T will have higher forward prices at date t than those given by the Black-Scholes model.

Proof:

First, from Theorem 2, the Black-Scholes formula holds if the pricing kernel has constant, non-state-dependent elasticity. Further, from Theorem

1 we know that, if the pricing kernel has declining elasticity, all options have higher prices than in the case of constant elasticity. Hence, the forward prices of options in the case of declining elasticity exceed the Black-Scholes prices.□

In Theorem 2 we show that the assumption of either a Brownian motion or a pricing kernel with constant, non-state-dependent elasticity is sufficient for the Black-Scholes model to hold. The prior work of Brennan (1979), who showed that, in a representative agent single-period economy, constant relative risk aversion is a necessary condition for Black-Scholes to price options on the market portfolio suggests that these conditions may also be necessary. However this is not the case. The Black-Scholes model does not require a pricing kernel with constant, state-independent elasticity, or a Brownian motion in the forward price. If, however, we add a mild restriction on the pricing kernel in an intertemporal setting, to the effect that the pricing kernel is path independent, we can show necessity of the Brownian motion. First, we define path-independence of the pricing kernel.

Definition [Path-independence of the pricing kernel]

A pricing kernel is path-independent if for any two outcomes of S_T: $S_{T,1}$, $S_{T,2}$, the ratio

$$\frac{\phi_{t,T}(S_{T,1})}{\phi_{t,T}(S_{T,2})}$$

does not depend on the state I_t, $\forall t < T$. We now establish

Corollary 2 *(Necessity of a Brownian Motion in the Forward Price for Black-Scholes Pricing)*

Assume the same information process as in Theorem 2 and path independence of the pricing kernel. Then the Black-Scholes formula correctly prices European-style options on an asset with price S_T at time T, only if the underlying asset has a forward price which follows a Brownian motion.

Proof:

If the Black-Scholes model holds at date t, the risk-adjusted density of S_T must be lognormal. This density equals the true density multiplied by $\phi_{t,T}(S_T)$. Since the true density is lognormal, by assumption, it follows that $\phi_{t,T}(S_T)$ has constant elasticity, $\eta_{t,T}$, which may, however, depend on I_t. Hence

$$\frac{\phi_{t,T}(S_{T,1})}{\phi_{t,T}(S_{T,2})} = \left(\frac{S_{T,1}}{S_{T,2}}\right)^{-\eta_{t,T}(I_t)},$$

so that the pricing kernel is path-dependent. But this path dependency is ruled out by assumption. Hence, by the equivalence of A) and B) in Theorem 2 it follows that lnF_τ is a Brownian motion.□

Corollary 2 shows that a geometric Brownian motion information process and path independence of the pricing kernel imply a Brownian motion of the asset forward price, if the Black-Scholes model is to hold. Many financial models assume time additive utility of a representative investor, an assumption which guarantees path independence of the pricing kernel. Hence, the Black-Scholes world is only slightly more general than a world where the asset forward price follows a Brownian motion.

4.2 Constant Elasticity: An Example in the Case of a Binomial Process.

In order to clarify the restrictions implied by constant elasticity of the pricing kernel, we now look at an example where the asset forward price follows a binomial process. The example allows us to specify the process followed by the conditional expectation of $\phi_{t,T}$. In order to be consistent, in the limit, with geometric Brownian motion, we assume that the information process of S_T follows a multiplicative binomial process.

Given a forward asset price $F_{t,T}$, we now assume an n-stage, stationary multiplicative binomial process for the forward price $F_{\tau,T}$, over the period from t to T. Specifically, let u and d be the proportionate up and down movements of the binomial process over each sub-interval, then

$$\frac{F_{\tau+1,T}}{F_{\tau,T}} = \left\{ \begin{array}{ccc} u & , & q \\ d & , & 1-q \end{array} \right\}, \forall\tau \tag{21}$$

where q is the probability of an up-movement in the forward price over any sub-interval. When n is large, the process in (21) converges to a Brownian motion process. We now show, consistent with Theorem 2, that the pricing kernel has constant elasticity.

First, we need to specify the pricing kernel process. Defining $F_{t+\Delta t,T} = F_{t,T} + \Delta F_t$ and noting that $E[\Delta F_t \theta_{t+1}] = 0]$ from the results in appendix B, it follows that

$$F_{t,T} = E_t[F_{t+1,T}\theta_{t+1}],$$

where θ_{t+1} is the conditional expectation, at time $t+1$, of the pricing kernel $\phi_{t,T}$. In the binomial case, there are only two states at time $t+1$, so we can write

$$F_{t,T} = qF_{t+1,T,u}\theta_{t+1,u} + (1-q)F_{t+1,T,d}\theta_{t+1,d} \tag{22}$$

where $\theta_{t+1,u}$ and $\theta_{t+1,d}$ are the values, of the conditional expectation $E_{t+1}(\phi_{t,T})$, in the up-state and down-state respectively.

However, since the forward price moves from t to $t+1$ as a two-state branching process we have a dynamically complete market economy. It follows that there exists a unique "risk neutral" probability measure under which the forward price of the asset is a martingale. Also the probability of an up movement under this measure over any sub-period is a constant:

$$p = \frac{1-d}{u-d}, \quad 0 \le p \le 1$$

The forward price of the risky asset at any point of time t must also therefore be given by the equation:

$$F_{t,T} = pF_{t+1,T,u} + (1-p)F_{t+1,T,d}$$

or

$$F_{t,T} = qF_{t+1,T,u}\left(\frac{p}{q}\right) + (1-q)F_{t+1,T,d}\left(\frac{1-p}{1-q}\right). \tag{23}$$

Equating (23) and (22) for the conditional expectation of the pricing kernel, $E_{t+1}(\phi_{t,T})$,

$$\theta_{t+1,u} = \frac{p}{q}, \qquad \theta_{t+1,d} = \frac{1-p}{1-q}.$$

in the up-states and down-states. Also, if j is the number of up movements of the asset price over the n sub-periods from t to τ,

$$\theta_{\tau,j} = (\theta_{t+1,u})^j(\theta_{t+1,d})^{n-j}$$

We show now that $\ln(F_{\tau,T})$ and $\ln\theta_\tau$ are perfectly correlated. First, the forward price, after j up-moves, is given by

$$F_{\tau,T,j} = F_{t,T}u^j d^{n-j}$$

Hence, taking the logarithm of the pricing kernel expectation and of the forward price, yields

$$\ln\theta_{\tau,j} = j\ln\theta_{t+1,u} + (n-j)\ln\theta_{t+1,d}$$

and

$$\ln F_{\tau,T,j} = \ln F_{t,T} + j\ln u + (n-j)\ln d$$

Thus, $\ln\theta_\tau$ and $\ln F_{\tau,T}$ are linear in j. It follows that we can write in general

$$\ln\theta_\tau = \alpha_\tau + \beta\ln F_{\tau,T}$$

for appropriate α_τ and β, and in particular:

$$\ln\phi_{t,T} = \alpha_T + \beta\ln S_T \tag{24}$$

Equation (24) establishes the perfect correlation of $\ln(S_T)$ and $\ln\phi_{t,T}$.

We can now investigate the elasticity of the pricing kernel. Equation (24) is the key to understanding the restrictions imposed on the pricing kernel by the assumption of the lognormal process for the asset price. It implies that the pricing kernel has the same stochastic properties as the asset price itself. In particular, in the limit as $n \to \infty$, the unconditional pricing kernel and the asset price are lognormally distributed, as in Rubinstein (1976) and Brennan (1979).

Although for a finite binomial process with n sub-periods, there exists only a finite number of S_T values, we can think of a large n so that, approximately, S_T may be considered a variable which is continuous on the range $(0,\infty)$. Then differentiating equation (24) with respect to $\ln S_T$ yields the elasticity of the pricing kernel,

$$\frac{\partial\ln\phi_{t,T}}{\partial\ln S_T} = -\eta_{t,T} = \beta \tag{25}$$

Hence, a stationary multiplicative binomial process of $F_{\tau,T}$ implies a constant and state-independent elasticity of the pricing kernel. This binomial example illustrates the result in Theorem 2, where a geometric Brownian motion for the asset forward price was shown to imply a constant, non-state-dependent elasticity of the pricing kernel. Here, starting with a multiplicative binomial distribution for the forward price, we have also shown that the pricing kernel is perfectly correlated with the asset price and has constant, non-state-dependent elasticity.[7] In the limit, both the asset price and the pricing kernel are lognormal and the Black-Scholes model holds for European-style claims on the asset.

[7]The results here relate closely to those in Stapleton and Subrahmanyam (1984b). They showed that, if the forward price is multiplicative binomial, a risk-neutral valuation relationship holds for the valuation of options on the asset, if the utility function of the representative agent is a power function. The results here are analogous to those, but in a multi-period setting.

5 Declining Elasticity and Excess Volatility

So far, we have shown, in section 3, that options have higher prices when the pricing kernel has declining elasticity than when it has constant elasticity. We have then shown, assuming that the information process follows a geometric Brownian motion, that the asset forward price follows a geometric Brownian motion if and only if the pricing kernel has constant elasticity. It remains to be shown exactly how declining elasticity affects the forward price process. We now derive the implications for the forward price process, of relaxing the assumption of constant elasticity of the pricing kernel. We show in the case of declining elasticity, that the variance of the forward price, $var_t(F_{\tau,T})$ increases relative to the constant elasticity case, and also that returns exhibit negative autocorrelation.[8]

Theorem 3 *Consider an economy for dates $\tau \in [t,T]$. Assume that the information process for the asset price at date T follows a geometric Brownian motion. Let $F_{\tau,T,1}$ and $F_{\tau,T,2}$ be the forward prices of the asset, at time τ, under the constant and declining elasticity pricing kernels respectively. Then,*

a) across states, the ratio of the two prices $F_{\tau,T,2}/F_{\tau,T,1}$ increases monotonically in $F_{\tau,T,1}$ $\forall \tau \in (t,T)$,

b) there exists a $F^*_{\tau,T,1}$, such that

$$\begin{array}{lllll} F_{\tau,T,2} & < & [=] & [>] & F_{\tau,T,1} & \text{if} \\ F_{\tau,T,1} & < & [=] & [>] & F^*_{\tau,T,1} & \forall \tau \in (t,T), \end{array}$$

c) the variance of the forward price is higher under the declining elasticity pricing kernel,

$$var_t(F_{\tau,T,2}) > var_t(F_{\tau,T,1}) \qquad \forall \tau \in (t,T).$$

d) For dates $\tau = t_1, t_2, ..., t_j, ..., T$, the price relatives $(F_{t_j,T,2}/F_{t_{j-1},T,2})$ exhibit negative autocorrelation.

Proof:

[8] In the case of increasing elasticity of the pricing kernel, the variance declines relative to the constant elasticity case, although the returns exhibit negative autocorrelation in this case also.

a) Constant elasticity and the lognormality of S_T imply that the expected return $E_\tau(S_T/F_{\tau,T,1})$ is independent of the state at time τ. In the case of declining elasticity, the rate of return in the high states (high $F_{\tau,T,1}$), is relatively low and the rate of return in the low states, is relatively high, compared to the constant elasticity case. Therefore, the forward price at time τ, $F_{\tau,T,2}$, for the declining elasticity case is relatively higher than $F_{\tau,T,1}$ in the high states and relatively lower in the low states. Since elasticity is monotonically declining, it follows that $F_{\tau,T,2}/F_{\tau,T,1}$ is monotonically increasing in $F_{\tau,T,1}$, $\forall \tau \in (t,T)$.

b) Given the same initial price $F^*_{t,T}$, it must be that the forward prices under the two pricing kernels do not dominate each other. Hence, given that $F_{\tau,T,1}/\ F_{\tau,T,2}$ increases monotonically in $F_{\tau,T,1}$, there can be only one value of $F_{\tau,T,1}$ where $F_{\tau,T,2} = F_{\tau,T,1}$. In other words, there is a $F^*_{\tau,T,1}$, such that $F_{\tau,T,2} = F_{\tau,T,1} = F^*_{\tau,T,1}$, and the result b) follows.

c) From a) and b), it follows that

$$F_{\tau,T,2} = F_{\tau,T,1} + E[F_{\tau,T,2} - F_{\tau,T,1}] + \epsilon \tag{26}$$

where $E(\epsilon) = 0$ and $\mathrm{cov}(\epsilon, F_{\tau,T,1}) > 0$ since $F_{\tau,T,2}$ gets larger relative to $F_{\tau,T,1}$ as $F_{\tau,T,1}$ increases. Hence,

$$var(F_{\tau,T,2}) = var(F_{\tau,T,1}) + var(\epsilon) + 2\mathrm{cov}(\epsilon, F_{\tau,T,1}) > var(F_{\tau,T,1}). \tag{27}$$

d) For the constant elasticity pricing kernel, the autocorrelation of returns is zero, since the forward price process is generated by a geometric Brownian motion. Now, assume non-constant elasticity of the pricing kernel. Consider dates t, t_1, T and the price relatives $F_{t_1,T}/F_{t,T}$ and $S_T/F_{t_1,T}$. If the price relative in the period $[t, t_1]$ is lower [higher] under non-constant elasticity, then the conditional expected price relative in the period $[t_1, T]$ must be higher [lower] implying negative autocorrelation. Second, we split the period $[t_1, T]$ into subperiods $[t_1, t_2]$ and $[t_2, T]$. By the same argument as before, given some state at t_1, the price relatives $F_{t_2,T}/F_{t_1,T}$ and $S_T/F_{t_2,T}$ must be negatively autocorrelated under non-constant elasticity. Similarly, the period $[t_2, T]$ can be split sequentially into arbitrarily many subperiods so that, by induction, negative autocorrelation of the price relatives is obtained for any number of subperiods. □

Theorem 3 shows that a geometric random walk for the forward price is ruled out by declining elasticity. Moreover, the forward price at any intermediate date is more volatile under the declining elasticity than under the constant elasticity pricing kernel.

6 Option Pricing and the Elasticity of the Pricing Kernel in a Representative Agent Economy

The analysis of option prices using the pricing kernel approach in a no-arbitrage setting is quite general. However, it is useful to relate the analysis to an equilibrium setting in order to interpret the pricing kernel in economic terms. For example, what kind of equilibrium would lead to pricing kernels with constant or declining elasticity? What restrictions on preferences would lead to such pricing kernels? In order to answer these questions, we now make the more traditional assumption of a representative investor economy, where the agent has utility for end of period consumption. The analysis below provides a set of restrictive, sufficient conditions, under which the pricing kernel for an asset has the characteristics assumed in previous sections of the paper.

We now assume that aggregate end-of-period consumption, C_T, and the spot price of the asset on the terminal date T, S_T, have a constant elasticity with respect to each other, but with an error.[9] In other words the two variables are log-linearly related with an independent error term as follows.[10]

$$\ln C_T = a + b \ln S_T + \epsilon, \tag{28}$$

where ϵ is independent of S_T. A special case is analysed by Rubinstein (1976), and Brennan (1979), who show that the Black-Scholes model holds in a single-period discrete-time economy where a representative investor exists with a utility that exhibits constant relative risk aversion, and where aggregate wealth is lognormally distributed.[11] The Rubinstein-Brennan assumptions imply a pricing kernel with constant elasticity. Now denoting the utility function of the representative investor as $u(C_T)$ we can establish:

[9] C_T can be literally interpreted as aggregate consumption or as aggregate wealth in a single period setting. More generally, it can be thought of as a state variable which is the argument in the pricing kernel function.

[10] We do not assume here that either the asset price or aggregate consumption is lognormally distributed. Joint lognormality of the variables is sufficient, but not necessary for the log-linear relationship to hold.

[11] Following up on a result in Merton (1973), Rubinstein (1976) and Brennan (1979), showed that the Black-Scholes model holds under these assumptions. Brennan shows that the constant relative risk aversion assumption is also a necessary condition. Stapleton and Subrahmanyam [(1984a) and (1984b)] and Heston (1993) have extended this work in various directions.

Theorem 4 *(Elasticity of the Pricing Kernel in a Representative Investor Economy)*

Consider an economy in which consumption takes place at time T. Assume that an asset with price S_T and aggregate consumption C_T are log-linearly related as in equation (28) above and that a representative investor exists with relative risk aversion $R(C_T)$. Then, at any date t, the pricing kernel, $\phi_{t,T}(S_T)$ for the asset has elasticity

$$\eta_t(S_T) = b\hat{R}_t(S_T)$$

where

$$\hat{R}_t(S_T) = E_t\left[R(C_T)\frac{u'(C_T)}{E_t[u'(C_T) \mid S_T]} \mid S_T\right]$$

Proof:

In a representative investor economy the pricing kernel is

$$\psi_{t,T} = \frac{u'(C_T)}{E_t[u'(C_T)]}.$$

Since t and T are fixed, we denote the pricing kernel as ψ, where $\psi = \psi(C_T), C_T > 0$. The asset-specific pricing kernel is

$$\phi = E_t[\psi \mid S_T]$$

where we can write $\phi = \phi(S_T)$.

The elasticity of the asset specific pricing kernel, ϕ, is $\eta = -\partial\text{ln}\phi/\partial\text{ln}S_T$. Using (28), and the fact that the partial derivative, $\partial\psi/\partial S_T = 0$, we have

$$\eta = \frac{-E\left[u''(C_T)C_T b \mid S_T\right]}{E\left[u'(C_T) \mid S_T\right]},$$

where for notational convenience we write $E_t(.)$ as $E(.)$. Hence, we can write

$$\eta = b\frac{E\left[R(C_T)u'(C_T) \mid S_T\right]}{E\left[u'(C_T) \mid S_T\right]} = b\hat{R}(S_T)$$

where

$$\hat{R}(S_T) = E\left[R(C_T)\frac{u'(C_T)}{E[u'(C_T) \mid S_T]} \mid S_T\right]$$

is the representative agent's asset specific relative risk aversion. □

Corollary 3 *(Declining Elasticity)*

Suppose that relative risk aversion of the representative investor, $R(C_T)$, is declining in C_T, then if $b \neq 0$, the elasticity of the asset specific pricing kernel, $\phi_{t,T}(S_T)$, declines in S_T.

Proof

See Appendix C.

The significance of Theorem 4 and Corollary 3 is as follows. In a representative investor economy, the elasticity of the pricing kernel is closely related to the relative risk aversion of the investor. However, for a specific asset, the elasticity depends on a 'risk adjusted' relative risk aversion, which accounts for the risk of aggregate consumption, given the asset price. In Corollary 3, we find that this risk adjusted relative risk aversion declines with the asset price, if the actual relative risk aversion declines with aggregate consumption.

The results in Theorem 4 and Corollary 3 allow us to generalize the conclusions of Brennan (1979) and Rubinstein (1976). They showed that the Black-Scholes formula priced European-style options if the asset price and aggregate consumption are joint-lognormally distributed and if a representative investor exists with CPRA utility. In our Theorem 4, we first show that lognormality of aggregate consumption is not required. In fact, if we have CPRA and the log-linear relationship between the asset price and consumption in equation (28), then the pricing kernel will have constant elasticity, and from Theorem 2, the Black-Scholes model will hold. Furthermore, if the representative investor has declining proportional risk aversion (DPRA), then this will translate into a declining elasticity pricing kernel. The result, in that case, is that all options on the asset have higher prices than those given by the Black-Scholes model.

7 Conclusions and Extensions

We have derived the main implications, for the asset price process and for option prices, of declining elasticity of the pricing kernel. Firstly, under declining elasticity, options have higher prices than under the more familiar assumption of constant elasticity. Secondly, in the special case where the information process of the asset price follows a geometric Brownian motion, the Black-Scholes model underprices European-style options. Also, given the terminal probability distribution of the asset price, the stochastic process of the asset forward price has higher volatility and exhibits negative autocorrelation under declining elasticity. Thirdly, declining elasticity is consistent, in a representative investor economy, with declining proportional risk aversion of the representative investor.

The model in which the asset (forward) price follows a geometric Brownian motion is one of the standard work-horses of finance. It has been useful in deriving many empirically testable propositions, but its characteristics and valuation implications are not always in line with the empirical evidence. Examples of such empirical anomalies include the high volatility of stock returns, their autocorrelation and the underpricing of contingent claims. The question, therefore, is whether the implicit assumption of constant elasticity of the pricing kernel can be modified for the resultant models to better fit the data. An alternative proposed and analyzed in this paper is to assume a pricing kernel that exhibits declining elasticity with respect to the payoff on the asset. This model could help explain a number of empirical anomalies relating to the return generating process and the pricing of contingent claims.

Several other directions of research can be pursued, based on the research reported in this paper. First, the properties of the pricing kernel that lead to a broader class of stochastic processes for returns than the standard geometric Brownian motion could be explored. These properties could be tested directly to assess their empirical validity as has been proposed in the literature on the term structure of interest rates. Second, the further implications of declining elasticity of the pricing kernel for option pricing, such as for the "smile" effect, that relates implied volatilities to strike prices, could be explored further. This could, in turn, provide a better theoretical justification for recent work on fitting binomial trees using observed option prices.

Appendix A: Proof that more than two intersections of the pricing kernel cannot exist

Suppose there are three or more intersections of the two pricing kernels. Consider the first three intersections at forward prices S_T^A, S_T^B and S_T^C respectively. Suppose that at S_T^A, ϕ_2 intersects ϕ_1 from above, i.e.,

$$-\frac{\partial \phi_1(S_T^A)}{\partial S_T} < -\frac{\partial \phi_2(S_T^A)}{\partial S_T}$$

Since, at the first intersection,

$$\phi_1(S_T^A) = \phi_2(S_T^A)$$

it follows that

$$\begin{aligned} \eta_1(S_T^A) &= -\frac{\partial \phi_1(S_T^A)}{\partial S_T} \cdot \frac{S_T^A}{\phi_1(S_T^A)} \\ < \eta_2(S_T^A) &= -\frac{\partial \phi_2(S_T^A)}{\partial S_T} \cdot \frac{S_T^A}{\phi_2(S_T^A)} \end{aligned} \tag{29}$$

Similarly at S_T^B, ϕ_2 intersects ϕ_1 from below, it follows that

$$\eta_1(S_T^B) > \eta_2(S_T^B) \tag{30}$$

Again, at S_T^C, since ϕ_2 intersects ϕ_1 from above, we must have

$$\eta_1(S_T^C) < \eta_2(S_T^C) \tag{31}$$

However, this would contradict inequality (8). Thus, three or more intersections of the two pricing kernels are not possible. In conclusion, the two pricing kernels must intersect twice and, in order to satisfy $\eta_2' \leq 0$, ϕ_2 must intersect ϕ_1 from above at the first intersection, S_T^A, and from below at the second intersection, S_T^B $\square$

Appendix B: Proof that $E_\tau(dF_\tau\theta_{\tau+d\tau}) = 0$

To establish this result, we will first consider the discrete quantity $E_\tau(\Delta F_\tau \theta_{\tau+\Delta\tau})$, where $F_{\tau+\Delta\tau} = F_\tau + \Delta F_\tau$, and then take the limit. Consider the quantity

$$\begin{aligned}
E_\tau(F_{\tau+\Delta\tau}\theta_{\tau+\Delta\tau}) &= E_\tau[F_{\tau+\Delta\tau}E_{\tau+\Delta\tau}(\phi_{t,T})] \\
&= E_\tau[E_{\tau+\Delta\tau}(S_T\phi_{\tau+\Delta\tau,T})E_{\tau+\Delta\tau}(\phi_{t,T})] \\
&= E_\tau[E_{\tau+\Delta\tau}(S_T\phi_{\tau+\Delta\tau,T})\phi_{t,\tau}\phi_{\tau,\tau+\Delta\tau}] \\
&= E_\tau[S_T\phi_{\tau+\Delta\tau,T}\phi_{t,\tau}\phi_{\tau,\tau+\Delta\tau}] \\
&= E_\tau[S_T\phi_{\tau,T}]\phi_{t,\tau} \\
&= F_\tau\phi_{t,\tau} \qquad (32)
\end{aligned}$$

By definition

$$F_{\tau+\Delta\tau} = F_\tau + \Delta F_\tau,$$

hence

$$\begin{aligned}
E_\tau[F_{\tau+\Delta\tau}\theta_{\tau+\Delta\tau}] &= E_\tau[F_\tau\theta_{\tau+\Delta\tau}] + E_\tau[\Delta F_\tau\theta_{\tau+\Delta\tau}] \\
&= E_\tau[F_\tau E_{\tau+\Delta\tau}(\phi_{t,T})] + E_\tau[\Delta F_\tau\theta_{\tau+\Delta\tau}] \\
&= F_\tau E_\tau(\phi_{t,T}) + E_\tau[\Delta F_\tau\theta_{\tau+\Delta\tau}] \\
&= F_\tau\phi_{t,\tau} + E_\tau[\Delta F_\tau\theta_{\tau+\Delta\tau}] \qquad (33)
\end{aligned}$$

Combining (32) and (33), it follows that

$$E_\tau[\Delta F_\tau\theta_{\tau+\Delta\tau}] = 0.$$

Hence, taking limits,

$$\lim_{\Delta\tau\to 0} E_\tau[\Delta F_\tau\theta_{\tau+\Delta\tau}] = E_\tau(dF_\tau\theta_{\tau+d\tau}) = 0.$$

□

Appendix C: Proof of Corollary 3

From Theorem 4, the elasticity of the pricing kernel is

$$\eta = bE\left[R(C_T)\frac{u'(C_T)}{E\left[u'(C_T) \mid S_T\right]} \mid S_T\right].$$

Hence, for $b \neq 0$,

$$\begin{aligned}\frac{1}{b^2}\frac{\partial\eta}{\partial \ln S_T} &= E\left[R'(C_T)C_T\frac{u'(C_T)}{E\left[u'(C_T) \mid S_T\right]} \mid S_T\right] - \frac{E\{R(C_T)^2 u'(C_T)E[u'(C_T) \mid S_T]\}}{E[u'(C_T) \mid S_T]^2} \\ &+ \frac{E[R(C_T)u'(C_T) \mid S_T]^2}{E[u'(C_T) \mid S_T]^2}\end{aligned}$$

and therefore

$$\frac{1}{b^2}\frac{\partial\eta}{\partial \ln S_T} = E[R'(C_T)C_T\frac{u'(C_T)}{E[u'(C_T) \mid S_T]} \mid S_T] - \frac{E[R(C_T)u'(C_T)\{R(C_T)-\hat{R}(S_T)\} \mid S_T]}{E[u'(C_T) \mid S_T]}. \tag{34}$$

As $E[u'(C_T)\{R(C_T)-\hat{R}(S_T)\} \mid S_T] = 0$, we can expand the second term to

$$-\frac{E[\{R(C_T)-\hat{R}(S_T)\}u'(C_T)\{R(C_T)-\hat{R}(S_T)\} \mid S_T]}{E[u'(C_T) \mid S_T]} < 0. \tag{35}$$

$R'(C_T) \leq 0$ means that the first term is negative. We have shown that the second term in (34) is also negative. Therefore, η declines in $\ln S_T$. □

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